



On geometric structure of symmetric spaces



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ABSTRACT

In this article we discuss local approach to strict K -monotonicity and local uniform rotundity in symmetric spaces. We prove several general results on local structure of symmetric spaces E showing relation between strict monotonicity and strict K -monotonicity and the Kadec–Klee property for global convergence in measure. We also present the full criteria for points of upper K -monotonicity in Lorentz spaces $\Gamma_{p,w}$ for degenerated weight function w . Next we characterize local uniform rotundity in symmetric spaces E proving several correspondences between $x \in E$ a point of local uniform rotundity and its decreasing rearrangement x^* and absolute value $|x|$. Finally, we apply these results to find complete criteria for local uniform rotundity of Lorentz spaces $\Gamma_{p,w}$.

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1. Introduction

Recently, the geometric structure of the symmetric space has been investigated expansively by many authors [4–6,10,12,13,19,18]. The studies of the local and global properties in symmetric spaces is a key for many different types of branches of mathematics. Indeed, it has been found many applications of the monotonicity and rotundity properties of symmetric spaces in approximation theory (see [9,10,12,20,27]). It is worth mentioning that the monotonicity properties play a similar significant role in the best dominated approximation problems in Banach lattices as the respective rotundity properties do in the best approximation problems in Banach spaces. Moreover, the structure of symmetric spaces plays a crucial role as interpolation spaces between Banach couple (L^1, L^∞) in an investigation of measurable operators (for more details see [1,13,26]). The natural question of existence of the monotonicity and rotundity properties is a keyword for many employable issues but not always easily solvable problems in the case of certain degenerated symmetric spaces. In [18], H. Hudzik, A. Kamińska and M. Mastyło presented the full criteria for local uniform rotundity (LUR) and strict K -monotonicity (SKM) in Orlicz–Lorentz spaces $\Lambda_{\phi,w}$ under some additional assumptions on the weight function w and Orlicz function ϕ . Moreover, in [18], it was proved that

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a symmetric rotund and K -monotone space is strictly K -monotone. In [30], A. Medzhitov and P. Sukochev researched in a symmetric space E a correspondence between LUR property and LUR^* property, that is local uniform rotundity considered on E^* the cone of nonnegative and nonincreasing functions of E . *It is worth noticing that LUR property with corresponding DGL property was researched intensively in K -interpolation spaces in [15,16] by P.G. Dodds, T.K. Dodds, A.A. Sedaev and F.A. Sukochev.* The next interesting results were published in [4], where authors discussed for a symmetric space E a relation between the global version of some rotundity properties P on E and its restricted version to E^* . Let's recall that in [6, Proposition 2.1] it was proved that a separable symmetric space E with the Kadec–Klee property is strictly K -monotone. Moreover, authors in [6] showed that in separable Lorentz spaces Λ_ϕ , strict K -monotonicity is equivalent to the Kadec–Klee property (see Theorem 2.11). Recently, in [11] it has been discussed some sufficient and necessary conditions for strict K -monotonicity of some important concrete symmetric spaces. Furthermore, in [11] authors have presented an example of symmetric spaces which are strictly K -monotone and are not rotund.

The recent studies of the monotonicity and rotundity properties in certain symmetric spaces have given a positive answer of the precise descriptions of the wanted properties although it required technical proofs (see [8,17,19,25]). In the spirit of finding an applicable local geometry, in this article it is characterized the local version of strict K -monotonicity and local uniform rotundity of symmetric spaces and organized as follows.

In Section 2 we recall the necessary notions.

In Section 3 we present several results in symmetric spaces devoted to local approach of strict K -monotonicity, strict monotonicity and the Kadec–Klee property for global convergence in measure. Namely, we investigate in a symmetric space a relation between a point of upper (lower) K -monotonicity and a point of upper (lower) monotonicity, respectively. We also show a characterization of a point of upper local uniform K -monotonicity in terms of points of upper K -monotonicity and the Kadec–Klee property for global convergence in measure.

In Section 4 we characterize the complete criteria for a point of upper K -monotonicity in Lorentz spaces $\Gamma_{p,w}$ with the degenerated weight function w .

Section 5 is devoted to an investigation of a point of local uniform rotundity in symmetric spaces. We answer the essential question whether a point of local uniform rotundity can be equivalently considered only on the positive cone E^+ of a Banach function space E . This problem is proved without any additional assumption and shows that $x \in E$ is a point of local uniform rotundity if and only if $|x|$ is an LUR point in E . We also deliberate the local version of a relation between local uniform rotundity of a symmetric space E and its positive cone E^* , of all decreasing rearrangement x^* of $x \in E$. Namely, we prove under some additional assumptions that a point x is an LUR point if and only if its decreasing rearrangement x^* is an LUR point. In the final result of this section we improve the answer for well known essential question whether LUR property can be researched only on E^* . This problem was establish previously under some additional conditions. In this significant problem we show that a symmetric space E is LUR if and only if the cone E^* is LUR .

The intention of the last Section 6 is to find a complete criteria of local uniform rotundity in Lorentz spaces $\Gamma_{p,w}$. Applying obtained results in the previous sections for local approach of strict K -monotonicity and LUR property in symmetric spaces we investigate the equivalent conditions for local uniform rotundity in particular classes of Lorentz spaces $\Gamma_{p,w}$. We show the class of Banach spaces constructed by the K -method of interpolation which possess LUR property. Finally, we present the necessary conditions for a point of local uniform rotundity in Lorentz spaces $\Gamma_{p,w}$ with the degenerated weight function w .

2. Preliminaries

Let \mathbb{R} and \mathbb{N} be the sets of reals and positive integers, respectively. As usual $S(X)$ (resp. $B(X)$) stands for the unit sphere (resp. the closed unit ball) of a Banach space $(X, \|\cdot\|_X)$.

Denote by L^0 the set of all (equivalence classes of) extended real-valued Lebesgue measurable functions on $[0, \alpha)$, where $\alpha = 1$ or $\alpha = \infty$.

A Banach lattice $(E, \|\cdot\|_E)$ is called a *Banach function space* (or a *Köthe space*) if it is a sublattice of L^0 satisfying the following conditions:

- (1) If $x \in L^0$, $y \in E$ and $|x| \leq |y|$ a.e., then $x \in E$ and $\|x\|_E \leq \|y\|_E$.
- (2) There exists a strictly positive $x \in E$.

By E^+ we denote the positive cone of E , i.e. $E^+ = \{x \in E : x \geq 0\}$. We use the notation $A^c = [0, \alpha) \setminus A$ for any measurable set A . By μ denote the Lebesgue measure on $[0, \alpha)$. A point $x \in E$ is said to be an *order continuous point* if for any sequence $(x_n) \subset E^+$ such that $x_n \leq |x|$ and $x_n \rightarrow 0$ a.e. we have $\|x_n\|_E \rightarrow 0$. A Köthe space E is called *order continuous* (shortly $E \in (OC)$) if every element x of E is an order continuous point (see [28]). Unless we say otherwise, we assume in the whole paper that E has *Fatou property*, i.e. if $(x_n) \subset E^+$, $\sup_{n \in \mathbb{N}} \|x_n\|_E < \infty$ and $x_n \uparrow x \in L^0$, then $x \in E$ and $\|x_n\|_E \uparrow \|x\|_E$.

A point $x \in E^+ \setminus \{0\}$ is called a *point of upper monotonicity* shortly a *UM point* (a *point of lower monotonicity* shortly an *LM point*) if for any $y \in E^+$ such that $x \leq y$ and $y \neq x$ ($y \leq x$ and $y \neq x$), we have $\|x\|_E < \|y\|_E$ ($\|y\|_E < \|x\|_E$). A point $x \in E^+$ is said to be a *point of upper local uniform monotonicity* (*lower local uniform monotonicity*) shortly a *ULUM point* (an *LLUM point*) if for any sequence $(x_n) \subset E$ such that $x \leq x_n$ and $\|x_n\|_E \rightarrow \|x\|_E$ ($0 \leq x_n \leq x$ and $\|x_n\|_E \rightarrow \|x\|_E$), it follows that $\|x_n - x\|_E \rightarrow 0$. Recall that if each $x \in E^+ \setminus \{0\}$ is a *UM point* or equivalently if each $x \in E^+ \setminus \{0\}$ is an *LM point*, then we say that E is *strictly monotone* ($E \in (SM)$) (see [2,19]). Similarly, if each point of $E^+ \setminus \{0\}$ is a *ULUM point* (an *LLUM point*), then we say that E is *upper locally uniformly monotone* shortly $E \in (ULUM)$ (*lower local uniform monotone* shortly $E \in (LLUM)$).

A point $x \in E$ is said to be a *point of local uniform rotundity* (shortly an *LUR point*) if for any sequence $(x_n) \subset E$ such that $\|x_n + x\|_E \rightarrow 2\|x\|_E$ and $\|x_n\|_E \rightarrow \|x\|_E$ we have $\|x_n - x\|_E \rightarrow 0$. A point $x \in E$ is said to be a *point of midpoint local uniform rotundity* (shortly an *MLUR point*) if for every sequence $(x_n) \subset E$ if $\|x_n + x\|_E \rightarrow \|x\|_E$ and $\|x_n - x\|_E \rightarrow \|x\|_E$, then $\|x_n\|_E \rightarrow 0$. A Banach function space E is said to be *locally uniformly rotund* shortly $E \in (LUR)$ (*midpoint locally uniformly rotund* shortly $E \in (MLUR)$) if each point $x \in E$ is an *LUR point* (an *MLUR point*) in E , respectively.

