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Non-standard shocks in the Buckley–Leverett equation

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ABSTRACT

It is shown how delta shock waves which consist of Dirac delta distributions and classical shocks can be used to construct non-monotone solutions of the Buckley–Leverett equation. These solutions are interpreted using a recent variational definition of delta shock waves in which the Rankine–Hugoniot deficit is explicitly accounted for [6]. The delta shock waves are also limits of approximate solutions constructed using a recent extension of the weak asymptotic method to complex-valued approximations [15]. Finally, it is shown how these non-standard shocks can be fitted together to construct similarity and traveling-wave solutions which are non-monotone, but still admissible in the sense that characteristics either enter or are parallel to the shock trajectories.

1. Introduction

The Buckley–Leverett (BL) equation describes two-phase flow in a porous medium in the limit where capillary forces can be neglected. In non-dimensional variables, the equation can be written in normalized form as

$$\partial_t u + \partial_x \left(\frac{u^2}{u^2 + a(u-1)^2} \right) = 0, \tag{1.1}$$

where the unknown u represents the saturation of the wetting phase, and the constant $a > 0$ in the case of homogeneous systems represents the viscosity ratio between the fluids. In terms of the mathematical theory of hyperbolic conservation laws, the physical situation modeled by the equation is described by entropy admissible solutions of (1.1). As shown in [1], the mathematical entropy for this equation is given by the capillary energy, and admissible solutions with discontinuities must have characteristic curves which enter

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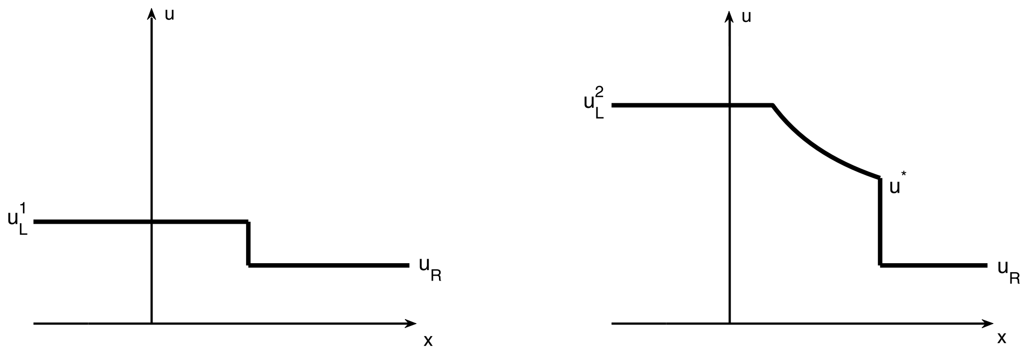


Fig. 1. The classical admissible solutions consisting of simple waves. In the left panel $u_L^1 < u^*$, and the Riemann problem is solved by a single shock. In the right panel $u_L^2 > u^*$, and the Riemann problem is solved by the combination of a rarefaction wave and a shock.

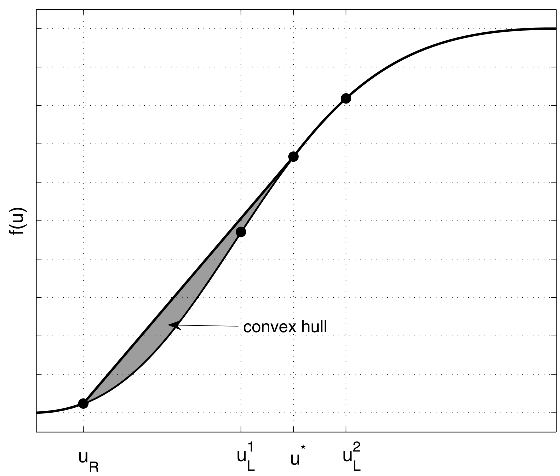


Fig. 2. The flux in the u–$f(u)$ plane. The right state is labeled u_R. The point at which the graph of f is tangent to the chord originating at $(u_R, f(u_R))$ is labeled $(u^*, f(u^*))$. Two possible left states are indicated as u_L^1 and u_L^2.

into or are parallel to the shock trajectories. For instance, one may consider the Riemann problem describing the evolution of two constant states separated by a single discontinuity. Such a configuration describes well the setup of many experiments started with initially homogeneous saturations, and where the proportion of the injected fluids is held constant during the experiment.

Mathematically, the Riemann problem consists of (1.1) supplemented with initial data given by

$$u(x,0) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0. \end{cases} \tag{1.2}$$

Depending on the size of u_R and u_L, the similarity solution to (1.1), (1.2) consists either of one shock wave, or of the combination of a shock wave followed by a rarefaction wave (see Figs. 1 and 2). The shock waves satisfy the usual Rankine–Hugoniot condition, and uniqueness of a solution to the Riemann problem can be established in the case of a combination of a shock and a rarefaction wave if it is required that the shock propagates at the same speed as the slowest part of the rarefaction wave [1,19,23].

In the current work, the focus is on solutions which consist of non-standard shocks which do not satisfy the Rankine–Hugoniot condition. These shocks feature a non-zero Rankine–Hugoniot deficit, and may be described with the help of Dirac delta distributions. The study of such measure-valued solutions goes back to the work of Korchinski [18] and Keyfitz and Kranzer [17]. There are a number of reasonable ways to multiply

Heaviside and Dirac distributions, such as [3–5], and using these, a number of definitions of measure-valued solutions of systems of conservation laws have been introduced, such as for example in [5,12,14,22].

