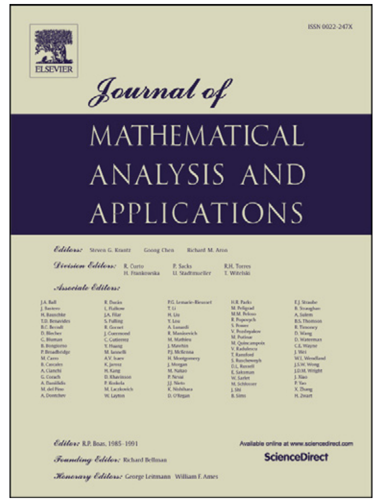

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Existence of anti-periodic mild solutions to semilinear nonautonomous evolution equations[†]

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Abstract In this paper, applying the theory of evolution family and Schauder's fixed point theorem, we prove the anti-periodic mild solutions for semilinear nonautonomous evolution equations in Banach space under conditions. Furthermore, an example is given to illustrate our results.

Keywords Anti-periodic function; Hille-Yosida operators; Evolution family; Mild solutions, Semilinear nonautonomous evolution equation.

1 Introduction

The study of the existence of anti-periodic solutions constitutes one of the most attractive topics in qualitative theory of differential equations due to its applications in engineering, physics, control theory and other subjects (see [1-4] and the references therein). Recently, the existence of anti-periodic solutions to differential equations in Banach spaces by using semigroup theory has been established by many authors. For example, we refer readers to [5-9].

In this paper, we study the existence of anti-periodic mild solutions to the semilinear nonautonomous evolution equations

$$\frac{d}{dt}u(t) = (A + B(t))u(t) + f(t, u(t)), t \in R, \tag{1}$$

where A is a Hille-Yosida operator having the domain $D(A)$ in Banach spaces X, $B(t), t \in R$ is a family of operators in $\mathcal{L}(\overline{D(A)}, X)$, the part of $(A+B(t))_{t\geq 0}$ in X_0 generates the evolution family $(U(t,s))_{t\geq s\geq 0}$ and $f : R \times X_0 \to X$ is a suitable function.

To the best of our knowledge, the literature concerning the existence of anti-periodic mild solutions to this problem is an untreated original problem, which constitutes one of the main motivations of this paper.

The paper is organized as follows: In Section 2, we give some definitions and fix notations which will be used in the sequel. In Section 3, the existence, uniqueness of anti-periodic mild solution to some semilinear nonautonomous evolution equations in Banach space are studied.

2 Preliminaries and basic results

We recall some definitions and fix notations which will be used in the sequel. Let X be a Banach space endowed with the norm $\|\cdot\|$ and $\mathcal{L}(X, Y)$ be the Banach space of all bounded linear operators from X to Y. $C_b(R, X)$ is the space of all bounded continuous functions from $R \to X$. $R^+ = [0, +\infty)$.

We give some basic results on extrapolation spaces of Hille-Yosida operators.

Definition 2.1 Let A be a linear operator with domain $D(A)$. We say that $(A, D(A))$ is a Hille-Yosida operator on X if there exists $\omega \in R$ and a positive constant $M \geq 1$ such that $(\omega, \infty) \subseteq \rho(A)$ and $\sup\{(\lambda-\omega)^n \|(\lambda - A)\|^{-n}\} \leq M$. The infimum of such a ω is called the type of A. If the constant ω can be chosen smaller than zero, A is said to be of negative type.

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From the Hille-Yosida theorem [10,Theorem II.3.8] we have the following result.

Lemma 2.1 Let $(A, D(A))$ be a Hille-Yosida operator on X, $X_0 = \overline{D(A)}$, $D(A_0) = \{x \in D(A) : Ax \in X_0\}$ and $A_0 : D(A_0) \subset X_0 \to X_0$ be the operator defined by $A_0 x = Ax$. The operator A_0 generates a C_0-semigroup $(T_0(t))_{t\geq 0}$ on X_0 with $\|T_0(t)\| \leq Me^{\omega t}$ for $t \geq 0$. Moreover, $\rho(A) \subset \rho(A_0)$ and $R(\lambda, A_0) = R(\lambda, A)|_{X_0}$, for $\lambda \in \rho(A)$.

