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On the determination of a function from an elliptical Radon transform

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ABSTRACT

In recent years, Radon-type transforms that integrate functions over various sets of ellipses/ellipsoids have been considered in synthetic aperture radar, ultrasound reflection tomography, and radio tomography. In this paper, we consider the transform that integrates a given function in $\mathbb{R}^n$ over a set of solid ellipses (when $n = 2$) or solid ellipsoids of rotation (when $n \geqslant 3$) with a fixed eccentricity and foci restricted to a hyperplane. Inversion formulas are obtained for appropriate classes of functions that are even with respect to the hyperplane. Stability estimates and local uniqueness results are also provided.

1. Introduction

Radon-type transforms that integrate functions over various sets of ellipses/ellipsoids have been arising in the recent decade, due to studies in Synthetic Aperture Radar (SAR) [2,4,8,9], Ultrasound Reflection Tomography (URT) [7,1], and radio tomography [15–17]. In particular, radio tomography is a new imaging method, which uses a wireless network of radio transmitters and receivers to image the distribution of attenuation within the network. The usage of radio frequencies brings in significant non-line-of-sight propagation, since waves propagate along many paths from a transmitter to a receiver. Given a transmitter and a receiver, wave paths observed for a given duration are all contained in an ellipsoid with foci at these two devices. It was thus suggested in [15–17] to approximate the obtained signal by the volume integral of the attenuation over this ellipsoid, which is the model we study in this article.

Due to these applications, there have been several papers devoted to such "elliptical Radon transform." The family of ellipses with one focus fixed at the origin and the other one moving along a given line was considered in [9]. In the same paper, the family of ellipses with a fixed focal distance was also studied. The authors of [7,1] dealt with the case of circular acquisition, when the two foci of ellipses with a given focal distance are located on a given circle. A family of ellipses with two moving foci was also handled in [4].

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In all these works, however, the ellipses have varying eccentricity, the ratio of the major axis to the focal distance. Also, their data were the line integrals of the function over ellipses rather than area integrals. The radio tomography application makes it interesting to consider integrals over solid ellipsoids. In this article, we consider the volume integrals of an unknown attenuation function over the family of ellipsoids of rotation in $\mathbb{R}^n$ with a fixed eccentricity and two foci located in a given hyperplane. We thus reserve the name **elliptical Radon transform** $R_E f$ for the volume integral of a function f over this family of ellipsoids. (When $n = 2$, the elliptical Radon transform $R_E f$ is the area integrals of a function over the family of ellipses with a fixed eccentricity and two foci located in a line.)

The volume integral of a function $f(x)$ over an ellipsoid (or an ellipse for $n = 2$) of the described type is equal to zero if the function is odd with respect to the chosen hyperplane. If the hyperplane is given by $x_n = 0$, we thus assume the function $f : \mathbb{R}^n \to \mathbb{R}$ to be even with respect to x_n: $f(x', x_n) = f(x', -x_n)$ where $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

Given a Radon-type transform, one is usually interested, among others, in the following questions: inversion formulas, uniqueness for a local data problem, and a stability estimate [10,11]. These are the issues we address below.

The problem is stated precisely in Section 2. Two inversion formulas are presented in Sections 3 and 4. Analogue of the Fourier slice theorem is obtained in Section 3 by taking the Fourier transform with respect to the center and a radial Fourier transform with respect to the half distance between two foci. This theorem plays a critical role in getting a stability estimate. The formula discussed in Section 4 is obtained by taking a Fourier type transform and needs less integration than the previous one in Section 3. A stability estimate is handled in Section 5. Section 6 is devoted to uniqueness for a local data problem.

2. Formulation of the problem

We consider all solid ellipses (when $n = 2$) or solid ellipsoids of rotation (when $n \geqslant 3$) in $\mathbb{R}^n$ with a fixed eccentricity $1/\lambda$, where $\lambda > 1$ and foci located in the hyperplane $x_n = 0$. We will identify this hyperplane with $\mathbb{R}^{n-1}$. While a function f depends on n parameters, the set of such ellipsoids depends on $2n - 2$ parameters. This means that when $n \geqslant 3$, the problem of inverting the elliptical Radon transform is $n - 2$-dimensions overdetermined. To reduce the overdeterminacy, we require that the focal axis is parallel to a given line, for instance, the x_1 coordinate axis. When $n = 2$, i.e., integral domain is an ellipse, there is no overdeterminacy and the focal axis is automatically parallel because the hyperplane is the line.

Let $u \in \mathbb{R}^{n-1}$ be the center of such an ellipse/ellipsoid and let $t > 0$ be the half of the focal distance. We denote this ellipse/ellipsoid by $E_{u,t}$. Then, the foci are

$$c_1 = (u_1 + t, u_2, \dots, u_{n-1}, 0) \quad \text{and} \quad c_2 = (u_1 - t, u_2, \dots, u_{n-1}, 0)$$

and the points $x \in E_{u,t}$ are described as follows:

$$\frac{(x_1 - u_1)^2}{\lambda^2} + \frac{(x_2 - u_2)^2}{\lambda^2 - 1} + \dots + \frac{x_n^2}{\lambda^2 - 1} \leqslant t^2.$$

To shorten the formulas, we are going to use the following notation:

$$\nu := \sqrt{\lambda^2 - 1}.$$

The elliptical Radon transform R_E maps a locally integrable function $f(x)$ into its integrals over the solid ellipses/ellipsoids $E_{u,t}$ for all $u \in \mathbb{R}^{n-1}$ and $t > 0$:

$$R_E f(u, t) = \int\limits_{E_{u,t}} f(x)\, dx.$$

Our goals are to reconstruct f from $R_E f$ and to study properties of this transform.