More recently, a variational definition for delta shocks for systems of two conservation laws was put forward in [6]. The definition laid down in [6] is generally used in tandem with the weak asymptotics method, such as defined in [4], and is able to directly keep track of the Rankine–Hugoniot deficit, thus circumventing the need to include singular terms in the definition of the weak solution, and avoiding the problem of multiplication of distributions. The technique of using complex-valued corrections in the method of weak asymptotics was introduced in [15,16] extends the range of applicability of the variational approach to a larger class of fluxes. The method introduced in [15,16] also turns out to be crucial in the treatment of equations such as (1.1) which feature quotients of singular terms, and was also used in [2,24,25].

Solutions with non-monotone profiles were also found if the Buckley–Leverett equation is regularized with a third-order derivative term representing a physical effect known as dynamic capillarity [9,10]. In these works, solutions were found which do not satisfy the usual entropy conditions, but which are given as the limit of a regularized problem in which both equilibrium and dynamic capillarity were taken into account, and which satisfy the classical Rankine–Hugoniot conditions. In contrast, in the current work, we find solutions which are entropy-admissible in the classical sense, but which feature a non-zero Rankine–Hugoniot deficit.

The works [9,10] were motivated in part by laboratory experiments which indicate that under certain conditions, the saturation exhibits a non-monotone behavior. Indeed, as discussed in [11] (and the references contained therein) large fluxes often feature a behavior which is characterized by an initial overshoot of the concentration at the wetting front, and subsequent drainage of the wetting phase behind the front. Such a phenomenon was also observed in recent experiments on water infiltration into different types of sand by DiCarlo [7] in which the situation is essentially one-dimensional.

In the experiments presented in [7], a constant flux is applied to an initially dry medium, a situation which can be described mathematically by requiring that the saturation function u satisfies

$$u = \begin{cases} u_L, & x \to -\infty, \\ u = 0, & x \to +\infty. \end{cases}$$

These experiments clearly show that non-monotone behavior is observed for large enough fluxes. A number of plausible mathematical explanations of this phenomenon have been put forward, including modifications of the Richards equation [8,11] and the regularization of the Buckley–Leverett equation by dynamic capillarity [9,10]. However, one particular feature of the experimental results, namely the constant width of the overshoot region has not been explained by any of the proposed mathematical models.

In the present note, it is *not* our purpose to offer an explanation of the non-monotone behavior in physical terms, but rather to demonstrate that non-monotone solutions may be found directly in the Buckley–Leverett equation. The solutions found here are based on the variational theory laid down in [6], which defines solutions featuring a positive Rankine–Hugoniot deficit. The solutions are admissible in the sense that they are limits of regularized solutions in the weak asymptotic limit, and in addition are entropy-admissible in the classical sense. While the Buckley–Leverett equation may not be precisely the correct model for the study of sand infiltration such as described in [7], our approach shows that non-monotone solutions exist – at least in a mathematical sense – directly in the hyperbolic theory. One may therefore speculate if variational definition of weak solutions is connected to possible modifications to conservation laws which break down due to the inadequacies of the continuum hypothesis.

The disposition of the paper is as follows. In the next section, we formulate a variational framework for delta-shock solutions of the Buckley–Leverett equation which mirrors the variational concept defined for 2×2 systems in [6,15]. In Section 3, we justify the δ-shock solution mentioned above by the weak asymptotic method. Using the solution concepts laid down in Sections 2 and 3, we show how to construct non-monotone solutions of the Riemann problem in Section 4. Finally, in Section 5, we construct non-monotone traveling-wave solutions of the Buckley–Leverett equation.

2. Variational formulation of delta-shock solutions

We define the flux

$$f(u) = \frac{u^2}{u^2 + a(u-1)^2}, \tag{2.1}$$

and write the equation in the form

$$u_t + f(u)_x = 0. \tag{2.2}$$

In order to allow solutions which are not admissible in the classical sense, we will make use of the concept of weak solutions which consist of shocks associated with Dirac masses placed on the path of the shock. The general definition follows the concept introduced in [6]. Suppose $\Gamma = \{\gamma_i \mid i \in I\}$ is a graph in the closed upper half plane, containing Lipschitz continuous arcs γ_i, $i \in I$, where I is a finite index set. Let I_0 be the subset of I containing all indices of arcs that connect to the x-axis, and let $\Gamma_0 = \{x_k^0 \mid k \in I_0\}$ be the set of initial points of the arcs γ_k with $k \in I_0$. Define the singular part by $\alpha(x,t)\delta(\Gamma) = \sum_{i \in I} \alpha_i(x,t)\delta(\gamma_i)$. Let u be a distribution of the form

$$u(x,t) = U(x,t) + \alpha(x,t)\delta(\Gamma),$$

where $U \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$. Let $\frac{\partial \varphi(x,t)}{\partial \mathbf{l}}$ denote the tangential derivative of a function φ on the graph γ_i, and let $\int_{\gamma_i}$ denote the line integral over the arc γ_i.

Definition 2.1. The distribution $u(x,t) = U(x,t) + \alpha(x,t)\delta(\Gamma)$ is called a generalized δ-shock wave solution of Eq. (1.1) with the initial data $u(x,0) = U_0(x) + \sum_{I_0} \alpha_k(x_0^k, 0)\delta(x - x_k^0)$ if the integral identity

$$\begin{aligned} &\int_{\mathbb{R}_+}\int_{\mathbb{R}} \left(U\partial_t\varphi + f(U)\partial_x\varphi\right)\, dxdt + \int_{\mathbb{R}} U_0(x)\varphi(x,0)\, dx \\ &\quad + \sum_{i\in I}\int_{\gamma_i} \alpha_i(x,t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}\, dx + \sum_{k\in I_0}\alpha_k(x_k^0,0)\varphi(x_k^0,0) = 0, \end{aligned} \tag{2.3}$$

holds for all test functions $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}_+)$.

