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Some properties of Lipschitz strongly p-summing operators

Khalil Saadi

University of M'sila, Laboratoire d'Analyse Fonctionnelle et Géométrie des Espaces, 28000 M'sila, Algeria

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ABSTRACT

We consider the space of molecules endowed with the transposed version of the Chevet–Saphar norm and we identify its dual space with the space of Lipschitz strongly p-summing operators. We also extend some old results to the category of Lipschitz mappings and we give a factorization result of Lipschitz (p, r, s)-summing operators.

0. Introduction

An area of research that is currently very active is the study of nonlinear geometry, of Banach spaces or even of general metric spaces, by borrowing ideas and insights from the linear theory of Banach spaces. A very powerful tool in the latter is given by the class of p-summing operators, so that naturally has led several authors to investigate Lipschitz versions of them starting with the seminal paper [7]. J.A. Chávez-Domínguez [2] has explored more properties of this class and has defined a norm on the space of molecules of which dual space coincides with the space of Lipschitz p-summing. The aim of this paper is to continue to study the same ideas developed in [2]. We try to give a similar treatment to the class of Cohen strongly p-summing operators. We consider the transposed version of the norm of Chevet–Saphar and we show that the dual of the space of molecules endowed with this norm coincides with the space of Lipschitz–Cohen strongly p-summing. Some old results have been established, namely a version of Grothendieck's theorem and the relationship between the Lipschitz mapping $T : X \to Y$ and its linearization $\widehat{T} : \mathcal{F}(X) \longrightarrow Y$ for the concepts of p-summing and Cohen strongly p-summing.

The paper is organized as follows.

E-mail address: kh_saadi@yahoo.fr.

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First, we recall some standard notations which will be used throughout. In Section 1, we define a norm on the space of molecules that is inspired by the Chevet–Saphar norms. We give and prove an integral characterization of a linear form on this new space. In Section 2, we give the definition of Lipschitz–Cohen strongly p-summing for maps from a metric space to a Banach space. We show that this space is precisely the dual of the space of molecules described in Section 1. Finally, in Section 3 we study some basic properties of these Lipschitz–Cohen strongly p-summing operators, drawing parallels to the linear theory. Some interesting results have been obtained namely the Grothendieck theorem and the relationship between the Lipschitz mapping and its linearization for certain concept of summability. The last part of this section is devoted to study a factorization result of Lipschitz (p, r, s)-summing operators like the one given in linear case. We show that the map T is Lipschitz (p^*, r, s)-summing if, and only if, T can be written as $T = T_2 \circ T_1$ where T_1 is Lipschitz r-summing and T_2 is Lipschitz–Cohen strongly s^*-summing.

Now, we recall briefly some basic notations and terminology. Throughout this paper we always consider metric spaces with a distinguished point (pointed metric spaces) which we denote by 0. Let X be a pointed metric space. We denote by $X^\#$ the Banach space of all Lipschitz functions $f : X \longrightarrow \mathbb{R}$ which vanish at 0 under the Lipschitz norm given by

$$Lip(f) = \sup\left\{ \frac{|f(x) - f(y)|}{d(x,y)} : x, y \in X,\ x \neq y \right\}.$$

Consider $1 \leq p \leq \infty$, for sequences of the same length $(\lambda_i)_{i=1}^n$ of real numbers and $(x_i)_{i=1}^n$, $(x_i')_{i=1}^n$ of points in X, we denote their weak Lipschitz p-norm by

$$w_p^{Lip}\big((\lambda_i, x_i, x_i')_i\big) = \sup_{f \in B_{X^\#}} \left(\sum_{i=1}^n \big|\lambda_i\big(f(x_i) - f\big(x_i'\big)\big)\big|^p \right)^{\frac{1}{p}}.$$

We denote by $\mathcal{F}(X)$ the free Banach space over X, i.e., $\mathcal{F}(X)$ is the completion of the space

$$AE = \left\{ \sum_{i=1}^n \lambda_i m_{x_i x_i'},\ (\lambda_i)_{i=1}^n \subset \mathbb{R},\ (x_i)_{i=1}^n, \big(x_i'\big)_{i=1}^n \subset X \right\},$$

with the norm

$$\|m\|_{\mathcal{F}(X)} = \inf\left\{ \sum_{i=1}^n |\lambda_i| d\big(x_i, x_i'\big) : m = \sum_{i=1}^n \lambda_i m_{x_i x_i'} \right\},$$

where the function $m_{x_i x_i'} : X^\# \to \mathbb{R}$ is defined as follows

$$m_{x_i x_i'}(f) = f(x_i) - f\big(x_i'\big).$$

We have

$$\mathcal{F}(X)^* = X^\#.$$

For the general theory of free Banach spaces, see [1,8,9,12,16]. Let X be a pointed metric space and Y be a Banach space, we denote by $Lip_0(X; Y)$ the Banach space of all Lipschitz functions $T : X \to Y$ such that $T(0) = 0$ with pointwise addition and Lipschitz norm. We also denote by $\mathcal{F}(X; Y)$ the vector space of all Y-valued molecules on X, i.e.,

$$\mathcal{F}(X; Y) = \left\{ \sum_{i=1}^n \lambda_i y_i m_{x_i x_i'},\ (\lambda_i)_i \subset \mathbb{R},\ (y_i)_i \subset Y,\ (x_i)_i, \big(x_i'\big)_i \subset X \right\}.$$

For any $T \in Lip_0(X;Y^*)$, we denote by φ_T its correspondent linear function on $\mathcal{F}(X;Y)$ defined by

$$\langle \varphi_T, m\rangle = \langle T, m\rangle,$$

where $\langle .,.\rangle$ is a pairing of $Lip_0(X;Y^*)$ and $\mathcal{F}(X;Y)$ defined by

$$\langle T, m\rangle = \sum_{x\in X} \langle T(x), m(x)\rangle.$$

Therefore, for a general molecule $m = \sum_{i=1}^{n} \lambda_i y_i m_{x_i x_i'}$,

$$\langle T, m\rangle = \sum_{i=1}^{n} \lambda_i \langle T(x_i) - T(x_i'), y_i\rangle. \tag{0.1}$$

Let X be a pointed metric space and Y be a Banach space, note that for any $T \in Lip_0(X;Y)$, then there exists a unique linear map (linearization of T) $\widehat{T} : \mathcal{F}(X) \longrightarrow Y$ such that $\widehat{T}\delta_X = T$ and $\|\widehat{T}\| = Lip(T)$, i.e., the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \delta_X \downarrow & \nearrow \widehat{T} & \\ \mathcal{F}(X) & & \end{array} \tag{0.2}$$

where δ_X is the canonical embedding so that $\langle \delta_X(x), f\rangle = \langle m_{x0}, f\rangle = f(x)$ for $f \in X^{\#}$. Let Y be a Banach space, then B_Y denotes its closed unit ball and Y^* its (topological) dual. Consider $1 \le p \le \infty$ and $n \in \mathbb{N}^*$. We denote by $l_p^n(Y)$ the Banach space of all sequences $(y_i)_{i=1}^n$ in Y with the norm

$$\|(y_i)_i\|_{l_p^n(Y)} = \left(\sum_{i=1}^{n} \|y_i\|^p\right)^{\frac{1}{p}},$$

and by $l_p^{n,\omega}(Y)$ the Banach space of all sequences $(y_i)_{i=1}^n$ in Y with the norm

$$\|(y_i)_i\|_{l_p^{n,\omega}(Y)} = \sup_{y^*\in B_{Y^*}} \left(\sum_{i=1}^{n} |\langle y_i, y^*\rangle|^p\right)^{\frac{1}{p}}.$$

We also have

$$\|(y_i)_i\|_{l_p^{n,\omega}(Y)} = \sup_{y^*\in B_{Y^*}} \left(\sum_{i=1}^{n} |\langle y_i, y^*\rangle|^p\right)^{\frac{1}{p}} = \|(y_i)_i\|_{l_p^{n,\omega}(Y^{**})}. \tag{0.3}$$

If $Y = \mathbb{R}$, we simply write l_p^n and $l_p^{n,w}$.