A point $x \in E$ is said to be an *H_g point* in E if for any $(x_n) \subset E$ such that $x_n \rightarrow x$ globally in measure and $\|x_n\|_E \rightarrow \|x\|_E$, we have $\|x_n - x\|_E \rightarrow 0$. We say that the space E has the *Kadec–Klee property for global convergence in measure* if each $x \in E$ is an *H_g point* in E .

For any function $x \in L^0$ we define its *distribution function* by

$$d_x(\lambda) = \mu \{s \in [0, \alpha) : |x(s)| > \lambda\}, \quad \lambda \geq 0,$$

and its *decreasing rearrangement* by

$$x^*(t) = \inf \{\lambda > 0 : d_x(\lambda) \leq t\}, \quad t \geq 0.$$

We say that a function $x \in L^0$ satisfies *condition (+)* if there exists $\tau_1 \geq 0$ such that $|x(t)| \geq \tau_1$ for a.e. $t \in [0, \alpha)$ and $d_{|x|-\tau_1}(\tau) < \infty$ for any $\tau > 0$ [21], i.e. $\mu(t \in [0, \alpha) : |x(t)| < x^*(\infty)) = 0$. The above notion is stated under the convention $x^*(\infty) = \lim_{t \rightarrow \infty} x^*(t)$ if $\alpha = \infty$ and $x^*(\infty) = 0$ if $\alpha = 1$.

Given $x \in L^0$ we denote the *maximal function* of x^* by

$$x^{**}(t) = \frac{1}{t} \int_0^t x^*(s) ds.$$

It is well known that $x^* \leq x^{**}$, x^{**} is decreasing and subadditive, i.e.

$$(x + y)^{**} \leq x^{**} + y^{**}$$

for any $x, y \in L^0$. For more properties of d_x , x^* and x^{**} see [1,26].

Two functions $x, y \in L^0$ are said to be *equimeasurable* (shortly $x \sim y$) if $d_x = d_y$. A Banach function space $(E, \|\cdot\|_E)$ is called *rearrangement invariant* (r.i. for short) or *symmetric* if whenever $x \in L^0$ and $y \in E$ with $x \sim y$, then $x \in E$ and $\|x\|_E = \|y\|_E$. Given an r.i. Banach function space E let ϕ_E denote its *fundamental function*, that is $\phi_E(t) = \|\chi_{(0,t)}\|_E$ for any $t \in [0, \alpha)$ (see [1]).

The notion \prec is defined for any x, y in $L^1 + L^\infty$ by

$$x \prec y \Leftrightarrow x^{**}(t) \leq y^{**}(t) \quad \text{for all } t > 0.$$

Let $0 < p < \infty$ and $w \in L^0$ be a nonnegative weight function, then the Lorentz space $\Gamma_{p,w}$ is a subspace of L^0 such that

$$\|x\|_{\Gamma_{p,w}} := \left(\int_0^\alpha x^{**p}(t) w(t) dt \right)^{1/p} < \infty.$$

Additionally, we assume that w is from class D_p , i.e.

$$W(s) := \int_0^s w(t) dt < \infty \quad \text{and} \quad W_p(s) := s^p \int_s^\alpha t^{-p} w(t) dt < \infty$$

for all $0 < s \leq 1$ if $\alpha = 1$ and for all $0 < s < \infty$ otherwise. These two conditions guarantee that Lorentz space $\Gamma_{p,w}$ is nontrivial. It is well known that $(\Gamma_{p,w}, \|\cdot\|_{\Gamma_{p,w}})$ is an r.i. quasi-Banach function space with Fatou property. It was proved in [23] that in the case when $\alpha = \infty$ the space $\Gamma_{p,w}$ has order continuous norm if and only if $\int_0^\infty w(t) dt = \infty$.

The spaces $\Gamma_{p,w}$ were introduced by A.P. Calderón in [3] in a similar way as the classical Lorentz spaces $\Lambda_{p,w}$ that is a subspace of L^0 with

$$\|x\|_{\Lambda_{p,w}} = \left(\int_0^\alpha x^{*p}(t) w(t) dt \right)^{1/p} < \infty,$$

where $p \geq 1$ and the weight function w is nonnegative and nonincreasing (see [29]). The space $\Gamma_{p,w}$ is an interpolation space between L^1 and L^∞ yielded by the Lions–Peetre K -method [1,26]. Clearly, $\Gamma_{p,w} \subset \Lambda_{p,w}$. The opposite inclusion $\Lambda_{p,w} \subset \Gamma_{p,w}$ is satisfied if and only if $w \in B_p$ (see [23]). It is worth mentioning that the spaces $\Gamma_{p,w}$ and $\Lambda_{p,w}$ are also connected by Sawyer’s result (Theorem 1 in [32]; see also [33]), which states that the Köthe dual of $\Lambda_{p,w}$, for $1 < p < \infty$ and $\int_0^\infty w(t) dt = \infty$, coincides with the space $\Gamma_{p',\tilde{w}}$, where $1/p + 1/p' = 1$ and $\tilde{w}(t) = \left(t / \int_0^t w(s) ds \right)^{p'} w(t)$.

It is easy to see that for $\alpha = 1$ by the Lebesgue Dominated Convergence Theorem, $\Gamma_{p,w}$ is order continuous. For more details about the properties of $\Gamma_{p,w}$ the reader is referred to [23].

A symmetric space E is called *K-monotone* (shortly $E \in (KM)$) if for any $x \in L^1 + L^\infty$ and $y \in E$ such that $x \prec y$, we have $x \in E$ and $\|x\|_E \leq \|y\|_E$. Recall that a symmetric space is *K-monotone* if and only if E is exact interpolation space between L^1 and L^∞ . It is well known that a symmetric spaces with Fatou property or with an order continuous norm is *K-monotone* (see [26]).

For any global property P and any symmetric space E the notation $E \in (P)^*$ means that E satisfies $(P)^*$, i.e. the cone $E^* = \{x^* : x \in E\}$ of all decreasing rearrangements of functions in E satisfies P property. Clearly, if $E \in (P)$, then $E \in (P)^*$. The natural question of the opposite conclusion was investigated by many authors in [4,10,12,30].

A point $x \in E$ is a *point of upper K-monotonicity* (*lower K-monotonicity*) shortly a *UKM point* (an *LKM point*) of E if and only if for any $y \in E$, $x^* \neq y^*$ with $x^{**} \leq y^{**}$ (with $y^{**} \leq x^{**}$), we have $\|x\|_E < \|y\|_E$ ($\|y\|_E < \|x\|_E$), respectively. Recall that a symmetric space E is said to be *strictly K-monotone* (shortly $E \in (SKM)$) if every point of E is a *UKM point* or equivalently if every point of E is an *LKM point*.

A point $x \in E$ is said to be a *point of upper local uniform K-monotonicity* of E (shortly a *ULUKM point*) if for any sequence $(x_n) \subset E$ such that $x^{**} \leq x_n^{**}$ for all $n \in \mathbb{N}$ and $\|x_n\|_E \rightarrow \|x\|_E$ we have $\|x^* - x_n^*\|_E \rightarrow 0$. A point $x \in E$ is called a *point of lower local uniform K-monotonicity* of E (shortly an *LLUKM point*) if and only if for any $(x_n) \subset E$ with $x_n^{**} \leq x^{**}$ for all $n \in \mathbb{N}$ and $\|x_n\|_E \rightarrow \|x\|_E$ we have $\|x^* - x_n^*\|_E \rightarrow 0$. Recall that a symmetric space E is called *upper locally uniformly K-monotone* shortly $E \in (ULUKM)$ (*lower locally uniformly K-monotone* shortly $(E \in (LLUKM))$) whenever each point of E is a *ULUKM point* (an *LLUKM point*), respectively. For more details the reader is referred to [6,18,11].

Remark 2.1. It is well known by the definition of *LUR* and *LLUM* points that for any symmetric space E if $x \in E$ is an *LUR point*, then $|x|$ is an *LLUM point*. Hence, by Theorem 2.1 in [10] it follows that x is a point of order continuity.

3. Local K-monotonicity

In this section we find a relation between local version of strict monotonicity and strict *K-monotonicity* and also the Kadec–Klee property for global convergence in measure. We start our discussion with some auxiliary result for a point of upper *K-monotonicity* which might be of interest independently for its application.

Remark 3.1. It is easy to notice that for any symmetric space E an element $x \in E$ is an *LKM point* (a *UKM point*) of E if and only if its decreasing rearrangement x^* or $|x|$ is an *LKM point* (a *UKM point*) of E , respectively. Analogously, it can be established a characterization of *ULUKM points* and *LLUKM points* in a symmetric space E . Moreover, it is clear that if $x \in E$ is a *ULUKM point* (an *LLUKM point*) of E , then x is also a *UKM point* (an *LKM point*) of E , respectively.

