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Some new convergent sequences and inequalities of Euler's constant

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ABSTRACT

In this paper, some new convergent sequences and inequalities of Euler's constant are provided. To demonstrate the superiority of our new convergent sequence over DeTemple's sequence, Vernescu's sequence and Mortici's sequences, some numerical computations are also given.

1. Introduction

In the theory of mathematical constants, an important concern is the definition of new sequences which converge to these fundamental constants with increasingly higher speed. These convergent sequences and constants play a key role in many areas of mathematics and science in general, as theory of probability, applied statistics, physics, special functions, number theory, or analysis.

One of the most useful convergent sequences in mathematics is

$$\gamma_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n, \tag{1.1}$$

which converges towards the well-known Euler's constant

$$\gamma = 0.57721566490115328\ldots.$$

Up until now, many researchers made great efforts in the area of concerning the rate of convergence of the sequence $(\gamma_n)_{n\geqslant 1}$ and establishing faster sequences to converge to Euler's constant and had a lot of inspiring results. For example, in [11–13,15], the following estimates are established

$$\frac{1}{2n+1} < \gamma_n - \gamma < \frac{1}{2n}, \tag{1.2}$$

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using interesting geometric interpretations. In [14], Vernescu provided the sequence

$$V_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{2n} - \ln n, \tag{1.3}$$

for which

$$\frac{1}{12(n+1)^2} < \gamma - V_n < \frac{1}{12n^2}. \tag{1.4}$$

In [1,2], DeTemple introduced a faster convergent sequence $(R_n)_{n \geqslant 1}$ to γ as follows,

$$R_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln\left(n + \frac{1}{2}\right), \tag{1.5}$$

which decreases to γ with the rate of convergence n^{-2}, since

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}. \tag{1.6}$$

Both (1.3) and (1.5) are slight modifications of Euler's sequences (1.1), but significantly improve the rate of convergence from n^{-1} to n^{-2}.

Recently, Mortici researched Euler's constant again, and provided some convergent sequences which are faster than (1.1), (1.3) and (1.5).

In [4], Mortici provided the following two sequences

$$u_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{(6-2\sqrt{6})n} - \ln\left(n + \frac{1}{\sqrt{6}}\right) \tag{1.7}$$

and

$$v_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{(6+2\sqrt{6})n} - \ln\left(n - \frac{1}{\sqrt{6}}\right). \tag{1.8}$$

Both sequences (1.7) and (1.8) were shown to converge to γ as n^{-3}.

Next, in [6], Mortici introduced the following class of sequences of the form

$$\mu_n(a,b) = \sum_{k=1}^{n} \frac{1}{k} + \ln\left(e^{a/(n+b)} - 1\right) - \ln a, \tag{1.9}$$

where a, b are real parameters, $a > 0$. Furthermore, they proved that among the sequences $(\mu_n(a,b))_{n \geqslant 1}$, the privileged one

$$\mu_n\left(\frac{\sqrt{2}}{2}, \frac{2+\sqrt{2}}{4}\right)$$

offers the best approximations of γ, since

$$\lim_{n\to\infty} n^3 \left(\mu_n\left(\frac{\sqrt{2}}{2}, \frac{2+\sqrt{2}}{4}\right) - \gamma\right) = \frac{\sqrt{2}}{96}. \tag{1.10}$$

Following their work, recently, in [3], we used continued fraction approximation to provide a new quicker sequence convergent to Euler's constant as follows,

$$r_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n - \cfrac{a_1}{n + \cfrac{a_2 n}{n + \cfrac{a_3 n}{n + \cfrac{a_4 n}{n + \ddots}}}}, \tag{1.11}$$

where $a_1 = \frac{1}{2}, a_2 = \frac{1}{6}, a_3 = -\frac{1}{6}, a_4 = \frac{3}{5}, \ldots$.

In this paper, using the same idea from the well-known sequence γ_n (1.1) to Vernescu's sequence (1.3) and Mortici's sequence (1.7)–(1.8), based on the early works of Mortici and DeTemple, we provide some more general convergent sequences for Euler's constant as follows:

Theorem 1.1. *For $r \in \mathbb{R}^+$, we have the following convergent sequence for Euler's constant, if $r \neq 2$,*

$$L_{r,n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{rn} - \ln n - \ln\left(1 + \frac{a_1}{n} + \frac{a_2}{n^2} + \frac{a_3}{n^3} + \cdots\right), \tag{1.12}$$

where

$$a_1 = \frac{2-r}{2r}, \qquad a_2 = \frac{r^2 - 12r + 12}{24r^2}, \qquad a_3 = \frac{r^3 + 2r^2 - 12r + 8}{48r^3}, \ldots.$$

