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Bounds for α-optimal partitioning of a measurable space based on several efficient partitions

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ABSTRACT

We provide a two-sided inequality for the α-optimal partition value of a measurable space according to a finite number of nonatomic finite measures. The result extends and often improves Legut [Inequalities for α-optimal partitioning of a measurable space, Proc. Amer. Math. Soc. 104 (1988)] since the bounds are obtained considering several partitions that maximize the weighted sum of the partition values with varying weights, instead of a single one. Furthermore, we show conditions that make these bounds sharper.

1. Introduction

Let $(C, \mathcal{C})$ be a measurable space, $\mu_1, \dots, \mu_n$ be n nonatomic finite measures defined on the same σ-algebra $\mathcal{C}$, and let $\mathscr{P}$ be the set of all measurable partitions $(A_1, \dots, A_n)$ of C ($A_i \in \mathcal{C}$ for all $i = 1, \dots, n$, $\bigcup_{i=1}^n A_i = C$, $A_i \cap A_j = \emptyset$ for all $i \neq j$). Let Δ_{n-1} denote the $(n-1)$-dimensional simplex. For this definition, and the many others taken from convex analysis, we refer to [9].

Definition 1. A partition $(A_1^*, \dots, A_n^*) \in \mathscr{P}$ is said to be α-optimal, for $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)^T \in \operatorname{ri} \Delta_{n-1}$, the relative interior of Δ_{n-1}, if

$$v^\alpha := \min_{i=1,\dots,n} \left\{ \frac{\mu_i(A_i^*)}{\alpha_i} \right\} = \sup \left\{ \min_{i=1,\dots,n} \left\{ \frac{\mu_i(A_i)}{\alpha_i} \right\} : (A_1, \dots, A_n) \in \mathscr{P} \right\}. \tag{1}$$

This problem has a consolidated interpretation in mathematical economics. We adopt the model considered in Dubins and Spanier [6]. C is a non-homogeneous, infinitely divisible good to be distributed

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among n agents with idiosyncratic preferences, represented by the measures. A partition $(A_1, \ldots, A_n) \in \mathscr{P}$ describes a possible division of the good, with portion A_i (not necessarily connected) given to agent i. A satisfactory compromise between the conflicting interests of the agents, each having a relative claim α_i, $i = 1, \ldots, n$, over the cake, is given by the α-optimal partition. It can be shown that the proposed solution coincides with the Kalai–Smorodinsky solution for bargaining problems (see Kalai and Smorodinsky [11] and Kalai [10]). When $\{\mu_i\}_{i=1,\ldots,n}$ are all probability measures, i.e. $\mu_i(C) = 1$ for all $i = 1, \ldots, n$, the claim vector $\boldsymbol{\alpha} = (1/n, \ldots, 1/n)^T$ describes a situation of perfect parity among agents. The necessity to consider finite measures stems from game theoretic extensions of the models. For such extensions we refer to Legut [13], Legut et al. [14] and Dall'Aglio et al. [4].

When all the μ_i are probability measures, Dubins and Spanier [6] showed that if $\mu_i \neq \mu_j$ for some $i \neq j$, then $v^\alpha > 1$. This bound was improved, together with the definition of an upper bound by Elton et al. [8]. A further improvement for the lower bound was given by Legut [12]. More recently, Legut and Wilczyński [16] gave an explicit formula for the value of v^α (and of the corresponding optimal partition) for the case $n = 2$, based on the Neyman–Pearson lemma.

The aim of the present work is twofold: We provide further refinements for Legut's bounds for any n, and we show conditions that make these bounds sharper. We consider here the same geometrical setting employed by Legut [12], i.e. the partition range, also known as Individual Pieces Set (IPS) (see Barbanel [2] for a thorough review of its properties), defined as

$$\mathcal{R} := \big\{\big(\mu_1(A_1), \ldots, \mu_n(A_n)\big) : (A_1, \ldots, A_n) \in \mathscr{P}\big\} \subset \mathbb{R}^n_+.$$

Let us consider some of its features. The set $\mathcal{R}$ is compact and convex (see Dvoretzky et al. [7]). The supremum in (1) is therefore attained. Moreover, as shown by Legut and Wilczyński [15],

$$v^\alpha = \max\big\{r \in \mathbb{R}_+ : (r\alpha_1, r\alpha_2, \ldots, r\alpha_n)^T \cap \mathcal{R} \neq \emptyset\big\}. \tag{2}$$

So, the vector $(v^\alpha\alpha_1, \ldots, v^\alpha\alpha_n)^T$ is the intersection between the Pareto frontier of $\mathcal{R}$ and the ray $r\boldsymbol{\alpha} = \{(r\alpha_1, \ldots, r\alpha_n)^T : r \geq 0\}$.