Let $\lambda \in \rho(A)$. we define a norm on space X_0 by

$$\|x\|_{-1} = \|R(\lambda, A_0)x\|, x \in X_0.$$

The completion of $(X_0, \|\cdot\|_{-1})$ will be called the extrapolation space of X_0 associated with A_0 and will be denoted by X_{-1}. One can show easily that, $T_0(t)$ has a unique bounded linear extension $T_{-1}(t)$ to X_{-1}. The operator family $(T_{-1}(t))_{t\geq 0}$ is a C_0-semigroup on X_{-1}, called the extrapolated semigroup of $(T_0(t))_{t\geq 0}$. The domain of its generator A_{-1} is equal to A_0.

For more details of Hille-Yosida operators and extrapolation spaces, we refer to [10-12] and the references therein.

From [13] and [14,Theorem 2.3], we have the following Lemma:

Lemma 2.2 Let $\Sigma_\theta = \{\lambda \in \mathbb{C} : |\arg \lambda| \leq \theta\} \bigcup \{0\} \subset \rho(A(t))$, $\theta \in (\frac{\pi}{2}, \pi)$. If there exist a constant k_0 and a set of real numbers $\alpha_1, \alpha_2, \cdots, \alpha_k, \beta_1, \cdots, \beta_k$ with $0 \leq \beta_i < \alpha_i \leq 2, i = 1, 2, \cdots, k$ such that

$$\|A(t)(\lambda - A(t))^{-1}(A(t)^{-1} - A(s)^{-1})\| \leq K_0 \sum_{i=1}^{k} (t-s)^{\alpha_i} |\lambda|^{\beta_i - 1},$$

for $t, s \in R$, $\lambda \in \Sigma_\theta \setminus \{0\}$ and there exists a constant $M \geq 0$ such that

$$\|(\lambda - A(t))^{-1}\| \leq \frac{M}{1 + |\lambda|}, \lambda \in \Sigma_\theta,$$

then there exists a unique evolution family $\{U(t, s), t \geq s; s \in R\}$.

Definition 2.2 A set $U = \{U(t, s) : t \geq s, t, s \in R\}$ of bounded linear operators on X is called an evolution family if

(i) $U(t, s) = U(t, r)U(r, s)$ and $U(s, s) = I$ for $t \geq r \geq s$ and

(ii) $(t, s) \mapsto U(t, s)$ is strongly continuous for $t > s$.

We also need to recall the following notation concerning exponential dichotomy. An evolution family U is called exponential dichotomy (or hyperbolic) if there are projections $P(t)$, $t \in R$ uniformly bounded and strongly continuous in t and constants $M, \omega > 0$ such that

(a) $U(t, s)P(s) = P(t)U(t, s)$ for all $t \geq s$;

(b) The restriction $U_Q(t, s) : Q(s)X \to Q(t)X$ is invertible for all $t \geq s$ (and we set $U_Q(s, t) = U_Q(t, s)^{-1}$). Here and below we let $Q = I - P$ for a projection P;

(c)$\|U(t, s)P(s)\| \leq Me^{-\omega(t-s)}$ and $\|U_Q(s, t)Q(t)\| \leq Me^{-\omega(t-s)}$.

If U is hyperbolic, then the operator family

$$\Gamma(t, s) = \begin{cases} U(t, s)P(s), & t \geq s, t, s \in R \\ -U_Q(t, s)Q(s), & t < s, t, s \in R \end{cases}$$

is called Green's function corresponding to U and $P(\cdot)$.

Definition 2.3 A function $f \in C_b(R, X)$ is called anti-periodic provided that

$$f(t + T) = -f(t), \forall t \in R.$$

Denote by $P_{TA}(R, X)$ the set of all anti-periodic functions.

Lemma 2.3[7] Let $f_n \in P_{TA}(R, X)$, such that $f_n \to f$ uniformly on R. Then $f \in P_{TA}(R, X)$.