Theorem 3.2. Let E be a symmetric space and $x, x_n \in E$ for any $n \in \mathbb{N}$. If x is a *UKM point* and $x^{**} \leq x_n^{**}$ and $\|x_n\|_E \rightarrow \|x\|_E$, then $x_n^{**} \rightarrow x^{**}$ and $x_n^* \rightarrow x^*$ in measure.

Proof. Since $\|x_n\|_E \rightarrow \|x\|_E$, there exists $M > 0$ such that $\|x_n\|_E \leq M$ for any $n \in \mathbb{N}$. Since E is symmetric space we have for any $n \in \mathbb{N}$ and $t > 0$ (see [1])

$$x_n^*(t)\phi(t) \leq x_n^{**}(t)\phi(t) \leq \|x_n\|_E \leq M.$$

Therefore, by Helly's selection principle passing to subsequence and relabeling if it is necessary there is $z \in E$ such that $z = z^*$ and $x_n^*(t)$ converges to $z^*(t)$ for a.a. $t \in [0, \alpha)$. Now we show that $z^* = x^*$. It is obvious that for all $0 < t < s < \alpha$,

$$\int_t^s x_n^* \rightarrow \int_t^s z^*. \quad (1)$$

We claim that

$$\int_0^t x_n^* \rightarrow \int_0^t x^* \quad (2)$$

for any $t > 0$. Indeed, if it is not true then there exist $\tau \in (0, \alpha)$, $\epsilon > 0$ and a subsequence $(x_{n_k}^*) \subset (x_n^*)$ such that

$$\int_0^\tau x^* + \epsilon \leq \int_0^\tau x_{n_k}^*.$$

By Corollary 2.2 in [6] there is $y = y^* \in E$ such that $x^{**} \leq y^{**} \leq x_{n_k}^{**}$ and

$$\int_0^\tau x^* + \epsilon \leq \int_0^\tau y^*.$$

By the last inequality we get $x^* \neq y^*$ and by assumption that x is a *UKM* point it follows that $\|x\|_E < \|y\|_E \leq \|x_{n_k}\|_E$. Thus, by assumption that $\|x_n\|_E \rightarrow \|x\|_E$ we conclude a contradiction which proves condition (2). Now according to (1) we obtain $x^* = z^*$ a.e. on $[0, \alpha)$. Hence, we have

$$x_n^* \rightarrow x^* \quad \text{for a.a. } t \in [0, \alpha). \quad (3)$$

Moreover, by condition (2) and by assumption $x^{**} \leq x_{n_k}^{**}$ and by monotonicity of the maximal function x^{**} it is easy to show that $x_{n_k}^{**}$ converges to x^{**} in measure. Now according to condition (3), $x_{n_k}^*$ converges to x^* locally in measure. Therefore, for $\delta > 0$ and $s < \alpha$,

$$\mu(t \in [0, s] : |x_{n_k}^*(t) - x^*(t)| > \delta) \rightarrow 0.$$

Clearly, it is enough to consider that $\alpha = \infty$. Now we continue the proof in two cases.

Case 1. Let $x^*(\infty) = 0$. By (3) and since x_n^* is decreasing, for any $\epsilon > 0$ there is $0 < t_\epsilon < \alpha$ and $N_\epsilon \in \mathbb{N}$ such that for all $n \geq N_\epsilon$,

$$\sup_{t \geq t_\epsilon} |x_n^*(t) - x^*(t)| \leq \epsilon.$$

Consequently,

$$\mu(t \in [t_\epsilon, \alpha) : |x_n^*(t) - x^*(t)| > \epsilon) \rightarrow 0,$$

whence $x_n^* \rightarrow x^*$ in measure.

Case 2. Let $x^*(\infty) > 0$. Since x_n^* is decreasing, by (3), we claim that $x_n^*(t) \geq x^*(\infty)$ for any $t \in (0, \alpha)$ and $n \in \mathbb{N}$. Indeed, assume that there exists $(n_k) \subset \mathbb{N}$ such that $x_{n_k}^*(t_k) < x^*(\infty)$ for some $t_k \in (0, \alpha)$. Then

for every $k \in \mathbb{N}$ and $t \geq t_k$ we have $x_{n_k}^*(t) < x^*(\infty)$. Therefore, for any $k \in \mathbb{N}$ there is $\epsilon_k \in (0, 1)$ such that for any $t \geq t_k$,

$$x^*(t) \geq x^*(\infty) > x_{n_k}^*(t_k) + \epsilon_k \geq x_{n_k}^*(t) + \epsilon_k.$$

Hence, for any $s > 0$ we have

$$\begin{aligned} \int_{t_k}^{t_k+s} x^*(t) dt &\geq \int_{t_k}^{t_k+s} x^*(\infty) dt \geq \int_{t_k}^{t_k+s} (x_{n_k}^*(t) + \epsilon_k) dt \\ &= \int_{t_k}^{t_k+s} x_{n_k}^*(t) dt + s\epsilon_k \end{aligned} \quad (4)$$

for every $k \in \mathbb{N}$. Clearly, by assumption we get

$$\int_{t_k}^{t_k+s} x_{n_k}^*(t) dt + \int_0^{t_k} x_{n_k}^*(t) dt \geq \int_{t_k}^{t_k+s} x_{n_k}^*(t) dt + \int_0^{t_k} x^*(t) dt. \quad (5)$$

Observe that for any $k \in \mathbb{N}$ there is $0 < s_k < \infty$ such that for each $s \geq s_k$,

$$\int_0^{t_k} x_{n_k}^*(t) dt < s\epsilon_k + \int_0^{t_k} x^*(t) dt.$$

Consequently, by assumption and by conditions (4) and (5) for any $s \geq s_k$ it follows that

$$\begin{aligned} \int_0^{t_k+s} x^*(t) dt &\geq \int_0^{t_k} x^*(t) dt + \int_{t_k}^{t_k+s} x_{n_k}^*(t) dt + s\epsilon_k \\ &> \int_0^{t_k} x_{n_k}^*(t) dt + \int_{t_k}^{t_k+s} x_{n_k}^*(t) dt \\ &= \int_0^{t_k+s} x_{n_k}^*(t) dt \geq \int_0^{t_k+s} x^*(t) dt. \end{aligned}$$

Hence we obtain a contradiction, which provides the claim. Now define

$$z_n(t) = x_n^*(t) - x^*(\infty) \quad \text{and} \quad z(t) = x^*(t) - x^*(\infty)$$

for every $n \in \mathbb{N}$. Clearly, $z = z^*$, $z(\infty) = 0$ and $z_n = z_n^*$ for any $n \in \mathbb{N}$. Moreover, by condition (3) we have

$$z_n^*(t) = x_n^*(t) - x^*(\infty) \rightarrow x^*(t) - x^*(\infty) = z^*(t)$$

for a.a. $t \in [0, \alpha)$. Now applying case 1 we obtain $z_n^* \rightarrow z^*$ in measure, so also $x_n^* \rightarrow x^*$ in measure. Finally, compounding both cases, in view of Lemma 2.3 in [6] we complete the proof. \square

Lemma 3.3. *Let E be a symmetric space. If $x \in E^+$ is a UKM point and satisfies condition (+), then x is a UM point.*

Proof. Let $x \leq y$ and $x \neq y$ where $y \in E^+$. Since x satisfies condition (+), there is $\tau_1 = x^*(\infty) \geq 0$ such that $|x| \geq \tau_1$ and $d_{(x-\tau_1)}(\tau) < \infty$ for all $\tau > 0$. By Lemma 2.1 in [10] there exists a positive measure set B such that $x^*(t) < y^*(t)$ for any $t \in B$. Clearly, $x^* \leq y^*$ and consequently $x^{**} \leq y^{**}$. Therefore, by assumption we have $\|x\|_E < \|y\|_E$, which implies that x is a *UM* point. \square

The immediate consequence of the previous theorem is the following corollary.

Corollary 3.4. *Let E be a symmetric space. If $x \in E^+$ is a *UKM* point, then x^* is a *UM* point.*

The immediate consequence of Theorem 2.2 in [10] and Lemma 3.3 is the following result.