To find both bounds, Legut locates the solution of the maxsum problem $\sup\{\sum_{i=1}^n \mu_i(A_i) : (A_1, \ldots, A_n) \in \mathscr{P}\}$ on the partition range. Then, he finds the convex hull of this point with the corner points of the partition range to find a lower bound, and uses a separating hyperplane argument to find the upper bound. We keep the same framework, but consider the solutions of several maxsum problems with weighted coordinates to find better approximations. Fix $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)^T \in \varDelta_{n-1}$ and consider

$$\sum_{i=1}^n \beta_i\mu_i\big(A_i^\beta\big) = \sup\left\{\sum_{i=1}^n \beta_i\mu_i(A_i) : (A_1, \ldots, A_n) \in \mathscr{P}\right\}. \tag{3}$$

Let η be a non-negative finite-valued measure with respect to which each μ_i is absolutely continuous (for instance we may consider $\eta = \sum_{i=1}^n \mu_i$). Then, by the Radon–Nikodym theorem, for each $A \in \mathcal{C}$,

$$\mu_i(A) = \int_A f_i d\eta \quad \forall i = 1, \ldots, n,$$

where f_i is the Radon–Nikodym derivative of μ_i with respect to η.

Finding a solution for (3) is straightforward:

Proposition 1. *(See [6, Theorem 2], [1, Theorem 2], [3, Proposition 4.3].) Let $\boldsymbol{\beta} \in \varDelta_{n-1}$ and let $B^\beta = (A_1^\beta, \ldots, A_n^\beta)$ be a partition of C. If*

$$\beta_k f_k(x) \geq \beta_h f_h(x) \quad \text{for all } h, k \leq n \text{ and for all } x \in A_k^\beta, \tag{4}$$

then $(A_1^\beta, \ldots, A_n^\beta)$ *is optimal for* (3).

Definition 2. Given $\boldsymbol{\beta} \in \Delta_{n-1}$, an *efficient value vector (EVV)* with respect to $\boldsymbol{\beta}$, $\boldsymbol{u}^\beta = (u_1^\beta, \ldots, u_n^\beta)^T$, is defined by

$$u_i^\beta = \mu_i\big(A_i^\beta\big), \quad \text{for each } i = 1, \ldots, n.$$

The EVV $\boldsymbol{u}^\beta$ is a point where the hyperplane

$$H_\beta = \big\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{\beta}^T \boldsymbol{x} = \boldsymbol{\beta}^T \boldsymbol{u}^\beta\big\} \tag{5}$$

touches the partition range $\mathcal{R}$, so $\boldsymbol{u}^\beta$ lies on the Pareto border of $\mathcal{R}$.

2. The main result

As proved in Legut [12], one EVV alone associated to the equitable $\boldsymbol{\beta}$ is enough to assure a lower bound. Here we give a general result for the case where several linearly independent EVVs are available. We derive this approximation result through a convex combination of these easily computable points in $\mathcal{R}$, which lie around $(v^\alpha \alpha_1, \ldots, v^\alpha \alpha_n)^T$.

Theorem 1. *Consider* $m \leq n$ *linearly independent vectors* $\boldsymbol{u}^1, \boldsymbol{u}^2, \ldots, \boldsymbol{u}^m$, *where* $\boldsymbol{u}^i = (u_{i1}, u_{i2}, \ldots, u_{in})^T$, $i = 1, \ldots, m$ *is the EVV associated to* $\boldsymbol{\beta}^i$, $\boldsymbol{\beta}^i = (\beta_{i1}, \beta_{i2}, \ldots, \beta_{in})^T \in \Delta_{n-1}$. *Assume*

$$\operatorname{rank}\big(\boldsymbol{u}^1, \ldots, \boldsymbol{u}^m, \boldsymbol{\alpha}\big) = m, \tag{6}$$