Lemma 2.4[7] Equipped with the supnorm, $P_{TA}(R, X)$ is a Banach space.

Definition 2.4 Let evolution family $U(t,s)$ be generated by the part of $(A+B(t))_{t\geq 0}$ in X_0. A function

$$u(t)=U(t,s)u(s)+\lim_{\lambda\to\infty}\int_s^t U(t,\tau)\lambda R(\lambda,A)f(\tau,u(\tau))d\tau,$$

is called a mild solution to Eq.(1) for $t\geq s$ and $s\in R$, where A , $B(t), t\in R$ are as in previous sections and $U(t,s)$ satisfies the variation-of-parameters formula

$$U(t,s)=T_0(t-s)+\int_s^t T_{-1}(t-\tau)B(\tau)U(\tau,s)d\tau.$$

For more details of mild solutions to Eq.(1), we refer to [15] and the references therein.

We give the famous Schauder's fixed point theorem as follows:

Lemma 2.5 (Schauder's fixed point theorem) Let D be a nonempty, closed, bounded, convex subset of a Banach space X. Let $F: D\to D$ be a continuous and compact operator, then the operator equation $Fu=u$ has a fixed point in D.

3 Existence of anti-periodic mild solutions

To study the existence of anti-periodic mild solutions to Eq.(1), we will assume that the following assumptions hold:

(H_1) The evolution family $U(t,s)$ is generated by the part of $(A+B(t))_{t\geq 0}$ in X_0 and satisfies the hypotheses of Lemma 2.2, also $U(t,s)$ has an exponential dichotomy with constants $\omega>0$, $M\geq 1$ and projections $P(t)$ for $t\in R$. $C=\sup_{t\in R}\{\|\lambda P(t)R(\lambda,A)\|, \|\lambda(I-P(t))R(\lambda,A)\|\}<\infty$;

(H_2) $(U(t,s))_{t\geq s\in R}$ is T-periodic, in the sense that there exists $T>0$ such that $U(t+T,s+T)=U(t,s)$ for $t\geq s$;

(H_3) The function $f: R\times X_0\to X$ is continuous and $f(t+T,-u)=-f(t,u)$ for all $t\in R$, $u\in X_0$;

(H_4) The function $f: R\times X_0\to X$ satisfies the Lipschitz condition:

$$\|f(t,x)-f(t,y)\|\leq L(t)\|x-y\|$$

for all $t\in R$, $x,y\in X_0$, where $L(t)$ satisfies

$$\|L\|^1_{loc}=\sup_{t\in R}\int_t^{t+1}L(s)ds<+\infty.$$

Theorem 3.1 Under assumptions $(H_1)-(H_3)$, for every $t\in R$, let

$$\Lambda u(t)=\lim_{\lambda\to\infty}\int_{-\infty}^{\infty}\Gamma(t,s)\lambda R(\lambda,A)f(s,u(s))ds.$$

If $u\in P_{TA}(R,X_0)$, then $\Lambda u(t)\in\overline{P_{TA}(R,X_0)}$.

Proof Firstly, it is easy to see that

$$\begin{aligned}\|\Lambda u(t)\| &= \|\lim_{\lambda\to\infty}\int_{-\infty}^{\infty}\Gamma(t,s)\lambda R(\lambda,A)f(s,u(s))ds\|\\ &\leq MC\int_{-\infty}^{\infty}e^{-\omega|t-s|}\|f(s,u(s))\|ds\\ &\leq \frac{2MC}{\omega}\|f\|_\infty.\end{aligned}$$

Thus Λ is well defined and Λu is bounded.