Corollary 3.5. *Let E be a symmetric space and $x \in E^+$ be a *UKM* point and an H_g point and satisfy condition (+), then x is a *ULUM* point.*

Lemma 3.6. *Let E be a symmetric space. If $x \in E^+$ is an *LKM* point and $\mu(t \in [0, \alpha) : 0 < x(t) \leq x^*(\infty)) = 0$, then x is an *LM* point.*

Proof. Let $y \leq x$ and $x \neq y$ where $y \in E^+$. So, there exists a positive measure set $A \subset [0, \alpha)$ such that $y(t) < x(t)$ for all $t \in A$. It is easy to see that $A \subset \text{supp}(x)$. If $x^*(\infty) = 0$, then by Lemma 3.2 in [22] we obtain $y^* \neq x^*$, by the inequality $y^{**} \leq x^{**}$, and by assumption that x is an *LKM* point we get $\|y\|_E < \|x\|_E$. Now consider $x^*(\infty) > 0$. Define $B = \{t : y(t) \geq x^*(\infty)\}$ and

$$z = \begin{cases} y\chi_{A \cap B} + x\chi_{\text{supp}(x) \setminus (A \cap B)}, & \text{if } \mu(A \cap B) > 0, \\ x^*(\infty)\chi_A + x\chi_{\text{supp}(x) \setminus A}, & \text{if } \mu(A \cap B) = 0. \end{cases}$$

Then, $z \neq x$ and $y \leq z \leq x$ and also

$$\mu(t : 0 < z(t) < x^*(\infty)) = 0.$$

Hence, by assumption $\mu(t : 0 < x(t) \leq x^*(\infty)) = 0$ and by Lemma 2.1 in [10] we obtain $z^* \neq x^*$. Consequently, by the inequality $y^{**} \leq z^{**} \leq x^{**}$ and by assumption that x is an *LKM* point we have $\|y\|_E \leq \|z\|_E < \|x\|_E$, which implies that x is an *LM* point. \square

Example 3.7. Now we show that the opposite conclusion in Lemmas 3.3 and 3.6 are not true in general. Observe that the condition (+) and the condition $\mu(t : 0 < x(t) \leq x^*(\infty)) = 0$ are necessary for $x \in E^+$ to be a *UM* point and an *LM* point, respectively (for more details see [10]). We claim that Lemmas 3.3 and 3.6 failed in the case when the condition (+) and the condition $\mu(t : 0 < x(t) \leq x^*(\infty)) = 0$ are omitted, respectively. Indeed, consider the Lorentz space $\Gamma_{p,w}$ with $W(t) = \int_0^t w$ strictly increasing. Define $x = \chi_{[1,\infty)}$. Then $x^*(\infty) = 1$ and $\mu(t : x(t) < x^*(\infty)) = 1$ as well as $\mu(t : 0 < x(t) \leq x^*(\infty)) = \infty$. By Lemmas 2.3 and 2.4 in [10] it follows that x is no *LM* point and no *UM* point in $\Gamma_{p,w}$. Moreover, by Theorem 12 in [11], the Lorentz space $\Gamma_{p,w}$ is strictly *K*-monotone, so x is an *LKM* point and a *UKM* point.

Theorem 3.8. *Let E be a symmetric space and $x, x_n \in E$ with $x^*(\infty) = 0$ and let:*

- (i) x^* be an H_g point and a *UKM* point.
- (ii) x be a *UKM* point and if $x_n^{**} \rightarrow x^{**}$ in measure and $\|x_n\|_E \rightarrow \|x\|_E$, then $\|x_n^* - x^*\|_E \rightarrow 0$.
- (iii) x be a *ULUM* point.

Then (i) \Rightarrow (ii) \Rightarrow (iii). If x^* is a point of order continuity then (iii) \Rightarrow (i).

Proof. (i) \Rightarrow (ii). Let $x, x_n \in E$ for $n \in \mathbb{N}$, $x_n^{**} \rightarrow x^{**}$ in measure and $\|x_n\|_E \rightarrow \|x\|_E$. Then, by property 11° in [26] we get that (x_n^{**}) converges to x^{**} at each point of continuity of x^{**} and so

$$\int_{\beta}^t (x_n^* - x^*)(s) ds \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for all $t > \beta > 0$. Denote $c = \sup_{n \in \mathbb{N}} \|x_n\|_E$. Since $\|x_n\|_E$ converges to $\|x\|_E$, we get for any $n \in \mathbb{N}$ and for any $t > 0$,

$$x_n^*(t)\phi(t) \leq \|x_n\|_E \leq c.$$

By Helly's selection principle, passing to subsequence and relabeling if it is necessary, we may assume x_n^* converges to a decreasing function z a.e. on $[0, \alpha)$. It is easy to notice that

$$\int_{\beta}^t (x_n^* - z^*)(s) ds \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for every $0 < \beta < t$. Consequently, for any $t > \beta > 0$ we have

$$\int_{\beta}^t x^* - z^* = 0,$$

which concludes $x^* = z^*$ a.e. on $[0, \alpha)$. Hence, we obtain that x_n^* converges to x^* a.e. on $[0, \alpha)$. By assumption that $x^*(\infty) = 0$ we get that x_n^* converges to x^* in measure on $[0, \alpha)$. Therefore, by assumption that x^* is an H_g point and $\|x_n\|_E \rightarrow \|x\|_E$ we have $\|x_n^* - x^*\|_E \rightarrow 0$.

(ii) \Rightarrow (iii). Assume $x^{**} \leq x_n^{**}$ and $\|x_n\|_E \rightarrow \|x\|_E$. Immediately, by Theorem 3.2 we have $x_n^{**} \rightarrow x^{**}$ in measure. Consequently, by condition (ii) we finish the proof.

(iii) \Rightarrow (i). Clearly, by Remark 3.1 we obtain that x^* is a UKM point. Now we show that x^* is an H_g point in E . Assume $x, x_n \in E$, $x_n \rightarrow x^*$ in measure and $\|x_n\|_E \rightarrow \|x\|_E$. Since x_n^* converges to x^* a.e. on $[0, \alpha)$ (see [26]) it follows that

$$\sup_{k \geq n} \{(x^* - x_k^*)^+\} \downarrow 0.$$

Hence, by order continuity of x^* we obtain

$$\|(x^* - x_n^*)^+\|_E \leq \left\| \sup_{k \geq n} \{(x^* - x_k^*)^+\} \right\|_E \downarrow 0. \quad (6)$$

By the inequalities

$$x_n^* \leq \max\{x_n^*, x^*\} \leq x_n^* + \sup_{k \geq n} (x^* - x_k^*)^+$$

for all $n \in \mathbb{N}$ and by condition (6) we conclude

$$\|\max\{x_n^*, x^*\}\|_E \rightarrow \|x\|_E. \quad (7)$$

Clearly, $x^* \leq \max\{x_n^*, x^*\}$ and so $x^{**} \leq (\max\{x_n^*, x^*\})^{**}$. Consequently, by condition (7) and by assumption (iii) we get

$$\|x^* - \max\{x_n^*, x^*\}\|_E \rightarrow 0.$$

Since $\|\max\{x_n^*, x^*\} - x^*\|_E = \|(x_n^* - x^*)^+\|_E$ for any $n \in \mathbb{N}$ we have

$$\|(x_n^* - x^*)^+\|_E \rightarrow 0.$$

Thus, by condition (6) we obtain

$$\|x_n^* - x^*\|_E \rightarrow 0.$$

Hence, by assumption that x_n converges to x^* in measure, by order continuity of x^* and by Theorem 2.4 in [13] we conclude

$$\|x_n - x^*\|_E \rightarrow 0,$$

which proves that x^* is an H_g point and completes the proof. \square

4. Points of upper K -monotonicity in Lorentz space $\Gamma_{p,w}$

Proposition 4.1. *Let $1 \leq p < \infty$, w be a weight function such that $w = 0$ on (a, b) and W is strictly increasing on $[0, \alpha) \setminus (a, b)$. If an element $x \in \Gamma_{p,w}$ is a UKM point, then*

$$\mu(t \in (a, b) : x^*(t) = x^*(a^-) \text{ or } x^*(t) = x^*(b)) = b - a.$$

Proof. Let $x \in \Gamma_{p,w}$ be a UKM point. Suppose for a contrary that

$$\mu(t \in (a, b) : x^*(t) = x^*(a^-) \text{ or } x^*(t) = x^*(b)) < b - a.$$

Then there exists $t \in (a, b)$ such that $x^*(b) < x^*(t) < x^*(a^-)$. Now we continue the proof in two cases.

Case 1. Consider $\gamma, \beta \in (a, b)$ such that $x^*(b) \leq x^*(\beta) < x^*(\gamma) \leq x^*(a^-)$. By the right-continuity of the decreasing rearrangement x^* there is $\xi \in (\gamma, \beta)$ such that $x^*(\beta) < x^*(\xi) \leq x^*(\gamma)$. Now we discuss case 1 in two parts.

Part 1. Assume that $x^*(\beta) < x^*(\xi) < x^*(\gamma)$. Denote

$$L = \int_{\gamma}^{\xi} (x^*(\gamma) - x^*), \quad P = \int_{\xi}^{\beta} (x^* - x^*(\beta)).$$

By right-continuity and monotonicity of x^* and since $\xi \in (\gamma, \beta)$ we may assume without loss of generality that $L > 0$ and $P > 0$. If $P \geq L$ then, by monotonicity and continuity of the maximal function x^{**} , we are able to find $\eta \in [\xi, \beta)$ such that

$$L = \int_{\eta}^{\beta} (x^* - x^*(\beta)). \quad (8)$$

Define

$$y = x^* \chi_{[0,\gamma] \cup [\xi,\eta] \cup [\beta,\alpha]} + x^*(\gamma) \chi_{(\gamma,\xi)} + x^*(\beta) \chi_{[\eta,\beta]}.$$

First observe that $y = y^*$ and $x^* \neq y^*$. We claim that $x^{**} \leq y^{**}$ and $\|x\|_{\Gamma_{p,w}} = \|y\|_{\Gamma_{p,w}}$. Notice that if $t \in (0, \gamma]$, then by the definition of y we get $x^{**}(t) = y^{**}(t)$. If $t \in (\gamma, \xi]$, then by monotonicity of x^* we obtain

$$\begin{aligned} y^{**}(t) &= \frac{1}{t} \int_0^t (x^* \chi_{[0,\gamma] \cup [\xi,\eta] \cup [\beta,\alpha]} + x^*(\gamma) \chi_{(\gamma,\xi)} + x^*(\beta) \chi_{[\eta,\beta]}) \\ &= \frac{1}{t} \int_0^t (x^* \chi_{[0,\gamma]} + x^*(\gamma) \chi_{(\gamma,\xi)}) = \frac{1}{t} \int_0^t x^* + \frac{1}{t} \int_0^t (x^*(\gamma) - x^*) \chi_{(\gamma,\xi)} \\ &\geq x^{**}(t). \end{aligned}$$