If $r = 2$, we have

$$L_{2,n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{2n} - \ln n - \ln\left(1 + \frac{b_1}{n^2} + \frac{b_2}{n^4} + \frac{b_3}{n^6} + \cdots\right), \tag{1.13}$$

where

$$b_1 = -\frac{1}{12}, \qquad b_2 = \frac{17}{1440}, \qquad b_3 = -\frac{1727}{362880}, \ldots.$$

In particular, let

$$\begin{aligned}
L_{r,n}^{(0)} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{rn} - \ln n; \\
L_{r,n}^{(1)} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{rn} - \ln n - \ln\left(1 + \frac{a_1}{n}\right); \\
L_{r,n}^{(2)} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{rn} - \ln n - \ln\left(1 + \frac{a_1}{n} + \frac{a_2}{n^2}\right); \\
L_{2,n}^{(1)} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{2n} - \ln n - \ln\left(1 + \frac{b_1}{n^2}\right).
\end{aligned}$$

It is easy to see that $L_{1,n}^{(0)} = \gamma_n$, $L_{2,n}^{(0)} = V_n$, $L_{1,n}^{(1)} = R_n$, $L_{6-2\sqrt{6},n}^{(1)} = u_n$ and $L_{6+2\sqrt{6},n}^{(1)} = v_n$ respectively. Comparing with DeTemple's sequence $(R_n)_{n\geqslant 1}$, Vernescu's sequence $(V_n)_{n\geqslant 2}$, Mortici's sequences $(u_n)_{n\geqslant 1}$, $(v_n)_{n\geqslant 1}$ and $\mu_n(\frac{\sqrt{2}}{2}, \frac{2+\sqrt{2}}{4})$, our new sequences $(L_{2\sqrt{2}-2,n}^{(2)})_{n\geqslant 1}$ and $(L_{2,n}^{(1)})_{n\geqslant 1}$ improve the rate of convergence from n^{-2} and n^{-3} to n^{-4}. In fact, if we need, using Theorem 1.1, we can obtain other convergent sequences which are faster than $L_{2\sqrt{2}-2,n}^{(2)}$ and $L_{2,n}^{(1)}$.

Furthermore, for $L_{6-2\sqrt{6},n}^{(1)}$, $L_{6+2\sqrt{6},n}^{(1)}$, $L_{2\sqrt{2}-2,n}^{(2)}$ and $L_{2,n}^{(1)}$, we also have the following inequalities:

Theorem 1.2. *For all natural numbers n and $n \neq 1$, we have*

$$\frac{\sqrt{6}}{108}\frac{1}{(n+1)^3} < \gamma - L^{(1)}_{6-2\sqrt{6},n} < \frac{\sqrt{6}}{108}\frac{1}{n^3}; \tag{1.14}$$

$$\frac{\sqrt{6}}{108}\frac{1}{(n+1)^3} < L^{(1)}_{6+2\sqrt{6},n} - \gamma < \frac{\sqrt{6}}{108}\frac{1}{(n-1)^3}; \tag{1.15}$$

$$\frac{1}{720}\frac{1}{(n+1)^4} < L^{(2)}_{2\sqrt{2}-2,n} - \gamma < \frac{1}{720}\frac{1}{(n-1)^4}; \tag{1.16}$$

$$\frac{17}{1440}\frac{1}{(n+1)^4} < L^{(1)}_{2,n} - \gamma < \frac{17}{1440}\frac{1}{(n-1)^4}. \tag{1.17}$$

To obtain Theorem 1.1, we need the following lemma which was used in [5–10] and is very useful for constructing asymptotic expansions.

Lemma 1.1. *If $(x_n)_{n\geqslant 1}$ is convergent to zero and there exists the limit*

$$\lim_{n\to\infty} n^s(x_n - x_{n+1}) = l \in [-\infty, +\infty], \tag{1.18}$$

with $s > 1$, then

$$\lim_{n\to\infty} n^{s-1}x_n = \frac{l}{s-1}. \tag{1.19}$$

Lemma 1.1 was first proved by Mortici in [8]. From Lemma 1.1, we can see that the speed of convergence of the sequence $(x_n)_{n\geqslant 1}$ increases together with the value s satisfying (1.18).

The rest of this paper is arranged as follows. In Section 2, we provide the proof of Theorem 1.1. In Section 3, the proof of Theorem 1.2 is given. In Section 4, we give some numerical computations which demonstrate the superiority of our new convergent sequences over DeTemple's sequence, Vernescu's sequence and Mortici's sequences.

