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Subspaces of codimension two with large projection constants

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This paper is dedicated to Eusebio Corbacho Rosas, advisor of the first author, on the occasion of his 65th birthday

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ABSTRACT

Let V be an n-dimensional real Banach space and let $\lambda(V)$ denote its absolute projection constant. For any $N \in \mathbb{N}$, $N \geq n$ define

$$\lambda_n^N = \sup\{\lambda(V)\colon \dim(V) = n,\ V \subset l_\infty^{(N)}\}.$$

The aim of this paper is to determine minimal projections with respect to l_1-norm as well as with respect to l_∞-norm for subspaces given by solutions of certain extremal problems. As an application we show that for any $n, N \in \mathbb{N}$, $N \geq n$ there exists an n-dimensional subspace $V_n \subset l_1^{(N)}$ such that

$$\lambda_n^N = \lambda(V_n, l_1^{(N)}).$$

Also we calculate relative and absolute projection constants of some subspaces of codimension two in $l_1^{(N)}$ and $l_\infty^{(N)}$ for $N \geq 3$ being odd natural number. Moreover, we show that for any odd natural number $n \geq 3$,

$$\lambda_n^{n+1} < \max_{x\in[0,1]} f_n(x) \leq \lambda_n^{n+2},$$

where

$$f_n(x) = \frac{2n}{n+1}(1-x) + \frac{1}{2}\left(x - 2\frac{1-x}{n+1} + \sqrt{\left(2\frac{1-x}{n+1} - x\right)^2 + 4(1-x)x}\right).$$

Also for any $n \in \mathbb{N}$ $x_n \in [0,1]$ satisfying

$$f_n(x_n) = \max_{x\in[0,1]} f_n(x)$$

will be calculated.

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1. Introduction

Let X be a real Banach space and let $V \subset X$ be a finite-dimensional subspace. A linear, continuous mapping $P : X \to V$ is called *a projection* if $P|_V = id|_V$. Denote by $\mathcal{P}(X, V)$ the set of all projections from X onto V. Set

$$\lambda(V, X) = \inf\big\{\|P\|:\ P \in \mathcal{P}(X, V)\big\}$$

and

$$\lambda(V) = \sup\big\{\lambda(V, X):\ V \subset X\big\}.$$

A projection $P_o \in \mathcal{P}(X, V)$ is called *minimal* if

$$\|P_o\| = \lambda(V, X).$$

The constant $\lambda(V, X)$ is called the *relative projection constant* and $\lambda(V)$ the *absolute projection constant.* Minimal projections in the context of functional analysis and approximation theory have been extensively studied by many authors (see e.g., [1–16,18–31,33–35]). Mainly the problems of existence of minimal projections, uniqueness of minimal projections, finding concrete formulas for minimal projections and estimates of the constant $\lambda(V, X)$ were considered.

General bounds for absolute projection constants were studied by many authors (see e.g. [4–7,18–21,32]). It is well-known (see e.g. [36]) that if V is a finite-dimensional space then

$$\lambda(V) = \lambda\big(I(V), l_\infty\big),$$

where $I(V)$ denotes any isometric copy of V in l_∞. Denote for any $n \in \mathbb{N}$

$$\lambda_n = \sup\big\{\lambda(V):\ \dim(V) = n\big\}$$

and for any $N \in \mathbb{N}$, $N \geq n$

$$\lambda_n^N = \sup\big\{\lambda(V):\ V \subset l_\infty^{(N)}\big\}.$$

By the Kadec–Snobar Theorem (see [17]) $\lambda(V) \leq \sqrt{n}$ for any $n \in \mathbb{N}$. However, determination of the constant λ_n seems to be difficult.

The aim of this paper is to determine minimal projections with respect to l_1-norm as well as for l_∞-norm for subspaces given by solutions of certain extremal problems. As an application we show that for any $n, N \in \mathbb{N}$, $N \geq n$ there exists an n-dimensional subspace $V_n \subset l_1^{(N)}$ such that

$$\lambda_n^N = \lambda\big(V_n, l_1^{(N)}\big).$$

Also we show that for any odd natural number $n \geq 3$,

$$\lambda_n^{n+1} < \max_{x \in [0,1]} f_n(x) \leq \lambda_n^{n+2},$$

where

$$f_n(x) = \frac{2n}{n+1}(1 - x) + \frac{1}{2}\left(x - 2\frac{1 - x}{n + 1} + \sqrt{\left(2\frac{1 - x}{n + 1} - x\right)^2 + 4(1 - x)x}\right).$$

Moreover we determine $x_n \in [0,1]$ such that

$$f_n(x_n) = \max_{x\in[0,1]} f_n(x).$$

2. Preliminary results

In this section mainly we consider the following problem. Maximize a function $f : \mathbb{R}^N \times (\mathbb{R}^N)^n \to R$ defined by:

$$f\big((u_1,\dots,u_N),x^1,\dots,x^n\big) = \sum_{i,j=1}^{N} u_i u_j \big|\langle x_i,x_j\rangle_n\big| \tag{1}$$

under constraints:

$$\langle x^i,x^j\rangle_N = \delta_{ij}, \quad 1 \le i \le j \le n; \tag{2}$$

$$\sum_{j=1}^{N} u_j^2 = 1. \tag{3}$$

Here for $j = 1,\dots,N$, $x_j = ((x^1)_j,\dots,(x^n)_j)$, $\langle w,z\rangle_n = \sum_{j=1}^{n} w_j z_j$ for any $w = (w_1,\dots,w_n), z = (z_1,\dots,z_n) \in \mathbb{R}^n$ and $\langle p,q\rangle_N = \sum_{j=1}^{N} p_j q_j$ for any $p = (p_1,\dots,p_N), q = (q_1,\dots,q_N) \in \mathbb{R}^N$. Also we will work with

$$f_A\big((u_1,\dots,u_N),x^1,\dots,x^n\big) = \sum_{i,j=1}^{N} u_i u_j a_{ij}\langle x_i,x_j\rangle_n, \tag{4}$$

where $A = \{a_{ij}\}$ is a fixed $N \times N$ symmetric matrix. In the sequel we need the following

Lemma 2.1. *(See e.g. [6].) Let $x^1,\dots,x^n \in \mathbb{R}^N$ and $u \in \mathbb{R}^N$ satisfy (2), (3). Set $V = \operatorname{span}[x^1,\dots,x^n]$. Assume $v^1,\dots,v^n$ is an orthonormal basis of V (with respect to $\langle\cdot,\cdot\rangle_N$). Then*

$$f_A\big((u_1,\dots,u_N),x^1,\dots,x^n\big) = f_A\big((u_1,\dots,u_N),v^1,\dots,v^n\big)$$

and

$$f\big((u_1,\dots,u_N),x^1,\dots,x^n\big) = f\big((u_1,\dots,u_N),v^1,\dots,v^n\big)$$

for any $N \times N$ matrix A.