Secondly, for any $t, h \in R$

$$\begin{aligned}
\|\Lambda u(t+h) - \Lambda u(t)\| &= \| \lim_{\lambda\to\infty} \int_{-\infty}^{\infty} \Gamma(t+h,s)\lambda R(\lambda,A) f(s,u(s))ds \\
&- \lim_{\lambda\to\infty} \int_{-\infty}^{\infty} \Gamma(t,s)\lambda R(\lambda,A) f(s,u(s))ds\| \\
&= \| \lim_{\lambda\to\infty} \int_{-\infty}^{\infty} \Gamma(t+h,s+h)\lambda R(\lambda,A) f(s+h,u(s+h))ds \\
&- \lim_{\lambda\to\infty} \int_{-\infty}^{\infty} \Gamma(t,s)\lambda R(\lambda,A) f(s,u(s))ds\| \\
&\leq C \int_{-\infty}^{\infty} \|\Gamma(t+h,s+h) f(s+h,u(s+h)) - f(s,u(s))\| ds \\
&+ C \int_{-\infty}^{\infty} \|(\Gamma(t+h,s+h) - \Gamma(t,s)) f(s,u(s))\| ds.
\end{aligned}$$

Thus, $\|\Lambda u(t+h) - \Lambda u(t)\| \to 0$ as $h \to 0$, which proves that Λu is continuous.

Finally, It follows from (H_3) that for any $u \in P_{TA}(R, X_0)$ and for each $t \in R$

$$\begin{aligned}
\Lambda u(t+T) &= \lim_{\lambda\to\infty} \int_{-\infty}^{\infty} \Gamma(t+T,s)\lambda R(\lambda,A) f(s,u(s))ds \\
&= \lim_{\lambda\to\infty} \int_{-\infty}^{\infty} \Gamma(t+T,s+T)\lambda R(\lambda,A) f(s+T,-u(s))ds \\
&= -\lim_{\lambda\to\infty} \int_{-\infty}^{\infty} \Gamma(t,s)\lambda R(\lambda,A) f(s,u(s))ds \\
&= -\Lambda u(t).
\end{aligned}$$

Therefore, Λu is anti-periodic. The proof is complete.

Theorem 3.2 Under assumptions $(H_1) - (H_4)$, if $0 < \frac{2MC\|L\|^1_{loc}}{1-e^{-\omega}} < 1$, then Eq.(1) has a unique anti-periodic mild solution.

Proof Define the operator Λ as in Theorem 3.1 by

$$\Lambda u(t) = \lim_{\lambda\to\infty} \int_{-\infty}^{\infty} \Gamma(t,s)\lambda R(\lambda,A) f(s,u(s))ds$$

for every $t \in R$. By Theorem 3.1, the operator Λ is well defined and maps $P_{TA}(R, X_0)$ into itself.

Next, we prove that the operator Λ has a unique fixed point in $P_{TA}(R, X_0)$.

Let $u, v \in P_{TA}(R, X_0)$,then

$$\begin{aligned}
\|\Lambda u(t) - \Lambda v(t)\| &= \| \lim_{\lambda\to\infty} \int_{-\infty}^{\infty} \Gamma(t,s)\lambda R(\lambda,A)[f(s,u(s)) - f(s,v(s))]ds\| \\
&\leq MC \int_{-\infty}^{\infty} e^{-\omega|t-s|} \|f(s,u(s)) - f(s,v(s))\| ds \\
&\leq MC\|u-v\|(\int_{-\infty}^{t} e^{-\omega(t-s)} L(s)ds + \int_{t}^{\infty} e^{-\omega(s-t)} L(s)ds) \\
&\leq MC\|u-v\|(\sum_{k\geq 0} e^{-\omega k} \int_{t-k-1}^{t-k} L(s)ds + \sum_{k\geq 0} e^{-\omega k} \int_{t+k}^{t+k+1} L(s)ds) \\
&\leq 2MC\|u-v\| \sum_{k\geq 0} e^{-\omega k} \|L\|^1_{loc} \\
&\leq \frac{2MC\|L\|^1_{loc}}{1-e^{-\omega}} \|u-v\|.
\end{aligned}$$

For $0 < \frac{2MC\|L\|^1_{loc}}{1-e^{-\omega}} < 1$, it follows from the Banach contraction mapping principle that Λ admits a unique fixed point in $P_{TA}(R, X_0)$.

To end of the proof, we will prove that $u \in P_{TA}(R, X_0)$ is a mild solution of Eq.(1).