2. Proof of Theorem 1.1

First, we deal with (1.12). Based on the argument of Theorem 2.1 in [9] or Theorem 5 in [10], we need to find the value $a_1 \in \mathbb{R}$ which produces the most accurate approximation of the form

$$L^{(1)}_{r,n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{rn} - \ln n - \ln\left(1 + \frac{a_1}{n}\right). \tag{2.1}$$

To measure the accuracy of this approximation, a method is to say that an approximation (2.1) is better if $L^{(1)}_{r,n} - \gamma$ converges to zero faster. Using (2.1) and developing the power series in $1/n$, we have

$$L^{(1)}_{r,n} - L^{(1)}_{r,n+1} = \frac{2-r-2a_1r}{2rn^2} + \frac{r-3+3a_1r+3a_1^2r}{3rn^3} + O\left(\frac{1}{n^4}\right). \tag{2.2}$$

From Lemma 1.1, we know that the speed of convergence of the sequence $(L^{(1)}_{r,n} - \gamma)_{n\geqslant 1}$ is even higher as the value s satisfies (1.18). Thus, using Lemma 1.1, we have:

(i) If $a_1 \neq (2-r)/2r$, then the rate of convergence of the sequence $(L^{(1)}_{r,n} - \gamma)_{n\geqslant 1}$ is n^{-1}, since

$$\lim_{n\to\infty} n\big(L^{(1)}_{r,n} - \gamma\big) = \frac{2-r-2a_1r}{2r} \neq 0.$$

(ii) If $a_1 = (2-r)/2r$, then from (2.2), we have

$$L^{(1)}_{r,n} - L^{(1)}_{r,n+1} = \frac{r^2-12r+12}{12r^2}\frac{1}{n^3} + O\left(\frac{1}{n^4}\right)$$

and the rate of convergence of the sequence $(L_{r,n}^{(1)} - \gamma)_{n\geqslant 1}$ is n^{-2}, since

$$\lim_{n\to\infty} n^2\big(L_{r,n}^{(1)} - \gamma\big) = \frac{r^2 - 12r + 12}{24r^2}.$$

We know that the fastest possible sequence $(L_{r,n}^{(1)})_{n\geqslant 1}$ is obtained only for $a_1 = (2-r)/2r$.

Next, we define the sequence $(L_{r,n}^{(2)})_{n\geqslant 1}$ by the relation

$$L_{r,n}^{(2)} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{rn} - \ln n - \ln\left(1 + \frac{\frac{2-r}{2r}}{n} + \frac{a_2}{n^2}\right). \tag{2.3}$$

Using the same method from (2.1) to (2.2), we have

$$L_{r,n}^{(2)} - L_{r,n+1}^{(2)} = \frac{r^2 - 24a_2r^2 + 12 - 12r}{12r^2n^3} + \frac{6a_2r^3 + 12a_2r^2 + 3r^2 - 4}{4r^3n^4} + O\left(\frac{1}{n^5}\right). \tag{2.4}$$

The fastest possible sequence $(L_{r,n}^{(2)})_{n\geqslant 1}$ is obtained only for $a_2 = (r^2 - 12r + 12)/24r^2$. Then, from (2.4), we have

$$L_{r,n}^{(2)} - L_{r,n+1}^{(2)} = \frac{r^3 + 2r^2 - 12r + 8}{16r^3}\frac{1}{n^4} + O\left(\frac{1}{n^5}\right)$$

and the rate of convergence of the sequence $(L_{r,n}^{(2)} - \gamma)_{n\geqslant 1}$ is n^{-3}, since

$$\lim_{n\to\infty} n^3\big(L_{r,n}^{(2)} - \gamma\big) = \frac{r^3 + 2r^2 - 12r + 8}{48r^3}.$$

Thirdly, we define the sequence $(L_{r,n}^{(3)})_{n\geqslant 1}$ by the relation

$$L_{r,n}^{(3)} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{rn} - \ln n - \ln\left(1 + \frac{\frac{2-r}{2r}}{n} + \frac{\frac{r^2-12r+12}{24r^2}}{n^2} + \frac{a_3}{n}\right). \tag{2.5}$$

Using the same method from (2.1) to (2.2), we have

$$\begin{aligned} L_{r,n}^{(3)} - L_{r,n+1}^{(3)} = {} & \frac{r^3 - 48a_3r^3 + 2r^2 - 12r + 8}{16r^3n^4} \\ & + \frac{5760a_3r^4 - 97r^4 - 240r^3 + 5760a_3r^3 + 1320r^2 - 720}{1440r^4n^5} + O\left(\frac{1}{n^6}\right). \end{aligned} \tag{2.6}$$

