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# Conservative solutions to a system of asymptotic variational wave equations ☆

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ABSTRACT

This paper is concerned with an asymptotic variational wave system which models weakly nonlinear waves for a system of variational wave equations arising in the theory of nematic liquid crystals and a few other physical contexts. By constructing a global semigroup, we establish the well-posedness of the initial–boundary value problem within the class of energy-conservative solutions for initial data of finite energy.



## 1. Introduction

In this paper, we investigate the problem described by the system

$$u_{kt} + f(u)u_{kx} = \frac{1}{2}\int_0^x f_{u_k}\sum_{i=1}^n u_{ix}^2\,\mathrm{d}x, \quad (k = 1, 2, \ldots, n) \tag{1.1}$$

for all $t \geq 0$, $x \geq 0$, with the initial–boundary value conditions

$$u(0, x) = u_0(x), \qquad u(t, 0) = \mathbf{0}, \tag{1.2}$$

where $u = (u_1, \ldots, u_n)$ is the unknown vector function defined for $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^+$, $u_0(x) = (u_{10}, \ldots, u_{n0})(x)$, $f : \mathbb{R}^n \mapsto \mathbb{R}$ is a smooth function satisfying

$$f(\mathbf{0}) \geq 0, \qquad |\nabla f(u) - \nabla f(v)| \leq L|u - v|, \quad \forall u, v \in \mathbb{R}^n \tag{1.3}$$

for a constant $L$.

System (1.1) can be derived from the following variational wave equations:

$$\psi_{ktt} - c(\psi)[c(\psi)\psi_{kx}]_x = c(\psi)\sum_{i=1}^n (c_{\psi_i}\psi_{kx} - c_{\psi_k}\psi_{ix})\psi_{ix}, \quad (k = 1, 2, \ldots, n), \tag{1.4}$$

which are the Euler–Lagrange equations of a variational principle arising in the theory of nematic liquid crystals; see [1,7,8]. As in [8], we look for a weakly nonlinear asymptotic solution of (1.4) of the form

$$\psi(t, x) = \psi_0 + \varepsilon u(\varepsilon t, x - c(\psi_0)t) + O(\varepsilon^2),$$

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where $\psi = (\psi_1, \dots, \psi_n)$, $\psi_0$ is a constant vector and $c(\psi_0) > 0$ is the unperturbed wave speed. Then the vector function $u(\cdot, \cdot)$ satisfies

$$\left(u_{kt} + \left(\sum_{i=1}^{n} c_{u_i}(u_0)u_i\right) u_{kx}\right)_x = \frac{1}{2} c_{u_k}(u_0) \sum_{i=1}^{n} u_{ix}^2 \quad (k = 1, 2, \dots, n) \tag{1.5}$$

up to a scaling and reflection of the independent variables, assuming that $\psi_0$ is such that $\sum_{i=1}^{n} |c_{u_i}(\psi_0)| \neq 0$. Integrating system (1.5) with respect to $x$ gives a special case of (1.1).

The case $n = 1$ for (1.5) yields the well-known Hunter–Saxton equation, which has been widely studied by many authors since it was introduced [8]. It possesses a number of nice properties [9,17] and also has an interesting geometric interpretation [11,12]. Since smooth solutions may not exist globally in time [8], it becomes necessary to consider the global existence of weak solutions. There are at least two natural distinct classes of admissible weak solutions, which are called dissipative and conservative solutions [10]. The dissipative solution loses all the energy while the conservative solution preserves its energy at the blow-up time. The existence of dissipative solutions and conservative solutions to the initial–boundary value problem of the Hunter–Saxton equation are presented among others in [2,3,13,18–21].

In [4], Bressan, Zhang and Zheng established the well-posedness of the initial–boundary value problem to the case $n = 1$ of (1.1) for initial data of finite energy. Moreover, they found that the dissipative solutions may not depend continuously on the initial data when $f$ is non-convex.

Recently, the two-component Hunter–Saxton system

$$\begin{cases} u_{txx} + uu_{xxx} + 2u_x u_{xx} = \rho\rho_x, \\ \rho_t + (\rho u)_x = 0, \end{cases} \tag{1.6}$$

arising from the two-component Camassa–Holm equation [5,6], has attracted much attention; see for example [14–16]. We notice that system (1.6) is a particular case of (1.1) for $n = 2$.