This definition applies to the case of general initial data. However, let us first look at the Riemann problem for (1.1) with initial data

$$U(x,0) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0. \end{cases} \tag{2.4}$$

Using Definition 2.1, it is not difficult to see that for any $c \in \mathbb{R}$, and any given u_L and u_R, a solution of the form $u(x,t) = U(x,t) + \alpha(t)\delta(x - ct)$ exists, where

$$U(x,t) = \begin{cases} u_L, & x < ct, \\ u_R, & x > ct, \end{cases} \tag{2.5}$$

and the amplitude of the singular part of the shock is given by

$$\alpha(t) = \big(c[u_R - u_L] - [f(u_R) - f(u_L)]\big)t. \tag{2.6}$$

Theorem 2.1. *Given any $u_L, u_R \in [0,1]$, and given any $c \in \mathbb{R}$, define the distribution $u(x,t) = U(x,t) + \alpha(t)\delta(x-ct)$, where $U(x,t)$ is given by (2.5) and $\alpha(t)$ is given by (2.6). Then $u(x,t)$ is a solution of the Riemann problem (2.2), (2.4) in the sense of Definition 2.1.*

Proof. The proof of the theorem follows by substituting u into (2.3). After standard transformations, the identity

$$\int_{\mathbb{R}_+} (c[U] - [f(U)])\,\varphi(ct,t)\;dt - \int_{\mathbb{R}_+} \alpha'(t)\varphi(ct,t)\;dt = 0$$

appears, where $[U] = u_R - u_L$ and $[f(U)] = f(u_R) - f(u_L)$, and since $\alpha(0) = 0$, the conclusion follows immediately. □

In the next section, we will justify the solution given above by the method of weak asymptotics. In general, if the weak asymptotic method, such as defined in [4,6] is used in tandem with the above definition, the solutions are thought to be admissible, although this admissibility concept does not yield uniqueness. The weak asymptotic method was recently extended to the case where complex-valued corrections are allowed [15,16], and it will appear in the next section that the use of complex-valued corrections plays a crucial role in the construction of weak asymptotic solutions to the Buckley–Leverett equation.

3. Weak asymptotics

In this section, we shall construct an approximative solution to the Buckley–Leverett equation posed with piecewise constant initial data. We shall show how to accomplish this in the case of the Riemann problem since the case of multiple steps is treated similarly. We begin with a general definition of what we shall mean by an approximate solution. First we define a vanishing family of distributions.

Definition 3.1. Let $f_\varepsilon(x) \in \mathcal{D}'(\mathbb{R})$ be a family of distributions depending on $\varepsilon \in (0,1)$. We say that $f_\varepsilon = o_{\mathcal{D}'}(1)$ if for any test function $\phi(x) \in \mathcal{D}(\mathbb{R})$, the estimate

$$\langle f_\varepsilon, \phi \rangle = o(1), \quad \text{as} \quad \varepsilon \to 0$$

holds.

The estimate on the right-hand side is understood in the usual Landau sense. Thus we may say that a family of distributions approach zero in the sense defined above if for a given test function ϕ, the pairing $\langle f_\varepsilon, \phi \rangle$ converges to zero as ε approaches zero.

Definition 3.2. We say that the family of complex-valued distributions (u_ε) represents a weak asymptotic solution to (1.1) if there exist real-valued distribution $u \in C\big((0,\infty); \mathcal{D}'(\mathbb{R})\big)$, such that for every fixed $t \in (0,\infty)$

$$u_\varepsilon \rightharpoonup u, \quad \text{as} \quad \varepsilon \to 0,$$

in the sense of distributions in $\mathcal{D}'(\mathbb{R})$, and

$$\partial_t u_\varepsilon + \partial_x f(u_\varepsilon), \;=\; o_{\mathcal{D}'}(1). \tag{3.1}$$

Let us remark that it is usually assumed that the relations $o_{\mathcal{D}'(\mathbb{R})}(1)$ hold uniformly with respect to $t \in (0, \infty)$. However, in the present case, Definition 3.2 is only used as an admissibility condition in tandem with Definition 2.1 defining singular solutions. Since we also use further admissibility conditions for the solutions constructed in Sections 4 and 5, we settle for the slightly weaker concept that the distributions (u_ε) converge to u pointwise in t. With these definitions in hand, the following theorem can be proved.

Theorem 3.1. *For any $u_L, u_R \in [0,1]$, and every $c \in \mathbb{R}$, there exists a family of functions (u_ε) representing the weak asymptotic solution to (1.1) such that*

$$u_\varepsilon \rightharpoonup U(x,t) + \alpha(t)\delta(x - ct), \tag{3.2}$$

where $U(x,t)$ is given by (2.5) and $\alpha(t)$ is given by (2.6).

Proof. In order to construct an approximate solution to (1.1) satisfying (3.2), we introduce a number of approximations. Denote by $\omega \in C^\infty(\mathbb{R})$ a smooth non-decreasing function such that $0 \le \omega \le 1$ and

$$\omega(z) = \begin{cases} 0, & z \le -1, \\ 1, & z \ge 1. \end{cases}$$

We denote the approximate Heaviside function and the approximate delta distribution, respectively by

$$H_\varepsilon(x) = \omega(z/\varepsilon), \quad \delta_\varepsilon(x) = \frac{1}{\varepsilon}\omega'(x/\varepsilon), \quad x \in \mathbb{R}.$$

In addition, we shall need the function χ_ε, defined by

$$\chi_\varepsilon(x) = \begin{cases} 1, & |x| \le \varepsilon/2, \\ 0, & |x| \ge \varepsilon/2 + 2\varepsilon^2, \\ 1 - \frac{1}{\sqrt{2}\varepsilon}\sqrt{|x| - \varepsilon/2}, & \text{otherwise}. \end{cases}$$