The fastest possible sequence $(L_{r,n}^{(3)})_{n\geqslant 1}$ is obtained only for $a_3 = (r^3 + 2r^2 - 12r + 8)/48r^3$. Then, from (2.6), we have

$$L_{r,n}^{(3)} - L_{r,n+1}^{(3)} = \frac{23r^4 + 120r^3 + 120r^2 - 480r + 240}{1440r^4n^5} + O\left(\frac{1}{n^6}\right)$$

and the rate of convergence of the sequence $(L_{r,n}^{(3)} - \gamma)_{n\geqslant 1}$ is n^{-4}, since

$$\lim_{n\to\infty} n^4\big(L_{r,n}^{(3)} - \gamma\big) = \frac{23r^4 + 120r^3 + 120r^2 - 480r + 240}{5760r^4}.$$

By induction, we have $a_4 = (23r^4 + 120r^3 + 120r^2 - 480r + 240)/5760r^4, \cdots$, the new sequence (1.12) is obtained.

Next, we deal with (1.13). We need to find the value $b_1 \in \mathbb{R}$ which produces the most accurate approximation of the form

$$L_{2,n}^{(1)} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{2n} - \ln n - \ln\left(1 + \frac{b_1}{n^2}\right). \tag{2.7}$$

Using (2.7) and developing the power series in $1/n$, we have

$$L_{2,n}^{(1)} - L_{2,n+1}^{(1)} = \frac{-1-12b_1}{6n^3} + \frac{1+12b_1}{4n^4} + \frac{-3-40b_1+20b_1^2}{10n^5} + O\left(\frac{1}{n^6}\right). \tag{2.8}$$

From Lemma 1.1, we know that the speed of convergence of the sequence $(L_{2,n}^{(1)} - \gamma)_{n \geqslant 1}$ is even higher as the value s satisfies (1.18). Thus, using Lemma 1.1, we have:
(i) If $b_1 \neq -1/12$, then the rate of convergence of the sequence $(L_{2,n}^{(1)} - \gamma)_{n \geqslant 1}$ is n^{-2}, since

$$\lim_{n\to\infty} n^2\bigl(L_{2,n}^{(1)} - \gamma\bigr) = \frac{-1-12b_1}{12} \neq 0.$$

(ii) If $b_1 = -1/12$, then from (2.8), we have

$$L_{2,n}^{(1)} - L_{2,n+1}^{(1)} = \frac{17}{360}\frac{1}{n^5} + O\left(\frac{1}{n^6}\right)$$

and the rate of convergence of the sequence $(L_{2,n}^{(1)} - \gamma)_{n \geqslant 1}$ is n^{-4}, since

$$\lim_{n\to\infty} n^4\bigl(L_{2,n}^{(1)} - \gamma\bigr) = \frac{17}{1440}.$$

We know that the fastest possible sequence $(L_{2,n}^{(1)})_{n \geqslant 1}$ is obtained only for $b_1 = -1/12$.

Next, we define the sequence $(L_{2,n}^{(2)})_{n \geqslant 1}$ by the relation

$$L_{2,n}^{(2)} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{2n} - \ln n - \ln\left(1 + \frac{-\frac{1}{12}}{n^2} + \frac{b_2}{n^4}\right). \tag{2.9}$$

Using the same method from (2.7) to (2.8), we have

$$L_{2,n}^{(2)} - L_{2,n+1}^{(2)} = \frac{17-1440b_2}{360n^5} + \frac{-17+1440b_2}{144n^6} + \frac{1291-12\,3984b_2}{6048n^7} + O\left(\frac{1}{n^8}\right). \tag{2.10}$$

The fastest possible sequence $(L_{2,n}^{(2)})_{n \geqslant 1}$ is obtained only for $b_2 = 17/1440$. Then, from (2.10), we have

$$L_{2,n}^{(2)} - L_{2,n+1}^{(2)} = -\frac{1727}{60\,480}\frac{1}{n^7} + O\left(\frac{1}{n^8}\right)$$

and the rate of convergence of the sequence $(L_{2,n}^{(2)} - \gamma)_{n \geqslant 1}$ is n^{-6}, since

$$\lim_{n\to\infty} n^6\bigl(L_{r,n}^{(2)} - \gamma\bigr) = -\frac{1727}{362\,880}.$$

By induction, we have $b_3 = -1727/362\,880, \cdots$, and the new sequence (1.13) is obtained.