The purpose of the present paper is to establish the global well-posedness of the problem (1.1)–(1.3) for conservative solutions. We use the method used in [4] to construct a global semigroup for conservative solutions to the problem. The uniqueness result follows directly from the constructive procedure. The global existence of dissipative solutions to system (1.1) will be addressed in a forthcoming paper.

We present the main theorem of this paper, Theorem 2.1, in Section 2. Section 3 is devoted to proving this theorem.

## 2. The main theorem

Before we state our main results, let us first recall the definition of solutions introduced by Bressan, Zhang and Zheng [4].

**Definition 1.** A vector function $u(t, x)$, defined on $[0, T] \times \mathbb{R}^+$, is a solution of problem (1.1)–(1.3) if, for $k = 1, 2, \dots, n$, the following hold.

(i) The function $u_k$ is locally Hölder continuous with respect to both $t$ and $x$. The initial and boundary conditions (1.2) hold pointwise. For each time $t$, the map $x \mapsto u_k(t, x)$ is absolutely continuous with $u_{kx}(t, \cdot) \in L^2_{\mathrm{loc}}(\mathbb{R}^+)$.

(ii) For any $M > 0$, the map $t \mapsto u_k(t, \cdot) \in L^2([0, M])$ is absolutely continuous and satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} u_k(t, \cdot) = -f(u)u_{kx} + \frac{1}{2}\int_0^{*} f_{u_k}(u) \sum_{i=1}^{n} u_{ix}^2 \,\mathrm{d}x \tag{2.1}$$

for a.e. $t \in [0, T]$. Here equality is understood in the sense of functions in $L^2([0, M])$.

We notice here that there is no need to consider weak solutions in the distributional sense by the local integrability assumptions $u_k(t, \cdot) \in L^2_{\mathrm{loc}}(\mathbb{R}^+)$ $(k = 1, 2, \dots, n)$.

For each smooth solution, we can easily check that it satisfies

$$\left(\sum_{i=1}^{n} u_{ix}^2\right)_t + \left(f(u) \sum_{i=1}^{n} u_{ix}^2\right)_x = 0,$$

which implies that the existence of energy-conservative solutions is possible. We say that a solution $u = u(t, x)$ is conservative if the family of absolutely continuous measures $\{\mu_{(t)}; t \geq 0\}$ defined by $\mathrm{d}\mu_{(t)} = \sum_{i=1}^{n} u_{ix}^2(t)\mathrm{d}x$ provides a measure-valued solution to

$$\omega_t + [f(u)\omega]_x = 0, \tag{2.2}$$

that is, for every $t_2 \geq t_1 \geq 0$ and any non-negative function $\phi \in C_c^1$, there holds

$$\int \phi(t, \cdot)\, \mathrm{d}\mu_{(t)}\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \left\{ \int \left(\phi_t(t, \cdot) + \phi_x(t, \cdot) f(u(t, \cdot))\right) \mathrm{d}\mu_{(t)} \right\} \mathrm{d}t. \tag{2.3}$$

We define an evolution semigroup on a domain $\mathcal{D}$ defined as follows. An element of $\mathcal{D}$ is a couple $(u, \mu)$, where $u : \mathbb{R}^+ \mapsto \mathbb{R}^n$ is a continuous vector function with $u(0) = \mathbf{0}$ and whose distributional derivative $u_x$ lies in $(L^2)^n$; while $\mu = \mu^a + \mu^s$, the sum of the absolutely continuous and the singular part (with respect to the Lebesgue measure), is a bounded non-negative Radon measure on $\mathbb{R}^+$, whose absolutely continuous part $\mu^a$ satisfies

$$\mathrm{d}\mu^a = \sum_{i=1}^n u_{ix}^2 \,\mathrm{d}x. \tag{2.4}$$

We consider the distance of the Radon measures on $\mathbb{R}^+$ as

$$\mathrm{d}(\mu, \tilde{\mu}) := \sup_{\varphi} \left| \int \varphi \,\mathrm{d}\mu - \int \varphi \,\mathrm{d}\tilde{\mu} \right|, \tag{2.5}$$

for all smooth functions $\varphi$ with $|\varphi| \le 1$, $|\varphi_x| \le 1$.