In the case when $t \in (\xi, \eta]$, $x^*(t) = y^*(t)$. In view of the previous inequality, it follows that $y^{**}(t) \geq x^{**}(t)$ for any $t \in (\xi, \eta]$. If $t \in (\eta, \beta]$, then by the definition of L and by (8) we have

$$\begin{aligned} y^{**}(t) &= \frac{1}{t} \int_0^t (x^* \chi_{[0,\gamma] \cup [\xi,\eta]} + x^*(\gamma) \chi_{(\gamma,\xi)} + x^*(\beta) \chi_{[\eta,\beta]}) \\ &= \frac{1}{t} \int_0^t x^* + \frac{1}{t} \int_0^t (x^*(\gamma) - x^*) \chi_{(\gamma,\xi)} + \frac{1}{t} \int_0^t (x^*(\beta) - x^*) \chi_{[\eta,\beta]} \\ &= x^{**}(t) + \frac{L}{t} - \frac{1}{t} \int_\eta^t (x^* - x^*(\beta)) \geq x^{**}(t). \end{aligned}$$

Hence, since $x^* = y^*$ on (β, α) , it is easy to see that $x^{**}(t) = y^{**}(t)$ for any $t \in (\beta, \alpha)$. Consequently, by assumption $w = 0$ on (a, b) and since $[\gamma, \beta] \subset (a, b)$ we conclude

$$\|y\|_{\Gamma_{p,w}}^p = \int_0^a y^{**p} w + \int_b^\alpha y^{**p} w = \int_0^a x^{**p} w + \int_b^\alpha x^{**p} w = \|x\|_{\Gamma_{p,w}}^p.$$

Now considering $P < L$ we may choose $\eta \in (\gamma, \xi)$ such that

$$P = \int_\eta^\xi (x^*(\eta) - x^*). \quad (9)$$

Next assuming

$$y = x^* \chi_{[0,\eta] \cup [\beta,\alpha]} + x^*(\eta) \chi_{(\eta,\xi)} + x^*(\beta) \chi_{[\xi,\beta]},$$

it is easy to show that $x^* \neq y^*$ and $\|y\|_{\Gamma_{p,w}} = \|x\|_{\Gamma_{p,w}}$. Now we prove that $y^{**} \geq x^{**}$. By the definition of y we have $y^{**}(t) = x^{**}(t)$ for any $t \in (0, \eta]$. In the case when $t \in (\eta, \xi]$, then by monotonicity of x^* it follows that

$$y^{**}(t) = \frac{1}{t} \int_0^t (x^* \chi_{[0,\eta]} + x^*(\eta) \chi_{(\eta,\xi)}) = x^{**}(t) + \frac{1}{t} \int_0^t (x^*(\eta) - x^*) \chi_{(\eta,\xi)} \geq x^{**}(t).$$

If $t \in (\xi, \beta]$, then by assumption $P > 0$ and by (9) we conclude

$$\begin{aligned} y^{**}(t) &= \frac{1}{t} \int_0^t (x^* \chi_{[0,\eta]} + x^*(\eta) \chi_{(\eta,\xi)} + x^*(\beta) \chi_{[\xi,\beta]}) \\ &= x^{**}(t) + \frac{1}{t} \int_0^t (x^*(\eta) - x^*) \chi_{(\eta,\xi)} + \frac{1}{t} \int_0^t (x^*(\beta) - x^*) \chi_{[\xi,\beta]} \\ &= x^{**}(t) + \frac{P}{t} - \frac{1}{t} \int_0^t (x^* - x^*(\beta)) \chi_{[\xi,\beta]} \geq x^{**}(t). \end{aligned}$$

Using the above inequality, since $x^* = y^*$ on (β, α) , it is easy to notice that $y^{**}(t) = x^{**}(t)$ for any $t \in (\beta, \alpha)$. Finally, combining the cases when $P \geq L$ and $P < L$, by assumption that x is a *UKM* point, we obtain a contradiction.

Part 2. Now we may assume that for any $s \in (a, b)$ we have either $x^*(s) = x^*(\gamma)$ or $x^*(s) = x^*(\beta)$, otherwise we proceed as in part 1. Denote $\eta = \inf\{s \in (\gamma, \beta) : x^*(s) = x^*(\beta)\}$. Then $x^*(\gamma) = x^*(\eta^-) > x^*(\eta) = x^*(\beta)$. If $x^*(a^-) > x^*(a) = x^*(\gamma)$, then we define $\delta = (\eta + a)/2$ and $\epsilon = \min\{x^*(a^-) - x^*(a), x^*(\eta^-) - x^*(\eta)\}$ and also

$$y = x^* + \epsilon \chi_{[a,\delta]} - \epsilon \chi_{[\delta,\eta]}.$$

Otherwise, if $x^*(\beta) = x^*(b^-) > x^*(b)$, then we take $\delta = (\eta + b)/2$, $\epsilon = \min\{x^*(b^-) - x^*(b), x^*(\eta^-) - x^*(\eta)\}$ and

$$y = x^* + \epsilon \chi_{[\eta,\delta]} - \epsilon \chi_{[\delta,b]}.$$

Clearly, $x^* \neq y^*$, $y^* = y$. Since $w = 0$ on (a, b) , by the definition of y it follows that $\|y\|_{\Gamma_{p,w}} = \|x\|_{\Gamma_{p,w}}$. Furthermore, it is easy to see that $x^{**} \leq y^{**}$. Finally, by assumption that x is a *UKM* point, we get a contradiction.

Case 2. Now assume that for all $t \in (a, b)$ we have $x^*(t) = c$ where $x^*(b) < c < x^*(a^-)$. Let $\epsilon = \frac{1}{2} \min\{x^*(a^-) - c, c - x^*(b)\}$ and $\delta = \frac{a+b}{2}$. Define

$$y = x^* \chi_{[0,a) \cup [b,\alpha)} + (c + \epsilon) \chi_{[a,\delta]} + (c - \epsilon) \chi_{[\delta,b]}.$$

Notice that $y = y^* \neq x^*$. We claim that $y^{**} \geq x^{**}$. Clearly, taking $t \in (0, a]$, by the definition of y , we obtain $x^{**}(t) = y^{**}(t)$. Next, for any $t \in (a, \delta]$ we get

$$\begin{aligned} y^{**}(t) &= \frac{1}{t} \int_0^t (x^* \chi_{[0,a)} + (c + \epsilon) \chi_{[a,\delta]}) = x^{**}(t) + \frac{1}{t} \int_0^t (c + \epsilon - x^*) \chi_{[a,\delta]} \\ &= x^{**}(t) + \frac{(t-a)\epsilon}{t} \geq x^{**}(t). \end{aligned}$$

Furthermore, if $t \in (\delta, b]$, then by the definition of δ we have

$$y^{**}(t) = \frac{1}{t} \int_0^t (x^* \chi_{[0,a) \cup [b,\alpha)} + (c + \epsilon) \chi_{[a,\delta]} + (c - \epsilon) \chi_{[\delta,b]})$$

$$\begin{aligned}
&= x^{**}(t) + \frac{1}{t} \int_0^t (c + \epsilon - x^*) \chi_{[a,\delta)} + \frac{1}{t} \int_0^t (c - \epsilon - x^*) \chi_{(\delta,b)} \\
&= x^{**}(t) + \frac{(b-t)\epsilon}{t} \geq x^{**}(t).
\end{aligned}$$

Thus, we can easily show that $x^{**}(t) = y^{**}(t)$ for $t \in (b, \alpha)$, which proves the claim. Therefore, by assumption $w = 0$ a.e. on (a, b) it follows that $\|y\|_{\Gamma_{p,w}} = \|x\|_{\Gamma_{p,w}}$. Finally, by assumption that x is a UKM point we conclude a contradiction. \square

Proposition 4.2. *Let $1 \leq p < \infty$, w be a weight function such that $w = 0$ on (a, b) and W is strictly increasing on $[0, \alpha) \setminus (a, b)$ and let $x \in \Gamma_{p,w}$. If the following condition*

$$\mu(t \in (a, b) : x^*(t) = x^*(a^-) \text{ or } x^*(t) = x^*(b)) = b - a$$

holds, then x is a UKM point.