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3. Proof of Theorem 1.2

First, we prove (1.14). Based on the argument of theorem in [1] or the method in [2], it is easy to obtain

$$\gamma - L^{(1)}_{6-2\sqrt{6},n} = \sum_{k=n}^{\infty}\left(L^{(1)}_{6-2\sqrt{6},k+1} - L^{(1)}_{6-2\sqrt{6},k}\right) = \sum_{k=n}^{\infty} f(k). \tag{3.1}$$

It is easy to obtain

$$f'(x) = -F(x)/G(x), \tag{3.2}$$

where

$$G(x) = (3-\sqrt{6})x^2(x+1)^2(2\sqrt{6}x - 6x - 4 + \sqrt{6})(2\sqrt{6}x - 6x + 2 - \sqrt{6})$$

and

$$F(x) = 36\sqrt{6}x - 88x + 29\sqrt{6} - 71.$$

For the upper bound in (1.14), combining

$$\begin{aligned}\sqrt{6}G(x) - 9(x+1/2)^5F(x) &= (891 - 363\sqrt{6})x^5 + \big(5379/2 - (2193/2)\sqrt{6}\,\big)x^4 \\ &\quad + \big(5655/2 - (2307/2)\sqrt{6}\,\big)x^3 + \big(4953/4 - (1011/2)\sqrt{6}\,\big)x^2 \\ &\quad + \big(3591/16 - (1467/16)\sqrt{6}\,\big)x + 639/32 - (261/32)\sqrt{6} \\ &> 0, \end{aligned} \tag{3.3}$$

as $x \geqslant 1$, we have

$$-f'(x) = \frac{F(x)}{G(x)} \leqslant \frac{\sqrt{6}}{9(x+1/2)^5}. \tag{3.4}$$

Since $f(\infty) = 0$, we have

$$f(k) = -\int_k^{\infty} f'(x)dx \leqslant \frac{\sqrt{6}}{9}\int_k^{\infty}\left(x+\frac{1}{2}\right)^{-5}dx = \frac{\sqrt{6}}{36}\left(k+\frac{1}{2}\right)^{-4} \leqslant \frac{\sqrt{6}}{36}\int_k^{k+1} x^{-4}dx, \tag{3.5}$$

where we use the following fact

$$\int_k^{k+1} x^{-4}dx - \left(k+\frac{1}{2}\right)^{-4} = \frac{40k^4 + 80k^3 + 51k^2 + 11k + 1}{3k^3(k+1)^3(2k+1)^4} > 0$$

in the last inequality in (3.5). Combining (3.1) and (3.5), for all natural numbers n, we have

$$\gamma - L^{(1)}_{6-2\sqrt{6},n} \leqslant \sum_{k=n}^{\infty}\frac{\sqrt{6}}{36}\int_k^{k+1} x^{-4}dx = \frac{\sqrt{6}}{36}\int_n^{\infty} x^{-4}dx = \frac{\sqrt{6}}{108n^3}. \tag{3.6}$$

For the lower bound, combining (3.2), we have

$$-f'(x) = \frac{F(x)}{G(x)} \geqslant \frac{\sqrt{6}}{9(x+1)^5}, \tag{3.7}$$

where we use the following fact, for $x \geqslant 1$,

$$\begin{aligned}\sqrt{6}G(x) - 9(x+1)^5 F(x) &= (2871 - 1173\sqrt{6})x^5 + (10\,227 - 4179\sqrt{6})x^4 \\ &\quad + (14\,550 - 5946\sqrt{6})x^3 + (10\,542 - 4308\sqrt{6})x^2 \\ &\quad + (3987 - 1629\sqrt{6})x + 639 - 261\sqrt{6} \\ &< 0. \end{aligned} \tag{3.8}$$

Combining (3.7), we have

$$f(k) = -\int_k^\infty f'(x)dx \geqslant \frac{\sqrt{6}}{9}\int_k^\infty (x+1)^{-5}dx = \frac{\sqrt{6}}{36}(k+1)^{-4} \geqslant \frac{\sqrt{6}}{36}\int_{k+1}^{k+2} x^{-4}dx. \tag{3.9}$$

Combining (3.1) and (3.9), we have

$$\gamma - L^{(1)}_{6-2\sqrt{6},n} \geqslant \sum_{k=n}^\infty \frac{\sqrt{6}}{36}\int_{k+1}^{k+2} x^{-4}dx = \frac{\sqrt{6}}{36}\int_{n+1}^\infty x^{-4}dx = \frac{\sqrt{6}}{108(n+1)^3}. \tag{3.10}$$

Combining (3.6) and (3.10), we complete the proof of (1.14).