For a convenience of a reader we state below [6, Lemma 2.4] with a proof.

Lemma 2.2. *Let $n,N \in \mathbb{N}$, $N \ge n$. Fix $u = (u_1,\dots,u_N) \in \mathbb{R}^N$ with nonnegative coordinates. Let us consider a function $f : \mathbb{R}^{nN} \to \mathbb{R}$ given by*

$$f\big(x^1,\dots,x^n\big) = \sum_{i,j=1}^{N} u_i u_j \big|\langle x_i,x_j\rangle\big|_n,$$

where $x^i \in \mathbb{R}^N$ for $i = 1,\dots,n$. Assume that $y^1,\dots,y^n \in \mathbb{R}^N$ are so chosen that

$$f\big(y^1,\dots,y^n\big) = \max\big\{f\big(x^1,\dots,x^n\big)\colon\ \big(x^1,\dots,x^n\big) \text{ satisfying (2)}\big\}.$$

Let $A \in \mathbb{R}^{N\times N}$ be a matrix defined by

$$a_{ij} = \operatorname{sgn}\bigl(\langle y_i, y_j\rangle_n\bigr) \tag{5}$$

for $i, j = 1, \ldots, N$ ($\operatorname{sgn}(0) = 1$ by definition). Define $B \in \mathbb{R}^{N\times N}$ by

$$b_{ij} = u_i u_j a_{ij} \tag{6}$$

for $i, j = 1, \ldots, N$. Let

$$b_1 \geq b_2 \geq \cdots \geq b_N$$

denote the eigenvalues of B. (Since B is symmetric all of them are real.) Then there exist orthogonal (with respect to $\langle\cdot,\cdot\rangle_N$) eigenvectors of B $w^1, \ldots, w^n \in \mathbb{R}^N$ corresponding to $b_1, \ldots, b_n$ and satisfying (2) such that

$$f\bigl(w^1, \ldots, w^n\bigr) = f\bigl(y^1, \ldots, y^n\bigr) = \sum_{j=1}^{n} b_j.$$

Set

$$f_1\bigl(x^1, \ldots, x^n\bigr) = \sum_{i,j=1}^{N} b_{ij}\langle x_i, x_j\rangle_n.$$

If $y^1, \ldots, y^n \in \mathbb{R}^N$ are such that

$$f_1\bigl(y^1, \ldots, y^n\bigr) = \max\{f_1,\ \textit{under constraint (2)}\} = \max\{f,\ \textit{under constraint (2)}\}$$

and $b_n > b_{n+1}$ then $\operatorname{span}[y^i : i = 1, \ldots, n] = \operatorname{span}[w^i : i = 1, \ldots, n]$.

Proof. Since u_j are nonnegative,

$$f_1\bigl(x^1, \ldots, x^n\bigr) \leq f\bigl(x^1, \ldots, x^n\bigr)$$

for any $x^1, \ldots, x^n \in \mathbb{R}^N$. Moreover,

$$f_1\bigl(y^1, \ldots, y^n\bigr) = f\bigl(y^1, \ldots, y^n\bigr).$$

Hence f_1 attains its maximum under constraints (2) at $(y^1, \ldots, y^n)$. We now apply the Lagrange Multiplier Theorem to the function f_1. This is possible since f_1 is a C^∞ function. Notice that by [22, p. 261] $\operatorname{rank}(G'(y^1, \ldots, y^n)) = n(n+1)/2$ where G' is the $n(n+1)/2 \times nN$ matrix associated with conditions (2). Consequently there exist Lagrange multipliers k_{ij}, $1 \leq i \leq j \leq n$, such that

$$\frac{\partial(f_1 - \sum_{1\leq i\leq j\leq n} k_{ij}G_{ij})}{\partial(x^l)_m}\bigl(y^1, \ldots, y^n\bigr) = 0 \tag{7}$$

for $l = 1, \ldots, n$, $m = 1, \ldots, N$, where $G_{ij}(x^1, \ldots, x^n) = \langle x^i, x^j\rangle_N - \delta_{ij}$. Let us define for $i, j \in \{1, \ldots, n\}$, $\gamma_{ij} = k_{ij}/2$ if $i < j$, $\gamma_{ij} = k_{ji}/2$, if $j < i$ and $\gamma_{ii} = k_{ii}$. Hence the system (7) can be rewritten (compare with [22, p. 262, formula (3.14)]) as:

$$B\bigl(y^m\bigr) = \sum_{i=1}^{n} \gamma_{mi} y^i \tag{8}$$

for $m = 1, \dots, n$. Let $\Gamma = \{\gamma_{ij},\ i,j = 1,\dots,n\}$. Observe that Γ is a symmetric $n \times n$ matrix. Hence it has real eigenvalues $a_1, \dots, a_n$. Without loss of generality we can assume that

$$a_1 \geq a_2 \geq \cdots \geq a_n. \tag{9}$$

Let $V = [v_{ij}]$ be the $n \times n$ orthonormal matrix consisting of eigenvectors of Γ. Then

$$V^T \Gamma V = D, \tag{10}$$

where D is a diagonal matrix with $d_{ii} = a_i$ for $i = 1, \dots, n$. Now we show that

$$a_i = b_i \tag{11}$$

for $i = 1, \dots, n$. First we prove that a_m, $m = 1, \dots, n$, are also eigenvalues of B. To do this, fix $m \in \{1, \dots, n\}$. Define

$$w^m = \sum_{j=1}^n v_{jm} y^j. \tag{12}$$

We show that $Bw^m = a_m w^m$. Note that

$$\begin{aligned} Bw^m &= B\left(\sum_{j=1}^n v_{jm} y^j\right) = \sum_{j=1}^n v_{jm} B\left(y^j\right) = \sum_{j=1}^n v_{jm}\left(\sum_{i=1}^n \gamma_{ji} y^i\right) \\ &= \sum_{i=1}^n \left(\sum_{j=1}^n v_{jm}\gamma_{ji}\right) y^i = \sum_{i=1}^n \left(\sum_{j=1}^n v_{jm}\gamma_{ij}\right) y^i = \sum_{i=1}^n (\Gamma V)_{im} y^i \end{aligned}$$