In fact, if $t \geq s$, then

$$\begin{aligned} u(t) - U(t,s)u(s) &= \lim_{\lambda\to\infty}\Big(\int_{-\infty}^{\infty}\Gamma(t,\tau)\lambda R(\lambda,A)f(\tau,u(\tau))d\tau - \int_{-\infty}^{s}U(t,\tau)P(\tau)\lambda R(\lambda,A)f(\tau,u(\tau))d\tau \\ &+ \int_{s}^{t}U(t,\tau)Q(\tau)\lambda R(\lambda,A)f(\tau,u(\tau))d\tau + \int_{t}^{\infty}U_Q(t,\tau)Q(\tau)\lambda R(\lambda,A)f(\tau,u(\tau))d\tau\Big) \\ &= \lim_{\lambda\to\infty}\Big(\int_{-\infty}^{\infty}\Gamma(t,\tau)\lambda R(\lambda,A)f(\tau,u(\tau))d\tau - \int_{-\infty}^{t}U(t,\tau)P(\tau)\lambda R(\lambda,A)f(\tau,u(\tau))d\tau \\ &+ \int_{s}^{t}U(t,\tau)\lambda R(\lambda,A)f(\tau,u(\tau))d\tau + \int_{t}^{\infty}U_Q(t,\tau)Q(\tau)\lambda R(\lambda,A)f(\tau,u(\tau))d\tau\Big) \\ &= \lim_{\lambda\to\infty}\int_{s}^{t}U(t,\tau)\lambda R(\lambda,A)f(\tau,u(\tau))d\tau, \end{aligned}$$

which implies that u is a mild solution to Eq.(1). The proof is complete.

Corollary 3.1 If assumptions $(H_1)-(H_3)$ hold true, the function f satisfies the Lipschitz condition

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\|$$

for all $t \in R$, $x, y \in X_0$, where $L > 0$ is a constant and If $0 < \frac{2MCL}{\omega} < 1$, then the Eq.(1) has a unique anti-periodic mild solution.

Let $h : R \to R$ be a continuous function such that $h(t) \geq 1$ for all $t \in R$, and $h(t) \to \infty$ as $|t| \to \infty$. We consider the space

$$C_h(X) = \{u \in C(R,X) : \lim_{|t|\to\infty}\frac{u(t)}{h(t)} = 0\}$$

endowed with the norm

$$\|u\|_h = \sup_{t\in R}\frac{\|u(t)\|}{h(t)}.$$

Lemma 3.1[16] A subset $K \subseteq C_h(X)$ is a relatively compact set if it verifies the following conditions:
(i) The set $K(t) = \{u(t) : u \in K\}$ is relatively compact in X for each $t \in R$;
(ii) The set K is equicontinuous;
(iii) For each $\varepsilon > 0$ there exists $L > 0$ such that $\|u(t)\| \leq \varepsilon h(t)$ for all $u \in K$ and all $|t| > L$.

To establish our next results, we consider the following assumptions.

(A_1) $f(t,x)$ is bounded continuous and there exists a continuous nondecreasing function $W : R^+ \to R^+$, such that

$$\|f(t,x)\| \leq W(\|x\|)$$

for all $t \in R$, $x \in X_0$;

(A_2) For each $\kappa \geq 0$, let $\beta(\kappa) = \int_{-\infty}^{\infty} e^{-\omega|t-s|}W(\kappa h(s))ds \in C_b(R)$ and $MC\beta(\kappa) < \infty$;

(A_3) For each $\epsilon > 0$, there is a $\delta > 0$, such that for every $u, v \in C_h(X_0)$, $\|u-v\|_h \leq \delta$ implies

$$\sup_{t\in R}\int_{-\infty}^{\infty} e^{-\omega|t-s|}\|f(s,u) - f(s,v)\|ds \leq \epsilon;$$

(A_4) $U(t,s)$ is a strongly continuous evolution family. Moreover, $U(t,s)$ is compact.