Next, we prove (1.15). It is easy to obtain

$$L^{(1)}_{6+2\sqrt{6},n} - \gamma = \sum_{k=n}^\infty \left(L^{(1)}_{6+2\sqrt{6},k} - L^{(1)}_{6+2\sqrt{6},k+1}\right) = \sum_{k=n}^\infty h(k). \tag{3.11}$$

It is easy to obtain

$$h'(x) = -U(x)/V(x), \tag{3.12}$$

where

$$V(x) = (3+\sqrt{6})x^2(x+1)^2(6x + 2\sqrt{6}x - 2 - \sqrt{6})(6x + 2\sqrt{6}x + 4 + \sqrt{6})$$

and

$$U(x) = 88x + 36\sqrt{6}x + 29\sqrt{6} + 71.$$

For the upper bound in (1.15), combining

$$\begin{aligned}\sqrt{6}V(x) - 9x^5U(x) &= (447\sqrt{6} + 1089)x^5 + (888 + 366\sqrt{6})x^4 - (240 + 96\sqrt{6})x^3 - (78\sqrt{6} + 192)x^2 \\ &> 0, \end{aligned} \tag{3.13}$$

for $x \geqslant 1$, we have

$$-h'(x) = \frac{U(x)}{V(x)} \leqslant \frac{\sqrt{6}}{9x^5}. \tag{3.14}$$

Since $h(\infty)=0$, we have

$$h(k)=-\int_k^\infty h'(x)dx\leqslant\frac{\sqrt{6}}{9}\int_k^\infty x^{-5}dx=\frac{\sqrt{6}}{36}k^{-4}\leqslant\frac{\sqrt{6}}{36}\int_{k-1}^{k}x^{-4}dx. \tag{3.15}$$

Combining (3.11) and (3.15), for all natural numbers $n\geqslant 2$, we have

$$L^{(1)}_{6+2\sqrt{6},n}-\gamma\leqslant\sum_{k=n}^{\infty}\frac{\sqrt{6}}{36}\int_{k-1}^{k}x^{-4}dx=\frac{\sqrt{6}}{36}\int_{n-1}^{\infty}x^{-4}dx=\frac{\sqrt{6}}{108(n-1)^3}. \tag{3.16}$$

For the lower bound, combining (3.12), we have

$$-h'(x)=\frac{U(x)}{V(x)}\geqslant\frac{\sqrt{6}}{9(x+1)^5}, \tag{3.17}$$

where we use the following fact, for $x\geqslant 1$,

$$\begin{aligned}\sqrt{6}V(x)-9(x+1)^5U(x)&=-(1173\sqrt{6}+2871)x^5-(4179\sqrt{6}+10\,227)x^4\\&\quad-(5946\sqrt{6}+14\,550)x^3-(4308\sqrt{6}+10\,542)x^2\\&\quad-(3987+1629\sqrt{6})x-639-261\sqrt{6}\\&<0.\end{aligned} \tag{3.18}$$

Combining (3.17), we have

$$h(k)=-\int_k^\infty h'(x)dx\geqslant\frac{\sqrt{6}}{9}\int_k^\infty(x+1)^{-5}dx=\frac{\sqrt{6}}{36}(k+1)^{-4}\geqslant\frac{\sqrt{6}}{36}\int_{k+1}^{k+2}x^{-4}dx. \tag{3.19}$$

Combining (3.11) and (3.19), we have

$$L^{(1)}_{6+2\sqrt{6},n}-\gamma\geqslant\sum_{k=n}^{\infty}\frac{\sqrt{6}}{36}\int_{k+1}^{k+2}x^{-4}dx=\frac{\sqrt{6}}{36}\int_{n+1}^{\infty}x^{-4}dx=\frac{\sqrt{6}}{108(n+1)^3}. \tag{3.20}$$

Combining (3.16) and (3.20), we complete the proof of (1.15).