3. Inversion of the elliptical Radon transform

In this section, we assume $f \in C_c^\infty(\mathbb{R}^n)$. Here is our strategy. First of all, we change the ellipse area or ellipsoid volume integral to the ellipse line or ellipsoid surface integral, differentiating with respect to t. Second, we take the Fourier transform of this derivative of $R_E f$ with respect to u. Next, taking a radial Fourier transform with respect to t, we obtain an analogue of the Fourier slice theorem (see [10–12]).

We introduce a back projection operator R_E^* for $g(u,t) \in C_c^\infty(\mathbb{R}^{n-1} \times \mathbb{R}_+)$ as

$$R_E^* g(x) = \int_{\mathbb{R}^{n-1}} g\left(u, \sqrt{\frac{|u_1 - x_1|^2}{\lambda^2} + \frac{|\tilde{u} - \tilde{x}|^2}{\nu^2} + \frac{x_n^2}{\nu^2}}\right) du, \tag{1}$$

where $u = (u_1, \tilde{u}) \in \mathbb{R}^{n-1}$ and $x = (x_1, \tilde{x}, x_n) \in \mathbb{R}^n$. In fact, $R_E^* g(x)$ is the dual transform not to $R_E f(u,t)$, but rather to $\frac{\partial}{\partial t} R_E f(u,t)$, i.e.,

$$\int_0^\infty \int_{\mathbb{R}^{n-1}} \frac{\partial}{\partial t} R_E f(u,t) g(u,t)\, du\, dt = \int_{\mathbb{R}^n} f(x) R_E^* g(x)\, dx. \tag{2}$$

Let χ_S denote the characteristic function of a set $S \subset \mathbb{R}^n$:

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Then the elliptical Radon transform can be written as

$$\begin{aligned} R_E f(u,t) &= \int_{\mathbb{R}^n} \chi_{E_{u,t}} f(x)\, dx = C(\lambda) \int_{\mathbb{R}^n} \chi_{|x|<t} f(\lambda x_1 + u_1, \nu \tilde{x} + \tilde{u}, \nu x_n)\, dx \\ &= C(\lambda) \int_0^t r^{n-1} \int_{S^{n-1}} f(\lambda r y_1 + u_1, \nu r \tilde{y} + \tilde{u}, \nu r y_n)\, d\sigma(y)\, dr, \end{aligned} \tag{3}$$

where $C(\lambda) = \lambda \nu^{n-1}$ and $\sigma(y)$ is the surface measure on S^{n-1}.

Formula (3) can be simplified by differentiation with respect to t and division by t^{n-1}, which yields

$$\begin{aligned} \frac{1}{t^{n-1}} \frac{\partial}{\partial t} R_E f(u,t) &= C(\lambda) \int_{|y|=1} f(\lambda t y_1 + u_1, \nu t \tilde{y} + \tilde{u}, \nu t y_n)\, d\sigma(y) \\ &= 2C(\lambda) \int_{|y'| \leqslant 1} f\left(u + (t\lambda y_1, t\nu \tilde{y}), t\nu \sqrt{1 - |y'|^2}\right) \frac{dy'}{\sqrt{1 - |y'|^2}}, \end{aligned} \tag{4}$$

where $y' = (y_1, \tilde{y}) \in \mathbb{R}^{n-1}$.

It is easy to check that R_E is invariant under the shift with respect to the first $n-1$ variables. That is, if $f_a(x) := f(x' + a, x_n)$ for $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and $a \in \mathbb{R}^{n-1}$, we have

$$(R_E f_a)(u,t) = (R_E f)(u + a, t).$$

Thus, application of the $(n-1)$-dimensional Fourier transform with respect to the center u seems reasonable. Doing this and changing the variable $y' \in \mathbb{R}^{n-1}$ to the polar coordinates $(\theta, s) \in S^{n-2} \times [0, \infty)$, we get

$$\frac{1}{t^{n-1}}\frac{\partial}{\partial t}\mathcal{F}_{n-1}(R_E f)(\xi',t) = 2C(\lambda)\int_0^1 \frac{s^{n-2}}{\sqrt{1-s^2}}\mathcal{F}_{n-1}f\left(\xi',t\nu\sqrt{1-s^2}\right)\int_{S^{n-2}} e^{its(\lambda\theta_1,\nu\tilde{\theta})\cdot\xi'}\,d\theta\,ds,$$

where $\mathcal{F}_{n-1}(R_E f)$ and $\mathcal{F}_{n-1}f$ are the Fourier transforms of $R_E f$ and f with respect to the first $n-1$ coordinates x' of x and u of (u,t), respectively, and $\theta = (\theta_1,\tilde{\theta}) \in S^{n-2}$.

To compute the inner integral, we use the known identity [3,6]

$$\int_{S^{n-2}} e^{i\xi'\cdot\theta}\,d\theta = (2\pi)^{(n-1)/2}\left|\xi'\right|^{(3-n)/2}J_{(n-3)/2}\left(\left|\xi'\right|\right).$$

We thus get

$$\begin{aligned}
&\frac{1}{t^{n-1}}\frac{\partial}{\partial t}\mathcal{F}_{n-1}(R_E f)\left(\frac{\xi_1}{\lambda},\frac{\tilde{\xi}}{\nu},t\right)\\
&= \omega_n\int_0^1 \frac{s^{n-2}}{\sqrt{1-s^2}}\mathcal{F}_{n-1}f\left(\frac{\xi_1}{\lambda},\frac{\tilde{\xi}}{\nu},t\nu\sqrt{1-s^2}\right)\left(ts\left|\xi'\right|\right)^{(3-n)/2}J_{(n-3)/2}\left(ts\left|\xi'\right|\right)ds,
\end{aligned}$$

where $\xi' = (\xi_1,\tilde{\xi}) \in \mathbb{R}\times\mathbb{R}^{n-2}$ and $\omega_n = 2(2\pi)^{(n-1)/2}C(\lambda)$.