let $\boldsymbol{U}$ *be the* $n \times m$ *matrix* $\boldsymbol{U} = (\boldsymbol{u}^1, \boldsymbol{u}^2, \ldots, \boldsymbol{u}^m)$ *and let* $\bar{\boldsymbol{U}}$ *be an* $m \times m$ *submatrix of* $\boldsymbol{U}$ *with* $\det(\bar{\boldsymbol{U}}) \neq 0$. *Let* $\bar{\boldsymbol{\alpha}}$ *be the vector obtained from* $\boldsymbol{\alpha}$ *by selecting the same rows as in* $\bar{\boldsymbol{U}}$. *Then,*

(*i*)

$$\boldsymbol{\alpha} \in \operatorname{cone}\big(\boldsymbol{u}^1, \boldsymbol{u}^2, \ldots, \boldsymbol{u}^m\big) \tag{7}$$

if and only if

$$\det(\bar{\boldsymbol{U}}) \det(\bar{\boldsymbol{U}}_{\alpha i}) \geq 0 \quad \text{for all } i = 1, \ldots, m, \tag{8}$$

where $\bar{\boldsymbol{U}}_{\alpha i}$ *is the* $m \times m$ *matrix obtained by replacing the* i*-th column of* $\bar{\boldsymbol{U}}$ *with* $\bar{\boldsymbol{\alpha}}$. *Moreover,* $\boldsymbol{\alpha} \in \operatorname{ri}(\operatorname{cone}(\boldsymbol{u}^1, \boldsymbol{u}^2, \ldots, \boldsymbol{u}^m))$ *if and only if all the inequalities in* (8) *are strict.*

(*ii*) *For any choice of* $\boldsymbol{u}^1, \boldsymbol{u}^2, \ldots, \boldsymbol{u}^m$,

$$v^\alpha \leq \min_{i=1,\ldots,m} \frac{(\boldsymbol{\beta}^i)^T \boldsymbol{u}^i}{(\boldsymbol{\beta}^i)^T \boldsymbol{\alpha}}. \tag{9}$$

Moreover, if (8) *holds, then*

$$\frac{1}{\boldsymbol{e}^T \bar{\boldsymbol{U}}^{-1} \bar{\boldsymbol{\alpha}}} \leq v^\alpha \tag{10}$$

where $\boldsymbol{e} = (1, 1, \ldots, 1)^T \in \mathbb{R}^m$.

Proof. To prove (i), let $\boldsymbol{t} = (t_1, t_2, \ldots, t_m)^T$ and consider, for any $r > 0$, the linear system

$$\boldsymbol{U}\boldsymbol{t} = r\boldsymbol{\alpha} \tag{11}$$

with variables in $\boldsymbol{t}$. By (6) this is equivalent to

$$\bar{\boldsymbol{U}}\boldsymbol{t} = r\bar{\boldsymbol{\alpha}}, \tag{12}$$

and, by Cramer's rule, it admits the unique solution

$$\boldsymbol{t} = r\left(\frac{\det(\bar{\boldsymbol{U}}_{\alpha 1})}{\det(\bar{\boldsymbol{U}})}, \cdots, \frac{\det(\bar{\boldsymbol{U}}_{\alpha m})}{\det(\bar{\boldsymbol{U}})}\right)^T. \tag{13}$$

Now, (7) holds if and only if $t_i \geq 0$ for every $i = 1, \ldots, m$, which in turn holds if and only if (8) holds. Moreover, $t_i > 0$ for every $i = 1, \ldots, m$ if and only if all the inequalities in (8) are strict.

To prove (ii), consider, for any $i = 1, \ldots, m$, the hyperplane (5) that intersects the ray $r\boldsymbol{\alpha}$ at the point $(\bar{r}_i\alpha_1, \ldots, \bar{r}_i\alpha_n)$, with

$$\bar{r}_i = \frac{(\boldsymbol{\beta}^i)^T\boldsymbol{u}^i}{(\boldsymbol{\beta}^i)^T\boldsymbol{\alpha}}.$$

Since $\mathcal{R}$ is convex, the intersection point is not internal to $\mathcal{R}$. So, $\bar{r}_i \geq v^\alpha$ for all $i = 1, \ldots, m$, and, therefore, $\min_{i=1,\ldots,m} \bar{r}_i \geq v^\alpha$.