Recall that a semigroup $\mathcal{S}$ on a domain $\mathcal{D}$ is a map $\mathcal{S} : \mathcal{D} \times [0, +\infty) \mapsto \mathcal{D}$ such that $\mathcal{S}_0 w = w$ and $\mathcal{S}_s(\mathcal{S}_t w) = \mathcal{S}_{s+t} w$ for every $s, t \ge 0$ and $w \in \mathcal{D}$. The main results can be stated as follows.

**Theorem 2.1.** *Assume that $u_0(0) = \mathbf{0}$ and $u_{0x} \in (L^2(\mathbb{R}^+))^n$. Then there exists a semigroup $\mathcal{S} : \mathcal{D} \times [0, +\infty) \mapsto \mathcal{D}$ with the following properties. Denote by $t \mapsto (u(t), \mu_{(t)}) := S_t(u_0, \mu_{(0)})$ the trajectory corresponding to the initial data $(u_0, \mu_{(0)}) \in \mathcal{D}$. Then*

(i) *The functions $u_k(t, x)$ $(k = 1, 2, \ldots, n)$ are locally Hölder continuous in $\mathbb{R}^+ \times \mathbb{R}^+$. The vector function $u = (u_1, \ldots, u_n)$ provides a solution of (1.1)–(1.3) in the sense of Definition 1.*

(ii) *The assignment $t \mapsto \mu_{(t)}$ provides a measure valued solution to the linear transport equation*

$$\omega_t + [f(u)\omega]_x = 0, \qquad \omega(0) = \mu_{(0)}.$$

*Moreover, for $k = 1, 2, \ldots, n$, the singular part of the measure $f_{u_k}(u(t)) \cdot \mu_{(t)}$ vanishes at almost every time $t \ge 0$, i.e.,*

$$\nabla f(u(t)) \cdot \mu^s_{(t)} = \mathbf{0}, \quad a.e.\ t. \tag{2.6}$$

(iii) (Temporal continuity) *For every $M > 0$, the couple $(u, \mu)$ satisfies the Lipschitz continuity property:*

$$\int_0^M \sum_{i=1}^n |u_i(t, x) - u_i(s, x)| \,\mathrm{d}x \le C|t - s|, \tag{2.7}$$

$$d(\mu_{(t)}, \mu_{(s)}) \le C|t - s|,$$

*where the constant $C$ depends only on $M, f$ and $\mu_{(0)}(\mathbb{R}^+) < \infty$.*

(iv) (Continuous dependence) *Assume that a sequence of initial conditions $(u_0^\nu, \mu^\nu_{(0)}) \in \mathcal{D}$ satisfy*

$$u_0^\nu \to u_0 \quad \text{uniformly on bounded sets},$$
$$d(\mu^\nu_{(0)}, \mu_{(0)}) \to 0$$

*for some $(u_0, \mu_{(0)}) \in \mathcal{D}$, as $\nu \to +\infty$. Then the corresponding solutions satisfy*

$$u^\nu(t, x) \to u(t, x) \quad \text{uniformly for } t, x \text{ in bounded sets},$$
$$d(\mu^\nu_{(t)}, \mu_{(t)}) \to 0, \quad \text{for every } t > 0.$$

## 3. Proof of Theorem 2.1

This section is devoted to the Proof of Theorem 2.1. We assume without loss of generality that $\mu_{(0)}$ has compact support, namely $u_0$ is a constant vector when $x > R$ for some constant $R > 0$. The general case follows directly from an approximation argument. We divide the proof into several steps.

### *3.1. Construction of the trajectory*

Let an initial data $(u_0, \mu_{(0)}) \in \mathcal{D}$ be given and denote $\xi_0 := \mu_{(0)}(\mathbb{R}^+) < \infty$. We construct a vector function $U = (U_1, \ldots, U_n)(t, \xi)$ on the semi-infinite strip $\{t \ge 0, \xi \in [0, \xi_0]\}$ by first setting

$$U(0, \xi) = U_0(\xi) := u_0(y_0(\xi)), \tag{3.1}$$

where

$$y_0(\xi) := \inf\{x \ge 0;\ \mu_{(0)}([0, x]) \ge \xi\}, \tag{3.2}$$

for $0 < \xi \le \xi_0$, while

$$y_0(0) = \sup\{x;\ \mu_{(0)}([0, x]) = 0\}, \tag{3.3}$$

which implies that $y_0(\xi)$ is continuous at the point $\xi = 0$. We comment that the vector function $\xi \mapsto u_0(y_0(\xi))$ is always continuous, despite the fact that the map $\xi \mapsto y_0(\xi)$ may have upward jumps.