1. The Chevet–Saphar norms on the space of molecules

Let E, F be Banach spaces, in [3,15], the Chevet–Saphar norms d_p and g_p are defined on tensor product $E \otimes F$ for $1 \le p \le \infty$ as follows

$$d_p(u) = \inf\left\{\|(x_i)_i\|_{l_{p^*}^{n,w}(E)} \|(y_i)_i\|_{l_p^n(F)}\right\},$$

taking the infimum over all representations of u of the form $u = \sum_{i=1}^{n} x_i \otimes y_i \in E \otimes F$. If we interchange the roles of the weak and strong norms in d_p, we obtain the transposed norm

$$g_p(u) = \inf\{\|(x_i)_i\|_{l_p^n(E)}\|(y_i)_i\|_{l_{p^*}^{n,w}(F)}\}.$$

For every p, we have

$$g_p = d_p^t,$$

where d_p^t is defined as follows

$$d_p^t(u; E \otimes F) = d_p(u^t; F \otimes E),$$

and the transpose, u^t, of $u = \sum_{i=1}^{n} x_i \otimes y_i$ is given by $u^t = \sum_{i=1}^{n} y_i \otimes x_i$. Inspired by the tensor norm g_p, we give a new norm on $\mathcal{F}(X;Y)$ like the one given by J.A. Chávez-Domínguez in [2] for the norm d_p. Note that the space $\mathcal{F}(X;Y)$ plays the role of the tensor product in the linear theory. Let $p \in [1, \infty]$ and $m \in \mathcal{F}(X;Y)$. We consider for $m \in \mathcal{F}(X;Y)$

$$\mu_p(m) = \inf\{\|(\lambda_i d(x_i, x_i'))_i\|_{l_p^n}\|(y_i)_i\|_{l_{p^*}^{n,w}(Y)}\},$$

where the infimum is taken over all representations of m of the form

$$m = \sum_{i=1}^{n} \lambda_i y_i m_{x_i x_i'},$$

with $x_i, x_i' \in X$, $y_i \in Y$, $\lambda_i \in \mathbb{R}$; $(1 \le i \le n)$ and $n \in \mathbb{N}^*$.

Proposition 1.1. *Let X be a pointed metric space and Y be a Banach space. Let $p \in [1, \infty]$, then μ_p is a norm on $\mathcal{F}(X;Y)$.*

Proof. It is clear that for any molecule $m \in \mathcal{F}(X;Y)$ and any scalar α we have

$$\mu_p(m) \ge 0 \quad \text{and} \quad \mu_p(\alpha m) = |\alpha|\mu_p(m).$$

Let $y^* \in Y^*$, $f \in X^{\#}$ and $m \in \mathcal{F}(X;Y)$. Using the pairing formula (0.1)

$$\begin{aligned}
|\langle y^* f, m\rangle| &= \left|\sum_{i=1}^{n} \lambda_i y^*(y_i)\big(f(x_i) - f(x_i')\big)\right| \quad \text{by Hölder inequality} \\
&\le \left(\sum_{i=1}^{n} \big|\lambda_i\big(f(x_i) - f(x_i')\big)\big|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y^*(y_i)|^{p^*}\right)^{\frac{1}{p^*}} \\
&\le \|y^*\| Lip(f) \|(\lambda_i d(x_i, x_i'))_i\|_{l_p^n} \|(y_i)_i\|_{l_{p^*}^{n,w}(Y)}.
\end{aligned}$$

By taking the infimum over all representations of m, we obtain

$$|\langle y^* f, m\rangle| \le \|y^*\| Lip(f) \mu_p(m).$$

Now, suppose that $\mu_p(m) = 0$, then for every $y^* \in Y^*$ and $f \in X^\#$

$$0 = \langle y^* f, m \rangle = \sum_{i=1}^{n} \langle f, \lambda_i y^*(y_i) m_{x_i x'_i} \rangle,$$

by the duality between $\mathcal{F}(X)$ and $X^\#$, the real-valued molecule $y^* \circ m$ is equal to 0 for all $y^* \in Y^*$ and consequently $m = 0$. Let now $m_1, m_2 \in \mathcal{F}(X; Y)$. By the definition of μ_p we can find a representation

$$m_1 = \sum_{i=1}^{l} \lambda_{1i} y_{1i} m_{x_{1i} x'_{1i}},$$

such that

$$\left\| \left(\lambda_{1i} d\left(x_{1i}, x'_{1i}\right) \right)_i \right\|_{l_p^l} \left\| (y_{1i})_i \right\|_{l_{p^*}^{l,w}(Y)} \le \mu_p(m_1) + \varepsilon.$$

Replacing (λ_{1i}) and (y_{1i}) by an appropriate multiple of them,

$$\lambda_{1i} = \lambda_{1i} \frac{\|(y_{1i})_i\|_{l_{p^*}^{l,w}(Y)}^{\frac{1}{p}}}{\|(\lambda_{1i} d(x_{1i}, x'_{1i}))_i\|_{l_p^l}^{\frac{1}{p^*}}}, \qquad y_{1i} = y_{1i} \frac{\|(\lambda_{1i} d(x_{1i}, x'_{1i}))_i\|_{l_p^l}^{\frac{1}{p^*}}}{\|(y_{1i})_i\|_{l_{p^*}^{l,w}(Y)}^{\frac{1}{p}}},$$

we can find

$$\left\| \left(\lambda_{1i} d\left(x_{1i}, x'_{1i}\right) \right)_i \right\|_{l_p^l} \le \left(\mu_p(m_1) + \varepsilon \right)^{\frac{1}{p}}, \qquad \left\| (y_{1i})_i \right\|_{l_{p^*}^{l,w}(Y)} \le \left(\mu_p(m_1) + \varepsilon \right)^{\frac{1}{p^*}}.$$

Similarly for m_2, we choose a representation

$$m_2 = \sum_{i=1}^{s} \lambda_{2i} y_{2i} m_{x_{2i} x'_{2i}},$$

such that

$$\left\| \lambda_{2i} d\left(x_{2i}, x'_{2i}\right) \right\|_{l_p^s} \left\| (y_{2i})_i \right\|_{l_{p^*}^{s,w}(Y)} \le \mu_p(m_2) + \varepsilon.$$

Again, replacing (λ_{2i}) and (y_{2i}) by an appropriate multiple of them as in the above, we find

$$\left\| \lambda_{2i} d\left(x_{2i}, x'_{2i}\right) \right\|_{l_p^s} \le \left(\mu_p(m_2) + \varepsilon \right)^{\frac{1}{p}}, \qquad \left\| (y_{2i})_i \right\|_{l_{p^*}^{s,w}(Y)} \le \left(\mu_p(m_2) + \varepsilon \right)^{\frac{1}{p^*}}.$$