Assuming now that (8) holds, we choose $r^* > 0$ so that the corresponding $\boldsymbol{t}^*$ in (13) satisfies

$$\boldsymbol{e}^T\boldsymbol{t}^* = 1. \tag{14}$$

$r^*\boldsymbol{\alpha}$ is the convex combination of the vectors in $\boldsymbol{U}$ with weights in $\boldsymbol{t}^*$, and is aligned with $\boldsymbol{\alpha}$. By the convexity of $\mathcal{R}$, r^* provides a lower bound for v^α.

System (12) implies $\boldsymbol{t}^*/r^* = \bar{\boldsymbol{U}}^{-1}\bar{\boldsymbol{\alpha}}$, and, by (14),

$$\frac{1}{r^*} = \frac{1}{r^*}\boldsymbol{e}^T\boldsymbol{t}^* = \boldsymbol{e}^T\bar{\boldsymbol{U}}^{-1}\bar{\boldsymbol{\alpha}}, \tag{15}$$

which, in turn, implies (10). □

Remark 1. In the corollaries and the examples that follow, we will consider the situation where $m = n$. In such case, (6) is trivially satisfied, and an easy geometric interpretation can be given to condition (8). For any $j \leq n$, consider the hyperplane

$$H_{-j} = \{\boldsymbol{x} \in \mathbb{R}^n : \det(\boldsymbol{u}^1, \ldots, \boldsymbol{u}^{j-1}, \boldsymbol{x}, \boldsymbol{u}^{j+1}, \ldots, \boldsymbol{u}^n) = 0\},$$

passing through the origin and all the EVVs but $\boldsymbol{u}^j$. H_{-j} separates $\boldsymbol{u}^j$ and $\boldsymbol{\alpha}$ (weakly or strictly, resp.) if and only if (8) (weakly or strictly, resp.) holds.

In what follows, $\boldsymbol{u}^1, \boldsymbol{u}^2, \ldots, \boldsymbol{u}^m$ will be sometimes referred to as the supporting set of EVVs for the lower bound. We next consider two corollaries that provide bounds in case only one EVV is available. The first one works with an EVV associated to an arbitrary vector $\boldsymbol{\beta} \in \Delta_{n-1}$.

Corollary 1. *(See [5, Proposition 3.4].) Let $\mu_1, \ldots, \mu_n$ be finite measures and let $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)^T$ be the EVV corresponding to $\boldsymbol{\beta} \in \Delta_{n-1}$ such that*

$$\alpha_j^{-1} u_j = \max_{i=1,\ldots,n} \alpha_i^{-1} u_i. \tag{16}$$

Then,

$$\frac{u_j}{\alpha_j + \sum_{i \neq j} [\mu_i^{-1}(C)(\alpha_i u_j - \alpha_j u_i)]} \leq v^\alpha \leq \frac{\boldsymbol{\beta}^T \boldsymbol{u}}{\boldsymbol{\beta}^T \boldsymbol{\alpha}}. \tag{17}$$

Proof. Consider the corner points of the partition range

$$\boldsymbol{e}^i = \big(0, \ldots, 0, \mu_i(C), 0, \ldots, 0\big)^T \in \mathbb{R}^n, \quad i = 1, \ldots, n$$

where $\mu_i(C)$ is placed on the i-th coordinate, and

$$\boldsymbol{U} = \big(\boldsymbol{e}^1, \ldots, \boldsymbol{e}^{j-1}, \boldsymbol{u}, \boldsymbol{e}^{j+1}, \ldots, \boldsymbol{e}^n\big).$$

Now

$$\det(\boldsymbol{U}) = u_j \prod_{i \neq j} \mu_i(C) > 0$$

$$\det(\boldsymbol{U}_{\alpha j}) = \alpha_j \prod_{i \neq j} \mu_i(C) > 0$$

and, for all $i \neq j$, by (16),

$$\det(\boldsymbol{U}_{\alpha i}) = (\alpha_i u_j - \alpha_j u_i) \prod_{k \neq i, k \neq j} \mu_k(C) \geq 0.$$