(by(10))

$$= \sum_{i=1}^n (VD)_{im} y^i = \sum_{i=1}^n v_{im} a_m y^i a_m \left(\sum_{i=1}^n v_{im} y^i\right) = a_m w^m.$$

Hence for $m = 1, \dots, n$ a_m are eigenvalues of B with the corresponding vectors w^m. By Lemma 2.1, the vectors w^j satisfy (2). Notice that by (12) and Lemma 2.1

$$f_1\left(y^1, \dots, y^n\right) = f_1\left(w^1, \dots, w^n\right).$$

Since for any $m = 1, \dots, n$ and $i = 1, \dots, N$,

$$\left(Bw^m\right)_i = a_m \left(w^m\right)_i,$$

multiplying each of the above equations by $(w^m)_i$ and summing them up we get that

$$\sum_{j=1}^n a_j = f_1\left(w^1, \dots, w^n\right) = f_1\left(y^1, \dots, y^n\right) = f\left(y^1, \dots, y^n\right).$$

If $a_i \neq b_i$ for some $i \in \{1, \dots, n\}$, let $v^1, \dots, v^n$ be the orthogonal eigenvectors of B satisfying (2) corresponding to $b_1, \dots, b_n$. Reasoning as above, we get

$$f\left(v^1,\dots,v^n\right) \geq \sum_{i,j=1}^{N} u_i u_j \operatorname{sgn}\left(\langle y_i, y_j\rangle_n\right)\langle v_i, v_j\rangle_n = \sum_{i=1}^{n} b_i > \sum_{i=1}^{n} a_i = f\left(y^1,\dots,y^n\right);$$

a contradiction. The fact that $\operatorname{span}[y^i : i = 1,\dots,n] = \operatorname{span}[w^i : i = 1,\dots,n]$ follows from (12) and invertibility of the matrix V. □

In the sequel we also need the following results.

Theorem 2.1. *(See [6].) Let $n, N \in \mathbb{N}$, $N \geq n$. Then*

$$\lambda_n^N = \max\left\{ \sum_{i,j=1}^{N} u_i u_j \left|\langle x_i, x_j\rangle\right|_n \colon \left(x^1,\dots,x^n\right) \text{ satisfying (2) and (3)} \right\}.$$

Lemma 2.3. *(See e.g. [6].) For any $n \geq 2$,*

$$\lambda_n^{n+1} = 2 - 2/(n+1).$$

Moreover, $\lambda_n^{n+1} = \lambda(\ker(f), l_\infty^{(n+1)})$ if and only if $f = c(\pm 1,\dots,\pm 1)$, where c is a positive constant.

Theorem 2.2. *(See [8] or [9].) Let X be a Banach space and let V be a finite-dimensional subspace of X. Let B_{X^*} ($B_{X^{**}}$ resp.) denote the closed unit ball in X^* (in X^{**} resp.). For $P \in \mathcal{P}(X,V)$ put*

$$\operatorname{Ext}_P = \left\{\left(x^*, x^{**}\right) \in B_{X^*} \times B_{X^{**}} \colon x^*\left(P^{**}x^{**}\right) = \|P\|\right\},$$

where P^{**} is the second adjoint extension of P. Then P is a minimal projection from X onto V if and only if there exists a Borel probability measure μ on $B_{X^*} \times B_{X^{**}}$ with $\operatorname{supp}(\mu) \subset \operatorname{Ext}(P)$ (we consider $B_{X^*} \times B_{X^{**}}$ with the weak $\times$ weak** topology) such that an operator $E_P : X \to X^{**}$ defined by*

$$E_P(z) = \int_{\operatorname{Ext}_P} \left(x^* \otimes x^{**}\right) z \, d\mu\left(x^*, x^{**}\right)$$

for $z \in X$ maps V into V. Here $x^* \otimes x^{**}$ denotes the one-dimensional operator from X into X^{**} defined by

$$\left(x^* \otimes x^{**}\right) z = x^*(z) x^{**}.$$

Now we reformulate Theorem 2.2 in the case of $X = l_1^{(n)}$ and $X = l_\infty^{(n)}$ in a simpler form.

Theorem 2.3. *Let $P \in \mathcal{P}(l_1^{(n)}, V)$, where V is a k-dimensional subspace of $l_1^{(n)}$. In this case, it is sufficient to consider*

$$\operatorname{Ext}_P = \left\{(x, e_i) \colon x(Pe_i) = \|P\|,\ x \in \{-1,+1\}^n,\ e_i = (0,\dots,0,1_i,0,\dots,0_n),\ i = 1,\dots,n\right\}.$$

Then P is a minimal projection onto V if and only if there exists a convex combination of one-dimensional operators $(x \otimes e_i)$, where $(x, e_i) \in \operatorname{Ext}_P$, which maps V into V. In other words, P is a minimal projection onto V if and only if there exists an operator $E_P : l_1^{(n)} \to l_1^{(n)}$ of the form

$$E_P(z) = \sum_{w=(x,e_i)\in \operatorname{Ext}_P} \lambda_w x(z) e_i,$$

where $z \in l_1^{(n)}$, $\lambda_w \geq 0$, $\sum_{w\in \operatorname{Ext}_P} \lambda_w = 1$ such that $E_P(V) \subset V$.

Theorem 2.4. *Let $P \in \mathcal{P}(l_\infty^{(n)}, V)$, where V is a k-dimensional subspace of $l_\infty^{(n)}$. In this case it is enough to consider*

$$\mathrm{Ext}_P = \bigl\{(e_i, x)\colon\ e_i(Px) = \|P\|,\ x \in \{-1,+1\}^n,\ e_i = (0,\dots,0,1_i,0,\dots,0_n),\ i = 1,\dots,n\bigr\}.$$

Then P is a minimal projection onto V if and only if there exists a convex combination of one-dimensional operators $(e_i \otimes x)$, where $(e_i,x) \in \mathrm{Ext}_P$, which maps V onto V. In other words, P is a minimal projection onto V if and only if there exists an operator $E_P : l_\infty^{(n)} \to l_\infty^{(n)}$ of the form

$$E_P(z) = \sum_{w=(e_i,x)\in \mathrm{Ext}_P} \lambda_w z_i x,$$

where $z \in l_\infty^{(n)}$, $\lambda_w \ge 0$, $\sum_{w\in \mathrm{Ext}_P} \lambda_w = 1$, such that $E_P(V) \subset V$.