This enables us to get an analogue of the Fourier slice theorem.

Theorem 1. *For a function $f \in C_c^\infty(\mathbb{R}^n)$ that is even with respect to x_n, the following formula holds:*

$$\mathcal{F}_n f(\xi) = \frac{|(\lambda\xi_1,\nu\tilde{\xi},\nu\xi_n)|^{n-2}|\nu\xi_n|}{2^{n+1}\pi^n C(\lambda)^2}\mathcal{F}_n\left(R_E^*\frac{1}{t^{n-1}}\frac{\partial}{\partial t}R_E f\right)(\xi), \tag{5}$$

where $\mathcal{F}_n f$ is the n-dimensional Fourier transform of f.

Proof. Let us denote the radial Fourier transform by $H_n f(\rho)$, i.e.,

$$H_n f(\rho) := \rho^{1-n/2}\int_0^\infty t^{n/2}J_{(n-2)/2}(t\rho)f(t)\,dt.$$

We recall that if f is a radial function on $\mathbb{R}^n$, then the Fourier transform $\mathcal{F}_n f$ of f is also radial and $\mathcal{F}_n f = (2\pi)^{n/2}H_n f_0$ where $f_0(|x|) = f(x)$ (cf. [14]). Taking this transform of $\frac{1}{t^{n-1}}\frac{\partial}{\partial t}\mathcal{F}_{n-1}(R_E f)$ as a function of t, we have for $\xi' = (\xi_1,\tilde{\xi}) \in \mathbb{R}^{n-1}$,

$$\begin{aligned}
&H_n\left(\frac{1}{t^{n-1}}\frac{\partial}{\partial t}\mathcal{F}_{n-1}(R_E f)\right)\left(\frac{\xi_1}{\lambda},\frac{\tilde{\xi}}{\nu},\rho\right)\\
&= \omega_n\rho^{1-n/2}\int_0^\infty\int_0^1 t^{\frac{n}{2}}J_{\frac{n-2}{2}}(t\rho)\left(s\left|\xi'\right|\right)^{\frac{3-n}{2}}J_{\frac{n-3}{2}}\left(ts\left|\xi'\right|\right)\mathcal{F}_{n-1}f\left(\frac{\xi_1}{\lambda},\frac{\tilde{\xi}}{\nu},t\nu\sqrt{1-s^2}\right)\frac{s^{n-2}\,ds\,dt}{\sqrt{1-s^2}}.
\end{aligned} \tag{6}$$

It is known [5, p. 59 (18) vol. 2 or for $n=2$, p. 55 (35) vol. 1] that for $a>0$, $\beta>0$, and, $\mu>\nu>-1$,

$$\begin{aligned}
&\int_0^\infty x^{\nu+1/2}\left(x^2+\beta^2\right)^{-1/2\mu}J_\mu\left(a\left(x^2+\beta^2\right)^{1/2}\right)J_\nu(xy)(xy)^{1/2}\,dx\\
&= \begin{cases} a^{-\mu}y^{\nu+1/2}\beta^{-\mu+\nu+1}(a^2-y^2)^{1/2\mu-1/2\nu-1/2}J_{\mu-\nu-1}(\beta(a^2-y^2)^{1/2}) & \text{if } 0<y<a,\\ 0 & \text{if } a<y<\infty. \end{cases}
\end{aligned}$$

To use the above identity, we make the change of variables $(s,t) \to (x,\beta)$, where $t = \sqrt{x^2+\beta^2}$ and $s = x/\sqrt{x^2+\beta^2}$ in Eq. (6), which gives

$$
\begin{aligned}
&H_n\left(\frac{1}{t^{n-1}}\frac{\partial}{\partial t}\mathcal{F}_{n-1}(R_E f)\right)\left(\frac{\xi_1}{\lambda},\frac{\tilde{\xi}}{\nu},\rho\right)\\
&\quad= \omega_n \rho^{\frac{2-n}{2}}\left|\xi'\right|^{\frac{3-n}{2}} \int_0^\infty\int_0^\infty |x| J_{\frac{n-2}{2}}\left(\rho\left(x^2+\beta^2\right)^{\frac{1}{2}}\right)\left(x^2+\beta^2\right)^{-\frac{n-2}{4}} J_{\frac{n-3}{2}}\left(x\left|\xi'\right|\right)\mathcal{F}_{n-1}f\left(\frac{\xi_1}{\lambda},\frac{\tilde{\xi}}{\nu},\nu\beta\right)dx\,d\beta\\
&\quad= \begin{cases} C(\lambda)\frac{2^{n/2+1}\pi^{n/2}\rho^{2-n}}{\sqrt{\rho^2-|\xi'|^2}}\int_0^\infty \mathcal{F}_{n-1}f(\frac{\xi_1}{\lambda},\frac{\tilde{\xi}}{\nu},\beta\nu)\cos(\beta\sqrt{\rho^2-|\xi'|^2})\,d\beta & \text{if } |\xi'|<\rho,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned} \tag{7}
$$

Substituting $\rho = |\xi| = |(\xi',\xi_n)|$ yields

$$
H_n\left(\frac{1}{t^{n-1}}\frac{\partial}{\partial t}\mathcal{F}_{n-1}(R_E f)\right)\left(\frac{\xi_1}{\lambda},\frac{\tilde{\xi}}{\nu},|\xi|\right) = C(\lambda)\frac{2(2\pi)^{n/2}|\xi|^{2-n}}{|\xi_n|}\int_0^\infty \mathcal{F}_{n-1}f\left(\frac{\xi_1}{\lambda},\frac{\tilde{\xi}}{\nu},\beta\nu\right)\cos(\xi_n\beta)\,d\beta.
$$