Therefore, $\boldsymbol{U}$ satisfies the hypothesis (8) of Theorem 1. Since $\boldsymbol{U}$ has inverse

$$\boldsymbol{U}^{-1} = \begin{pmatrix} \frac{1}{\mu_1(C)} & 0 & \cdots & -\frac{u_1}{\mu_1(C) u_j} & \cdots & 0 \\ 0 & \frac{1}{\mu_2(C)} & \cdots & -\frac{u_2}{\mu_2(C) u_j} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{u_j} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\frac{u_n}{\mu_n(C) u_j} & \cdots & \frac{1}{\mu_n(C)} \end{pmatrix},$$

the following lower bound is guaranteed for v^α:

$$v^\alpha \geq r^* = \frac{u_j}{\alpha_j + \sum_{i \neq j} [\mu_i^{-1}(C)(\alpha_i u_j - \alpha_j u_i)]}.$$

The upper bound is, again, a direct consequence of part (ii) in Theorem 1. $\square$

In case all measures μ_i are normalized to one and the only EVV considered is the one corresponding to the equitable $\boldsymbol{\beta}$, we obtain Legut's result.

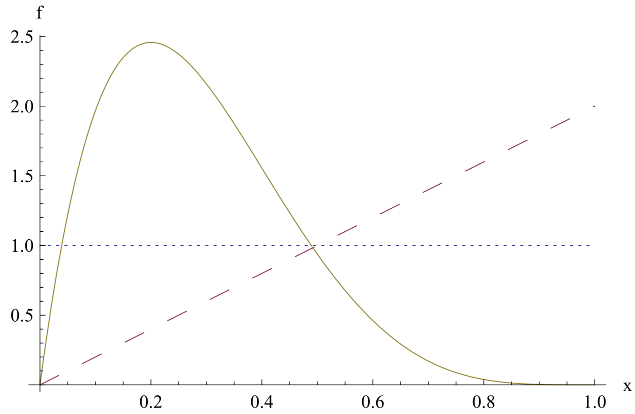

Fig. 1. The density functions in Example 1. Agent 1: tiny dashing; Agent 2: large dashing; Agent 3: continuous line.

Corollary 2. *(See [12, Theorem 3].) Let $\mu_1, \ldots, \mu_n$ be probability measures and let $\boldsymbol{u}^{eq} = (u_1^{eq}, u_2^{eq}, \ldots, u_n^{eq})^T$ be an EVV corresponding to $\boldsymbol{\beta}^{eq} = (1/n, \ldots, 1/n)^T$. Let $u_j^{eq} = \max_{i=1,\ldots,n} u_i^{eq}$. Then,*

$$\frac{u_j^{eq}}{u_j^{eq} - \alpha_j(K-1)} \leq v^\alpha \leq K, \tag{18}$$

where $K = \sum_{i=1}^n u_i^{eq}$.

Proof. Simply apply Corollary 1 with $\mu_i(C) = 1$, for all $i = 1, \ldots, n$, and $\boldsymbol{\beta}^{eq}$. Then

$$v^\alpha \geq r^* = \frac{u_j^{eq}}{\alpha_j + \sum_{i \neq j}(\alpha_i u_j^{eq} - \alpha_j u_i^{eq})} = \frac{u_j^{eq}}{u_j^{eq} - \alpha_j(K-1)}.$$

Finally, by part (ii) of Theorem 1, we have

$$v^\alpha \leq \frac{(\boldsymbol{\beta}^{eq})^T \boldsymbol{u}^{eq}}{(\boldsymbol{\beta}^{eq})^T \boldsymbol{\alpha}} = \sum_{i=1}^n u_i^{eq}. \qquad \square$$

It is important to notice that the lower bound provided by Theorem 1 does not necessarily improve on Legut's lower bound, but it certainly does so when

$$\mathrm{cone}\big(\boldsymbol{u}^1, \boldsymbol{u}^2, \ldots, \boldsymbol{u}^m\big) \subseteq \mathrm{cone}\big(\boldsymbol{e}^1, \ldots, \boldsymbol{e}^{j-1}, \boldsymbol{u}^{eq}, \boldsymbol{e}^{j+1}, \ldots, \boldsymbol{e}^n\big), \tag{19}$$

for, in such case, $\mathrm{conv}(\boldsymbol{u}^1, \boldsymbol{u}^2, \ldots, \boldsymbol{u}^m)$ lies above $\mathrm{conv}(\boldsymbol{e}^1, \ldots, \boldsymbol{e}^{j-1}, \boldsymbol{u}^{eq}, \boldsymbol{e}^{j+1}, \ldots, \boldsymbol{e}^n)$, and the first set of EVVs provides a better bound then the latter.