Proof. Let the assumption be satisfied and let $y \in \Gamma_{p,w}$ be such that $y^* \neq x^*$ and $x^{**} \leq y^{**}$. Then, there exist $t > 0$ and $\epsilon > 0$ such that

$$\epsilon + \int_0^t x^* \leq \int_0^t y^*. \quad (10)$$

If $t \in [0, \alpha) \setminus [a, b]$ then by continuity of the maximal function there is $\delta > 0$ such that for any $s \in (t - \delta, t + \delta) \subset [0, \alpha) \setminus [a, b]$ we have

$$\frac{\epsilon}{2} + \int_0^s x^* \leq \int_0^s y^*. \quad (11)$$

Clearly, by assumption that W is strictly increasing on $[0, \alpha) \setminus (a, b)$ it follows that $\mu((c, d) \cap \text{supp}(w)) > 0$ for any $(c, d) \subset [0, \alpha) \setminus (a, b)$. Therefore, by condition (11) and by superadditivity of the power function u^p for $p \geq 1$ we get

$$\begin{aligned}
\int_{t-\delta}^{t+\delta} y^{**p}(s) w(s) ds &\geq \int_{t-\delta}^{t+\delta} \left(x^{**}(s) + \frac{\epsilon}{2s}\right)^p w(s) ds \\
&\geq \int_{t-\delta}^{t+\delta} x^{**p}(s) w(s) ds + \int_{t-\delta}^{t+\delta} \frac{\epsilon^p w(s)}{(2s)^p} ds \\
&> \int_{t-\delta}^{t+\delta} x^{**p}(s) w(s) ds.
\end{aligned}$$

Hence, since $x^{**} \leq y^{**}$ we conclude $\|y\|_{\Gamma_{p,w}} > \|x\|_{\Gamma_{p,w}}$. Now assume that $t = a$ or $t = b$. Then, replacing the interval $(t - \delta, t + \delta)$ by the interval $(a - \delta, a)$ for $t = a$ and by $(b, b + \delta)$ for $t = b$ we can show analogously $\|y\|_{\Gamma_{p,w}} > \|x\|_{\Gamma_{p,w}}$. Now suppose that $t \in (a, b)$. Without loss of generality we may assume

$$x^{**} = y^{**} \quad \text{on } [0, \alpha) \setminus (a, b), \quad (12)$$

which concludes that $y^* = x^*$ a.e. on $[0, a]$. We present the proof in two cases.

Case 1. Assume that x^* is constant on (a, b) . Then, since y^* is decreasing on (a, b) and $x^{**}(b) = y^{**}(b)$ and by condition (10), there exists $t_0 \in (a, b)$ such that

$$x^*(t_0) > y^*(t_0). \quad (13)$$

We claim that for any $s \in (a, b)$,

$$x^*(s) = x^*(b). \quad (14)$$

Indeed, considering the case when $x^*(s) = x^*(a^-)$ for all $s \in (a, b)$, since $x^* = y^*$ a.e. on $[0, a]$, by condition (10) we get a contradiction. Now according to conditions (13) and (14) we have for all $s \in [t_0, b]$,

$$y^*(s) \leq y^*(t_0) < x^*(t_0) = x^*(b),$$

which implies $y^*(b) < x^*(b)$. Thus, by right-continuity of the decreasing rearrangement there is $\delta > 0$ and $\epsilon_0 > 0$ such that for all $s \in [b, b + \delta]$,

$$x^*(s) > y^*(s) + \epsilon_0.$$

Hence, by condition (12) we obtain

$$\int_0^{b+\delta} y^* = \int_0^b x^* + \int_b^{b+\delta} x^* > \int_0^b y^* + \int_b^{b+\delta} (y^* + \epsilon_0) = \epsilon_0 \delta + \int_0^{b+\delta} y^*,$$

which gives a contradiction.

Case 2. Now we consider that for any $t \in (a, b)$ either $x^*(t) = x^*(a^-)$ or $x^*(t) = x^*(b)$. Denote $c = \inf\{s \in (a, b) : x^*(s) = x^*(b)\}$. Clearly, $a < c < b$. By right-continuity of the decreasing rearrangement x^* we have $x^*(c^-) > x^*(c)$. Furthermore, since $x^{**} \leq y^{**}$ on $[0, \alpha)$ and $x^* = y^*$ a.e. on $[0, a]$ we obtain $x^* = y^*$ a.e. on $[0, c)$. Since $x^* = x^*(b)$ on $[c, b)$, we may proceed similarly as in case 1 and easily get a contradiction which finishes the proof. \square

The immediate consequence of Propositions 4.1 and 4.2 is the following theorem.

Theorem 4.3. *Let $1 \leq p < \infty$, w be a weight function such that $w = 0$ on (a, b) and W is strictly increasing on $[0, \alpha) \setminus (a, b)$. The point $x \in \Gamma_{p,w}$ is a UKM point if and only if*

$$\mu(t \in (a, b) : x^*(t) = x^*(a^-) \text{ or } x^*(t) = x^*(b)) = b - a.$$

5. Local uniform rotundity in symmetric spaces

Recently, H. Hudzik, A. Kamińska and M. Mastyło in Theorem 3 (see [19]) have proved that a Banach function lattice E is locally uniformly rotund if and only if E^+ possesses the same property. The proof of this result was established, in view of Theorem 1.2 in [14], applying an equivalent norm that is local uniform rotund in the case of the finite measure. Now we present a local approach of Theorem 3 in [19] for the finite and infinite measure which requires a completely different techniques.

Theorem 5.1. *Let E be a Banach function space. An element $x \in E$ is an LUR point if and only if $|x|$ is an LUR point.*

Proof. Necessity. Let $(x_n) \subset E$ and

$$\|x_n + |x|\|_E \rightarrow 2\|x\|_E \quad \text{and} \quad \|x_n\|_E \rightarrow \|x\|_E. \quad (15)$$

Define $y_n = \text{sign}(x)x_n + x_n\chi_{\text{supp}(x)^c}$ for any $n \in \mathbb{N}$. Notice that $|y_n| = |x_n|$ for every $n \in \mathbb{N}$. Therefore,

$$\begin{aligned} \| |x| + x_n \|_E &= \| \text{sign}(x)x + x_n \|_E = \| \text{sign}(x)x + \text{sign}(x)y_n + x_n\chi_{\text{supp}(x)^c} \|_E \\ &= \| \text{sign}(x)(x + y_n) + y_n\chi_{\text{supp}(x)^c} \|_E \end{aligned}$$

for any $n \in \mathbb{N}$. Hence, since

$$| \text{sign}(x)(x + y_n) + y_n\chi_{\text{supp}(x)^c} | = | (x + y_n)\chi_{\text{supp}(x)} + y_n\chi_{\text{supp}(x)^c} |$$

it follows that

$$\| |x| + x_n \|_E = \| (x + y_n)\chi_{\text{supp}(x)} + y_n\chi_{\text{supp}(x)^c} \|_E = \| x + y_n \|_E$$

for each $n \in \mathbb{N}$. Consequently, by condition (15) we obtain

$$\|x + y_n\|_E \rightarrow 2\|x\|_E \quad \text{and} \quad \|y_n\|_E \rightarrow \|x\|_E.$$

Thus, by assumption that x is an *LUR* point we get

$$\|x - y_n\|_E \rightarrow 0. \quad (16)$$

Moreover, for any $n \in \mathbb{N}$,

$$\begin{aligned} \|x - y_n\|_E &= \| \text{sign}(x)|x| - y_n\chi_{\text{supp}(x)} - y_n\chi_{\text{supp}(x)^c} \|_E \\ &= \| \text{sign}(x)|x| - y_n\chi_{\text{supp}(x)} - x_n\chi_{\text{supp}(x)^c} \|_E \\ &= \| \text{sign}(x)|x| - \text{sign}(x)x_n - x_n\chi_{\text{supp}(x)^c} \|_E \\ &= \| \text{sign}(x)(|x| - x_n) - x_n\chi_{\text{supp}(x)^c} \|_E. \end{aligned}$$

Now observe that

$$| \text{sign}(x)(|x| - x_n) - x_n\chi_{\text{supp}(x)^c} | = | (|x| - x_n)\chi_{\text{supp}(x)} - x_n\chi_{\text{supp}(x)^c} |$$

and so

$$\begin{aligned} \|x - y_n\|_E &= \| \text{sign}(x)(|x| - x_n) - x_n\chi_{\text{supp}(x)^c} \|_E \\ &= \| (|x| - x_n)\chi_{\text{supp}(x)} - x_n\chi_{\text{supp}(x)^c} \|_E = \| |x| - x_n \|_E \end{aligned}$$

which implies the end of the proof of necessity, in view of condition (16).