Theorem 2.5. *(See [3] or [31, pp. 104, 105].) Fix $N \in \mathbb{N}$, $N \ge 3$. Let $f = (f_1,\dots,f_N) \in l_1^{(N)}$, $\|f\|_1 = 1$. Assume that $\frac{1}{2} > f_i > 0$ for $i = 1,\dots,N$. Let*

$$V = \ker(f) = \Biggl\{x = (x_1,\dots,x_N) \in l_\infty^{(N)}\colon\ \sum_{j=1}^N f_j x_j = 0\Biggr\}.$$

Then

$$\lambda\bigl(V, l_\infty^{(N)}\bigr) = 1 + \Biggl(\sum_{j=1}^N \frac{f_j}{1-2f_j}\Biggr)^{-1}.$$

Also by [31, p. 105, Theorem III.3.1] there exists exactly one minimal projection $P_o \in \mathcal{P}(l_\infty^{(N)}, V)$ determined by a formula

$$P_o x = x - f(x) y,$$

where $y \in \mathbb{R}^N$ is defined by

$$y = \biggl(\frac{\lambda(V, l_\infty^{(N)}) - 1}{1-2f_1}, \dots, \frac{\lambda(V, l_\infty^{(N)}) - 1}{1-2f_N}\biggr).$$

Theorem 2.6. *(See [3] or [31, Theorem II.4.9, p. 67].) Fix $N \in \mathbb{N}$, $N \ge 3$. Let $f = (f_1,\dots,f_N) \in l_\infty^{(N)}$, $\|f\|_\infty = 1$. Assume that $1 = f_1 \ge f_2 \ge \dots \ge f_N \ge 0$ and $f_3 > 0$. Let*

$$V = \ker(f) = \Biggl\{x = (x_1,\dots,x_N) \in l_1^{(N)}\colon\ \sum_{j=1}^N f_j x_j = 0\Biggr\}.$$

Let us define for $i = 1,\dots,N$

$$a_i = \sum_{j=1}^i f_i, \qquad b_i = \sum_{j=1}^i f_i^{-1} \quad \text{and} \quad \text{for } i \ge 3, \quad \beta_i = \frac{b_i}{i-2}.$$

Also set for $j = 3,\dots,N$

$$c_j = \min\{f_j b_{j-1}, a_{j-1}\} \quad \text{and} \quad i = i(f) = \max\{j \ge 3\colon\ c_j \ge j-3\}.$$

Then

$$\lambda\big(V, l_1^{(N)}\big) = 1 + x,$$

where

$$x = 2\big(\big(\beta_i - f_i^{-1}\big)(i-2) + a_i f_i^{-1} - i\big)^{-1}$$

if $a_i < i - 2$ *and*

$$x = 2(a_i\beta_i - i)^{-1}$$

if $a_i \geq i - 2$.

3. Determination of minimal projections

We start with the following result.

Theorem 3.1. *Let* $n, N \in \mathbb{N}$, $N \geq n$. *Denote by* $\mathcal{A}$ *the set of all* $N \times N$ *symmetric matrices* (a_{ij}) *such that* $a_{ij} = \pm 1$ *and* $a_{ii} = 1$ *for* $i, j = 1, \ldots, N$. *Fix* $A = \{a_{ij}\} \in \mathcal{A}$. *Let us consider a function* $f_A : \mathbb{R}^{(n+1)N} \to \mathbb{R}$ *given by*

$$f_A\big(u, x^1, \ldots, x^n\big) = \sum_{i,j=1}^{N} u_i u_j a_{ij} \langle x_i, x_j \rangle_n,$$

where $x^i \in \mathbb{R}^N$ *for* $i = 1, \ldots, n$, *and* $u = (u_1, \ldots, u_N) \in \mathbb{R}^N$. *Let*

$$D = \max\big\{f_A\big(u, x^1, \ldots, x^n\big)\colon\ \big(x^1, \ldots, x^n\big), u \textit{ satisfying (2), (3)}\big\}.$$

Assume that $y^1, \ldots, y^n \in \mathbb{R}^N$ *and* $u = (u_1, \ldots, u_N)$ *are so chosen that*

$$f\big(u, y^1, \ldots, y^n\big) = D.$$

Assume furthermore that $u_j > 0$ *for* $j = 1, \ldots, N$. *Let* $z^1, \ldots, z^n$ *be the orthonormal eigenvectors associated with the* n *biggest eigenvalues of the matrix* $B = \{u_i u_j a_{ij}\}$. *Now define an* $n \times N$ *matrix* $\{f_{ij}\}$ *by*

$$f_{ij} = u_j z_j^i$$

and an $n \times N$ *matrix* $\{w_{ij}\}$ *by*

$$w_{ij} = z_j^i / u_j.$$

Let for $i = 1, \ldots, n$, $f^i \in \mathbb{R}^N$ *denote the* n*-th row of* F *and* $w^i \in \mathbb{R}^N$ *the* n*-th row of* W. *Put*

$$F = \operatorname{span}\big[f^i : i = 1, \ldots, n\big] \tag{13}$$

and

$$W = \operatorname{span}\big[w^i : i = 1, \ldots, n\big].$$

If for any $i, j \in \{1, \dots, N\}$, $a_{ij} = \operatorname{sgn}(\langle z_i, z_j\rangle_n)$ *if* $\langle z_i, z_j\rangle_n \neq 0$, *then*

$$\lambda(W) = \lambda\big(W, l_\infty^{(N)}\big) = D$$

and a minimal projection $P_W \in \mathcal{P}(l_\infty^{(N)}, W)$ *is given by*

$$P_W x = \sum_{i=1}^{n} f^i(x) w^i.$$

Analogously,

$$\lambda\big(F, l_1^{(N)}\big) = D$$

and a minimal projection $P_F \in \mathcal{P}(l_1^{(N)}, F)$ *is given by*

$$P_F x = \sum_{i=1}^{n} w^i(x) f^i.$$

In particular, $(P_F)^* = P_W$.