Example 1. We consider a $[0,1]$ good that has to be divided among three agents (see Fig. 1) with equal claims, $\boldsymbol{\alpha} = (1/3, 1/3, 1/3)^T$, and preferences given as density functions of probability measures w.r.t. the Lebesgue measure

$$f_1(x) = 1 \qquad f_2(x) = 2x \qquad f_3(x) = 30x(1-x)^4 \quad x \in [0,1],$$

f_3 being the density function of a Beta$(2,5)$ distribution. The preferences of the players are not concentrated (following Definition 12.9 in Barbanel [2]) and therefore there is only one EVV associated to each $\boldsymbol{\beta} \in \Delta_2$ (cf. [2], Theorem 12.12).

The EVV corresponding to $\boldsymbol{\beta}^{eq} = (1/3, 1/3, 1/3)^T$ is

$$\boldsymbol{u}^{eq} = (0.0501, 0.75, 0.8594)^T.$$

Consequently, the bounds provided by Legut are

$$1.3437 \leq v^\alpha \leq 1.6594.$$

Consider now two other vectors in Δ_2, $\boldsymbol{\beta}^1 = (13/24, 6/24, 5/24)^T$ and $\boldsymbol{\beta}^2 = (3/12, 8/12, 1/12)^T$, which generate the following EVVs

$$\boldsymbol{u}^1 = (1, 0, 0)^T \quad \text{and} \quad \boldsymbol{u}^2 = (0.1875, 0.9648, 0)^T.$$

The vectors $\boldsymbol{u}_1, \boldsymbol{u}_2$ and $\boldsymbol{u}_{eq}$ satisfy the hypotheses of Theorem 1 and the inclusion (19). The improved bounds are

$$1.3559 \leq v^\alpha \leq 1.625.$$

The next example shows that linearly independent (dependent, resp.) vectors $\{\boldsymbol{\beta}^i\}_{i=1,\dots,m}$ do not necessarily lead to linearly independent (dependent, resp.) EVVs $\{\boldsymbol{u}^i\}_{i=1,\dots,m}$.

Example 2. Consider again a $[0, 1]$ good to be divided among three agents, and preferences given by the following density functions w.r.t. the Lebesgue measure

$$\begin{aligned} f_1(x) &= (2/3)I_{[0,1/2)}(x) + (4/3)I_{(1/2,1]}(x), \\ f_2(x) &= 2I_{[0,2/5)}(x) + (1/3)I_{(2/5,1]}(x), \\ f_3(x) &= (1/2)I_{[0,3/4)(x)} + (5/2)I_{(3/4,1]}(x), \end{aligned}$$

$I_A(x)$ being the indicator function of the set A.

To the following three linearly independent vectors in Δ_2

$$\boldsymbol{\beta}^1 = (1/3, 1/3, 1/3)^T \qquad \boldsymbol{\beta}^2 = (2/5, 1/5, 2/5)^T \qquad \boldsymbol{\beta}^3 = (1/4, 1/3, 5/12)^T$$

we associate, respectively, the optimal partitions

$$\begin{aligned} B^1 = B^2 &= \big((2/5, 3/4), [0, 2/5), (3/4, 1]\big), \\ B^3 &= \big((1/2, 3/4), \big[0, 2/5\big), (2/5, 1/2) \cup \big(3/4, 1\big]\big). \end{aligned}$$

Consequently,

$$\boldsymbol{u}^1 = \boldsymbol{u}^2 = (2/5, 4/5, 5/8)^T \qquad \boldsymbol{u}^3 = (1/3, 4/5, 27/40)^T,$$

which are linearly dependent. On the other hand, considering

$$\boldsymbol{\beta}^4 = (0, 0, 1)^T \qquad \boldsymbol{\beta}^5 = (1/6, 1/6, 2/3)^T$$

we have

$$B^4 = \big(\varnothing, \varnothing, [0, 1]\big) \quad \text{and} \quad B^5 = B^3$$

and

$$\boldsymbol{u}^4 = (0, 0, 1) \quad \text{and} \quad \boldsymbol{u}^5 = \boldsymbol{u}^3$$

Now $\boldsymbol{\beta}^1, \boldsymbol{\beta}^4, \boldsymbol{\beta}^5$ are linearly dependent, while the corresponding EVVs are not.