This is a function which approximates the characteristic function of the interval $(-\varepsilon/2, \varepsilon/2) \subset \mathbb{R}$ in the sense that it is equal to unity in the interval $(-\varepsilon/2, \varepsilon/2)$, and which has support on the interval $(-\varepsilon/2 - 2\varepsilon^2, \varepsilon/2 + 2\varepsilon^2)$. We start with the following ansatz:

$$\begin{aligned} u_\varepsilon(x,t) &= u_R H_\varepsilon(x - ct - 10\varepsilon) + u_L H_\varepsilon(-x + ct - 10\varepsilon) \\ &\quad + \frac{\alpha(t)}{2}\delta_\varepsilon(x - ct - 20\varepsilon) + \frac{\alpha(t)}{2}\delta_\varepsilon(x - ct + 20\varepsilon) + u_{p(t)/\varepsilon}\chi_\varepsilon(x - ct), \end{aligned} \tag{3.3}$$

where $u_{p(t)/\varepsilon}$ is chosen such that

$$f(u_{p(t)/\varepsilon}) = \frac{p(t)}{\varepsilon}.$$

It is not difficult to see that

$$u_{p(t)/\varepsilon} = \frac{-a \pm \sqrt{\frac{a\varepsilon}{p(t)} - a}}{\frac{\varepsilon}{p(t)} - (a+1)} = \frac{a}{a + \sqrt{\frac{\varepsilon a}{p(t)} - a}}, \tag{3.4}$$

if the plus sign is chosen. In this case, we clearly have $|u(x,t)| \le 1$. Moreover, we may define $u_{p(t)/\varepsilon}(p = 0) = 0$. As will come to light, the function $p(t)$ will have to be chosen as $p(t) = c\alpha(t)$, and, for such a choice, $u_{p(t)/\varepsilon}$ will have a non-zero imaginary part for small enough ε. Notice also that

$$\begin{aligned}
&H_\varepsilon(x-ct-10\varepsilon)\,H_\varepsilon(-x+ct-10\varepsilon)=0,\\
&H_\varepsilon(x-ct-10\varepsilon)\,\delta_\varepsilon(x-ct-20\varepsilon)=\delta_\varepsilon(x-ct-20\varepsilon),\\
&H_\varepsilon(-x+ct-10\varepsilon)\,\delta_\varepsilon(x-ct+20\varepsilon)=\delta_\varepsilon(x-ct+20\varepsilon),\\
&H_\varepsilon(-x+ct-10\varepsilon)\,\chi_\varepsilon(x-ct)=0,\\
&H_\varepsilon(x-ct-10\varepsilon)\,\chi_\varepsilon(x-ct)=0.
\end{aligned}$$

Moreover, we have

$$f\big(u_{p(t)/\varepsilon}\chi_\varepsilon(x-ct)\big)=\frac{p(t)}{\varepsilon},\ x\in(ct-\varepsilon/2,ct+\varepsilon/2),\quad t>0.$$

Now, it remains to insert u_ε into (3.1). Accordingly, for an arbitrary test function $\varphi\in C_c^\infty(\mathbb{R})$, we have

$$\begin{aligned}
&\int_{\mathbb{R}}\big\{\partial_t u_\varepsilon\varphi-f(u_\varepsilon)\partial_x\varphi\big\}dx\\
&\quad=\int_{-\infty}^{ct-30\varepsilon}\big\{0-f(u_L)\varphi'(x)\big\}dx+\int_{ct+30\varepsilon}^{\infty}\big\{0-f(u_R)\varphi'(x)\big\}dx\\
&\qquad+\int_{ct-30\varepsilon}^{ct-15\varepsilon}\Big\{\Big[\tfrac{\alpha'(t)}{2}\delta_\varepsilon(x-ct+20\varepsilon)-\tfrac{\alpha(t)}{2}c\delta'_\varepsilon(x-ct+20\varepsilon)\Big]\varphi(x)-f(u_\varepsilon)\varphi'(x)\Big\}\,dx\\
&\qquad+\int_{ct+15\varepsilon}^{ct+30\varepsilon}\Big\{\Big[\tfrac{\alpha'(t)}{2}\delta_\varepsilon(x-ct-20\varepsilon)-\tfrac{\alpha(t)}{2}c\delta'_\varepsilon(x-ct-20\varepsilon)\Big]\varphi(x)-f(u_\varepsilon)\varphi'(x)\Big\}\,dx\\
&\qquad+\int_{ct-15\varepsilon}^{ct-5\varepsilon}\Big\{cu_L\delta_\varepsilon(-x+ct-10\varepsilon)\varphi(x)-f\big(u_LH_\varepsilon(-x+ct-10\varepsilon)\big)\varphi'(x)\Big\}dx\\
&\qquad+\int_{ct+5\varepsilon}^{ct+15\varepsilon}\Big\{-cu_R\delta_\varepsilon(x-ct-10\varepsilon)\varphi(x)-f\big(u_RH_\varepsilon(x-ct-10\varepsilon)\big)\varphi'(x)\Big\}dx\\
&\qquad+\int_{ct-5\varepsilon}^{ct+5\varepsilon}\partial_t u_{p(t)/\varepsilon}\chi_\varepsilon(x-ct)\varphi(x)dx-cu_{p(t)/\varepsilon}\int_{ct-5\varepsilon}^{ct+5\varepsilon}\partial_x\chi_\varepsilon(x-ct)\varphi(x)dx\\
&\qquad-\int_{ct-\varepsilon/2}^{ct+\varepsilon/2}\frac{p(t)}{\varepsilon}\varphi'(x)dx-\int\limits_{(ct-\varepsilon/2-2\varepsilon^2,ct+\varepsilon/2+2\varepsilon^2)\setminus(ct-\varepsilon/2,ct+\varepsilon/2)}f(u_{p(t)/\varepsilon}\chi_\varepsilon(x-ct))\varphi'(x)dx.
\end{aligned}$$