Sufficiency. Let $(x_n) \subset E$ and

$$\|x_n + x\|_E \rightarrow 2\|x\|_E \quad \text{and} \quad \|x_n\|_E \rightarrow \|x\|_E. \quad (17)$$

Notice,

$$\|x_n + x\|_E \leq \| |x_n| + |x| \|_E \leq \|x_n\|_E + \|x\|_E.$$

Hence, by condition (17) we get $\| |x| + |x_n| \|_E \rightarrow 2\|x\|_E$ and by assumption that $|x|$ is an *LUR* point in E we conclude

$$\| |x_n| - |x| \|_E \rightarrow 0. \quad (18)$$

Define for any $n \in \mathbb{N}$,

$$\begin{aligned} A_1^{(n)} &= \{t : x(t) \geq 0 \text{ and } x_n(t) \geq 0\}, & A_2^{(n)} &= \{t : x(t) < 0 \text{ and } x_n(t) < 0\}, \\ A_3^{(n)} &= \{t : x(t) \geq 0 \text{ and } x_n(t) < 0\}, & A_4^{(n)} &= \{t : x(t) < 0 \text{ and } x_n(t) \geq 0\}. \end{aligned}$$

We have

$$\begin{aligned} 0 &\leq \max \left\{ \left\| (x - x_n) \chi_{A_1^{(n)} \cup A_2^{(n)}} \right\|_E, \left\| (x + x_n) \chi_{A_3^{(n)} \cup A_4^{(n)}} \right\|_E \right\} \\ &\leq \left\| (x - x_n) \chi_{A_1^{(n)} \cup A_2^{(n)}} + (x + x_n) \chi_{A_3^{(n)} \cup A_4^{(n)}} \right\|_E \\ &= \| |x| - |x_n| \|_E. \end{aligned}$$

Hence, by condition (18) we obtain

$$\left\| (x - x_n) \chi_{A_1^{(n)} \cup A_2^{(n)}} \right\|_E \rightarrow 0, \quad \text{and} \quad \left\| (x + x_n) \chi_{A_3^{(n)} \cup A_4^{(n)}} \right\|_E \rightarrow 0. \quad (19)$$

Furthermore,

$$\begin{aligned} \|x + x_n\|_E &\leq \left\| (x + x_n) \chi_{A_1^{(n)} \cup A_2^{(n)}} \right\|_E + \left\| (x + x_n) \chi_{A_3^{(n)} \cup A_4^{(n)}} \right\|_E \\ &\leq \|x\|_E + \|x_n\|_E + \left\| (x + x_n) \chi_{A_3^{(n)} \cup A_4^{(n)}} \right\|_E. \end{aligned}$$

Thus, by conditions (17) and (19) it follows that

$$\left\| (x + x_n) \chi_{A_1^{(n)} \cup A_2^{(n)}} \right\|_E \rightarrow 2\|x\|_E. \quad (20)$$

Moreover, by (19) and by the triangle inequality for a norm we have

$$\left\| x \chi_{A_1^{(n)} \cup A_2^{(n)}} \right\|_E - \left\| x \chi_{A_1^{(n)} \cup A_2^{(n)}} \right\|_E \rightarrow 0. \quad (21)$$

Since,

$$\left\| (x + x_n) \chi_{A_1^{(n)} \cup A_2^{(n)}} \right\|_E \leq \left\| x \chi_{A_1^{(n)} \cup A_2^{(n)}} \right\|_E + \left\| x_n \chi_{A_1^{(n)} \cup A_2^{(n)}} \right\|_E \leq \|x\|_E + \|x_n\|_E,$$

according to conditions (17) and (20) we get

$$\left\| x \chi_{A_1^{(n)} \cup A_2^{(n)}} \right\|_E + \left\| x_n \chi_{A_1^{(n)} \cup A_2^{(n)}} \right\|_E \rightarrow 2\|x\|_E.$$

In consequence, by condition (21) it follows that

$$\left\| x \chi_{A_1^{(n)} \cup A_2^{(n)}} \right\|_E \rightarrow \|x\|_E \quad \text{and} \quad \left\| x_n \chi_{A_1^{(n)} \cup A_2^{(n)}} \right\|_E \rightarrow \|x\|_E.$$

Moreover, observe that

$$2 \left\| x \chi_{A_1^{(n)} \cup A_2^{(n)}} \right\|_E \leq \left\| |x| + |x| \chi_{A_1^{(n)} \cup A_2^{(n)}} \right\|_E \leq \|x\|_E + \left\| x \chi_{A_1^{(n)} \cup A_2^{(n)}} \right\|_E \leq 2 \|x\|_E.$$

Hence,

$$\left\| |x| + |x| \chi_{A_1^{(n)} \cup A_2^{(n)}} \right\|_E \rightarrow 2 \|x\|_E \quad \text{and} \quad \left\| x \chi_{A_1^{(n)} \cup A_2^{(n)}} \right\|_E \rightarrow \|x\|_E.$$

Since $|x|$ is an *LUR* point in E we obtain

$$\left\| x \chi_{A_3^{(n)} \cup A_4^{(n)}} \right\|_E = \left\| |x| - |x| \chi_{A_1^{(n)} \cup A_2^{(n)}} \right\|_E \rightarrow 0.$$

Furthermore, by the triangle inequality we have

$$\left| \left\| x \chi_{A_3^{(n)} \cup A_4^{(n)}} \right\|_E - \left\| x_n \chi_{A_3^{(n)} \cup A_4^{(n)}} \right\|_E \right| \leq \left\| (x + x_n) \chi_{A_3^{(n)} \cup A_4^{(n)}} \right\|_E.$$

Now applying condition (19) we get

$$\left\| x \chi_{A_3^{(n)} \cup A_4^{(n)}} \right\|_E \rightarrow 0 \quad \text{and} \quad \left\| x_n \chi_{A_3^{(n)} \cup A_4^{(n)}} \right\|_E \rightarrow 0.$$

Finally, by condition (19) we finish the proof in view of the following inequality,

$$\|x - x_n\|_E \leq \left\| (x - x_n) \chi_{A_1^{(n)} \cup A_2^{(n)}} \right\|_E + \left\| x \chi_{A_3^{(n)} \cup A_4^{(n)}} \right\|_E + \left\| x_n \chi_{A_3^{(n)} \cup A_4^{(n)}} \right\|_E. \quad \square$$

Theorem 5.2. *Let E be a symmetric space and let $x \in E^+$ be an *LUR* point, then x^* is an *LUR* point.*

Proof. Let $(x_n) \subset E$ and

$$\|x_n + x^*\|_E \rightarrow 2 \|x^*\|_E \quad \text{and} \quad \|x_n\|_E \rightarrow \|x^*\|_E.$$

Since x is an *LUR* point, by Remark 2.1 it follows that x is a point of order continuity. Hence, by Lemma 2.6 in [10] (see also Proposition 2.3 in [13]) we obtain that x^* is a point of order continuity and so by Lemma 2.5 in [10] we have $x^*(\infty) = 0$. Moreover, by Ryff's theorem [1] there exists a measure preserving transformation $\sigma : \text{supp}(x) \rightarrow \text{supp}(x^*)$ such that $x^* \circ \sigma = x$ a.e. on $\text{supp}(x)$. Now we consider two cases.

Case 1. Assume that $\mu(\text{supp}(x^*)) < \infty$. Let β be a measure preserving transformation from $\text{supp}(x)^c$ to $[\mu(\text{supp}(x)), \infty)$ (for more details of construction see [31]). Define

$$\gamma = \begin{cases} \sigma & \text{on } \text{supp}(x), \\ \beta & \text{on } \text{supp}(x)^c, \end{cases} \quad y_n = x_n \circ \gamma = \begin{cases} x_n \circ \sigma & \text{on } \text{supp}(x), \\ x_n \circ \beta & \text{on } \text{supp}(x)^c. \end{cases}$$

Clearly, y_n and x_n are equimeasurable. Therefore,

$$\|y_n\|_E = \|x_n\|_E \rightarrow \|x^*\|_E = \|x\|_E$$

and

$$\|y_n + x\|_E = \|x_n \circ \gamma + x^* \circ \gamma\|_E = \|x_n + x^*\|_E \rightarrow 2 \|x\|_E.$$

Since x is an *LUR* point we obtain that

$$\|x_n - x^*\|_E = \|y_n - x\|_E \rightarrow 0, \quad (22)$$

which implies that x^* is an *LUR* point.

Case 2. Assume that $\mu(\text{supp}(x^*)) = \infty$. Now we proceed analogously as for case 1 replacing γ by σ . Indeed, taking $y_n = x_n \circ \sigma$ we get

$$\|y_n + x\|_E = \|x_n \circ \sigma + x^* \circ \sigma\|_E = \|x_n + x^*\|_E \rightarrow 2\|x\|_E$$

and

$$\|y_n\|_E = \|x_n \circ \sigma\|_E = \|x_n\|_E \rightarrow \|x\|_E.$$

Hence, by assumption that x is an *LUR* point it follows that x^* is an *LUR* point. \square

A. Medzhitov and P. Sukochev proved in Theorem 1 of [30] that $E \in (LUR^*)$ implies $E \in (LUR)$ for a symmetric space E under some additional assumptions. We will show that the above conclusion holds without any additional conditions. First we present the result which is a localization of Theorem 1 in [30].