Establishing sufficient conditions that guarantee the linear independence (or dependence) of the EVVs remains an open issue.

3. Improving the bounds

The bounds for v^α depend on the choice of the EVVs that satisfy the hypotheses (6) and (8) of Theorem 1. Any additional EVV yields a new term in the upper bound. Since we consider the minimum of these terms, this addition is never harmful. Improving the lower bound is a more delicate task, since we should modify the set of supporting EVVs for the lower bound. When we examine a new EVV we should verify whether replacing an EVV in the old set will bring to an improvement.

The following theorem provides simple tests to verify whether such replacement will bring an improvement in the bound and indicates how to make the replacement.

Theorem 2. *Let $\boldsymbol{u}^*, \boldsymbol{u}^1, \ldots, \boldsymbol{u}^m$ be $m+1$ EVVs, $m \le n$, with*

$$\operatorname{rank}\bigl(\boldsymbol{u}^*, \boldsymbol{u}^1, \ldots, \boldsymbol{u}^m\bigr) = m \tag{20}$$

and the last m vectors linearly independent and satisfying conditions (6) and (8). Let $\boldsymbol{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$, $x_k \ge 0$ for every $k = 1, \ldots, m$, and $\boldsymbol{y} = (y_1, \ldots, y_m) \in \mathbb{R}^m$, respectively, be the unique solutions of the following linear systems of equations

$$\boldsymbol{U}\boldsymbol{x} = \sum_{k=1}^m x_k \boldsymbol{u}^k = \boldsymbol{\alpha} \tag{21}$$

$$\boldsymbol{U}\boldsymbol{y} = \sum_{i=1}^m y_i \boldsymbol{u}^i = \boldsymbol{u}^* \tag{22}$$

with $\boldsymbol{U} = (\boldsymbol{u}^1, \ldots, \boldsymbol{u}^m)$. Take $j \le m$ such that $y_j \ne 0$. Then, replacing $\boldsymbol{u}^j$ with $\boldsymbol{u}^*$ in $\boldsymbol{u}^1, \ldots, \boldsymbol{u}^m$, the EVVs are linearly independent and satisfy assumption (6) of Theorem 1. Moreover, the same EVVs satisfy condition (7) in Theorem 1 if and only if

$$y_j > 0 \quad \textit{for some } j \le m \qquad x_k \ge \frac{y_k}{y_j} x_j \quad \textit{for all } k \ne j. \tag{23}$$

When (23) holds, the same replacement also yields a sharper lower bound if and only if

$$x_j > 0 \quad \textit{and} \quad \sum_{k=1}^m y_k > 1. \tag{24}$$

Proof. Let $\bar{\boldsymbol{U}}$ be an $m \times m$ submatrix of $\boldsymbol{U}$ with $\det(\bar{\boldsymbol{U}}) \ne 0$ and let $\boldsymbol{U}^*$ denote the matrix obtained from $\boldsymbol{U}$ by replacing $\boldsymbol{u}^j$, the j-th column of $\boldsymbol{U}$, with $\boldsymbol{u}^*$. Finally, let $\bar{\boldsymbol{U}}^*$ be the submatrix of $\boldsymbol{U}^*$ with the same selection of rows operated in $\bar{\boldsymbol{U}}$. Since

$$\det\bigl(\bar{\boldsymbol{U}}^*\bigr) = y_j \det(\bar{\boldsymbol{U}})$$

then $y_j \ne 0$ implies that the vectors $\boldsymbol{u}^1, \ldots, \boldsymbol{u}^{j-1}, \boldsymbol{u}^*, \boldsymbol{u}^{j+1}, \ldots, \boldsymbol{u}^m$ are linearly independent.