Now, we have

$$\begin{aligned} w_p(m_1 + m_2) &\le \left(\left\| \lambda_{1i} d\left(x_{1i}, x'_{1i}\right) \right\|_{l_p^l}^p + \left\| \lambda_{2i} d\left(x_{2i}, x'_{2i}\right) \right\|_{l_p^s}^p \right)^{\frac{1}{p}} \left(\left\| (y_{1i})_i \right\|_{l_{p^*}^{l,w}(Y)}^{p^*} + \left\| (y_{2i})_i \right\|_{l_{p^*}^{s,w}(Y)}^{p^*} \right)^{\frac{1}{p^*}} \\ &\le \left(\mu_p(m_1) + \mu_p(m_2) + 2\varepsilon \right)^{\frac{1}{p}} \left(\mu_p(m_1) + \mu_p(m_2) + 2\varepsilon \right)^{\frac{1}{p^*}} \\ &\le \mu_p(m_1) + \mu_p(m_2) + 2\varepsilon. \end{aligned}$$

By letting ε tend to zero we obtain the triangle inequality for μ_p. $\square$

We denote by $\mathcal{F}_{\mu_p}(X; Y)$ the completion of $\mathcal{F}(X; Y)$ for the norm μ_p.

Proposition 1.2. *Let X be a pointed metric space and Y be a Banach space. We have*

$$\mathcal{F}_{\mu_p}(X;Y) = \mathcal{F}(X)\hat{\otimes}_{g_p} Y,$$

where g_p is the Chevet–Saphar norm defined as above.

Proof. We can establish the identification via the next linear map

$$\varphi(m) = \varphi\left(\sum_{i=1}^{n} \lambda_i y_i m_{x_i x_i'}\right) = u = \sum_{i=1}^{n} (\lambda_i m_{x_i x_i'}) \otimes y_i.$$

Indeed, we have

$$\begin{aligned}
\mu_p(m) &= \inf\left\{\left\|\left(\lambda_i d(x_i, x_i')\right)_i\right\|_{l_p^n} \left\|(y_i)_i\right\|_{l_{p^*}^{n,w}(Y)}\right\} \\
&= \inf\left\{\left(\sum_{i=1}^{n} \|\lambda_i m_{x_i x_i'}\|_{\mathcal{F}(X)}^p\right)^{\frac{1}{p}} \left\|(y_i)_i\right\|_{l_{p^*}^{n,w}(Y)}\right\} \\
&= \inf\left\{\left\|(\lambda_i m_{x_i x_i'})_i\right\|_{l_p^n(\mathcal{F}(X))} \left\|(y_i)_i\right\|_{l_{p^*}^{n,w}(Y)}\right\} = g_p(u).
\end{aligned}$$

Now, it suffices to show that φ is onto. Let $u = \sum_{i=1}^{n} v_i \otimes y_i \in \mathcal{F}(X)\hat{\otimes}_{g_p} Y$ where $v_i = \sum_{s_i=1}^{k_i} \lambda_{s_i} m_{x_{s_i} x'_{s_i}}$. We put for $1 \le i \le n$, $m_i = \sum_{s_i=1}^{k_i} \lambda_{s_i} y_i m_{x_{s_i} x'_{s_i}}$ and $m = \sum_{i=1}^{n} m_i$. We will verify that $\varphi(m) = u$. Indeed,

$$\begin{aligned}
\varphi(m) &= \sum_{i=1}^{n} \varphi(m_i) = \sum_{i=1}^{n} \varphi\left(\sum_{s_i=1}^{k_i} \lambda_{s_i} y_i m_{x_{s_i} x'_{s_i}}\right) \\
&= \sum_{i=1}^{n} \sum_{s_i=1}^{k_i} \lambda_{s_i} m_{x_{s_i} x'_{s_i}} \otimes y_i = \sum_{i=1}^{n} v_i \otimes y_i = u. \qquad \square
\end{aligned}$$

In the next result we give a characterization of an element of the dual of the space $\mathcal{F}_{\mu_p}(X;Y)$. For the proof, we need the following lemma which is due to Ky Fan (see [5, p. 190] for more detail about this lemma).

Lemma 1.3 *(Ky Fan). Let E be a Hausdorff topological vector space and $\mathcal{C}$ be a compact convex subset of E. Let M be a set of functions on $\mathcal{C}$ with values in $(-\infty, \infty]$ having the following:*

(a) *each $f \in M$ is convex and lower semicontinuous;*
(b) *if $g \in conv(M)$, there is an $f \in M$ such that $g(x) \le f(x)$, for every $x \in \mathcal{C}$;*
(c) *there is an $r \in \mathbb{R}$ such that each $f \in M$ has a value not greater than r.*

Then, there is an $x_0 \in \mathcal{C}$ such that $f(x_0) \le r$ for all $f \in M$.

Theorem 1.4. *Let X be a pointed metric space, Y be a Banach space and $C > 0$. The following properties are equivalent.*

(1) *The function φ is μ_p-continuous on $\mathcal{F}(X;Y)$, i.e.,*

$$\left|\varphi(m)\right| \le C\mu_p(m) \quad \textit{for all } m \in \mathcal{F}(X;Y). \tag{1.1}$$

(2) *For any representation of m of the form $m = \sum_{i=1}^{n} \lambda_i y_i m_{x_i x'_i}$, we have*

$$\sum_{i=1}^{n} \left|\varphi(\lambda_i y_i m_{x_i x'_i})\right| \leq C\mu_p(m). \tag{1.2}$$

(3) *There exists a Radon probability μ on B_{Y^*} such that for every atom of the form $ym_{xx'}$*

$$\left|\varphi(ym_{xx'})\right| \leq Cd\big(x, x'\big)\|y\|_{L_{p^*}(\mu)}. \tag{1.3}$$

Proof. (1) $\Rightarrow$ (2): Let $(\alpha_i)_{1\leq i\leq n}$ be a scalar sequence. By (1.1), we have

$$\begin{aligned}
\left|\varphi\left(\sum_{i=1}^{n} \alpha_i \lambda_i y_i m_{x_i x'_i}\right)\right| &= \left|\sum_{i=1}^{n} \alpha_i \varphi(\lambda_i y_i m_{x_i x'_i})\right| \\
&\leq C\left\|\left(\alpha_i \lambda_i d\big(x_i, x'_i\big)\right)_i\right\|_{l_p^n} \left\|(y_i)_i\right\|_{l_{p^*}^{n,w}(Y)} \\
&\leq C\left\|(\alpha_i)_i\right\|_{l_\infty^n} \left\|\left(\lambda_i d\big(x_i, x'_i\big)\right)_i\right\|_{l_p^n} \left\|(y_i)_i\right\|_{l_{p^*}^{n,w}(Y)}.
\end{aligned}$$

Taking the supremum over all sequences $(\alpha_i)_{1\leq i\leq n}$ with $\|(\alpha_i)_i\|_{l_\infty^n} \leq 1$, we obtain (1.2).