Since f is even with respect to x_n, the last integral is the Fourier transform of f with respect to x_n, so we get

$$
H_n\left(\frac{1}{t^{n-1}}\frac{\partial}{\partial t}\mathcal{F}_{n-1}(R_E f)\right)\left(\frac{\xi_1}{\lambda},\frac{\tilde{\xi}}{\nu},|\xi|\right) = \frac{2(2\pi)^{n/2}|\xi|^{2-n}}{|\xi_n|}\lambda\nu^{n-2}\mathcal{F}_n f\left(\frac{\xi_1}{\lambda},\frac{\tilde{\xi}}{\nu},\frac{\xi_n}{\nu}\right). \tag{8}
$$

Taking the Fourier transform of $R_E^* g$ with respect to x yields

$$
\begin{aligned}
\mathcal{F}_n\left(R_E^* g\right)\left(\frac{\xi_1}{\lambda},\frac{\tilde{\xi}}{\nu},\frac{\xi_n}{\nu}\right) &= \int_{\mathbb{R}^n} e^{-ix\cdot(\frac{\xi_1}{\lambda},\frac{\tilde{\xi}}{\nu},\frac{\xi_n}{\nu})} R_E^* g(x)\,dx\\
&= \int_{\mathbb{R}^n} e^{-ix\cdot(\frac{\xi_1}{\lambda},\frac{\tilde{\xi}}{\nu},\frac{\xi_n}{\nu})}\int_{\mathbb{R}^{n-1}} g\left(u,\sqrt{\frac{|u_1-x_1|^2}{\lambda^2}+\frac{|\tilde{u}-\tilde{x}|^2}{\nu^2}+\frac{x_n^2}{\nu^2}}\right)du\,dx\\
&= \int_{\mathbb{R}^{n-1}} e^{-iu\cdot(\frac{\xi_1}{\lambda},\frac{\tilde{\xi}}{\nu})}\int_{\mathbb{R}^n} e^{-i(x'-u,x_n)\cdot(\frac{\xi_1}{\lambda},\frac{\tilde{\xi}}{\nu},\frac{\xi_n}{\nu})} g\left(u,\sqrt{\frac{|u_1-x_1|^2}{\lambda^2}+\frac{|\tilde{u}-\tilde{x}|^2}{\nu^2}+\frac{x_n^2}{\nu^2}}\right)dx\,du\\
&= C(\lambda)\int_{\mathbb{R}^{n-1}} e^{-iu\cdot(\frac{\xi_1}{\lambda},\frac{\tilde{\xi}}{\nu})}\int_{\mathbb{R}^n} e^{-ix\cdot\xi} g\left(u,|x|\right)dx\,du\\
&= (2\pi)^{n/2}C(\lambda)\int_{\mathbb{R}^{n-1}} e^{-iu\cdot(\frac{\xi_1}{\lambda},\frac{\tilde{\xi}}{\nu})}\left(H_n g(u,\cdot)\right)\left(|\xi|\right)du\\
&= (2\pi)^{n/2}C(\lambda)H_n\mathcal{F}_{n-1}g\left(\frac{\xi_1}{\lambda},\frac{\tilde{\xi}}{\nu},|\xi|\right),
\end{aligned} \tag{9}
$$

where $x = (x',x_n) = (x_1,\tilde{x},x_n) \in \mathbb{R}^n, u = (u_1,\tilde{x}) \in \mathbb{R}^{n-1}$ and $\xi = (\xi',\xi_n) = (\xi_1,\tilde{\xi},\xi_n) \in \mathbb{R}^n$. Combining Eq. (8) and Eq. (9), we get Eq. (5). □

Remark 2. Theorem 1 leads naturally to a Fourier type inversion formula for even functions, if one supplements Eq. (5) with the inverse Fourier transform.

One can also obtain a useful relation with a convolution.

Proposition 3. *Let $\phi \in C_c^\infty(\mathbb{R}^{n-1} \times [0,\infty))$ and $f \in C_c^\infty(\mathbb{R}^n)$ be even with respect to x_n. If $\psi = R_E^* \phi$ and $g = \frac{1}{t^{n-1}} \frac{\partial}{\partial t} R_E f$, then we have*

$$g * \phi = \frac{1}{C(\lambda)t^{n-1}} \frac{\partial}{\partial t} R_E(f * \psi),$$

where

$$g * \phi\big(u, |\omega|\big) = \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^{n-1}} g\big(u - u', \big|\omega - \omega'\big|\big)\phi\big(u', \big|\omega'\big|\big)\, du'\, d\omega'.$$

Proof. Note that since $(2\pi)^{n/2} H_n f_0 = \mathcal{F}_n f$ for a radial function f on $\mathbb{R}^n$ and $f_0(|x|) = f(x)$, we get $H_n(f * g) = (2\pi)^{n/2} H_n f H_n g$. Taking the Fourier transform of $g * \phi$ with respect to u and H_n with respect to t, we get

$$\begin{aligned} H_n \mathcal{F}_{n-1}(g * \phi)\left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, |\xi|\right) &= (2\pi)^{n/2} H_n \mathcal{F}_{n-1} g\left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, |\xi|\right) H_n \mathcal{F}_{n-1} \phi\left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, |\xi|\right) \\ &= \frac{1}{C(\lambda)} H_n \mathcal{F}_{n-1} g\left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, |\xi|\right) \mathcal{F}_n \psi\left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, \frac{\xi_n}{\nu}\right). \end{aligned}$$