For $t > 0$, we define $U$ as the solution of

$$\frac{\partial}{\partial t}U(t,\xi) = \frac{1}{2}\int_0^{\xi} \nabla f(U(t,\eta))\,\mathrm{d}\eta \tag{3.4}$$

with initial data (3.1). The Lipschitz continuity of vector function $\nabla f$ indicates that a unique solution $U$ of the above system can be constructed though the fixed point theorem. In fact, consider the space of continuous functions $(C([0,\infty)\times[0,\xi_0]))^n$ with the equivalent weighted norm

$$\|U\|_* := \sup_{t\geq 0,\xi\in[0,\xi_0]} e^{-L\xi_0 t}|U(t,\xi)|. \tag{3.5}$$

Define the transformation $U \mapsto \mathcal{T}U$ as

$$\mathcal{T}U := u_0(y_0(\xi)) + \frac{1}{2}\int_0^t\int_0^{\xi} \nabla f(U(s,\eta))\,\mathrm{d}\eta\mathrm{d}s. \tag{3.6}$$

We thus find by (1.3) that

$$\begin{aligned}|(\mathcal{T}U-\mathcal{T}V)(t,\xi)| &\leq \frac{1}{2}\int_0^t\int_0^{\xi} |(\nabla f(U)-\nabla f(V))(s,\eta)|\,d\eta\mathrm{d}s\\ &\leq \frac{1}{2}\int_0^t\int_0^{\xi} L|(U-V)(s,\eta)|\,d\eta\mathrm{d}s \leq \frac{1}{2}e^{L\xi_0 t}\delta,\end{aligned}$$

if $\|U-V\|_* = \delta$. Therefore we have

$$\|\mathcal{T}U-\mathcal{T}V\|_* \leq \frac{1}{2}\|U-V\|_*,$$

which means that the map $\mathcal{T}$ is contractive and then there exists a unique fixed point $U = U(t,\xi)$, defined on $\mathbb{R}^+\times[0,\xi_0]$.

Next we construct the characteristic curves, which are determined by the following equation:

$$\frac{\partial}{\partial t}y(t,\xi) = f(U(t,\xi)), \qquad y(0,\xi) = y_0(\xi), \tag{3.7}$$

that is,

$$y(t,\xi) = y_0(\xi) + \int_0^t f(U(\tau,\xi))\,\mathrm{d}\tau. \tag{3.8}$$

Now we define the vector function $u = u(t,x)$ by

$$u(t,x) := U(t,\xi(t,x)) \tag{3.9}$$

for any given $(t,x)$, where

$$\xi(t,x) := \sup\{\xi;\, y(t,\xi)\leq x\},$$

and the corresponding measure $\mu_{(t)}$ at time $t$ by

$$\mu_{(t)}(J) := \text{meas}\,\{\xi\in[0,\xi_0];\, y(t,\xi)\in J\} \tag{3.10}$$

for each Borel set $J\subset\mathbb{R}^+$, that is, $\mu_{(t)}$ is defined as the push-forward of Lebesgue measure on $[0,\xi_0]$ through the mapping $\xi\mapsto y(t,\xi)$.

### *3.2. Absolute continuity*

For $k = 1,2,\ldots,n$, we assert that the map $\xi\mapsto U_k(t,\xi)$ is absolutely continuous for each time $t\geq 0$. We first consider the case $t = 0$. Let $[\xi_i,\xi_i']\subset[0,\xi_0]$, $i = 1,\ldots,N$, be disjoint intervals satisfying