Proof. Fix $y^1, \dots, y^n \in \mathbb{R}^N$ and $u = (u_1, \dots, u_N)$ with $u_i > 0$ for $i = 1, \dots, N$ such that

$$f\big(u, y^1, \dots, y^n\big) = D.$$

By the proof of Lemma 2.2, $f(u, z^1, \dots, z^n) = D$, where $z^1, \dots, z^n$ are the orthonormal eigenvectors associated to the n biggest eigenvalues of $B = \{u_i u_j a_{ij}\}$. Since the vectors z^i are orthonormal, $P_F \in \mathcal{P}(l_1^{(N)}, F)$. Now for $i = 1, \dots, N$ we calculate $\|P_F e_i\|_1$, where $e_i = (0, \dots, 1_i, 0, \dots, 0)$. Notice that

$$\|P_F e_i\|_1 = \left\| \sum_{j=1}^{n} w_i^j f^j \right\|_1 = \sum_{k=1}^{N} \left| \sum_{j=1}^{n} w_i^j f_k^j \right| = \sum_{k=1}^{N} \left| \sum_{j=1}^{n} \frac{u_k z_i^j z_k^j}{u_i} \right| = \frac{\sum_{k=1}^{N} u_k |\langle z_i, z_k\rangle|}{u_i}.$$

Since $a_{ij} = \operatorname{sgn}(\langle z_i, z_j\rangle)$, if $\langle z_i, z_j\rangle_n \neq 0$, for $i, j = 1, \dots, N$,

$$\|P_F e_i\| = \frac{\sum_{k=1}^{N} u_k |\langle z_i, z_k\rangle|}{u_i} = \frac{\sum_{k=1}^{N} u_k a_{ij} \langle z_i, z_k\rangle}{u_i}. \tag{14}$$

Since $f(u, z^1, \dots, z^n) = D$, and the orthonormality condition on z^i does not depend on u, by the Lagrange Multipliers Theorem and (3) there exists $\lambda \in \mathbb{R}$ with

$$\frac{\partial f}{\partial u_i}\big(u, z^1, \dots, z^n\big) - 2\lambda u_i = 0,$$

for $i = 1, \dots, N$. Since $u_i > 0$ for $i = 1, \dots, N$, this leads to equations

$$\frac{\sum_{k=1}^{N} u_k a_{ij} \langle z_i, z_k\rangle}{u_i} = \lambda.$$

Hence $\|P_F\| = \|P_F e_i\|_1$ for $i = 1, \dots, N$.

Now we construct an operator $E_P : l_1^{(N)} \to l_1^{(N)}$ satisfying the assumptions of Theorem 2.3. Let for $i = 1, \dots, N$ a^i denote the i-th row of A. By (14), $a^i(P_F e_i) = \|P\|$ for $i = 1, \dots, n$. Define $E_P : l_1^{(N)} \to l_1^{(N)}$ by

$$E_P(x) = \sum_{j=1}^{N} u_j^2 a^j(x) e_j. \tag{15}$$

Notice that for $i = 1, \dots, n$ and $j = 1, \dots, N$

$$\left(E_P f^i\right)_j = u_j^2 a^j\left(f^i\right) = u_j^2 \left(\sum_{k=1}^{N} a_{jk} z_k^i u_k\right) = u_j \left(\sum_{k=1}^{N} a_{jk} z_k^i u_j u_k\right) = u_j \lambda_i \left(z^i\right)_j = \lambda_i \left(f^i\right)_j,$$

where $\lambda_1 \geq \dots \geq \lambda_n$ denote the biggest n eigenvalues of $B = \{u_i u_j a_{ij}\}$. Consequently, $E_P(F) \subset F$. Since $\sum_{j=1}^{N} u_j^2 = 1$, by (15) and Theorem 2.3 P_F is a minimal projection.

Now we show that P_W is a minimal projection too. Since $P_W = (P_F)^*$, by the previous considerations,

$$\left(P_W a^i\right)_i = \|P_W\| \tag{16}$$

for $i = 1, \dots, N$. Define $E_P : l_\infty^{(N)} \to l_\infty^{(N)}$ by

$$E_P(x) = \sum_{j=1}^{N} u_j^2 x_j a^j. \tag{17}$$

Notice that for $i = 1, \dots, n$ and $j = 1, \dots, N$

$$\left(E_P w^i\right)_j = \sum_{k=1}^{N} \left(w^i\right)_k u_k^2 a_{jk} = \sum_{k=1}^{N} \left(z^i\right)_k u_k a_{jk} = \frac{\sum_{k=1}^{N} (z^i)_k u_j u_k a_{jk}}{u_j} = \lambda_i \frac{(z^i)_j}{u_j} = \lambda_i \left(w^i\right)_j.$$

Consequently, $E_P(W) \subset W$. Since $\sum_{j=1}^{N} u_j^2 = 1$, by (15) and Theorem 2.4 P_W is a minimal projection. It is clear from our considerations that $\|P_F\| = \|P_W\| = D$. The proof is complete. $\square$

Remark 3.1. Theorem 3.1 remains true if we maximize f_A under additional condition which says that there exist nonempty and pairwise disjoint subsets $A_1, \dots, A_k$ of $\{1, \dots, N\}$ and positive constants $c_1, \dots, c_k$ such that $u_j = c_i$ for $j \in A_i$ and $i = 1, \dots, k$.