In the last line we used Eq. (9). Eq. (8) implies

$$\begin{aligned} H_n \mathcal{F}_{n-1}(g * \phi)\left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, |\xi|\right) &= \frac{2^{n/2+1}\pi^{n/2}|\xi|^{2-n}\nu^{n-2}\lambda}{C(\lambda)|\xi_n|} \mathcal{F}_n f\left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, \frac{\xi_n}{\nu}\right) \mathcal{F}_n \psi\left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, \frac{\xi_n}{\nu}\right) \\ &= \frac{2^{n/2+1}\pi^{n/2}|\xi|^{2-n}\nu^{n-2}\lambda}{C(\lambda)|\xi_n|} \mathcal{F}_n(f * \phi)\left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, \frac{\xi_n}{\nu}\right) \\ &= \frac{1}{C(\lambda)} H_n\left(\frac{1}{t^{n-1}} \frac{\partial}{\partial t} \mathcal{F}_{n-1}(R_E f * \phi)\right)\left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, |\xi|\right), \end{aligned}$$

which proves our assertion. $\square$

4. A different inversion method

In this section, we provide a different inversion formula for the elliptical Radon transform. To obtain this formula, we start to take a transform, which is like the Fourier transform, but with kernel $e^{i\omega t^2}$ instead of $e^{i\omega t}$, of the derivative of $R_E f$ in t, where ω is the transform variable (cf. [13]). To get f from this transform, we change variables.

Theorem 4. *Let $f \in C_c^\infty(\mathbb{R}^n)$ be even with respect to x_n. If $G(u,w) = \int_0^\infty \partial_t R_E f(u,t) e^{iwt^2}\, dt$, then $f(x)$ can be reconstructed as follows:*

$$\frac{|x_n|}{(2\pi)^n C(\lambda)^2} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}^{n-1}} e^{-i\frac{|\alpha|^2}{4\gamma}} e^{i\alpha\cdot(\frac{x_1}{\lambda}, \frac{\tilde{x}}{\nu})} e^{-i\gamma(\frac{x_1^2}{\lambda^2} + \frac{|\tilde{x}|^2}{\nu^2} + \frac{x_n^2}{\nu^2})} G\left(\frac{\lambda\alpha_1}{2\gamma}, \frac{\nu\tilde{\alpha}}{2\gamma}, \gamma\right) d\alpha\, d\gamma,$$ [1]

where $\alpha = (\alpha_1, \tilde{\alpha}) \in \mathbb{R} \times \mathbb{R}^{n-2}$ and $C(\lambda) = \lambda\nu^{n-1}$, as before.

[1] A function G depends on n-dimensional variables $(u,w) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and for simplicity, we sometimes represent these variables as $(u_1, \tilde{u}, w) \in \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}$.

Proof. By the definition of $G(u,w)$, we have

$$\begin{aligned}
G(u,w) &:= \int_0^\infty \frac{\partial}{\partial t} R_E f(u,t) e^{iwt^2}\, dt \\
&= C(\lambda) \int_0^\infty t^{n-1} \int_{|y|=1} f(\lambda t y_1 + u_1, \nu t\tilde{y} + \tilde{u}, t\nu y_n) e^{iwt^2}\, d\sigma(y)\, dt \\
&= C(\lambda) \int_{\mathbb{R}^n} f(\lambda y_1 + u_1, \nu\tilde{y} + \tilde{u}, \nu y_n) e^{iw|y|^2}\, dy,
\end{aligned}$$

where in the second equality, we used Eq. (4) and in the last equality we switched from the polar to the Cartesian coordinates. Making the change of variables $x_1 = \lambda y_1 + u_1$, $\tilde{x} = \nu\tilde{y} + \tilde{u}$, and $x_n = \nu y_n$, we get

$$\begin{aligned}
G(u,w) &= \int_{\mathbb{R}^n} f(x) e^{iw(\frac{(x_1-u_1)^2}{\lambda^2} + \frac{|\tilde{x}-\tilde{u}|^2}{\nu^2} + \frac{x_n^2}{\nu^2})}\, dx \\
&= e^{iw\frac{u_1^2}{\lambda^2}} e^{iw\frac{|\tilde{u}|^2}{\nu^2}} \int_{\mathbb{R}^n} f(x) e^{iw(\frac{x_1^2}{\lambda^2} + \frac{|\tilde{x}|^2}{\nu^2} + \frac{x_n^2}{\nu^2})} e^{-2iwu_1\frac{x_1}{\lambda^2}} e^{-2iw\frac{\tilde{u}\cdot\tilde{x}}{\nu^2}}\, dx,
\end{aligned}$$

where $x = (x_1, \tilde{x}, x_n) \in \mathbb{R}^n$ and $u = (u_1, \tilde{u}) \in \mathbb{R}^{n-1}$. Next, make the change of variables

$$\mathsf{x}_1 = \frac{x_1}{\lambda}, \qquad \tilde{\mathsf{x}} = \frac{\tilde{x}}{\nu}, \quad \text{and} \quad r = \frac{x_1^2}{\lambda^2} + \frac{|\tilde{x}|^2}{\nu^2} + \frac{x_n^2}{\nu^2},$$

so that

$$x_1 = \mathsf{x}_1\lambda, \qquad \tilde{x} = \tilde{\mathsf{x}}\nu, \quad \text{and} \quad x_n = \nu\sqrt{r - \mathsf{x}_1^2 - |\tilde{\mathsf{x}}|^2}.$$

The Jacobian of this transformation is

$$J = \begin{vmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \nu & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-\mathsf{x}_1\nu}{2\sqrt{r-\mathsf{x}_1^2-\tilde{\mathsf{x}}^2}} & \frac{-\tilde{\mathsf{x}}\nu}{2\sqrt{r-\mathsf{x}_1^2-|\tilde{\mathsf{x}}|^2}} & \cdots & \frac{\nu}{2\sqrt{r-\mathsf{x}_1^2-|\tilde{\mathsf{x}}|^2}} \end{vmatrix} = \frac{C(\lambda)}{2\sqrt{r-\mathsf{x}_1^2-|\tilde{\mathsf{x}}|^2}},$$

so that

$$dx = \frac{C(\lambda)}{2\sqrt{r-\mathsf{x}_1^2-|\tilde{\mathsf{x}}|^2}}\, d\mathsf{x}_1\, d\tilde{\mathsf{x}}\, dr.$$