(2) $\Rightarrow$ (3): Let φ be a μ_p-continuous function on $\mathcal{F}(X;Y)$. Let $K = B_{Y^*}$. We consider the set $\mathcal{C}$ of probability measures on K. It is a convex and compact subset of $C(K)^*$ endowed with its weak*-topology. Let M be the set of all functions on $\mathcal{C}$ with values in $\mathbb{R}$ of the form

$$\Psi_{((\lambda_i),(x_i),(x'_i),(y_i))}(\mu) = \sum_{i=1}^{n} \left|\varphi(\lambda_i y_i m_{x_i x'_i})\right| - \sum_{i=1}^{n} \left(\frac{C}{p}\left\|\left(\lambda_i d\big(x_i, x'_i\big)\right)_i\right\|_{l_p^n}^p + \frac{C}{p^*}\|y_i\|_{L_{p^*}(\mu)}^{p^*}\right),$$

where $(x_i)_{1\leq i\leq n}, (x'_i)_{1\leq i\leq n} \subset X$, $(y_i)_{1\leq i\leq n} \subset Y$ and $(\lambda_i)_{1\leq i\leq n} \subset \mathbb{R}$. We will verify the assumptions of Ky Fan's lemma:

(a) It is easy to see that each element of M is convex and continuous on $\mathcal{C}$.

(b) It suffices to show that M is convex. Let Ψ_1, Ψ_2 in M such that

$$\begin{aligned}
&\Psi_{1((\lambda_{1i}),(x_{1i}),(x'_{1i}),(y_{1i}))}(\mu) \\
&\quad = \sum_{i=1}^{l} \left|\varphi(\lambda_{1i} y_{1i} m_{x_{1i} x'_{1i}})\right| - \sum_{i=1}^{l} \left(\frac{C}{p}\left\|\left(\lambda_{1i} d\big(x_{1i}, x'_{1i}\big)\right)_i\right\|_{l_p^l}^p + \frac{C}{p^*}\|y_{1i}\|_{L_{p^*}(\mu)}^{p^*}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\Psi_{2((\lambda_{2i}),(x_{2i}),(x'_{2i}),(y_{2i}))}(\mu) \\
&\quad = \sum_{i=1}^{s} \left|\varphi(\lambda_{2i} y_{2i} m_{x_{2i} x'_{2i}})\right| - \sum_{i=1}^{s} \left(\frac{C}{p}\left\|\left(\lambda_{2i} d\big(x_{2i}, x'_{2i}\big)\right)_i\right\|_{l_p^s}^p + \frac{C}{p^*}\|y_{2i}\|_{L_{p^*}(\mu)}^{p^*}\right).
\end{aligned}$$

It follows that

$$\alpha\Psi_1 + (1-\alpha)\Psi_2 = \sum_{i=1}^{n} \left|\varphi(\lambda_i y_i m_{x_i x'_i})\right| - \sum_{i=1}^{n} \left(\frac{C}{p}\left\|\left(\lambda_i d\big(x_i, x'_i\big)\right)_i\right\|_{l_p^n}^p + \frac{C}{p^*}\|y_i\|_{L_{p^*}(\mu)}^{p^*}\right),$$

with $n = l + s$, and

$$x_i = \begin{cases} x_{1i} & \text{if } 1 \le i \le l \\ x_{2(i-l)} & \text{if } l+1 \le i \le n \end{cases},$$

$$x'_i = \begin{cases} x'_{1i} & \text{if } 1 \le i \le l \\ x'_{2(i-l)} & \text{if } l+1 \le i \le n \end{cases},$$

$$y_i = \begin{cases} \alpha^{\frac{1}{p^*}} y_{1i} & \text{if } 1 \le i \le l \\ (1-\alpha)^{\frac{1}{p^*}} y_{2(i-l)} & \text{if } l+1 \le i \le n \end{cases},$$

$$\lambda_i = \begin{cases} \alpha^{\frac{1}{p}} \lambda_{1i} & \text{if } 1 \le i \le l \\ (1-\alpha)^{\frac{1}{p}} \lambda_{2(i-l)} & \text{if } l+1 \le i \le n \end{cases}.$$

(c) Let us show that $r = 0$ verifies the condition (c) of Ky Fan's lemma. There exists $y_0^* \in B_{Y^*}$ such that

$$\sup_{\|y^*\|_{Y^*}=1} \left(\sum_{i=1}^{n} \left| \langle y_i, y^* \rangle \right|^{p^*} \right)^{\frac{1}{p^*}} = \left(\sum_{i=1}^{n} \left| \langle y_i, y_0^* \rangle \right|^{p^*} \right)^{\frac{1}{p^*}}.$$

Let $\delta_{y_0^*}$ be Dirac's measure supported by y_0^*. Using the elementary identity

$$\forall \alpha, \beta \in \mathbb{R}_+^*: \quad \alpha\beta = \inf_{\epsilon>0} \left\{ \frac{1}{p} \left(\frac{\alpha}{\epsilon} \right)^p + \frac{1}{p^*} (\epsilon\beta)^{p^*} \right\},$$

and by (1.2), we find that by taking

$$\alpha = \left\| \left(\lambda_i d(x_i, x'_i) \right)_i \right\|_{l_p^n}, \quad \beta = \left(\sum_{i=1}^{n} \left| \langle y_i, y_0^* \rangle \right|^{p^*} \right)^{\frac{1}{p}}, \quad \epsilon = 1.$$

$$\begin{aligned} \Psi(\delta_{y_0^*}) &= \sum_{i=1}^{n} \left| \varphi(\lambda_i y_i m_{x_i x'_i}) \right| - \sum_{i=1}^{n} \left(\frac{C}{p} \left\| \left(\lambda_i d(x_i, x'_i) \right)_i \right\|_{l_p^n}^p + \frac{C}{p^*} \|y_i\|_{L_{p^*}(\delta_{y_0^*})}^{p^*} \right) \\ &= \sum_{i=1}^{n} \left| \varphi(\lambda_i y_i m_{x_i x'_i}) \right| - \sum_{i=1}^{n} \left(\frac{C}{p} \left\| \left(\lambda_i d(x_i, x'_i) \right)_i \right\|_{l_p^n}^p + \frac{C}{p^*} \left| \langle y_i, y_0^* \rangle \right|^{p^*} \right) \\ &\le \sum_{i=1}^{n} \left| \varphi(\lambda_i y_i m_{x_i x'_i}) \right| - C \left\| \left(\lambda_i d(x_i, x'_i) \right)_i \right\|_{l_p^n} \left(\sum_{i=1}^{n} \left| \langle y_i, y_0^* \rangle \right|^{p^*} \right)^{\frac{1}{p}} \\ &\le 0. \end{aligned}$$

By Ky Fan's lemma, there is $\mu \in \mathcal{C}$ such that $\Psi(\mu) \le 0$ for every $\Psi \in M$. If we consider $\lambda \in \mathbb{R}_+^*$, $x, x' \in X$ and $y \in Y$, we obtain

$$\Psi(\mu) = \Psi_{(\lambda,x,x',y)}(\mu) = \left| \varphi(\lambda y m_{xx'}) \right| - \frac{C}{p} |\lambda|^p d(x,x')^p - \frac{C}{p^*} \|y\|_{L_{p^*}(\mu)}^{p^*} \le 0.$$

Thus

$$|\lambda| \left| \varphi(y m_{xx'}) \right| \le \frac{C}{p} |\lambda|^p d(x,x')^p + \frac{C}{p^*} \|y\|_{L_{p^*}(\mu)}^{p^*}.$$