$$\sum_{i=1}^{N}|\xi_i'-\xi_i| \leq \varepsilon.$$

Denote $I_1$ as the set of indices $i$ such that

$$\frac{|U_k(\xi_i')-U_k(\xi_i)|}{y(\xi_i'-y(\xi_i))} \leq \sqrt{\varepsilon}$$

and $I_2$ as the set of other indices. Then we get

$$\sum_{i\in I_1} |U_k(\xi_i') - U_k(\xi_i)| \le \sqrt{\varepsilon} \sum_{i\in I_1} |y(\xi_i') - y(\xi_i)| \le R\sqrt{\varepsilon},$$

and

$$\sum_{i\in I_2} |U_k(\xi_i') - U_k(\xi_i)| \le \frac{1}{\sqrt{\varepsilon}} \sum_{i\in I_2} \frac{|U_k(\xi_i') - U_k(\xi_i)|^2}{y(\xi_i') - y(\xi_i)} \le \frac{1}{\sqrt{\varepsilon}} \sum_{i\in I_2} \int_{y(\xi_i)}^{y(\xi_i')} u_{kx}^2\,\mathrm{d}x$$
$$\le \frac{1}{\sqrt{\varepsilon}} \sum_{i\in I_2} |\xi_i' - \xi_i| \le \sqrt{\varepsilon}.$$

Combining the two above inequalities gives

$$\sum_{i=1}^{N} |U_k(\xi_i') - U_k(\xi_i)| \le (1+R)\sqrt{\varepsilon}, \quad (k = 1, 2, \ldots, n),$$

which concludes the assertion in the case $t = 0$.

For $t > 0$, we obtain by (3.4)

$$|U_k(t,\xi') - U_k(t,\xi)| \le |U_k(0,\xi') - U_k(0,\xi)| + |\xi' - \xi| \cdot \frac{t}{2} \cdot \sup_u |f_{u_k}(u)|$$

for $k = 1, 2, \ldots, n$, from which follows the desired result.

For future reference, we here state a useful consequence, that is, the vector function $U_\xi$ exists at a.e. $(t,\xi)$ and satisfies the evolution system by (3.4)

$$\frac{\partial}{\partial t} U_\xi(t,\xi) = \frac{1}{2} \nabla f(U(t,\xi)). \tag{3.11}$$

### 3.3. Measure transformations

For convenience of later application, we first analyze the absolutely continuous and the singular part of the measure $\mu$ defined as (3.10), under a continuous non-decreasing mapping $\xi \mapsto y(\xi)$.

**Lemma 1.** *For $k = 1, 2, \ldots, n$, let $U_k : [0, \xi_0] \mapsto \mathbb{R}$ be absolutely continuous with a square integrable derivative. Assume that the map $\xi \mapsto y(\xi)$ satisfies*

$$y(\xi) = y(0) + \int_0^\xi \sum_{i=1}^n U_{ix}^2(\eta)\,\mathrm{d}\eta. \tag{3.12}$$

*Define the vector function $u = u(x)$ implicitly by*

$$u(y(\xi)) := U(\xi) \tag{3.13}$$

*for $x \in [y(0), y(\xi_0)]$ and the measure $\mu$ by*

$$\mu(J) := \mathrm{meas}\{\xi \in [0,\xi_0];\, y(\xi) \in J\}. \tag{3.14}$$

*Then the absolutely continuous and the singular part of $\mu$ with respect to the Lebesgue measure are, respectively, given by*

$$\mu^a(A) = \mathrm{meas}\left\{\xi \in [0,\xi_0];\, y(\xi) \in A, \sum_{i=1}^n |U_{i\xi}(\xi)| \ne 0\right\}, \tag{3.15}$$

$$\mu^s(A) = \mathrm{meas}\{\xi \in [0,\xi_0];\, y(\xi) \in A, U_{k\xi}(\xi) = 0\ (k = 1, 2, \ldots, n)\}. \tag{3.16}$$

*Moreover, one has*

$$\mathrm{d}\mu^a = \sum_{i=1}^n u_{ix}^2\,\mathrm{d}x \tag{3.17}$$

*on the set $[y(0), y(\xi_0)]$. Conversely, if both $U$ and $y$ are absolutely continuous and (3.13), (3.14) and (3.17) are valid, then (3.12) must hold.*