The first two integrals on the right-hand side of the above equality converge to $-f(u_L)\varphi(ct)$ and $f(u_R)\varphi(ct)$, respectively, as $\varepsilon\to0$. The sum of the third and fourth integral converges to $\alpha'(t)\varphi(ct)+c\alpha(t)\varphi'(ct)$ as $\varepsilon\to0$. The fifth and sixth integrals converge to $cu_L\varphi(ct)$ and $-cu_R\varphi(ct)$, respectively as $\varepsilon\to0$. As for the seventh integral, we note that both $u_{\frac{p(t)}{\varepsilon}}$ and its time derivative are finite for any fixed $t>0$ for small enough ε. Thus, since the size of the domain of integration of the seventh integral is 10ε, the integral tends to zero as $\varepsilon\to0$. It will be shown in Appendix A that the eighth integral and the very last integral both tend to zero as $\varepsilon\to0$. Then, it can be concluded that

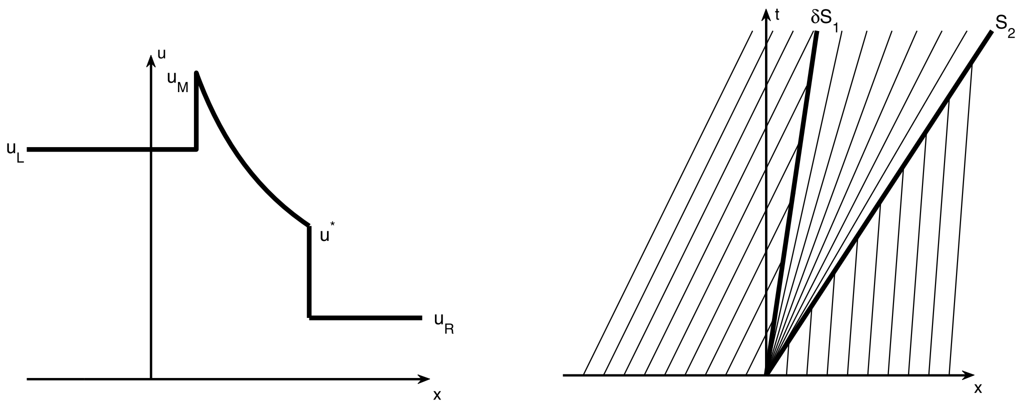


Fig. 3. Schematic picture of a non-monotone admissible solution to the Riemann problem. The shock δS_1 has been slowed to match the characteristic speed of u_M. Between u_M and u^*, there is a rarefaction wave, and from u^* to u_R, there is another admissible (non-delta) shock S_2.

$$\int_{\mathbb{R}} \{\partial_t u_\varepsilon \varphi - f(u_\varepsilon)\partial_x \varphi\} dx = \big(-c[u_R - u_L] + [f(u_R) - f(u_L)]\big)\varphi(ct)$$

$$+ \alpha'(t)\varphi(ct) + c\alpha(t)\varphi'(ct) - p(t) \int_{ct-\varepsilon/2}^{ct+\varepsilon/2} \frac{\varphi'(x)}{\varepsilon}\, dx + o_{\mathcal{D}'}(1),$$

where $o_{\mathcal{D}'}(1) \to 0$ as $\varepsilon \to 0$. The proof of the theorem is concluded by taking $p(t) = c\alpha(t)$ and letting $\varepsilon \to 0$. $\square$

It should be noted that multiplication of distributions can also be studied in the context of Colombeau algebras, such as explained in [3,13,21]. In fact, Definition 3.2 can be understood as a variant of appropriate definitions in [4,20]. The main difference is that in the present case, a solution is found pointwise with respect to $t \in \mathbb{R}_+$, and it is required that the distributional limit of the weak asymptotic solution be a distribution.

Another comment is in order. Note that we have used the functional form of the flux given by (2.1) for values of u on the real line. However, from a physical point of view, only the values of the flux for $0 \le u \le 1$ are important, and the definition of the flux for values other than these is essentially arbitrary. For instance, one could define $f(u)$ to be monotone and unbounded outside of the interval $u \in [0, 1]$. While such a definition would not change the physical applicability of Eq. (2.2), it would change the definition of delta-shock solutions. In particular, if $f(u)$ were unbounded, the use of imaginary corrections in (3.3) could possibly be avoided.

4. Riemann problem

The delta-shock solutions described in the previous section may be used to construct non-monotone solutions of the Riemann problem for (1.1). Let $u_I \in (0, 1)$ denote the value of u at which the inflection point of the graph of $f(u)$ is located. Given initial data (1.2), with $0 \le u_R < u_I$, a solution can be constructed by taking a delta shock δS_1 from u_L to a value u_M. Thanks to the Rankine–Hugoniot defect, this shock is slowed to have the same speed as the characteristic speed $f'(u_M)$. The solution continues with a rarefaction wave from u_M to u^*, where u^* denotes the value of u where the extension of the region under the graph of $f(u)$ to the convex hull to the point $\{u_R, f(u_R)\}$ begins, and a classical shock from u^* to u_R with shock speed determined by the usual Rankine–Hugoniot condition. This solution is depicted in Fig. 3, and the features are summarized in the following theorem.