Fix $\epsilon > 0$. Replacing λ by $\frac{1}{\epsilon^{p^*}}$

$$\frac{1}{\epsilon^{p^*}} \left| \varphi(y m_{xx'}) \right| \le C \left(\frac{1}{p \epsilon^{pp^*}} d(x,x')^p + \frac{1}{p^*} \|y\|_{L_{p^*}(\mu)}^{p^*} \right).$$

Then

$$\begin{aligned}\left|\varphi(ym_{xx'})\right| &\le C\left(\frac{1}{p\epsilon^p}d(x,x')^p + \frac{\epsilon^{p*}}{p^*}\|y\|^{p*}_{L_{p^*}(\mu)}\right) \\ &\le C\left(\frac{1}{p}\left(\frac{d(x,x')}{\epsilon}\right)^p + \frac{1}{p^*}\left(\epsilon\|y\|_{L_{p^*}(\mu)}\right)^{p*}\right).\end{aligned}$$

We take the infimum over all $\epsilon > 0$, we find

$$\left|\varphi(ym_{xx'})\right| \le Cd(x,x')\|y\|_{L_{p^*}(\mu)}.$$

(3) $\Rightarrow$ (1): Let $m \in \mathcal{F}(X;Y)$ such that

$$m = \sum_{i=1}^{n} \lambda_i y_i m_{x_i x_i'}.$$

By (1.3)

$$\begin{aligned}\left|\varphi(m)\right| &\le \sum_{i=1}^{n}\left|\varphi(\lambda_i y_i m_{x_i x_i'})\right| \\ &\le C\sum_{i=1}^{n}\lambda_i d(x_i,x_i')\|y_i\|_{L_{p^*}(\mu)} \quad \text{(by Hölder inequality)} \\ &\le C\left\|\left(\lambda_i d(x_i,x_i')\right)_i\right\|_{l_p^n}\left(\sum_{i=1}^{n}\int_{B_{Y^*}}\left|y^*(y_i)\right|^{p^*}d\mu\right)^{\frac{1}{p^*}} \\ &\le C\left\|\left(\lambda_i d(x_i,x_i')\right)_i\right\|_{l_p^n}\left(\int_{B_{Y^*}}\sum_{i=1}^{n}\left|y^*(y_i)\right|^{p^*}d\mu\right)^{\frac{1}{p^*}} \\ &\le C\left\|\left(\lambda_i d(x_i,x_i')\right)_i\right\|_{l_p^n}\left\|(y_i)_i\right\|_{l_{p^*}^{n,w}(Y)},\end{aligned}$$

as m is arbitrary, we find

$$\left|\varphi(m)\right| \le C\mu_p(m),$$

thus, φ is μ_p-continuous function on $\mathcal{F}(X;Y)$. □

2. The space of Lipschitz–Cohen strongly p-summing

Pietsch in [14] has published an interesting monograph on operator ideals. Recently, several generalizations of certain operator ideals to the space of Lipschitz maps have been investigate. In this section, we consider the ideal of (Cohen) strongly p-summing operators and we give its extension to Lipschitz mappings. In [4] Cohen has introduced the following concept: A linear operator $u : E \to F$ between Banach spaces is strongly p-summing (or Cohen strongly p-summing) $(1 < p \le \infty)$ if there is a positive constant C such that for all $n \in \mathbb{N}^*$, $x_1, ..., x_n \in E$ and $y_1^*, ..., y_n^* \in F^*$, we have

$$\sum_{i=1}^{n}\left|\langle u(x_i), y_i^*\rangle\right| \le C\left(\sum_{i=1}^{n}\|x_i\|^p\right)^{\frac{1}{p}}\left\|\left(y_i^*\right)_i\right\|_{l_{p^*}^{n,w}(F^*)}. \tag{2.1}$$

The smallest constant C, which is noted by $d_p(u)$, such that the inequality (2.1) holds, is called the strongly p-summing norm on the space $\mathcal{D}_p(E;F)$ of all Cohen strongly p-summing linear operators from E into F, which is a Banach space. If $p=1$, we have $\mathcal{D}_1(E;F)=\mathcal{B}(E;F)$, the space of all bounded linear operators from E to F.

We give the same definition to the category of Lipschitz mappings.

Definition 2.1. Let X be a pointed metric space and Y be a Banach space. Let $T:X\to Y$ be a Lipschitz map. T is Lipschitz–Cohen strongly p-summing $(1<p\le\infty)$ if there is a constant $C>0$ such that for any $n\in\mathbb{N}^*$, $(x_i)_i,(x'_i)_i$ in X; $(y^*_i)_i$ in Y^* and $(\lambda_i)_i$ in $\mathbb{R}^*_+$ $(1\le i\le n)$, we have

$$\sum_{i=1}^{n}\lambda_i\bigl|\bigl\langle T(x_i)-T\bigl(x'_i\bigr),y^*_i\bigr\rangle\bigr|\le C\bigl\|\bigl(\lambda_i d\bigl(x_i,x'_i\bigr)\bigr)_i\bigr\|_{l_p^n}\bigl\|\bigl(y^*_i\bigr)_i\bigr\|_{l_{p^*}^{n,w}(Y^*)}. \tag{2.2}$$

We denote by $\mathcal{D}_p^L(X;Y)$ the Banach space of all Lipschitz–Cohen strongly p-summing and $d_p^L(T)$ its norm

$$d_p^L(T)=\inf\bigl\{C>0,\ C\text{ verifying (2.2)}\bigr\}.$$

As in the linear case, if $p=1$ we have $\mathcal{D}_1^L(X;Y)=Lip_0(X;Y)$.

It is easy to show the next proposition.

Proposition 2.2. *Let u be a bounded linear operator from E into F and $1\le p\le\infty$. Then*

$$d_p(u)=d_p^L(u).$$

In the next result, we give the Pietsch domination theorem for the class of Lipschitz–Cohen strongly p-summing. The linear version of this theorem has been given by Pietsch [13] for p-summing linear operators.

Theorem 2.3 *(Pietsch's domination). Let X be a pointed metric space and Y be a Banach space. The following properties are equivalent.*

(1) *The mapping T belongs to $\mathcal{D}_p^L(X;Y)$.*
(2) *For $(x_i)_i,(x'_i)_i$ in X; $(y^*_i)_i$ in Y^* and $(\lambda_i)_i$ in $\mathbb{R}$ $(1\le i\le n)$, we have*

$$\left|\sum_{i=1}^{n}\lambda_i\bigl\langle T(x_i)-T\bigl(x'_i\bigr),y^*_i\bigr\rangle\right|\le d_p^L(T)\bigl\|\bigl(\lambda_i d\bigl(x_i,x'_i\bigr)\bigr)_i\bigr\|_{l_p^n}\bigl\|\bigl(y^*_i\bigr)_i\bigr\|_{l_{p^*}^{n,w}(Y^*)}. \tag{2.3}$$

(3) *There exist a constant $C>0$ and a Radon probability μ on $B_{Y^{**}}$ such that for all $x,x'\in X$ and $y^*\in Y^*$, we have*

$$\bigl|\bigl\langle T(x)-T\bigl(x'\bigr),y^*\bigr\rangle\bigr|\le Cd\bigl(x,x'\bigr)\bigl\|y^*\bigr\|_{L_{p^*}(\mu)}. \tag{2.4}$$

In this case,

$$d_p^L(T)=\inf\bigl\{C>0,\ C\ \textit{verifying (2.4)}\bigr\}.$$

Proof. (1) $\Rightarrow$ (2): Immediate.