**Proof.** In view of (3.12), it is easy to see that the image of a set $I \subseteq [0,\xi_0]$ under the mapping $\xi \mapsto y(\xi)$ has the Lebesgue measure

$$\mathrm{meas}(y(I)) = \int_I \sum_{i=1}^n U_{i\xi}^2(\eta)\,\mathrm{d}\eta.$$

Clearly, the singular part of $\mu$ with respect to the Lebesgue measure is given by (3.16). In order to establish (3.15) and (3.17), we consider a measurable set $A \subset [0, \xi_0]$ satisfying

$$\sum_{i=1}^{n} U_{i\xi}^2(\xi) \geq \varepsilon \quad \text{for all } \xi \in A$$

for any fixed $\varepsilon > 0$. Then we have

$$\begin{aligned}\int_{y(A)} \sum_{i=1}^{n} u_{ix}^2(x)\,\mathrm{d}x &= \int_A \sum_{i=1}^{n} \left(U_{i\xi} \cdot \frac{\mathrm{d}\xi}{\mathrm{d}y}\right)^2 \cdot \frac{\mathrm{d}y}{\mathrm{d}\xi}\,\mathrm{d}\xi \\ &= \int_A \sum_{i=1}^{n} \left(U_{i\xi} \cdot \frac{1}{\sum_{j=1}^{n} U_{j\xi}^2}\right)^2 \cdot \sum_{j=1}^{n} U_{j\xi}^2\,\mathrm{d}\xi = \mathrm{meas}(A),\end{aligned}$$

which concludes (3.15) and (3.17) by the arbitrariness of $\varepsilon$.

We now prove the last part of the lemma. Denote

$$I_\varepsilon := \{\xi \in [0, \xi_0];\, y_\xi(\xi) \geq \varepsilon\}.$$

For $\xi \in I_\varepsilon$, we get the chain rule due to (3.13)

$$u_{kx}(y(\xi))y_\xi(\xi) = U_{k\xi}(\xi), \quad (k = 1, 2, \ldots, n). \tag{3.18}$$

On the other hand, we find for $0 < a < b < \xi_0$ that

$$\mathrm{meas}([a, b] \cap I_\varepsilon) = \int_{[y(a),y(b)]\cap y(I_\varepsilon)} \sum_{i=1}^{n} u_{ix}^2(x)\,\mathrm{d}x = \int_{[a,b]\cap I_\varepsilon} \sum_{i=1}^{n} u_{ix}^2(y(\xi))y_\xi(\xi)\,\mathrm{d}\xi, \tag{3.19}$$

which implies, by the arbitrariness of $a < b$, that for $\xi \in I_\varepsilon$

$$y_\xi(\xi) = \frac{1}{\sum_{i=1}^{n} u_{ix}^2(y(\xi))}. \tag{3.20}$$

Combining (3.18) and (3.20) leads to

$$y_\xi(\xi) = \sum_{i=1}^{n} U_{i\xi}^2(\xi), \qquad u_{kx}(y(\xi)) = \frac{U_{k\xi}(\xi)}{\sum_{i=1}^{n} U_{i\xi}^2(\xi)}, \quad (k = 1, 2, \ldots, n) \tag{3.21}$$

for all $\xi \in I_\varepsilon$. Therefore, we obtain

$$\begin{aligned} y(\xi) &= y(0) + \int_0^\xi y_\xi(\eta)\,\mathrm{d}\eta = y(0) + \lim_{\varepsilon\to 0} \int_{[0,\xi]\cap I_\varepsilon} y_\xi(\eta)\,\mathrm{d}\eta \\ &= y(0) + \int_0^\xi \sum_{i=1}^{n} U_{i\xi}^2(\eta)\,\mathrm{d}\eta.\end{aligned}$$

The proof of Lemma 1 is complete. $\square$

### 3.4. *Existence of solutions*

We now use Lemma 1 to prove that the trajectory $t \mapsto (u(t, \cdot), \mu_{(t)})$ satisfies system (1.1) and the initial–boundary conditions (1.2).

We first assume that the map $\xi \mapsto y_0(\xi)$ is absolutely continuous. This assumption will be removed later.