Let the function $k(x_1, \tilde{x}, r)$ be defined by

$$k(x_1,\tilde{x},r) = \begin{cases} \frac{f(\lambda x_1, \nu\tilde{x}, \nu\sqrt{r-x_1^2-|\tilde{x}|^2})}{2\sqrt{r-x_1^2-|\tilde{x}|^2}} & 0 < |x_1|^2 + |\tilde{x}|^2 < r, \\ 0 & \text{otherwise.} \end{cases}$$

Since f is even in x_n, it is sufficient to consider the positive root of $\sqrt{r - x_1^2 - |\tilde{x}|^2}$. Then we can rewrite $G(u, w)$ as

$$\begin{aligned} G(u,w) &= C(\lambda) e^{iw\frac{u_1^2}{\lambda^2}} e^{iw\frac{|\tilde{u}|^2}{\nu^2}} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-2}} \int_{\mathbb{R}} k(x_1, \tilde{x}, r) e^{iwr} e^{-2i\frac{wx_1u_1}{\lambda}} e^{-2i\frac{w\tilde{x}\cdot\tilde{u}}{\nu}}\, dx_1\, d\tilde{x}\, dr \\ &= C(\lambda) e^{iw\frac{u_1^2}{\lambda^2}} e^{iw\frac{|\tilde{u}|^2}{\nu^2}} K\left(2\frac{wu_1}{\lambda}, 2\frac{w\tilde{u}}{\nu}, -w\right), \end{aligned}$$

where for $(\alpha_1, \tilde{\alpha}) \in \mathbb{R} \times \mathbb{R}^{n-2}$,

$$\begin{aligned} K(\alpha_1, \tilde{\alpha}, \gamma) &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-2}} \int_{\mathbb{R}} e^{-i(\alpha_1, \tilde{\alpha}, \gamma)\cdot(x_1, \tilde{x}, r)} k(x_1, \tilde{x}, r)\, dx_1\, d\tilde{x}\, dr \\ &= \frac{1}{C(\lambda)} e^{i\frac{|(\alpha_1,\tilde{\alpha})|^2}{4\gamma}} G\left(\frac{-\lambda\alpha_1}{2\gamma}, \frac{-\nu\tilde{\alpha}}{2\gamma}, -\gamma\right). \end{aligned}$$

Since $k(x_1, \tilde{x}, r)$ is

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-2}} \int_{\mathbb{R}} e^{i\alpha_1 x_1} e^{i\tilde{\alpha}\cdot\tilde{x}} e^{i\gamma r} K(\alpha_1, \tilde{\alpha}, \gamma)\, d\alpha\, d\gamma,$$

we get for $x_n > 0$,

$$\begin{aligned} f(x) &= \frac{x_n}{C(\lambda)} k\left(\frac{x_1}{\lambda}, \frac{\tilde{x}}{\nu}, \frac{x_1^2}{\lambda^2} + \frac{|\tilde{x}|^2}{\nu^2} + \frac{x_n^2}{\nu^2}\right) \\ &= \frac{x_n}{(2\pi)^n C(\lambda)} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-2}} \int_{\mathbb{R}} e^{i\alpha_1\frac{x_1}{\lambda}} e^{i\tilde{\alpha}\cdot\frac{\tilde{x}}{\nu}} e^{i\gamma(\frac{x_1^2}{\lambda^2} + \frac{|\tilde{x}|^2}{\nu^2} + \frac{x_n^2}{\nu^2})} K(\alpha_1, \tilde{\alpha}, \gamma)\, d\alpha_1\, d\alpha\, d\gamma \\ &= \frac{x_n}{(2\pi)^n C(\lambda)^2} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-2}} \int_{\mathbb{R}} e^{-i\frac{|\alpha|^2}{4\gamma}} e^{i\alpha\cdot(\frac{x_1}{\lambda}, \frac{\tilde{x}}{\nu})} e^{-i\gamma(\frac{x_1^2}{\lambda^2} + \frac{|\tilde{x}|^2}{\nu^2} + \frac{x_n^2}{\nu^2})} G\left(\frac{\lambda\alpha_1}{2\gamma}, \frac{\nu\tilde{\alpha}}{2\gamma}, \gamma\right) d\alpha_1\, d\tilde{\alpha}\, d\gamma, \end{aligned} \tag{10}$$

where $\alpha = (\alpha_1, \tilde{\alpha}) \in \mathbb{R} \times \mathbb{R}^{n-2}$. The evenness of f in x_n completes our proof. □

5. A stability estimate

In this section, we obtain a stability estimate for the elliptical Radon transform. Let $\mathcal{H}^\gamma(\mathbb{R}^n)$ be a regular Sobolev space with a norm

$$\|f\|_\gamma^2 := \int_{\mathbb{R}^n} \left|\mathcal{F}_n f(\xi)\right|^2 \left(1 + |\xi|^2\right)^\gamma d\xi.$$

Let us define $\mathcal{H}_e^\gamma(\mathbb{R}^n) = \{f \in \mathcal{H}^\gamma(\mathbb{R}^n)\colon f \text{ is even with respect to } x_n\}$ and let $L_{n-1}^2(\mathbb{R}^{n-1} \times [0, \infty))$ be the set of a function g on $\mathbb{R}^{n-1} \times [0, \infty)$ with

$$\|g\|^2 := \int_{\mathbb{R}^{n-1}} \int_0^\infty \left|g(u,t)\right|^2 t^{n-1}\, dt\, du < \infty.$$