(2) $\Rightarrow$ (3): Let $T\in\mathcal{D}_p^L(X;Y)$. First, we can see T as a mapping defined from X into Y^{**}. Let φ_T its correspondent linear function on $\mathcal{F}(X;Y^*)$. Let $m\in\mathcal{F}(X;Y^*)$, by (2.3) we have

$$\begin{aligned}\left|\varphi_T(m)\right| &= \left|\sum_{i=1}^{n} \lambda_i \left\langle T(x_i) - T(x_i'), y_i^*\right\rangle\right| \\ &\leq C \left\| \left(\lambda_i d(x_i, x_i')\right)_i \right\|_{l_p^n} \left\| \left(y_i^*\right)_i \right\|_{l_{p^*}^{n,w}(Y^*)},\end{aligned}$$

hence, as m is arbitrary,

$$\left|\varphi_T(m)\right| \leq C\mu_p(m),$$

then φ_T is μ_p-continuous on $\mathcal{F}(X; Y^*)$. By Theorem 1.4 (3), there exist a constant $C > 0$ and a Radon probability μ on $B_{Y^{**}}$ such that for all $x, x' \in X$ and $y^* \in Y^*$

$$\begin{aligned}\left|\left\langle T(x) - T(x'), y^*\right\rangle\right| &= \left|\varphi_T\left(y^* m_{xx'}\right)\right| \\ &\leq Cd\left(x, x'\right)\left\|y^*\right\|_{L_{p^*}(\mu)}.\end{aligned}$$

(3) ⇒ (1): Let T be a Lipschitz mapping verifies (2.4). For $x, x' \in X$ and $y^* \in Y^*$,

$$\begin{aligned}\left|\varphi_T\left(y^* m_{xx'}\right)\right| &= \left|\left\langle T(x) - T(x'), y^*\right\rangle\right| \\ &\leq Cd\left(x, x'\right)\left\|y^*\right\|_{L_{p^*}(\mu)}.\end{aligned}$$

So, by Theorem 1.4 (1) φ_T is μ_p-continuous on $\mathcal{F}(X; Y^*)$ and by (1.2)

$$\begin{aligned}\sum_{i=1}^{n} \lambda_i \left|\left\langle T(x_i) - T(x_i'), y_i^*\right\rangle\right| &= \sum_{i=1}^{n} \left|\varphi_T\left(\lambda_i y_i^* m_{x_i x_i'}\right)\right| \\ &\leq C\mu_p(m) \\ &\leq C \left\| \left(\lambda_i d(x_i, x_i')\right)_i \right\|_{l_p^n} \left\| \left(y_i^*\right)_i \right\|_{l_{p^*}^{n,w}(Y^*)}.\end{aligned}$$

Therefore T is in $\mathcal{D}_p^L(X; Y)$ and

$$d_p^L(T) \leq C. \qquad \square$$

The main result of this section is the following identification.

Theorem 2.4. *Let X be a pointed metric space and Y be a Banach space. Let $p \in [1, \infty]$. We have the isometric identification*

$$\mathcal{D}_p^L\left(X; Y^*\right) = \mathcal{F}_{\mu_p}(X; Y)^*. \tag{2.5}$$

Proof. Let $T \in \mathcal{D}_p^L(X; Y^*)$ and φ_T its correspondent linear function on $\mathcal{F}(X; Y)$. We will show that φ_T is μ_p-continuous. Let $m = \sum_{i=1}^{n} \lambda_i y_i m_{x_i x_i'} \in \mathcal{F}(X; Y)$. As y_i is an element in Y^{**}, we have by Theorem 2.3

$$\begin{aligned}\left|\varphi_T(m)\right| &= \left|\sum_{i=1}^{n} \lambda_i \left\langle T(x_i) - T(x_i'), y_i\right\rangle\right| \\ &\leq d_p^L(T) \left\| \left(\lambda_i d(x_i, x_i')\right)_i \right\|_{l_p^n} \sup_{y^{***} \in B_{Y^{***}}} \left(\sum_{i=1}^{n} \left|y^{***}(y_i)\right|^{p^*}\right)^{\frac{1}{p^*}} \\ &\leq d_p^L(T) \left\| \left(\lambda_i d(x_i, x_i')\right)_i \right\|_{l_p^m} \left\|(y_i)_i\right\|_{l_{p^*}^{n,w}(Y)}.\end{aligned}$$

Hence, as m is arbitrary,

$$\left|\varphi_T(m)\right| \leq d_p^L(T)\mu_p(m),$$

then φ_T is μ_p-continuous on $\mathcal{F}(X;Y)$ and $\|\varphi_T\|_{\mu_p} \leq d_p^L(T)$.

Conversely, let $\varphi \in \mathcal{F}_{\mu_p}(X;Y)^*$. Note that φ can be identified with a mapping $T_\varphi : X \longrightarrow Y^*$ via the formula

$$\langle T_\varphi(x), y\rangle = \varphi(ym_{x0}).$$

It is clear that T_φ is Lipschitz. Indeed,

$$\begin{aligned}\left\|T_\varphi(x) - T_\varphi(x')\right\| &= \sup_{y\in B_Y}\left|\langle T_\varphi(x) - T_\varphi(x'), y\rangle\right| \\ &= \sup_{y\in B_Y}\left|\langle \varphi(ym_{x0}) - \varphi(ym_{x'0})\rangle\right| \\ &= \sup_{y\in B_Y}\left|\varphi(ym_{xx'})\right| \\ &\leq \sup_{y\in B_Y}\|\varphi\| d(x,x')\|y\| \\ &\leq \|\varphi\|_{\mu_p} d(x,x').\end{aligned}$$

Now, let $(x_i)_i, (x_i')_i \subset X$, $(y_i^{**})_i \subset Y^{**}$ and $(\lambda_i)_i \subset \mathbb{R}_+^*$ $(1 \leq i \leq n)$. Consider the finite-dimensional subspaces

$$V = span(y_i^{**})_{i=1}^n \subset Y^{**},$$

and $U = span(T_\varphi(x_i) - T_\varphi(x_i'))_{i=1}^n \subset Y^*$. Let $\varepsilon > 0$. By the principle of local reflexivity [5, p. 178], there is an injective linear map $\phi : V \to Y$ such that

$$\max\{\|\phi\|, \|\phi\|\|\phi^{-1}\|\} \leq 1 + \varepsilon,$$

and $\langle \phi(y^{**}), u^*\rangle = \langle y^{**}, u^*\rangle$ for any $y^{**} \in V$ and $u^* \in U$. Letting $y_i = \phi(y_i^{**})$, the latter condition together with the continuity of φ imply

$$\begin{aligned}\left|\sum_{i=1}^n \lambda_i\langle T_\varphi(x_i) - T_\varphi(x_i'), y_i^{**}\rangle\right| &= \left|\sum_{i=1}^n \lambda_i\langle T_\varphi(x_i) - T_\varphi(x_i'), y_i\rangle\right| \\ &= \left|\varphi\left(\sum_{i=1}^n \lambda_i y_i m_{x_i x_i'}\right)\right| \\ &\leq \|\varphi\|_{\mu_p}\mu_p\left(\sum_{i=1}^n \lambda_i y_i m_{x_i x_i'}\right) \\ &\leq \|\varphi\|_{\mu_p}\left\|\left(\lambda_i d(x_i,x_i')\right)_i\right\|_{l_p^n}\left\|(y_i)_i\right\|_{l_{p^*}^{n,w}(Y)}.\end{aligned}$$