For each $t \geq 0$ and $\xi \in [0, \xi_0]$, we define

$$y(t, \xi) := y(t, 0) + \int_0^\xi \sum_{i=1}^{n} U_{i\xi}^2(t, \eta)\,\mathrm{d}\eta, \tag{3.22}$$

where $y(t,0) = y_0(0) + tf(\mathbf{0})$. Recalling (3.11), we find that

$$\frac{\partial}{\partial t} y_\xi(t,\xi) = \frac{\partial}{\partial t} \sum_{i=1}^{n} U_{i\xi}^2(t,\xi) = U_\xi(t,\xi) \cdot \nabla f(U(t,\xi)), \tag{3.23}$$

which, together with (3.4), (3.7) and the last assertion of Lemma 1, implies that the function $y = y(t,\xi)$ defined at (3.22) coincides with the one defined at (3.8).

It is easily seen that the initial–boundary conditions (1.2) hold by the definitions (3.1)–(3.2) and the construction (3.4). Hence, it remains to prove that the vector function $u$ satisfies (1.1). Fix a time $t > 0$; it suffices to check it for $x \in [y(t,0), y(t,\xi_0)]$ by the fact that $u(t,x) \equiv 0$ in $x \in [0, y(t,0)]$. Notice that there exists a unique $\xi \in [0,\xi_0]$ such that $x = y(t,\xi)$, for almost every $x \in [y(t,0), y(t,\xi_0)]$. Then we compute

$$\begin{aligned} u_t + f(u)u_x &= U_t(t,\xi) = \frac{1}{2}\int_0^\xi \nabla f(U(t,\eta))\,\mathrm{d}\eta \\ &= \frac{1}{2}\int_0^{y(t,\xi)} \nabla f(u(t,\cdot))\,\mathrm{d}\mu_{(t)}. \end{aligned}$$

Comparing the above with (1.1), we see that it only needs to establish the identity of measures

$$\nabla f(u) \sum_{i=1}^{n} u_{ix}^2 \mathrm{d}x = \nabla f(u) \mathrm{d}\mu_{(t)} \tag{3.24}$$

for almost every time $t \geq 0$. We notice that the open region

$$\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^+;\ \nabla f(u(t,x)) \neq 0\}$$

can be covered by countably many sets of the form

$$\Gamma := \{(t,x);\ t \in [t_1,t_2], x \in [y(t,a), y(t,b)]\}.$$

Thus it suffices to prove the following statement: assume that there exists $\hat{k} \in \{1,2,\ldots,n\}$ such that

$$f_{u_{\hat{k}}}(u(t,x)) > \delta > 0, \quad (t,x) \in \Gamma,$$

then, for a.e. $t \in [t_1,t_2]$, the restriction of the measure $\mu_{(t)}$ to the interval $[y(t,a), y(t,b)]$ satisfies $\mathrm{d}\mu_{(t)} = \sum_{i=1}^{n} u_{ix}^2 \mathrm{d}x$. We now prove this statement. It follows from (3.4) that

$$\frac{\partial}{\partial t} U_{\hat{k}\xi}(t,\xi) > \frac{\delta}{2}, \tag{3.25}$$

from which one has

$$\mathrm{meas}(\{(t,x) \in \Gamma;\ |U_{\hat{k}\xi}(t,\xi)| < \varepsilon\}) < \frac{4\xi_0}{\delta}\varepsilon,$$

which indicates, by the arbitrariness of $\varepsilon$, that there exists a set of times $\mathcal{N}$ of measure zero such that

$$\mathrm{meas}(\{\xi \in [a,b];\ U_{\hat{k}\xi}(t,\xi) = 0\}) = 0$$

for all times $t \notin \mathcal{N}$. In particular, we have

$$\mathrm{meas}(\{\xi \in [a,b];\ U_{k\xi}(t,\xi) = 0\ (k = 1,2,\ldots,n)\}) = 0$$

for all times $t \notin \mathcal{N}$, which completes the proof by using Lemma 1.