In the next theorem we show that for any $n, N \in \mathbb{N}$, $N \geq n$ $V_n \subset l_\infty^{(N)}$ maximizing the constant λ_n^N, i.e., $\lambda(V_n) = \lambda_n^N$, can be chosen as a subspace of $l_1^{(N)}$. Moreover we can choose V_n in such a way, that $l_1^{(N)}$ is a maximal overspace for V_n, i.e. $\lambda(V_n) = \lambda(V_n, l_1^{(N)})$. Note that in general for $W \subset l_1^{(N)}$, $\lambda(W) > \lambda(W, l_1^{(N)})$. Take for example $W_n = l_1^{(n)}$. Then $\lambda(W_n, l_1^{(N)}) = 1$ for any $N \geq n$ and by [20]

$$\lim_n \frac{\lambda(W_n)}{\sqrt{n}} = \sqrt{\frac{2}{\pi}}.$$

As a consequence of Lemma 2.2, Theorem 2.1 and Theorem 3.1 we get

Theorem 3.2. *Let $n, N \in \mathbb{N}$, $N \geq n$. Then there exists an n-dimensional space $V_n \subset l_1^{(N)}$ such that*

$$\lambda_n^N = \lambda\left(V_n, l_1^{(N)}\right) = \lambda(V_n).$$

Proof. If $n = 1$ or $n = N$, then the statement is obvious. Hence fix $n, N \in \mathbb{N}$ with $N > n > 1$. Let $u^o, y^1, \dots, y^n \in \mathbb{R}^N$ be so chosen that

$$M = \sum_{i,j=1}^{N} u_i^o u_j^o \big|\langle y_i, y_j\rangle_n\big| = \max\left\{ \sum_{i,j=1}^{N} u_i u_j \big|\langle x_i, x_j\rangle_n\big|:\ u, \big(x^1, \dots, x^n\big) \text{ satisfying (2) and (3)} \right\}.$$

First assume that $u_j^o > 0$ for $j = 1, \dots, N$. Let for $i, j = 1, \dots, N$ $a_{ij} = \operatorname{sgn}(\langle y_i, y_j\rangle_n)$ if $\langle y_i, y_j\rangle_n \neq 0$ and $a_{ij} = 1$ if $\langle y_i, y_j\rangle_n = 0$. By Theorem 2.1

$$M = \sum_{i,j=1}^{N} u_i^o u_j^o a_{ij}\langle y_i, y_j\rangle_n = \max\left\{ \sum_{i,j=1}^{N} u_i u_j a_{ij}\langle x_i, x_j\rangle_n:\ u, \big(x^1, \dots, x^n\big) \text{ satisfying (2) and (3)} \right\}.$$

Let $z^1, \dots, z^n$ be the orthonormal eigenvectors associated to the n biggest eigenvalues of $B = \{u_i^o u_j^o a_{ij}\}$. By the proof of Lemma 2.2,

$$M = \sum_{i,j=1}^{N} u_i^o u_j^o a_{ij}\langle z_i, z_j\rangle_n.$$

Let V_n be given by (13). By Theorem 3.1 and Theorem 2.1 and the definition of $\lambda(V_n)$,

$$M = \lambda_n^N \geq \lambda(V_n) \geq \lambda\big(V_n, l_1^{(N)}\big) = M,$$

which shows our claim.

By [6, Lemma 2.7], if $\lambda_n^N > \lambda_n^{N-1}$ then $u_o^j > 0$ for $j = 1, \dots, N$ and the above reasoning applies. If $\lambda_n^N = \lambda_n^{N-1}$, set

$$K = \min\big\{M \in \mathbb{N}:\ n \leq M \leq N,\ \lambda_n^N = \lambda_n^M\big\}.$$

Notice that by Lemma 2.3, for any $n \geq 2$

$$\lambda_n^{n+1} = 2 - \frac{2}{n+1} > 1 = \lambda_n^n.$$

Hence, since $N > n > 1$, $K > n > 1$, and $\lambda_n^K > \lambda_n^{K-1}$. Applying the above reasoning to K we get our claim. The proof is complete. □

Remark 3.2. Notice that Theorem 3.2 cannot be proved applying the fact that $(l_1^{(N)})^* = l_\infty^{(N)}$. Indeed, if $P \in \mathcal{P}(l_\infty^{(N)}, W)$,

$$P = \sum_{j=1}^{n} f_j(\cdot) w_j,$$

where $W = \operatorname{span}[w_i : i = 1, \dots, n]$, is a minimal projection then the adjoint operator

$$P^* = \sum_{j=1}^{n} w_j(\cdot) f_j \in \mathcal{P}\big(l_1^{(N)}, F\big)$$

is not necessarily a minimal projection, where $F = \operatorname{span}[f_j : j = 1, \dots, n]$. To show an example we apply Theorem 2.5 and Theorem 2.6.

Example 3.1. Let $N = 5$ and let $f = (f_1, \dots, f_5) = (3/7, 1/7, 1/7, 1/7, 1/7)$. Set

$$W = \left\{ x = (x_1, \dots, x_5) \in \mathbb{R}^5 \colon \sum_{j=1}^{5} f_j x_j = 0 \right\}.$$

By simple calculations and Theorem 2.5

$$\lambda\bigl(W, l_\infty^{(5)}\bigr) = 1 + 5/19.$$

Moreover, by Theorem 2.5, the minimal projection P_o is unique and defined by

$$P_o x = x - f(x) y,$$

where $y = (35/19, 7/19, 7/19, 7/19, 7/19)$. By the definition of the adjoint operator,

$$P_o^* x = x - y(x) f.$$

Set $z = (z_1, \dots, z_5) = (1, 1/5, 1/5, 1/5, 1/5) = (19/35) y$. It is clear that P_o is a projection from $l_1^{(5)}$ onto $F = \ker(z)$, i.e.,

$$F = \left\{ x = (x_1, \dots, x_5) \in \mathbb{R}^5 \colon \sum_{j=1}^{5} z_j x_j = 0 \right\}.$$

By simple calculations and Theorem 2.6 we get

$$\lambda\bigl(F, l_1^{(5)}\bigr) = 1 + 1/5 < 1 + 5/19 = \|P_o\| = \bigl\|P_o^*\bigr\|,$$

which shows that P_o^* is not a minimal projection in $\mathcal{P}(l_1^{(5)}, F)$.

4. Applications

First we will apply Lemma 2.2 and Theorem 2.1 to get an estimate of λ_n^{n+2} for odd numbers from below.

Theorem 4.1. *For any odd natural number $n \geq 3$,*

$$\lambda_n^{n+1} < \max_{x \in [0,1]} f_n(x) \leq \lambda_n^{n+2},$$

where

$$f_n(x) = \frac{2n}{n+1}(1 - x) + \frac{1}{2}\left(x - 2\frac{1-x}{n+1} + \sqrt{\left(2\frac{1-x}{n+1} - x\right)^2 + 4(1-x)x} \right).$$

Proof. To get this estimate we will calculate the n biggest eigenvalues of a specially chosen $(n+2) \times (n+2)$ matrix B satisfying the requirements of Lemma 2.2. The matrix B is constructed in the following manner.