Noting that

$$\left\|(y_i)_i\right\|_{l_{p^*}^{n,w}(Y)} = \sup_{\|y^*\|\leq 1}\left(\sum_{i=1}^n \left|y^*(y_i)\right|^{p^*}\right)^{\frac{1}{p^*}}$$

$$
\begin{aligned}
&= \sup_{\|y^*\|\le 1} \left(\sum_{i=1}^{n} \left| y^*\left(\phi\left(y_i^{**}\right)\right)\right|^{p^*} \right)^{\frac{1}{p^*}} \\
&\le \|\phi\| \left\| \left(y_i^{**}\right)_i \right\|_{l_{p^*}^{n,w}(Y^{**})} \\
&\le (1+\varepsilon) \left\| \left(y_i^{**}\right)_i \right\|_{l_{p^*}^{n,w}(Y^{**})}.
\end{aligned}
$$

Since $\varepsilon > 0$ was arbitrary, letting it go to zero proves that T_φ is Lipschitz–Cohen strongly p-summing and $d_p^L(T_\varphi) \le \|\varphi\|_{\mu_p}$. $\square$

3. Some inclusion and coincidence properties

The aim of this section is to explore more properties of the class of Lipschitz–Cohen strongly p-summing operators. We start by showing the relationship between the Lipschitz mapping and its linearization for the concept of strongly p-summing. A similar characterization holds for Lipschitz compact operators (see [11]).

Proposition 3.1. *The following properties are equivalent.*

(1) *The mapping T belongs to $\mathcal{D}_p^L(X;Y)$.*
(2) *The linear operator $\widehat{T}$ belongs to $\mathcal{D}_p(\mathcal{F}(X);Y)$.*

Even more, $\mathcal{D}_p^L(X;Y) = \mathcal{D}_p(\mathcal{F}(X);Y)$ holds isometrically.

Proof. First, suppose that $T \in \mathcal{D}_p^L(X;Y)$. Let $m \in \mathcal{F}(X)$ and $y^* \in Y^*$. Then

$$
\begin{aligned}
\left|\left\langle \widehat{T}(m), y^*\right\rangle\right| &\le \sum_{i=1}^{n} |\lambda_i| \left|\left\langle T(x_i) - T\left(x_i'\right), y^*\right\rangle\right| \\
&\le d_p^L(T) \sum_{i=1}^{n} |\lambda_i| d\left(x_i, x_i'\right) \left\|y^*\right\|_{L_{p^*}(\mu)},
\end{aligned}
$$

as m is arbitrary, we obtain

$$
\left|\left\langle \widehat{T}(m), y^*\right\rangle\right| \le d_p^L(T) \|m\|_{\mathcal{F}(X)} \left\|y^*\right\|_{L_{p^*}(\mu)},
$$

hence $\widehat{T}$ verifies Pietsch's domination for (Cohen) strongly p-summing operators [4, Theorem 2.3.1], then $\widehat{T} \in \mathcal{D}_p(\mathcal{F}(X);Y)$ and

$$
d_p(\widehat{T}) \le d_p^L(T).
$$

Conversely, suppose that $\widehat{T} \in \mathcal{D}_p(\mathcal{F}(X);Y)$. Let $x, x' \in X$ and $y^* \in Y^*$, by Pietsch's domination of $\widehat{T}$

$$
\begin{aligned}
\left|\left\langle T(x) - T\left(x'\right), y^*\right\rangle\right| &= \left|\left\langle \widehat{T}(m_{xx'}), y^*\right\rangle\right| \\
&\le d_p(\widehat{T}) \|m_{xx'}\| \left\|y^*\right\|_{L_{p^*}(\mu)} \\
&\le d_p(\widehat{T}) d\left(x, x'\right) \left\|y^*\right\|_{L_{p^*}(\mu)},
\end{aligned}
$$

by Theorem 2.3, T is in $\mathcal{D}_p^L(X;Y)$ and

$$
d_p^L(T) \le d_p(\widehat{T}). \qquad \square
$$

One of the nice results of Cohen is that a linear map $u : E \to F$ between Banach spaces is strongly p-summing if and only if the adjoint map $u^* : F^* \to E^*$ is p^*-summing. It would be interesting to point out that an analogous situation holds in the nonlinear case: if X is a metric space and Y is a Banach space, $T : X \to Y$ is Lipschitz–Cohen strongly p-summing if and only if the "adjoint" map $T^{\#} \mid_{Y^*}: Y^* \to X^{\#}$ is p^*-summing (this map is actually just the linear adjoint of the linearization $\widehat{T} : \mathcal{F}(X) \to Y$).

For Lipschitz p-summing operators we have the following result.

Proposition 3.2. *Let $1 \le p < \infty$. Let $T : X \to Y$ be a Lipschitz map and $\widehat{T}$ its linearization. Suppose that $\widehat{T}$ is p-summing, then T is Lipschitz p-summing.*

Proof. If $\widehat{T}$ is p-summing then it is Lipschitz p-summing, and by (0.2) T Lipschitz factors through $\widehat{T}$, so T is Lipschitz p-summing by the ideal property of Lipschitz p-summing operators. $\square$

Remark 3.3. The converse of the precedent Proposition is not true. Indeed, the canonical inclusion

$$\delta_{\mathbb{R}} : \mathbb{R} \to \delta_{\mathbb{R}}(\mathbb{R}) \subset \mathcal{F}(\mathbb{R})$$

is Lipschitz p-summing since it is Lipschitz equivalent to the identity $id_{\mathbb{R}}$; hence $\delta_{\mathbb{R}} : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ is Lipschitz p-summing. But the linearization of this map is the identity on $\mathcal{F}(\mathbb{R})$, which cannot be p-summing because $\mathcal{F}(\mathbb{R})$ is infinite-dimensional (isometric to $L_1(\mathbb{R})$, in fact).

Corollary 3.4. *Let X be a pointed metric space and Y is an $\mathcal{L}_p$-space $(1 \le p < \infty)$. Then*

$$\mathcal{D}^L_{p^*}(X;Y) \subset \Pi^L_p(X;Y).$$

Proof. If T is in $\mathcal{D}^L_{p^*}(X;Y)$, the Proposition 3.1 implies that $\widehat{T} : \mathcal{F}(X) \to Y$ is Cohen strongly p^*-summing. By a result of Cohen [4, Theorem 3.2.3], $\widehat{T}$ is p-summing and by Proposition 3.2, T is Lipschitz p-summing with

$$\pi^L_p(T) \le d^L_{p^*}(T). \qquad \square$$

We recall that (see [2]) $cs_p(X;Y)$ is the space of molecules $\mathcal{F}(X;Y)$ endowed with the next norm

$$cs_p(m) = \inf\left\{\left\|\left(\lambda_i\|y_i\|\right)_i\right\|_{l_p^n} w_{p^*}^{Lip}\left(\left(\lambda_i^{-1}, x_i, x_i'\right)_i\right)\right\},$$

where the infimum is taken over all representations of m of the form

$$m = \sum_{i=1}^{n} y_i m_{x_i x_i'},$$

with $x_i, x_i' \in X$, $y_i \in Y$, $\lambda_i \in \mathbb{R}_+^*$; $(1 \le i \le n)$ and $n \in \mathbb{N}^*$.