Now we consider the case that the mapping $\xi \mapsto y_0(\xi)$ is non-decreasing but only left continuous. In this case, the distributional derivative of this map is thus a measure, which can be denoted as $D_\xi y_0(\xi) = \sigma = \sigma^a + \sigma^s$. By the assumptions, the absolutely continuous part $\sigma^a$ satisfies $\mathrm{d}\sigma^a = |U_{0\xi}|^2 \mathrm{d}\xi$ and then the map $\xi \mapsto y_0(\xi)$ can be written as

$$y_0(\xi) = y_0(0) + \int_0^\xi |U_{0\xi}|^2(\eta)\,\mathrm{d}\eta + \sigma^s([0,\xi)).$$

Define

$$\bar{y}_0(\xi) := y_0(0) + \int_0^\xi |U_{0\xi}|^2(\eta)\,\mathrm{d}\eta,$$

$$\bar{u}_0(\bar{y}_0(\xi)) := U_0(\xi), \qquad \bar{\mu}_{(0)}(J) := \mathrm{meas}\{\xi;\ \bar{y}_0(\xi) \in J\}.$$

Then, for the new initial data $(\bar{u}_0, \bar{\mu}_{(0)})$, the map $\xi \mapsto \bar{y}_0(\xi)$ is absolutely continuous. Thus the previous analysis is valid and then there exists a vector function $\bar{u}(t,x)$ satisfying problem (1.1)–(1.3) for this new initial data. One can easily check that the function constructed in (3.4), (3.7)–(3.9) for the original initial data $u_0$ satisfies

$$u(t, y(t,\xi) + \sigma^s([0,\xi))) = U(t,\xi),$$

that is,

$$u(t,x) = U(t,\xi(t,x)), \quad \text{where } \xi(t,x) := \sup\{\xi;\ y(t,\xi) + \sigma^s([0,\xi)) \leq x\},$$

which provides a solution by its relationship with $\bar{u}(t,x)$.

### 3.5. Properties of solutions

We first see by the assumption $\xi_0 = \mu_{(0)}(\mathbb{R}^+) < \infty$ that

$$\text{Tot.Var.}\{u_k(t,\cdot);[0,M]\} \le \sqrt{\xi_0 M}, \quad (k = 1, 2, \ldots, n),$$

which means, by the boundary condition, that for any $x \in [0, M]$

$$|u_k(t,x)| \le \sqrt{\xi_0 M}, \quad (k = 1, 2, \ldots, n),$$

from which we get

$$\int_0^M |u_k(t,x) - u_k(s,x)|\,\mathrm{d}x \le |t-s| \cdot \left(\sqrt{\xi_0 M} \sup_{|u_i| \le \sqrt{\xi_0 M}} f(u) + \frac{\xi_0 M}{2} \sup_{|u_i| \le \sqrt{\xi_0 M}} f_{u_k}(u)\right).$$

This proves (2.7). The Lipschitz continuity property of the measure $\mu_{(t)}$ with respect to time can be easily obtained by applying the expression of $y(t, \xi)$ and the definition (2.5).

We now show that the vector function $u(t, x)$ is Hölder continuous locally in $(t, x)$. For $k = 1, 2, \ldots, n$, we see that $u_k$ is Hölder continuous in $x$ for each fixed time $t$ with exponent $1/2$ by the Sobolev embedding theorem. On the other hand, $u_k$ is Lipschitz continuous in time along each characteristic by the equation of $u_k$ in (1.1). Since the characteristic speed $f(u)$ is locally bounded, we obtain that $u_k$ is also Hölder continuous locally in the time direction.

The continuous dependence of solutions can be established by the fact that the corresponding vector functions $U^\nu(t, \xi)$ satisfy $U^\nu(t, \xi) \to U(t, \xi)$ uniformly on $[0, T] \times [0, \xi_0]$ for any $T > 0$.

So far, we have completed the proof of Theorem 2.1.

### 3.6. Uniqueness of solutions

Since the solution of the initial–boundary value problem (1.1)–(1.3) is constructed by the fixed point theorem, then one directly obtains the uniqueness result, which can be stated as follows.

**Theorem 3.1.** *Suppose that there exist a vector function $u = u(t, x)$ and a family of measures $\mu_{(t)}$ satisfying* (i) *and* (ii) *in Theorem* 2.1. *Assume moreover that the map $t \mapsto U(t, \xi)$, defined by*

$$U(t,\xi) := u(t, y(t,\xi)), \quad y(t,\xi) := \inf\{x \ge 0;\ \mu_{(t)}([0,x]) \ge \xi\},$$

*is absolutely continuous and satisfies system (3.4) for a.e. $\xi$. Then there is the identity $(u(t), \mu_{(t)}) = \mathcal{S}_t(u_0, \mu_{(0)})$. In other words, the solution which satisfies the above conditions is unique.*

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