Let $A = (a_{ij})$ be an $(n+2) \times (n+2)$ symmetric matrix defined as:

$$a_{ii} = 1,$$
$$a_{1j} = 1 \quad \text{for } j = 2, \dots, \frac{n+3}{2},$$

$$a_{1j} = -1 \quad \text{for } j = \frac{n+3}{2} + 1, \dots, n+2,$$
$$a_{ij} = -1 \quad \text{for } i, j = 2, \dots, n+2, \ i \neq j.$$

If $n = 3$ the matrix A has the following shape

$$A = \begin{pmatrix} 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix}. \tag{18}$$

Define for any $u \in [0, 1]$:

$$B_u = w^T A w$$

where $w = (u, c, \dots, c)$ and $c = \sqrt{\frac{(1-u^2)}{(n+1)}}$.

Observe that for any $u \in [0, 1]$, $2c^2$ is an eigenvalue of B_u with multiplicity $n - 1$. Indeed, if we define for $j = 1, \dots, (n-1)/2$

$$y^j = (0, 1, 1, \dots, 1_{j+1}, -j, 0, \dots, 0) \tag{19}$$

and for $j = 1, \dots, (n-1)/2$

$$y^{j+(n-1)/2} = (0, \dots, 0_{(n+3)/2}, 1, 1, \dots, 1_{((n+3)/2)+j}, -j, 0, \dots, 0) \tag{20}$$

then it is easy to verify that $\{y^j\colon\ j = 1, \dots, n-1\}$ form a basis of the eigenspace for $2c^2$. Notice that another eigenvalue of B_u is $-(n-1)c^2$ with an eigenvector $(0, 1, 1, \dots, 1)$.

Now we find two lacking eigenvalues. To do this, we consider vectors of the form

$$v_a = (a, 1, \dots, 1_{\frac{(n+3)}{2}}, -1, \dots, -1) \tag{21}$$

and we show that v_a for properly chosen a are eigenvectors of B_u. Notice that v_a as an eigenvector should satisfy the following system of equations:

$$u^2 a + (n+1)c = \lambda a$$

and

$$uca + 2c^2 = \lambda$$

with unknowns a and λ. One can get the first equation multiplying the vector v_a by the first row of B and the second one multiplying the vector v_a by other rows. After elementary calculations one can get the solutions for a and λ as

$$a_1 = \frac{(u^2 - 2c^2) + \sqrt{(2c^2 - u^2)^2 + 4(n+1)u^2c^2}}{2uc}$$

and

$$a_2 = \frac{(u^2 - 2c^2) - \sqrt{(2c^2 - u^2)^2 + 4(n+1)u^2c^2}}{2uc}$$

and, correspondingly,

$$\lambda_1 = 2c^2 + \frac{(u^2 - 2c^2) + \sqrt{(2c^2 - u^2)^2 + 4(n+1)u^2c^2}}{2}$$

and

$$\lambda_2 = 2c^2 + \frac{(u^2 - 2c^2) - \sqrt{(2c^2 - u^2)^2 + 4(n+1)u^2c^2}}{2}.$$

Observe that $\lambda_1 > 2c^2$. Hence, the sum of the n-biggest eigenvalues (replacing u^2 by x) is

$$f_n(x) = \frac{2n}{n+1}(1-x) + \frac{1}{2}\left(x - 2\frac{1-x}{n+1} + \sqrt{\left(2\frac{1-x}{n+1} - x\right)^2 + 4(1-x)x}\right),$$

where $x \in [0,1]$. The next step is to calculate $\max\{f_n(x)\colon x \in [0,1]\}$. After elementary calculations (it is necessary to solve a quadratic equation with respect to x) we get that $f_n'(x) = 0$ if and only if

$$x = x_{o,n} = 1 - (n+1)\frac{3\sqrt{3n^2 + 2n - 1} + 3n - 1}{9n^2 + 12n - 5}.$$

In particular this means that $c_{o,n}$ corresponding to $u_{o,n} = \sqrt{x_{o,n}}$ should be equal to $\sqrt{\frac{3\sqrt{3n^2+2n-1}+3n-1}{9n^2+12n-5}}$. Now we show that $0 < x_{o,n} < 1$. This is equivalent to $(n+1)c_{o,n}^2 < 1$ for $n \in \mathbb{N}$, $n \geq 3$. It is easy to check that the last inequality reduces to

$$-9n^4 - 48n^3 + 2n^2 + 80n - 25 < 0.$$

Observe that $-9n^4 + 34n^3 < 0$ if and only if $n \geq 34/9$, which shows our claim for $n \geq 4$. Also one can easily check that the above inequality holds true for $n = 3$. Moreover, it is not difficult to check that $f'(0) > 0$ which shows that $f'(x) > 0$ for $x \in [0, x_o)$ and $f'(x) < 0$ for $x \in (x_o, 1]$. This means that f attains its global maximum on $[0,1]$ at x_o. Observe that by Lemma 2.3, $\lambda_n^{n+1} = 2 - \frac{2}{n+1}$. Since $f(0) = 2 - \frac{2}{n+1}$, $\max_{x\in[0,1]} f(x) > \lambda_n^{n+1}$, as required. □

Notice that for $n = 3$ Theorem 4.1 has been proven in [6].

Now we apply Theorem 4.1 and Theorem 3.1 to determine minimal projections onto some subspaces of $l_\infty^{(N)}$ and $l_1^{(N)}$ of codimension two. First we recall a well-known

Lemma 4.1. *For any $n \in \mathbb{N} \setminus \{0\}$*

$$\sum_{j=1}^{n} \frac{1}{j(j+1)} = 1 - \frac{1}{n+1}.$$

Theorem 4.2. *Fix $n \in \mathbb{N}$, $n \geq 3$. Let $A = \{a_{ij}\}$ be an $(n+2) \times (n+2)$ matrix defined in Theorem 4.1. Let us consider a function $f_A : \mathbb{R}^{(n+1)(n+2)} \to \mathbb{R}$ given by*

$$f_A(u, x^1, \ldots, x^n) = \sum_{i,j=1}^{n+2} u_i u_j a_{ij} \langle x_i, x_j \rangle_n,$$

where $x^i \in \mathbb{R}^{(n+2)}$ for $i = 1, \ldots, n$, and $u = (u_1, \ldots, u_{n+2}) \in \mathbb{R}^{(n+2)}$. Let

$$D = \max\{f_A(u, x^1, \dots, x^n): (x^1, \dots, x^n), u \text{ satisfying (2), (3)}, \ u_2 = u_3 = \dots = u_{n+2} = c\}.$$