Then $L_{n-1}^2(\mathbb{R}^{n-1} \times [0, \infty))$ is a Hilbert space. Also, by the Plancherel formula, we have $\|g\| = 2^{2n-2}\pi^{3n/2-1}\Gamma(n/2)\|\tilde{g}\|$, where

$$\tilde{g}(\xi,|\zeta|)=\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^n} g(u,|w|)e^{-i(u,w)\cdot(\xi,\zeta)}\,du\,dw.$$

Let $\mathcal{H}^\gamma(\mathbb{R}^{n-1}\times[0,\infty))$ be the set of a function $g\in L^2_{n-1}(\mathbb{R}^{n-1}\times[0,\infty))$ with $\|g\|_\gamma<\infty$, where

$$\|g\|_\gamma^2:=\int_{\mathbb{R}^{n-1}}\int_0^\infty \left|\tilde{g}(\xi',\eta)\right|^2\left(1+\left|\xi'\right|^2+|\eta|^2\right)^\gamma \eta^{n-1}\,d\eta\,d\xi'.$$

Theorem 5. *For $\gamma\geqslant 0$, there is a constant C_n such that $f\in\mathcal{H}^\gamma_e(\mathbb{R}^n)$,*

$$\|f\|_\gamma\leqslant C_n\left\|t^{1-n}\partial_t R_E f\right\|_{\gamma+(n-1)/2}.$$

Proof. Let $g=t^{1-n}\partial_t R_E f$. Note that from Eq. (9), we have

$$\mathcal{F}_n\left(R_E^* g\right)\left(\frac{\xi_1}{\lambda},\frac{\tilde{\xi}}{\nu},\frac{\xi_n}{\nu}\right)=C(\lambda)\int_{\mathbb{R}^{n-1}} e^{-iu\cdot(\frac{\xi_1}{\lambda},\frac{\tilde{\xi}}{\nu})}\int_{\mathbb{R}^n} e^{-ix\cdot\xi} g\left(u,|x|\right)dx\,du=C(\lambda)\tilde{g}\left(\frac{\xi_1}{\lambda},\frac{\tilde{\xi}}{\nu},|\xi|\right).\tag{11}$$

Combining Eq. (11) and Theorem 1, we have

$$\mathcal{F}_n f(\xi)=\frac{|(\lambda\xi_1,\nu\tilde{\xi},\nu\xi_n)|^{n-2}|\nu\xi_n|}{2^{n+1}\pi^n C(\lambda)}\tilde{g}\left(\xi_1,\tilde{\xi},\left|(\lambda\xi_1,\nu\tilde{\xi},\nu\xi_n)\right|\right).$$

Hence, we have

$$\begin{aligned}
\|f\|_\gamma^2&=\int_{\mathbb{R}^n}\left(1+|\xi|^2\right)^\gamma\left|\mathcal{F}_{n-1}f(\xi)\right|^2 d\xi\\
&=\frac{1}{2^{2n+2}\pi^{2n}C(\lambda)^2}\int_{\mathbb{R}^n}\left|(\lambda\xi_1,\nu\tilde{\xi},\nu\xi_n)\right|^{2n-4}|\nu\xi_n|^2\left(1+|\xi|^2\right)^\gamma\left|\tilde{g}\left(\xi_1,\tilde{\xi},\left|(\lambda\xi_1,\nu\tilde{\xi},\nu\xi_n)\right|\right)\right|^2 d\xi\\
&\leqslant C_n\int_{\mathbb{R}^n}\left|(\lambda\xi_1,\nu\tilde{\xi},\nu\xi_n)\right|^{2n-4}|\nu\xi_n|^2\left(1+\left|(\lambda\xi_1,\nu\tilde{\xi},\nu\xi_n)\right|^2\right)^\gamma\left|\tilde{g}\left(\xi_1,\tilde{\xi},\left|(\lambda\xi_1,\nu\tilde{\xi},\nu\xi_n)\right|\right)\right|^2 d\xi\\
&\leqslant C_n\int_{\mathbb{R}^{n-2}}\int_{\mathbb{R}}\int_0^\infty\left(\eta^2-\lambda^2\xi_1^2-\nu^2|\tilde{\xi}|^2\right)^{\frac{1}{2}}\eta^{2n-3}\left(1+\eta^2\right)^\gamma\left|\tilde{g}(\xi_1,\tilde{\xi},\eta)\right|^2 d\eta\,d\xi_1\,d\tilde{\xi}.
\end{aligned}$$

In the last line, we change the variable ξ_n to $\eta=|(\lambda\xi_1,\nu\tilde{\xi},\nu\xi_n)|$. □

6. Uniqueness for the local problem

Theorem 1 implies that an even function $f\in C_c^\infty(\mathbb{R}^n)$ is uniquely determined by $R_E f$. The question arises if f is uniquely determined by some partial information. The approach in this section is similar to the one in [3].

Theorem 6. *Let $u^0\in\mathbb{R}^{n-1}$, $\epsilon>0$, and $T>0$ be arbitrary. Let $f\in C_c^\infty(\mathbb{R}^n)$ be even with respect to x_n and suppose $g=R_E f$ is equal to zero on the open set*

$$U_{T,\epsilon}=\left\{(u,t)\in\mathbb{R}^{n-1}\times[0,\infty):\ \left|u-u^0\right|<\epsilon,0\leqslant t<T\right\}.$$

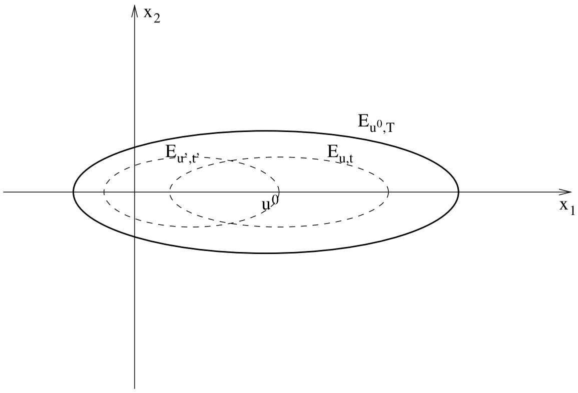

Fig. 1. Ellipses $E_{u^0,T}$, $E_{u,t}$, $E_{u',t'}$.