Next, we prove (1.16). It is easy to obtain

$$L^{(2)}_{2\sqrt{2}-2,n}-\gamma=\sum_{k=n}^{\infty}\left(L^{(2)}_{2\sqrt{2}-2,k}-L^{(2)}_{2\sqrt{2}-2,k+1}\right)=\sum_{k=n}^{\infty}l(k). \tag{3.21}$$

It is easy to obtain

$$l'(x)=-W(x)/K(x), \tag{3.22}$$

where

$$\begin{aligned}K(x)&=2(\sqrt{2}-1)x^2(x+1)^2\left(12\sqrt{2}x^2-18x^2-24x+15\sqrt{2}x-9+5\sqrt{2}\right)\\&\quad\times\left(12\sqrt{2}x^2-18x^2+12x-9\sqrt{2}x-3+2\sqrt{2}\right)\end{aligned}$$

and

$$W(x) = 58\sqrt{2}x^2 - 82x^2 - 270\sqrt{2}x + 382x + 273 - 193\sqrt{2}.$$

For the upper bound in (1.16), combining

$$\begin{aligned}
&K(x) - 36(x-1/2)^6 W(x) \\
&\quad = (21\,384\sqrt{2} - 30\,240)x^7 + (36\,942 - 26\,106\sqrt{2})x^6 + (20\,622\sqrt{2} - 29\,166)x^5 \\
&\qquad + \big(3133/2 - (2197/2)\sqrt{2}\,\big)x^4 + (10\,996 - 7774\sqrt{2})x^3 + \big((38\,549/8)\sqrt{2} - 54\,521/8\big)x^2 \\
&\qquad + \big(13\,023/8 - (9207/8)\sqrt{2}\,\big)x + (1737/16)\sqrt{2} - 2457/16 \\
&\quad > 0,
\end{aligned} \tag{3.23}$$

for $x \geqslant 1$, we have

$$-l'(x) = \frac{W(x)}{K(x)} \leqslant \frac{1}{36(x-1/2)^6}. \tag{3.24}$$

Since $l(\infty) = 0$, we have

$$l(k) = -\int_k^\infty l'(x)dx \leqslant \frac{1}{36}\int_k^\infty (x-1/2)^{-6}dx = \frac{1}{180}(k-1/2)^{-5} \leqslant \frac{1}{180}\int_{k-1}^{k} x^{-5}dx. \tag{3.25}$$

Combining (3.21) and (3.25), for all natural numbers $n \geqslant 2$, we have

$$L^{(2)}_{2\sqrt{2}-2,n} - \gamma \leqslant \sum_{k=n}^\infty \frac{1}{180}\int_{k-1}^{k} x^{-5}dx = \frac{1}{180}\int_{n-1}^\infty x^{-5}dx = \frac{1}{720(n-1)^4}. \tag{3.26}$$

For the lower bound, combining (3.12), we have

$$-l'(x) = \frac{W(x)}{K(x)} \geqslant \frac{1}{36(x+1)^6}, \tag{3.27}$$

where we use the following fact, for $x \geqslant 1$,

$$\begin{aligned}
&K(x) - 36(x+1)^6 W(x) \\
&\quad = (2592\sqrt{2} - 3672)x^7 + (37\,884\sqrt{2} - 53\,616)x^6 + (145\,524\sqrt{2} - 205\,908)x^5 \\
&\qquad + (266\,404\sqrt{2} - 376\,906)x^4 + (272\,324\sqrt{2} - 385\,256)x^3 + (160\,612\sqrt{2} - 227\,206)x^2 \\
&\qquad + (51\,408\sqrt{2} - 72\,720)x + 6948\sqrt{2} - 9828 \\
&\quad < 0.
\end{aligned} \tag{3.28}$$

Combining (3.27), we have

$$l(k) = -\int_k^\infty l'(x)dx \geqslant \frac{1}{36}\int_k^\infty (x+1)^{-6}dx = \frac{1}{180}(k+1)^{-5} \geqslant \frac{1}{180}\int_{k+1}^{k+2} x^{-5}dx. \tag{3.29}$$

Combining (3.21) and (3.29), we have

$$L_{2\sqrt{2}-2,n}^{(2)} - \gamma \geqslant \sum_{k=n}^{\infty} \frac{1}{180} \int_{k+1}^{k+2} x^{-5} dx = \frac{1}{180} \int_{n+1}^{\infty} x^{-5} dx = \frac{1}{720(n+1)^4}. \tag{3.30}$$

Combining (3.26) and (3.30), we complete the proof of (1.16).