Corollary 3.5. *Let X be a pointed metric space, $1 < p < \infty$ and Y be an $\mathcal{L}_p$-space. The identity mapping*

$$id : cs_p(X;Y) \longrightarrow \mathcal{F}_{\mu_p}(X;Y)$$

is continuous with $\|id\| \le 1$.

Proof. Let $m \in cs_p(X;Y)$ and $\varphi \in \mathcal{F}_{\mu_p}(X;Y)^*$ such that

$$\|\varphi\|_{\mu_p} \leq 1.$$

By Theorem 2.4, we can identify φ with a function $T_\varphi \in \mathcal{D}_p^L(X;Y^*)$ with $\|\varphi\|_{\mu_p} = d_p^L(T_\varphi)$. By the above result, $T_\varphi \in \Pi_{p^*}^L(X;Y^*)$, and Theorem 4.3 in [2] asserts that $\varphi \in cs_p(X;Y)^*$ with

$$\begin{aligned}\|\varphi\|_{cs_p(X;Y)^*} &= \pi_{p^*}^L(T_\varphi) \\ &\leq d_p^L(T_\varphi) = \|\varphi\|_{\mu_p},\end{aligned}$$

consequently,

$$\mu_p(m) = \sup_{\|\varphi\|_{\mu_p}\leq 1} \big\|\varphi(m)\big\| \leq \sup_{\|\varphi\|_{cs_p(X;Y)^*}\leq 1} \big\|\varphi(m)\big\| = \|m\|_{cs_p(X;Y)}. \qquad \square$$

In the next result, we give a version of Grothendieck's theorem (this famous result is due to Grothendieck [10]). We mention that other nonlinear versions have already appeared in the literature (for example in [7]).

Corollary 3.6 *(Grothendieck's theorem). Let $X = l_1$ and H be a Hilbert space. Then*

$$\Pi_1^L(X;H) = Lip_0(X;H).$$

Proof. In this case, the free Banach space $\mathcal{F}(X)$ is isometrically isomorphic to $L_1(\mathbb{R})$ (see [6, Corollary 8]), then $\widehat{T} : \mathcal{F}(X) \to H$ is 1-summing, consequently T is 1-summing. $\square$

In the last result, we consider Lipschitz (p, r, s)-summing linear operators and we combine with Theorems 5.2 and 5.4 in [2] for giving a factorization result using the language of Lipschitz–Cohen strongly p-summing operators.

We recall the following definition as stated in [2].

Definition 3.7. Let X be a pointed metric space and Y be a Banach space. Let $T : X \to Y$ be a Lipschitz map. T is Lipschitz (p, r, s)-summing if there is a constant $C > 0$ such that for any $n \in \mathbb{N}^*$, $(x_i)_i, (x'_i)_i$ in X; $(y_i^*)_i$ in Y^* and $(\lambda_i)_i, (k_i)_i$ in $\mathbb{R}_+^*$ $(1 \leq i \leq n)$, we have

$$\big\|\big(\lambda_i\big\langle T(x_i) - T\big(x'_i\big), y_i^*\big\rangle\big)_i\big\|_{l_p^n} \leq C w_r^{Lip}\big(\big(\lambda_i k_i^{-1}, x_i, x'_i\big)_i\big)\big\|\big(k_i y_i^*\big)_i\big\|_{l_s^{n,w}(Y^*)}. \tag{3.1}$$

We denote by $\Pi_{p,r,s}^L(X;Y)$ the Banach space of all Lipschitz (p, r, s)-summing operators.

Theorem 3.8. *Let $p, r, s \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{r} + \frac{1}{s} = 1$. Let $T \in Lip_0(X;Y)$, the following are equivalent.*

1. *The mapping T belongs to $\Pi_{p^*,r,s}^L(X;Y)$.*
2. *There exist a constant $C > 0$ and regular Borel probability measures μ and ν on the weak* compact unit balls $B_{X^\#}, B_{Y^{**}}$ such that for all $x, x' \in X$ and $y^* \in Y^*$*

$$\big|\big\langle T(x) - T\big(x'\big), y^*\big\rangle\big| \leq C\Big(\int_{B_{X^\#}} \big|f(x) - f\big(x'\big)\big|^r d\mu(f)\Big)^{\frac{1}{r}} \Big(\int_{B_{Y^{**}}} \big|y^{**}\big(y^*\big)\big|^s d\nu\big(y^{**}\big)\Big)^{\frac{1}{s}}. \tag{3.2}$$

(3) *There exist a metric space* $\tilde{X}$ *and two Lipschitz mappings* T_1, T_2 *such that* $T_1 \in \Pi_r^L(X; \tilde{X})$, $T_2 \in \mathcal{D}_{s^*}^L(\tilde{X}; Y)$ *and*

$$T = T_2 \circ T_1. \tag{3.3}$$

Proof. (1) $\Leftrightarrow$ (2): First, by (0.3) the norm of T as an element of $\Pi^L_{p^*,r,s}(X;Y)$ is the same as its norm in $\Pi^L_{p^*,r,s}(X;Y^{**})$. Then, the equivalence follows from [2, Theorems 5.2 and 5.4] (specialized to the case $E = Y^*$).

(2) $\Rightarrow$ (3): Suppose that T verifies (3.2). Then, we have the following diagram which is commutative

$$\begin{array}{lll} X & \xrightarrow{T} & Y \\ \downarrow j_X & \nearrow \overline{T} & \\ \tilde{X} & \subset L_r(\mu) & \end{array}$$

where $j_X : X \to L_r(\mu)$ is the isometric injection (which is Lipschitz r-summing), $\tilde{X} = j_X(X)$ is a pointed metric space of which the metric is defined by

$$\text{For } \tilde{x}, \tilde{x}' \in \tilde{X}: \quad d(\tilde{x}, \tilde{x}') = \|\tilde{x} - \tilde{x}'\|_{L_r(\mu)},$$

and its origin is $j_X(0)$. We have $T = \overline{T} \circ j_X$. The mapping $\overline{T}$ is well defined and is Lipschitz–Cohen strongly s^*-summing. Indeed,

$$\begin{aligned} \left|\langle \overline{T}(\tilde{x}) - \overline{T}(\tilde{x}'), y^* \rangle\right| &= \left|\langle T(x) - T(x'), y^* \rangle\right| \\ &\le C\Big(\int_{B_{X^\#}} \left|f(x) - f(x')\right|^r d\mu\Big)^{\frac{1}{r}} \|y^*\|_{L_s(\nu)} \\ &\le C\Big(\int_{B_{X^\#}} \left|(\tilde{x} - \tilde{x}')(f)\right|^r d\mu\Big)^{\frac{1}{r}} \|y^*\|_{L_s(\nu)} \\ &\le C\|\tilde{x} - \tilde{x}'\|_{L_r(\mu)} \|y^*\|_{L_s(\nu)}, \end{aligned}$$

therefore by Theorem 2.3, $\overline{T}$ is Lipschitz–Cohen strongly s^*-summing.

(3) $\Rightarrow$ (2): Given a factorization (3.3), consider the map $\delta_{\tilde{X}} \circ T_1 : X \longrightarrow \mathcal{F}(\tilde{X})$ and $\widehat{T_2} : \mathcal{F}(\tilde{X}) \longrightarrow Y \subset Y^{**}$, then $T = \widehat{T_2} \circ \delta_{\tilde{X}} \circ T_1$, so, we obtain what we needed for [2, Theorem 5.4 (c)]. $\square$

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