Theorem 3.3 Under the previous assumptions, let $V \subseteq P_{TA}(R, X_0)$ and $\Lambda(V)(t) := \{\Lambda u(t) : u \in V\}$, where

$$\Lambda u(t) = U(t,s)u(s) + \lim_{\lambda\to\infty}\int_{s}^{t}U(t,\tau)\lambda R(\lambda,A)f(\tau,u(\tau))d\tau,$$

then $\Lambda(V)(t)$ is a relatively compact subset of X_0 for each $t \in R$ and V is equicontinuous.

Proof Firstly, we will prove that $\Lambda(V)(t)$ is a relatively compact subset of X_0 for each $t \in R$.
For each $s < t$ and $\varepsilon > 0$ is such that $s \leq t - \varepsilon$,

$$\begin{aligned}
\Lambda u &= U(t,s)u(s) + \lim_{\lambda\to\infty}\int_s^{t-\varepsilon} U(t,\tau)\lambda R(\lambda,A)f(\tau,u(\tau))d\tau \\
&+ \lim_{\lambda\to\infty}\int_{t-\varepsilon}^{t} U(t,\tau)\lambda R(\lambda,A)f(\tau,u(\tau))d\tau \\
&= U(t,t-\varepsilon)[U(t-\varepsilon,s)u(s) + \lim_{\lambda\to\infty}\int_s^{t-\varepsilon} U(t-\varepsilon,\tau)\lambda R(\lambda,A)f(\tau,u(\tau))d\tau] \\
&+ \lim_{\lambda\to\infty}\int_{t-\varepsilon}^{t} U(t,\tau)\lambda R(\lambda,A)f(\tau,u(\tau))d\tau \\
&= U(t,t-\varepsilon)\Lambda u(t-\varepsilon) + \lim_{\lambda\to\infty}\int_{t-\varepsilon}^{t} U(t,\tau)\lambda R(\lambda,A)f(\tau,u(\tau))d\tau.
\end{aligned}$$

Since $\{\Lambda u(t-\varepsilon)\}$ is bounded and $U(t,t-\varepsilon)$ is compact, $\{U(t,t-\varepsilon)\Lambda u(t-\varepsilon), t > \varepsilon\}$ is a relatively compact subset of X_0. Secondly, for $U(t,s)$ is strongly continuous and f is bounded continuous,

$$\lim_{\lambda\to\infty}\int_{t-\varepsilon}^{t} U(t,\tau)\lambda R(\lambda,A)f(\tau,u(\tau))d\tau \to 0$$

as $\varepsilon \to 0$. Thus, $\Lambda(V)(t) := \{\Lambda u(t) : u \in V\}$ is a relatively compact subset of X_0 for each $t \in R$.
Finally, we will show that the set V is equicontinuous.
In fact, proceeding as above, for $t_1 < t_2$, $t_1, t_2 \in R$, we can decompose

$$\begin{aligned}
\Lambda u(t_2) - \Lambda u(t_1) &= (U(t_2,s) - U(t_1,s))u(s) + \lim_{\lambda\to\infty}\int_s^{t_2} U(t_2,\tau)\lambda R(\lambda,A)f(\tau,u(\tau))d\tau \\
&- \lim_{\lambda\to\infty}\int_s^{t_1} U(t_1,\tau)\lambda R(\lambda,A)f(\tau,u(\tau))d\tau \\
&= (U(t_2,t_1) - I)[U(t_1,s))u(s) + \lim_{\lambda\to\infty}\int_s^{t_1} U(t_1,\tau)\lambda R(\lambda,A)f(\tau,u(\tau))d\tau] \\
&+ \lim_{\lambda\to\infty}\int_{t_1}^{t_2} U(t_2,\tau)\lambda R(\lambda,A)f(\tau,u(\tau))d\tau \\
&= (U(t_2,t_1) - I)u(t_1) + \lim_{\lambda\to\infty}\int_{t_1}^{t_2} U(t_2,\tau)\lambda R(\lambda,A)f(\tau,u(\tau))d\tau.
\end{aligned}$$

Since the set $\Lambda(V)(t)$ is relatively compact in X_0 and $U(t,s)$ is strongly continuous, then

$$\lim_{t_2-t_1\to 0}\|(U(t_2,t_1) - I)u(t_1)\| = 0.$$

For $U(t,s)$ is strongly continuous and f is bounded continuous, we obtain

$$\lim_{t_2-t_1\to 0}\|\lim_{\lambda\to\infty}\int_{t_1}^{t_2} U(t_2,\tau)\lambda R(\lambda,A)f(\tau,u(\tau))d\tau\| = 0.$$

Therefore, combining the estimates, we have

$$\lim_{t_2-t_1\to 0}\|\Lambda u(t_2) - \Lambda u(t_1)\| = 0$$

uniformly for $u \in V$. The proof is complete.