Theorem 4.1. *Suppose we are given Riemann data* (1.2)*, such that* $0 < u_R < u_I$*, and* $0 < u_L \le 1$*. If* $u_M > u^*$ *is such that* $f'(u_M) < f'(u_L)$*, then there is a solution of the Riemann problem given by*

$$u(x,t) = U(x,t) + \alpha(t)\delta(x - c_1 t), \tag{4.1}$$

where $U(x,t)$ *is given by*

$$U(x,t) = \begin{cases} u_L, & x < c_1 t, \\ u_M, & x = c_1 t, \\ [f']^{-1}(x/t), & c_1 t < x < c_2 t \\ u_R, & c_2 t < x. \end{cases}$$

The shock velocity of the delta shock δS_1 *is given by* $c_1 = f'(u_M)$*, and the strength of* δS_1 *is given by* $\alpha(t) = (c_1[u_M - u_L] - [f(u_M) - f(u_L)])t$*. The shock velocity of the shock* S_2 *is given by* $c_2 = \frac{[f(u_R)-f(u^*)]}{[u_R - u^*]} = f'(u^*)$*. The solution is admissible in the sense that both shocks are compressive.*

The proof of the theorem follows easily from piecing together the delta shock δS_1, the rarefaction wave, and the classical shock S_2. It is straightforward to check that this is a solution both in the sense of Definition 2.1 and in the sense of Definition 3.2 if the strength of the delta shock δS_1 is given by (2.6) as indicated. Moreover, as shown in the right panel of Fig. 3, the solution is admissible in the sense that characteristics are either entering the shock, or are parallel to the shock. Thus the solution is compressive, and it may also be shown that the solution is entropy admissible in the sense defined in [1].

It should be noted that the above construction works for both $u_L < u^*$ and $u_L > u^*$. Note also that the admissibility condition is not strong enough to yield uniqueness, as the value of u_M is not uniquely determined from the initial data.

5. Traveling waves

The solution in the previous section may be modified by extending the region in which the solution u takes the value u_M to nonzero width. This region will then be sandwiched between the delta shock δS_1 on the left and the rarefaction wave on the right. However, in this case the solution is not the solution of a Riemann problem. On the other hand, inclusion of the rarefaction wave precludes the possibility of constant width of the overshoot region, which is observed experimentally. In this final section, we shall consider the possibility of traveling-wave solutions which are steady waves which propagate without altering the solution profile in time.

Suppose the values u_R and u_L are given such that $0 \le u_R < u_I < u_L < 1$ and $f'(u_R) < f'(u_L)$. As before, let $\{u^*, f(u^*)\}$ be the point on the graph of $f(u)$ which marks the right endpoint of the part of the convex hull which lies above the graph, as shown in Fig. 2. Suppose we have $u_R < u_L < u^*$. Then since $u_I < u_L$, we have $f'(u_L) > f'(u^*)$, and the solution consists of a delta shock, a constant region $u = u^*$ and a regular shock connecting u^* and u_R. The solution is illustrated in Fig. 4. Defining the L^∞-part of the solution by

$$U(x,t) = \begin{cases} u_L, & x < ct \\ u^*, & ct \le x < ct + m \\ u_R, & ct + m \le x, \end{cases} \tag{5.1}$$

for an arbitrary $m \in \mathbb{R}$, the following theorem can be formulated.

Theorem 5.1. *Suppose we are given* u_R *and* u_L*, such that* $0 \le u_R < u_I < u_L < u^*$*, and* $f'(u_R) < f'(u_L)$*. Then there exists a solution of* (1.1) *given by*

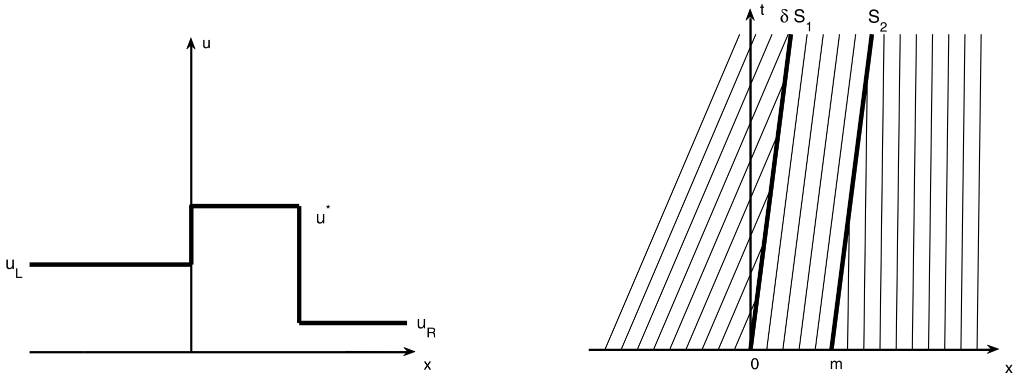


Fig. 4. Schematic picture of a traveling-wave solution. The shock δS_1 has been slowed to match the characteristic speed of u^*. The shock S_2 also matches the characteristic speed of u^*.

$$u(x,t) = U(x,t) + \alpha(t)\delta(x - ct), \tag{5.2}$$

where $U(x,t)$ *is given by* (5.1)*, the shock velocity of the delta shock* δS_1 *and the classical shock* S_2 *is given by* $c = f'(u^*)$*, and the strength of the* δS_1 *is given by* $\alpha(t) = (c[u^* - u_L] - [f(u^*) - f(u_L)])t$*.*

The solution is admissible in the sense that all characteristics either enter into or are collinear to shock trajectories.

In this theorem, c represents the velocity of the profile, and m denotes the width of the overshoot region. The proof follows immediately from the proofs of Theorem 2.1 and Theorem 3.1 since the two shocks are separated by a region of nonzero width m.