For any $u = (u_1, u_2, \dots, u_{n+2}) \in \mathbb{R}^{n+2}$ as above with $u_1 > 0$ and $u_2 = c > 0$ let $z^1, \dots, z^n \in \mathbb{R}^{n+2}$ be the orthonormal eigenvectors determined in Theorem 4.1 associated to the n biggest eigenvalues of $B = \{u_i u_j a_{ij}\}$. Now define (as in Theorem 3.1) the $n \times (n+2)$ matrix $\{f_{ij}\}$ by

$$f_{ij} = u_j z^i_j$$

and the $n \times (n+2)$ matrix $\{w_{ij}\}$ by

$$w_{ij} = z^i_j / u_j.$$

Let for $i = 1, \dots, n$, $f^i \in \mathbb{R}^{n+2}$ denote the n-th row of F and $w^i \in \mathbb{R}^{n+2}$ the n-th row of W. Put

$$F = \operatorname{span}\left[f^i : i = 1, \dots, n\right] \tag{22}$$

and

$$W = \operatorname{span}\left[w^i : i = 1, \dots, n\right].$$

If $u = (u_1, c, \dots, c) \in \mathbb{R}^{n+2}$ and $z^1, \dots, z^n \in \mathbb{R}^{n+2}$ defined above are so chosen that $D = f_A(u, z^1, \dots, z^n)$, then

$$\lambda(W) = \lambda\left(W, l_\infty^{(n+2)}\right) = D$$

and a minimal projection $P_W \in \mathcal{P}(l_\infty^{(n+2)}, W)$ is given by

$$P_W x = \sum_{i=1}^n f^i(x) w^i.$$

Analogously,

$$\lambda\left(F, l_1^{(n+2)}\right) = D$$

and a minimal projection $P_F \in \mathcal{P}(l_1^{(n+2)}, F)$ is given by

$$P_F x = \sum_{i=1}^n w^i(x) f^i.$$

In particular, $(P_F)^* = P_W$. Moreover,

$$D = \|P_F\| = \|P_W\| = \max_{x \in [0,1]} f_n(x) = f_n(x_{o,n}),$$

where f_n is given in Theorem 4.1 and $x_{o,n} = 1 - (n+1)\frac{3\sqrt{3n^2+2n-1}+3n-1}{9n^2+12n-5}$.

Proof. Fix $u = (u_1, c, \dots, c) \in \mathbb{R}^{n+2}$, satisfying (3), i.e. $c = \sqrt{\frac{1-u_1^2}{n+1}}$. Let $f_{A,u} : \mathbb{R}^{n(n+2)} \to \mathbb{R}$ be given by

$$f_{A,u}\left(x^1, \dots, x^n\right) = \sum_{i,j=1}^{n+2} u_i u_j a_{ij} \langle x_i, x_j \rangle_n,$$

where $x^i \in \mathbb{R}^{(n+2)}$ for $i = 1, \dots, n$. Let

$$D_{u_1} = \max\{f_{A,u}(x^1, \dots, x^n)\colon (x^1, \dots, x^n) \text{ satisfying (2)}\}.(19)$$

Let $y^1, \dots, y^n \in \mathbb{R}^{n+2}$ denote the orthogonal eigenvectors, defined by (19), (20) and (21), corresponding to the n-biggest eigenvalues of the matrix $B = \{u_i u_j a_{ij}\}$. Let for $j = 1, \dots, n$, $z^j = y^j / \|y^j\|_e$. By the proof of Lemma 2.2,

$$D_{u_1} = f_{A,u}(z^1, \dots, z^n).$$

It is clear that

$$D = \max_{u_1 \in [0,1]} D_{u_1}.$$

By Theorem 4.1, $D = D_{v_1}$ for $v_1 = \sqrt{1 - (n+1)\frac{3\sqrt{3n^2+2n-1}+3n-1}{9n^2+12n-5}}$. Let $v = (v_1, c_o, \dots, c_o)$, where $c_o = \sqrt{\frac{1-v_1^2}{n+1}}$. It is clear that $v_1 > 0$ and $c_o > 0$.

Now we show that for any $i, j = 1, \dots, n+2$, $\operatorname{sgn}(\langle z_i, z_j\rangle_n) = a_{ij}$. If $i = j$, then $\langle z_i, z_j\rangle_n = \|(z_i^1, \dots, z_i^n)\|_e^2$. Hence

$$1 = \operatorname{sgn}(\langle z_i, z_i\rangle_n) = a_{ii}.$$

Notice that by (21),

$$z^n = (d, e, \dots, e_{(n+3)/2}, -e, \dots, -e) \tag{23}$$

where $d^2 + (n+1)e^2 = 1$, $d > 0$ and $e > 0$. Hence it is easy to see that for $i = 1, j = 2, \dots, (n+3)/2$

$$1 = \operatorname{sgn}(de) = \operatorname{sgn}(\langle z_1, z_j\rangle_n) = a_{1j},$$

and

$$-1 = \operatorname{sgn}(-de) = \operatorname{sgn}(\langle z_1, z_j\rangle_n) = a_{1j},$$

for $j = (n+5)/2, \dots, n+2$. Now assume that $i, j \in \{2, \dots, (n+3)/2\}$ and $i \neq j$. By Lemma 4.1,

$$\langle z_i, z_j\rangle_n = \frac{-2}{n+1} + e^2 \le \frac{-1}{n+1},$$

since $e^2 \le \frac{1}{n+1}$. Hence

$$-1 = \operatorname{sgn}(\langle z_i, z_j\rangle_n) = a_{ij}.$$

The same reasoning applies to the case $i, j \in \{\frac{n+3}{2} + 1, \dots, n+2\}$ and $i \neq j$. Finally if $i \in \{2, \dots, (n+3)/2\}$ and $j \in \{\frac{n+3}{2} + 1, \dots, n+2\}$ then it is easy to see that

$$\langle z_i, z_j\rangle_n = -e^2,$$

which shows our claim. Hence all the assumptions of Theorem 3.1 and Remark 3.1 are satisfied. The proof is complete. □

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