Theorem 3.4 If assumptions $(A_1) - (A_4)$ hold true, then Eq.(1) has an anti-periodic mild solution.
Proof Let $D = \{u \in P_{TA}(R,X_0) \cap C_h(X_0)\}$, and $\Lambda(D)(t) := \{\Lambda u(t) : u \in D\}$, where

$$\Lambda u(t) = U(t,s)u(s) + \lim_{\lambda\to\infty}\int_s^t U(t,\tau)\lambda R(\lambda,A)f(\tau,u(\tau))d\tau.$$

From the proof of Theorem 3.2, we have

$$\Lambda u(t)=\lim_{\lambda\to\infty}\int_{-\infty}^{\infty}U(t,s)\lambda R(\lambda,A)f(s,u(s))ds.$$

We divide the proof in several steps.

Step1. For $u\in D$, we have that

$$\begin{aligned}\|\Lambda u(t)\| &= \|\lim_{\lambda\to\infty}\int_{-\infty}^{\infty}\Gamma(t,s)\lambda R(\lambda,A)f(s,u(s))ds\|\\ &\leq MC\int_{-\infty}^{\infty}e^{-\omega|t-s|}\|f(s,u(s))\|ds\\ &\leq MC\int_{-\infty}^{\infty}e^{-\omega|t-s|}W(\|u(s)\|)ds\\ &\leq MC\int_{-\infty}^{\infty}e^{-\omega|t-s|}W(\|u\|_h h(s))ds\\ &\leq MC\beta(\|u\|_h).\end{aligned}$$

It follows from condition (A_2) that $\Lambda : C_h(X_0)\to C_h(X_0)$.

Step2. The map Λ is continuous. In fact, for $\epsilon>0$, we take δ involved in condition (A_3). If $u,v\in C_h(X_0)$ and $\|u-v\|_h\leq\delta$, then

$$\begin{aligned}\|\Lambda u(t)-\Lambda v(t)\|_h &= \frac{\|\lim_{\lambda\to\infty}\int_{-\infty}^{\infty}\Gamma(t,s)\lambda R(\lambda,A)[f(s,u(s))-f(s,v(s))]ds\|}{h(t)}\\ &\leq \frac{MC\int_{-\infty}^{\infty}e^{-\omega|t-s|}\|f(s,u(s))-f(s,v(s))\|ds}{h(t)}\\ &\leq \epsilon,\end{aligned}$$

which shows the assertion.

Step3. Applying condition (A_2), we can show that

$$\frac{\|\Lambda u\|}{h(t)}\leq\frac{MC}{h(t)}\int_{-\infty}^{\infty}e^{-\omega|t-s|}W(\|u\|_h h(s))ds\to 0,|t|\to\infty$$

and this convergence is independent of $u\in D$.

From the proof of the Theorem 3.3 and Step 3, D satisfies conditions (i)-(iii) of Lemma 3.1, so D is a relatively compact set in $C_h(X_0)$. It follows from the proof of step 1-step 3 that Λ is a compact operator.

Step4. Applying Theorem 3.1, we obtain that $\Lambda(P_{TA}(R,X_0))\subseteq P_{TA}(R,X_0)$. Consequently, combining with step 1 and step 2 we infer that $\Lambda(P_{TA}(R,X_0)\cap C_h(X_0))\subseteq P_{TA}(R,X_0)\cap C_h(X_0)$, and also

$$\Lambda(\overline{P_{TA}(R,X_0)\cap C_h(X_0)}^h)\subseteq\overline{\Lambda(P_{TA}(R,X_0)\cap C_h(X_0))}^h\subseteq\overline{P_{TA}(R,X_0)\cap C_h(X_0)}^h,$$

where $\overline{D}^h$ denotes the closure of D in $C_h(X_0)$. Applying Lemma 2.5, we deduce that Λ has a fixed point $u\in\overline{P_{TA}(R,X_0)\cap C_h(X_0)}^h$.