Note that this solution is steady, but is not a similarity solution such as the weak solution of a Riemann problem. A certain measure of uniqueness for the above solution follows from the principle of minimizing the number of delta shocks (cf. [16]) because choosing a value other than u^* for the overshoot region would necessitate the inclusion of two delta shocks. However, this principle is not strong enough to provide overall uniqueness, as the width m is still undetermined.

Next, we treat the case where $u^* < u_L$. In this case, the solution contains two delta shocks. First, for an arbitrary m, define $U(x,t)$ by

$$U(x,t) = \begin{cases} u_L, & x < ct \\ u_M, & ct \le x < ct + m \\ u_R, & ct + m \le x. \end{cases} \tag{5.3}$$

Then the following theorem holds.

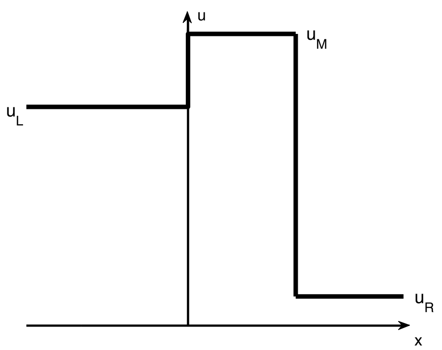
Theorem 5.2. *Suppose we are given* u_R *and* u_L *such that* $0 \le u_R < u^* < u_L < 1$*, and* $f'(u_R) < f'(u_L)$*. If* u_M *can be chosen such that* $f'(u_R) < f'(u_M) < f'(u_L)$*, then there exists a solution of* (1.1) *given by*

$$u(x,t) = U(x,t) + \alpha_1(t)\delta(x - ct) + \alpha_2(t)\delta(x - ct - m), \tag{5.4}$$

where $U(x,t)$ *is given by* (5.3)*. The shock velocity of the delta shocks* δS_1 *and* δS_2 *is given by* $c = f'(u_M)$*. The strength of* δS_1 *is given by* $\alpha_1(t) = (c[u_M - u_L] - [f(u_M) - f(u_L)])t$*, and the strength of* δS_2 *is given by* $\alpha_2(t) = (c[u_R - u_M] - [f(u_R) - f(u_M)])t$*.*

The solution is admissible in the sense that all characteristics either enter into or are collinear to shock trajectories.

As explained above, the proof of this theorem also follows immediately from the proofs of Theorem 2.1 and Theorem 3.1 since the two shocks are separated by a region of nonzero width m. Note that there are



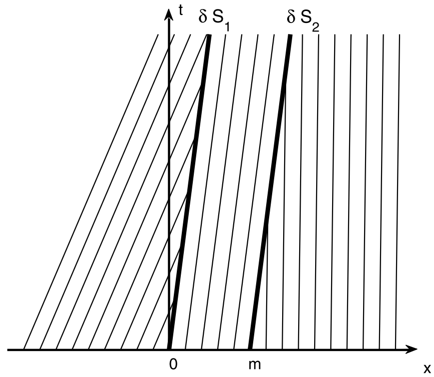


Fig. 5. Schematic picture of a non-monotone traveling-wave solution. The shock δS_1 has been slowed to match the characteristic speed of u_M. The shock δS_2 also matches the characteristic speed of u_M.

two possibilities for choosing u_M. One is in the interval (u_R, U_l), where $U_l < u_L$ and $f'(U_l) = f'(u_L)$, and the other is in the interval (u_L, U_r), where $U_r > u_L$ is such that $f'(U_r) = f'(u_r)$. The former choice yields a monotone solution while the latter yields a non-monotone solution (see Fig. 5).

As already mentioned in the introduction, experimental results show that if a saturation overshoot develops, then the speed of the two fronts will generally be the same, so that the width of the domain of maximum saturation should remain unchanged in time. While the solutions constructed in this section exhibit such a behavior, the link to the physical modeling of an infiltration problem is unclear since the non-zero Rankine–Hugoniot deficit appears to upset the principle of mass conservation. Nevertheless, it has been shown in this paper that it is at least mathematically possible to construct non-monotone admissible solutions of the Buckley–Leverett equation with various requisite properties.

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Appendix A. Convergence of integrals

In this appendix, it will be shown that the eighth and tenth integrals in the expression for $\int_{\mathbb{R}} \big\{\partial_t u_\varepsilon \varphi - f(u_\varepsilon)\partial_x\varphi\big\} dx$ in the proof of Theorem 3.1 converge to zero. Let us first look at the eighth integral.

Lemma A.1.

$$c u_{p(t)/\varepsilon} \int_{ct-5\varepsilon}^{ct+5\varepsilon} \partial_x \chi_\varepsilon(x-ct)\varphi(x)dx \tag{A.1}$$

approaches zero as $\varepsilon \to 0$.

Proof. Note first of all that it follows from (3.4) that $u_{p(t)/\varepsilon}$ is finite and uniformly bounded for all ε. Next, recall the definition of χ_ε,

$$\chi_\varepsilon(x) = \begin{cases} 1, & |x| \le \varepsilon/2, \\ 0, & |x| \ge \varepsilon/2 + 2\varepsilon^2, \\ 1 - \frac{1}{\sqrt{2}\varepsilon}\sqrt{|x| - \varepsilon/2}, & \text{otherwise,} \end{cases}$$

and note that we have

$$\int\limits_{\varepsilon/2}^{\varepsilon/2+2\varepsilon^2} |\partial_x \chi_\varepsilon(x)|\, dx = - \int\limits_{\varepsilon/2}^{\varepsilon/2+2\varepsilon^2} \partial_x \chi_\varepsilon(x)\, dx = \chi(\varepsilon/2) - \chi(\varepsilon/2 + 2\varepsilon^2) = 1. \tag{A.2}$$