Then f equals zero on the open set

$$V_T = \left\{x \in \mathbb{R}^n\colon \ \frac{(x_1 - u_1^0)^2}{\lambda^2} + \frac{(\tilde{x} - \tilde{u}^0)^2}{\nu^2} + \frac{x_n^2}{\nu^2} < T^2\right\}.$$

Here $x = (x_1, \tilde{x}, x_n) \in \mathbb{R}^n$ and $u^0 = (u_1^0, \tilde{u}^0) \in \mathbb{R}^{n-1}$. Also, g is equal to zero on the open cone

$$W_T = \big\{(u,t) \in \mathbb{R}^{n-1} \times [0,T)\colon \ |u - u_0| + t < T\big\} \quad (cf.\ Fig.\ 1).$$

Proof. Without loss of generality, we may assume $u^0 = 0$. Let $f \in C^\infty(\mathbb{R}^n)$. Clearly, g is also differentiable. Differentiating $R_E f(u,t)$ with respect to u_1 yields

$$\begin{aligned}
\frac{\partial}{\partial u_1} R_E f(u,t) &= C(\lambda) \int\limits_{\mathbb{R}^n} \chi_{|x|<t} \frac{\partial}{\partial u_1} f(\lambda x_1 + u_1, \nu\tilde{x} + \tilde{u}, \nu x_n)\, dx \\
&= \lambda C(\lambda) \int\limits_{\mathbb{R}^n} \chi_{|x|<t} \frac{\partial}{\partial x_1} f(\lambda x_1 + u_1, \nu\tilde{x} + \tilde{u}, \nu x_n)\, dx \\
&= \lambda C(\lambda) \frac{1}{t} \int\limits_{|x|=t} x_1 f(\lambda x_1 + u_1, \nu\tilde{x} + \tilde{u}, \nu x_n)\, d\sigma(x).
\end{aligned}$$

Here we used Eq. (3) and the divergence theorem. Similarly, we have for $i = 2, \ldots, n-1$

$$\frac{\partial}{\partial u_i} R_E f(u,t) = \nu C(\lambda) \frac{1}{t} \int\limits_{|x|=t} x_i f(\lambda x_1 + u_1, \nu\tilde{x} + \tilde{u}, \nu x_n)\, d\sigma(x).$$

Using Eq. (4), we get

$$\begin{aligned}
\frac{\partial}{\partial t} R_E(x_i f)(u,t) &= C(\lambda) \int\limits_{|x|=t} (u_i + \nu x_i) f(\lambda x_1 + u_1, \nu\tilde{x} + \tilde{u}, \nu x_n)\, d\sigma(x) \\
&= u_i \frac{\partial}{\partial t} g(u,t) + t \frac{\partial}{\partial u_i} g(u,t).
\end{aligned}$$

Let the linear operator D_i be defined by $D_i g(u,t) = u_i \partial_t g(u,t) + t\partial_{u_i} g(u,t)$. Then $\frac{\partial}{\partial t} R_E(x_i f)(u,t)$ is $D_i g(u,t)$. By iteration, we obtain $\frac{\partial}{\partial t} R_E(p(x')f) = p(D)g$ where p is an $n-1$-variable polynomial. If g is zero in $U_{T,\epsilon}$, then $p(D)g$ is also zero in $U_{T,\epsilon}$. Then we have for any point $(u,t) \in U_{T,\epsilon}$,

$$\begin{aligned}\frac{\partial}{\partial t} R_E\big(p(x')f\big)(u,t) &= C(\lambda) \int\limits_{|x|=t} p(u_1+\lambda x_1, \tilde{u}+\nu\tilde{x}) f(\lambda x_1+u_1, \nu\tilde{x}+\tilde{u}, \nu x_n)\, d\sigma(x) \\ &= C(\lambda) \int\limits_{|y|<t} p\big(u+(\lambda y_1, \nu\tilde{y})\big) f\big(u+(\lambda y_1,\nu\tilde{y}), \nu\sqrt{t^2-|y|^2}\big) \frac{dy}{\sqrt{t^2-|y|}} = 0.\end{aligned}$$

The Stone–Weierstrass theorem implies that for fixed u and t, we can choose a sequence of polynomials p_j such that $p_j(u+(\lambda y_1,\nu\tilde{y}))$ converge to $f(u+(\lambda y_1,\nu\tilde{y},\nu\sqrt{t^2-|y|^2}))$ uniformly for $|y| \leqslant t$ and $y=(y_1,\tilde{y}) \in \mathbb{R}^{n-1}$. It follows that $f=0$ in V_T and that $g=0$ in W_T. $\square$

7. Conclusion

Several types of elliptical Radon transforms have been considered in SAR, URT, and radio tomography. Among these, we study the elliptical Radon transform $R_E f$ arising in radio tomography imaging [15–17]. This transform R_E maps a given function f on $\mathbb{R}^n$ into the set of its integrals over the solid ellipses/ellipsoids $E_{u,t}$. We describe two different ways of determining a function f from $R_E f$ and present a stability estimate and a local uniqueness result for R_E.

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