Consider now a solution $\boldsymbol{x}^* = (x_1^*, \ldots, x_m^*)$ of the system of linear equations

$$\boldsymbol{U}^* \boldsymbol{x}^* = \sum_{k \neq j} x_k^* \boldsymbol{u}^k + x_j^* \boldsymbol{u}^* = \boldsymbol{\alpha}. \tag{25}$$

Since (22) holds, we can write (25) as

$$\sum_{k \neq j} x_k^* \boldsymbol{u}^k + x_j^* \left(\sum_{i=1}^{m} y_i \boldsymbol{u}^i \right) = \boldsymbol{\alpha}$$

which, when $y_j \neq 0$, has the unique solution $x_k^* = x_k - \frac{y_k}{y_j} x_j$ for $k \neq j$, and $x_j^* = \frac{x_j}{y_j}$, with $\operatorname{rank}(\boldsymbol{U}^*, \boldsymbol{\alpha}) = m$. Moreover, $x_k^* \geq 0$ for every $k = 1, \ldots, m$, and $\boldsymbol{\alpha}$ belongs to the cone generated by $\boldsymbol{u}^1, \ldots, \boldsymbol{u}^{j-1}, \boldsymbol{u}^*, \boldsymbol{u}^{j+1}, \ldots, \boldsymbol{u}^m$, if and only if (23) holds.

A comparison of the linear system (21) with (11) and (14) shows that the lower bound r^* provided by Theorem 1 can be written as

$$r^* = \frac{1}{\sum_{k=1}^{m} x_k}.$$

Now $\sum_{k=1}^{m} x_k^* < \sum_{k=1}^{m} x_k$ if and only if

$$\left(\sum_{k=1}^{m} y_k - 1 \right) \frac{x_j}{y_j} > 0.$$

Therefore, (23) and (24) imply that the new set of EVVs provides a strictly sharper lower bound. □

Theorem 2 could, in principle, be applied iteratively by verifying the assumptions of the theorem for each new EVV. It must be noted, however, that, when $m < n$, we do not know about general reasonable conditions to generate a new EVV in the linear span of the current supporting set of EVVs, so to make (20) hold (see Example 2 in the previous section).

The same assumption, however, is trivially satisfied when $m = n$. Moreover, Theorem 2 guarantees that the new set of EVVs $\boldsymbol{u}^*, \{\boldsymbol{u}^i\}_{i \neq j}$, which provides an improved lower bound, is linearly independent, and this new supporting set can be compared with a new EVV for a further application of the theorem. In the example that follows we consider an instance of the iterative procedure.

Example 1 *(Continued).* We consider a list of 1000 random vectors in Δ_2 and, starting from the supporting set $\boldsymbol{e}^1, \boldsymbol{e}^2$ and $\boldsymbol{e}^3$, we iteratively pick each vector in the list. If this satisfies conditions (23) and (24), then the supporting set is updated. The update occurs 22 times and the resulting EVVs are

$$\begin{aligned} \boldsymbol{u}^1 &= (0.5356, 0.5128, 0.3857)^T \\ \boldsymbol{u}^2 &= (0.4592, 0.4887, 0.5780)^T \\ \boldsymbol{u}^3 &= (0.5562, 0.4384, 0.4524)^T \end{aligned}$$

corresponding, respectively, to

$$\begin{aligned} \boldsymbol{\beta}^1 &= (0.4612, 0.3304, 0.2084)^T \\ \boldsymbol{\beta}^2 &= (0.4484, 0.3136, 0.2380)^T \\ \boldsymbol{\beta}^3 &= (0.4674, 0.3119, 0.2207)^T \end{aligned}$$

with bounds shrinking to

$$1.48514 \leq v^{\alpha} \leq 1.48978.$$

The previous example shows that updating the supporting set through a random selection of the new candidates is rather inefficient, since it takes little less than 50 new random vectors, on average, to find a valid replacement for supporting EVVs.

A more efficient method picks the candidate EVVs through some accurate choice of the corresponding values of $\boldsymbol{\beta}$. In [5] a subgradient method is considered to find the value of v^{α} up to any specified level of precision. In that algorithm, the bounds provided by Corollary 1 are used, but these can be replaced by the sharper bounds suggested by Theorem 1.

Example 1 *(Continued).* Considering the improved subgradient algorithm, we obtain the following sharper bounds

$$1.48771 \leq v^{\alpha} \leq 1.48772$$

after 25 iterations of the algorithm in which, at each repetition, a new EVV is considered. Bounds with the same precision ($< 10^{-5}$) would have required 30 iterations using the algorithm described in [5].

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