Finally, we prove (1.17). It is easy to obtain

$$L_{2,n}^{(1)} - \gamma = \sum_{k=n}^{\infty} \left(L_{2,k}^{(1)} - L_{2,k+1}^{(1)}\right) = \sum_{k=n}^{\infty} g(k), \tag{3.31}$$

where

$$g'(x) = -\frac{68x^2 + 68x + 11}{2x^2(12x^2 - 1)(x+1)^2(12x^2 + 24x + 11)}. \tag{3.32}$$

For the upper bound in (1.17), it is easy to obtain

$$-g'(x) \leqslant \frac{17}{72x^6}, \tag{3.33}$$

for $x \geqslant 1$. Since $g(\infty) = 0$, combining (3.33), we have

$$g(k) = -\int_{k}^{\infty} g'(x) dx \leqslant \frac{17}{72} \int_{k}^{\infty} x^{-6} dx = \frac{17}{360} k^{-5} \leqslant \frac{17}{360} \int_{k-1}^{k} x^{-5} dx. \tag{3.34}$$

Combining (3.31) and (3.34), for all natural numbers $n \geqslant 2$, we have

$$L_{2,n}^{(1)} - \gamma \leqslant \sum_{k=n}^{\infty} \frac{17}{360} \int_{k-1}^{k} x^{-5} dx = \frac{17}{360} \int_{n-1}^{\infty} x^{-5} dx = \frac{17}{1440(n-1)^4}. \tag{3.35}$$

For the lower bound, we have

$$-g'(x) \geqslant \frac{17}{72(x+1)^6}, \tag{3.36}$$

for $x \geqslant 1$. Combining (3.36), we have

$$g(k) = -\int_{k}^{\infty} g'(x) dx \geqslant \frac{17}{72} \int_{k}^{\infty} (x+1)^{-6} dx = \frac{17}{360} (k+1)^{-5} \geqslant \frac{17}{360} \int_{k+1}^{k+2} x^{-5} dx. \tag{3.37}$$

Combining (3.31) and (3.37), we have

$$L_{2,n}^{(1)} - \gamma \geqslant \sum_{k=n}^{\infty} \frac{17}{360} \int_{k+1}^{k+2} x^{-5} dx = \frac{17}{360} \int_{n+1}^{\infty} x^{-5} dx = \frac{17}{1440(n+1)^4}. \tag{3.38}$$

Combining (3.35) and (3.38), we complete the proof of (1.17).

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Table 1
Simulations for $L_{2,n}^{(1)}$, R_n, V_n, $\mu_n(\frac{\sqrt{2}}{2}, \frac{2+\sqrt{2}}{4})$.

n	$L_{2,n}^{(1)} - \gamma$	$R_n - \gamma$	$\gamma - V_n$	$\mu_n(\frac{\sqrt{2}}{2}, \frac{2+\sqrt{2}}{4}) - \gamma$
10	1.1768×10^{-6}	3.7733×10^{-4}	8.3250×10^{-4}	1.1807×10^{-5}
25	3.0207×10^{-8}	6.4061×10^{-5}	1.3331×10^{-4}	8.6183×10^{-7}
50	1.8886×10^{-9}	1.6337×10^{-5}	3.3332×10^{-5}	1.1265×10^{-7}
100	1.1805×10^{-10}	4.1252×10^{-6}	8.3333×10^{-6}	1.4402×10^{-8}
250	3.0222×10^{-12}	6.6401×10^{-7}	1.3333×10^{-6}	9.3431×10^{-10}
1000	1.1806×10^{-14}	4.1625×10^{-8}	8.3333×10^{-8}	1.4698×10^{-11}

Table 2
Simulations for $L_{2\sqrt{2}-2,n}^{(2)}$, u_n and v_n.

n	$L_{2\sqrt{2}-2,n}^{(2)} - \gamma$	$\gamma - u_n$	$v_n - \gamma$
10	1.7275×10^{-7}	2.1179×10^{-5}	2.4228×10^{-5}
25	3.9353×10^{-9}	1.4127×10^{-6}	1.4909×10^{-6}
50	2.3444×10^{-10}	1.7901×10^{-7}	1.8390×10^{-7}
100	1.4276×10^{-11}	2.2528×10^{-8}	2.2833×10^{-8}
250	3.5956×10^{-13}	1.4476×10^{-9}	1.4555×10^{-9}
1000	1.3929×10^{-15}	2.2665×10^{-11}	2.2696×10^{-11}

4. Numerical computation

In this section, we give two tables to demonstrate the superiority of our new convergent sequences $L_{2\sqrt{2}-2,n}^{(2)}$ and $L_{2,n}^{(1)}$ over DeTemple's sequence $(R_n)_{n\geqslant 1}$, Vernescu's sequence $(V_n)_{n\geqslant 1}$, Mortici's sequences $(\mu_n(\frac{\sqrt{2}}{2}, \frac{2+\sqrt{2}}{4}))_{n\geqslant 1}$, $(u_n)_{n\geqslant 1}$ and $(v_n)_{n\geqslant 1}$, respectively.

Combining Theorem 1.1 and Theorem 1.2, we have Table 1 and Table 2.

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