Step5. We prove that $u\in P_{TA}(R,X_0)$.

Let $(u_n)_n$ be a sequence in $P_{TA}(R,X_0)\cap C_h(X_0)$ that converges to u for the topology in $C_h(X_0)$. It follows from condition (A_3) that $\Gamma u_n\to\Gamma u$ as $n\to\infty$, uniformly on R. This implies that $u\in P_{TA}(R,X_0)$, which completes the proof.

Remark 3.1 If assumption (A_3) of Theorem 3.4 is fulfilled in the following situation:

$$\|f(t,h(t)x)-f(t,h(t)y)\|\leq W(\|x-y\|),$$

for all $t \in R$, $x, y \in X_0$, $W(0) = 0$, then Theorem 3.4 is still true.

In fact, we use the same notations as in Theorem 3.4.

$$\begin{aligned}\|\Lambda u(t) - \Lambda v(t)\| &= \|\int_{-\infty}^{\infty} \Gamma(t,s)\lambda R(\lambda, A)[f(s,u(s)) - f(s,v(s))]ds\| \\ &\leq MC\int_{-\infty}^{\infty} e^{-\omega|t-s|}W(\frac{\|u(s)-v(s)\|}{h(s)})ds \\ &\leq \frac{2MC}{\omega}W(\|u-v\|_h).\end{aligned}$$

Since W is continuous, the above estimate shows that (A_3) hold, the remains of proof is essentially the same of Theorem 3.4.

4 Application

In this section we give an example to illustrate the above results. Consider the following retarded partial differential equation [17]:

$$\partial_t u(t,x) = \partial_x^2 u(t,x) + au(t,x) - b(t)u(t-1,x) + f(t,u(t,x)),\ t \in R,\ x \in [0,\pi], \tag{2}$$

with boundary initial conditions

$$u(t,0) = u(t,\pi) = 0,\ t \in R, \tag{3}$$

where $a, b \in R$ and $f : R \times [0,\pi] \to R$ for all $t \in R$.

Let $Y := C([0,\pi], R)$ and the operator B be defined on Y by $Bv = v'' + av$, with domain

$$D(B) := \{v \in C^2([0,\pi], R) : v(0) = v(\pi) = 0\}.$$

Set $X = Y \times E$, where $E = C([-1,0], Y)$, $w(t) = u(t,\cdot)$, $L(t)w = -b(t)w(-1)$ and $F(t,w) = f(t,u(t,\cdot))$. It is well known(see [15]) that Eq.(2)-(3) can be formulated as an abstract Cauchy problem

$$w'(t) = Aw(t) + B(t)w(t) + F(t,w)$$

where

$$A := \begin{pmatrix} 0 & B\delta_0 - \delta_0' \\ 0 & \frac{d}{d\tau} \end{pmatrix}, D(A) = \{0\} \times \{\phi \in C^1([-1,0], Y) : \phi(0) \in D(B)\}$$

with $\delta_0'\phi := \phi'(0)$ for $\phi \in C^1([-1,0], Y)$ and

$$B(t) := \begin{pmatrix} 0 & L(t) \\ 0 & 0 \end{pmatrix}.$$

It is shown in [18, Lemma 3.1] that A is a Hille-Yosida operator, the part A_0 of A in $X_0 := \overline{D(A)} = \{0\} \times E$ generates a C_0-semigroup $(T_0(t))_{t\geq 0}$ on X_0. Let $(T_{-1}(t))_{t\geq 0}$ denote the extrapolated semigroup of $(T_0(t))_{t\geq 0}$ on X. For more details, see [15, 18]. Under the similar conditions of Section 3, we can prove the Eq.(2)-(3) has a unique anti-periodic mild solutions.

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