Since the test function φ is smooth, we can use a finite Taylor expansion with the Lagrangian form of the remainder to write

$$\varphi(x) = \varphi(ct) + [x - ct]\varphi'(\xi),$$

where $\xi \in (ct - x, ct + x)$. Since the function $\partial_x \chi_\varepsilon(\cdot - ct)$ is odd around ct, we have

$$\begin{aligned} \int\limits_{ct-5\varepsilon}^{ct+5\varepsilon} \partial_x \chi_\varepsilon(x - ct)\varphi(x) dx &= 2 \int\limits_{ct+\varepsilon/2}^{ct+\varepsilon/2+2\varepsilon^2} \partial_x \chi_\varepsilon(x - ct)\, [x - ct]\varphi'(\xi)\, dx \\ &= 2 \int\limits_{\varepsilon/2}^{\varepsilon/2+2\varepsilon^2} \partial_x \chi_\varepsilon(x)\, x\varphi'(\xi + ct)\, dx. \end{aligned}$$

Thus it follows that

$$\left| \int\limits_{ct-5\varepsilon}^{ct+5\varepsilon} \partial_x \chi_\varepsilon(x - ct)\varphi(x)\, dx \right| \le 2 \left[\varepsilon/2 + 2\varepsilon^2\right] \max|\varphi'| \int\limits_{\varepsilon/2}^{\varepsilon/2+2\varepsilon^2} |\partial_x \chi_\varepsilon(x)|\, dx.$$

Using (A.2), we see that (A.1) approaches zero as $\varepsilon \to 0$. □

Next, we show that the tenth integral in the expression for $\int_{\mathbb{R}} \left\{\partial_t u_\varepsilon \varphi - f(u_\varepsilon)\partial_x \varphi\right\} dx$ in the proof of Theorem 3.1 approaches zero.

Lemma A.2.

$$\int\limits_{(ct-\varepsilon/2-2\varepsilon^2, ct+\varepsilon/2+2\varepsilon^2)\setminus(ct-\varepsilon/2, ct+\varepsilon/2)} f(u_{p(t)/\varepsilon}\chi_\varepsilon(x - ct))\varphi'(x) dx \tag{A.3}$$

approaches zero as $\varepsilon \to 0$.

Proof. Note first that

$$\int\limits_{\varepsilon/2}^{\varepsilon/2+2\varepsilon^2} \frac{1}{1 - \chi_\varepsilon(x)}\, dx = 2\sqrt{2}\varepsilon\, \sqrt{x - \varepsilon/2}\, \Big|_{\varepsilon/2}^{\varepsilon/2+2\varepsilon^2} \to 0 \tag{A.4}$$

as $\varepsilon \to 0$. The integral in question can be estimated as follows:

$$\left| \int\limits_{(ct-\varepsilon/2-2\varepsilon^2, ct+\varepsilon/2+2\varepsilon^2)\setminus(ct-\varepsilon/2, ct+\varepsilon/2)} f(u_{p(t)/\varepsilon}\chi_\varepsilon(x-ct))\varphi'(x)dx \right|$$
$$\le 2\max|\varphi'| \int\limits_{ct+\varepsilon/2}^{ct+\varepsilon/2+2\varepsilon^2} \left| f\left(\chi_\varepsilon(x-ct)u_{p(t)/\varepsilon}\right)\right| dx$$
$$= 2\max|\varphi'| \int\limits_{c\varepsilon/2}^{\varepsilon/2+2\varepsilon^2} \left| \frac{1}{1+a\left(1-\frac{1}{\chi_\varepsilon(x)u_{p(t)/\varepsilon}}\right)^2} \right| dx.$$

Now note that for any complex number, we have $|\mathrm{Im}(z)| \le |z|$, so that the denominator of the integrand may be replaced by its imaginary part. Substituting the expression for $u_{p(t)/\varepsilon}$, we see that

$$\mathrm{Im}\left\{1+a\left(1-\frac{1}{\chi_\varepsilon(x)u_{p(t)/\varepsilon}}\right)^2\right\} = 2\sqrt{a}\sqrt{1-\varepsilon/p(t)}\,\frac{1}{\chi_\varepsilon(x)}\left[\frac{1}{\chi_\varepsilon(x)}-1\right].$$

Hence the integral (A.3) can be estimated by

$$\frac{2\max|\varphi'|}{2\sqrt{a}\sqrt{1-\varepsilon/p(t)}} \int\limits_{\varepsilon/2}^{\varepsilon/2+2\varepsilon^2} \frac{\chi_\varepsilon(x)}{\left[\frac{1}{\chi_\varepsilon(x)}-1\right]} dx.$$

Finally, we have

$$\lim_{\varepsilon\to 0} \frac{2\max|\varphi'|}{2\sqrt{a}\sqrt{1-\varepsilon/p(t)}} \int\limits_{\varepsilon/2}^{\varepsilon/2+2\varepsilon^2} \frac{\chi_\varepsilon(x)}{\frac{1}{\chi_\varepsilon(x)}-1} dx = \lim_{\varepsilon\to 0} \frac{2\max|\varphi'|}{2\sqrt{a}\sqrt{1-\varepsilon/p(t)}} \int\limits_{\varepsilon/2}^{\varepsilon/2+2\varepsilon^2} \frac{\chi_\varepsilon^2(x)}{1-\chi_\varepsilon(x)} dx$$
$$\le \lim_{\varepsilon\to 0} \frac{2\max|\varphi'|}{2\sqrt{a}} \int\limits_{\varepsilon/2}^{\varepsilon/2+2\varepsilon^2} \frac{1}{1-\chi_\varepsilon(x)} dx,$$

since $\chi_\varepsilon(x)$ is at most equal to unity. Invoking (A.4), we see that the integral does indeed approach zero. □

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