Theorem 5.3. *Let E be a symmetric space such that $y^*(\infty) = 0$ for any $y \in E$ and let $x \in E^+$. If x^* is an *LUR* point, then x is an *LUR* point.*

Proof. Let $(x_n) \subset E$, $\|x_n\|_E \rightarrow \|x\|_E$ and $\|x_n + x\|_E \rightarrow 2\|x\|_E$. It is easy to notice that

$$\|x_n^*\|_E \rightarrow \|x^*\|_E. \quad (23)$$

Moreover, for any $n \in \mathbb{N}$,

$$x_n + x \prec x_n^* + x^*,$$

whence for every $n \in \mathbb{N}$,

$$\|x_n + x\|_E \leq \|x_n^* + x^*\|_E \leq \|x_n^*\|_E + \|x^*\|_E.$$

Therefore,

$$\|x_n^* + x^*\|_E \rightarrow 2\|x^*\|_E. \quad (24)$$

Hence, by assumption that x^* is an *LUR* point we obtain

$$\|x_n^* - x^*\|_E \rightarrow 0. \quad (25)$$

Moreover, by Remark 2.1 it follows that x^* is a point of order continuity and by Lemma 2.6 in [10] (or Proposition 2.3 in [13]) we conclude that x is a point of order continuity. Now, we claim that x_n converges to x in measure. First notice that

$$x_n + x \prec \frac{(x_n + x)^*}{2} + \frac{x_n^* + x^*}{2} \prec x_n^* + x^*$$

for any $n \in \mathbb{N}$. Thus, by condition (24) we have

$$\left\| \frac{(x_n + x)^*}{2} + \frac{x_n^* + x^*}{2} \right\|_E \rightarrow 2\|x^*\|_E. \quad (26)$$

Furthermore, by the triangle inequality we get

$$\begin{aligned} \left\| \frac{(x_n + x)^*}{2} + \frac{x_n^* + x^*}{2} \right\|_E &\leq \frac{1}{2} \|(x_n + x)^* + 2x^*\|_E + \frac{1}{2} \|x_n^* - x^*\|_E \\ &\leq \frac{1}{2} \|x_n + x\|_E + \frac{1}{2} \|x_n^* - x^*\|_E + \|x^*\|_E \end{aligned}$$

for any $n \in \mathbb{N}$. Consequently, by the assumption and by conditions (25) and (26) we obtain

$$\left\| \frac{(x_n + x)^*}{2} + x^* \right\|_E \rightarrow 2 \|x^*\|_E.$$

Hence, by assumption $\frac{1}{2} \|x_n + x\|_E \rightarrow \|x\|_E$ and by the fact that x^* is an *LUR* point we have

$$\left\| \frac{(x_n + x)^*}{2} - x^* \right\|_E \rightarrow 0.$$

Therefore, by Lemma 2.2 in [13] and by condition (25) we prove the claim. Finally, by Proposition 2.4 in [13] and by condition (25) it follows that $\|x_n - x\|_E \rightarrow 0$ and completes the proof. \square

Theorem 5.4. *Let E be a symmetric space. Then E is *LUR* if and only if E is *LUR*^{*}.*

Proof. Immediately, by Theorems 5.1 and 5.2 we complete the proof of the necessity. Now assume that E is *LUR*^{*}. Then x^* is an *LUR* point in E for every $x \in E$ and by Lemma 2.5 in [10] we obtain $x^*(\infty) = 0$. Consequently, Theorems 5.1 and 5.3 imply E is *LUR*. \square

Now we investigate under which condition a point of local uniform rotundity of a symmetric space E might be considered equivalently on E^* , i.e. on the cone of all decreasing rearrangements of elements in a symmetric space E . Let us notice that at the beginning of the proof of Theorem 5.5 we use some similar techniques as in the proof of Theorem 2.1 in [7].

Theorem 5.5. *Let E be a symmetric space and let $(x_n) \subset E$, $x \in E$. Then the following conditions:*

- (i) $\|x_n\|_E \rightarrow \|x\|_E$ and $\|x_n + x^*\|_E \rightarrow 2\|x\|_E \Rightarrow \|x_n - x^*\|_E \rightarrow 0$,
- (ii) $\|x_n\|_E \rightarrow \|x\|_E$ and $\|x_n^* + x^*\|_E \rightarrow 2\|x\|_E \Rightarrow \|x_n^* - x^*\|_E \rightarrow 0$,

satisfy the implication (i) \Rightarrow (ii). If E is order continuous then (ii) \Rightarrow (i).

Proof. It is obvious that the conclusion (i) \Rightarrow (ii) is fulfilled immediately. Now we prove (ii) \Rightarrow (i). Let $x_n, x \in E$ for all $n \in \mathbb{N}$ and

$$\|x_n\|_E \rightarrow \|x\|_E \quad \text{and} \quad \|x_n + x^*\|_E \rightarrow 2\|x\|_E. \quad (27)$$

Notice that for every $n \in \mathbb{N}$ we have

$$x_n + x^* \prec x_n^* + x^*$$

which concludes

$$\|x_n + x^*\|_E \leq \|x_n^* + x^*\|_E \leq \|x_n\|_E + \|x\|_E$$

for any $n \in \mathbb{N}$. Consequently, by condition (27) we have

$$\|x_n^* + x^*\|_E \rightarrow 2\|x\|_E \quad \text{and} \quad \|x_n\|_E \rightarrow \|x\|_E.$$

Hence, by (ii) we obtain

$$\|x_n^* - x^*\|_E \rightarrow 0. \quad (28)$$

Moreover, observe that

$$x_n + x^* \prec \frac{(x_n + x^*)^*}{2} + \frac{x_n^* + x^*}{2}$$

for each $n \in \mathbb{N}$. Thus, for all $n \in \mathbb{N}$ we get

$$\|x_n + x^*\|_E \leq \left\| \frac{(x_n + x^*)^*}{2} + \frac{x_n^* + x^*}{2} \right\|_E \leq \|x_n\|_E + \|x\|_E$$

which implies

$$\left\| \frac{(x_n + x^*)^*}{2} + \frac{x_n^* + x^*}{2} \right\|_E \rightarrow 2\|x\|_E.$$

So we have

$$\left\| \frac{(x_n + x^*)^*}{2} + \frac{x_n^* - x^*}{2} + x^* \right\|_E \rightarrow 2\|x\|_E. \quad (29)$$

Moreover, by the triangle inequality we get

$$\begin{aligned} \left\| \frac{(x_n + x^*)^*}{2} + \frac{x_n^* - x^*}{2} + x^* \right\|_E &\leq \left\| \frac{(x_n + x^*)^*}{2} + x^* \right\|_E + \frac{1}{2}\|x_n^* - x^*\|_E \\ &\leq \frac{1}{2}\|x_n\|_E + \frac{3}{2}\|x\|_E + \frac{1}{2}\|x_n^* - x^*\|_E \end{aligned}$$

for all $n \in \mathbb{N}$. Consequently, by (27), (28) and (29) we have

$$\left\| \frac{(x_n + x^*)^*}{2} + x^* \right\|_E \rightarrow 2\|x\|_E.$$

Clearly, by (27) we obtain

$$\left\| \frac{(x_n + x^*)^*}{2} \right\|_E \rightarrow \|x\|_E,$$

whence by condition (ii) we conclude

$$\left\| \frac{(x_n + x^*)^*}{2} - x^* \right\|_E \rightarrow 0. \quad (30)$$

Since E is order continuous, by Theorem 4.8 in [16] (see also Corollary 4.10 in [16] in the case when (\mathcal{M}, τ) is $L_\infty(0, \infty)$ with the trace given by the Lebesgue measure) it follows that E has an equivalent symmetric locally uniformly rotund norm $\|\cdot\|_E$. Thus, by (30) and by the triangle inequality we get

$$\left| \frac{1}{2} \| \| (x_n + x^*)^* \| \|_E - \| \| x^* \| \|_E \right| \leq \left\| \left\| \frac{(x_n + x^*)^*}{2} - x^* \right\| \right\|_E \rightarrow 0,$$

which implies

$$\| \| x_n + x^* \| \|_E \rightarrow 2 \| \| x \| \|_E. \quad (31)$$

Furthermore, by condition (28) and by the inequality

$$| \| \| x_n \| \|_E - \| \| x \| \|_E | \leq \| \| x_n^* - x^* \| \|_E$$

it follows that

$$\| \| x_n \| \|_E \rightarrow \| \| x \| \|_E.$$

Hence, by (31) and by local uniform convexity of the norm $\| \| \cdot \| \|_E$ we obtain

$$\| \| x_n - x^* \| \|_E \rightarrow 0.$$

Finally, by an equivalence of norms $\| \cdot \|_E$ and $\| \| \cdot \| \|_E$ we complete the proof. \square

Lemma 5.6. *Let E be a symmetric space. If $x \in E$ is an LUR point in E then x is a lower (an upper) locally uniformly K -monotone point.*

Proof. Let $x \in E$ be an LUR point. Assume $x_n, y_n \in E$ with $y_n^* \prec x^* \prec x_n^*$ for all $n \in \mathbb{N}$ and

$$\| x_n \|_E \rightarrow \| x \|_E \quad \text{and} \quad \| y_n \|_E \rightarrow \| x \|_E. \quad (32)$$

Observe that

$$x^* \prec \frac{x^* + x_n^*}{2} \prec x_n^* \quad \text{and} \quad y_n^* \prec \frac{x^* + y_n^*}{2} \prec x^*$$

for any $n \in \mathbb{N}$. Therefore, since E is a symmetric space we obtain

$$\| x^* \|_E \leq \frac{\| x^* + x_n^* \|_E}{2} \leq \| x_n^* \|_E \quad \text{and} \quad \| y_n^* \|_E \leq \frac{\| x^* + y_n^* \|_E}{2} \leq \| x^* \|_E$$

for every $n \in \mathbb{N}$. Hence,

$$\| x_n^* + x^* \|_E \rightarrow 2 \| x \|_E \quad \text{and} \quad \| y_n^* + x^* \|_E \rightarrow 2 \| x \|_E.$$

Thus, in view of Theorem 5.2, by condition (32) and by assumption that x is an LUR point we conclude

$$\| x_n^* - x^* \|_E \rightarrow 0 \quad \text{and} \quad \| y_n^* - x^* \|_E \rightarrow 0. \quad \square$$

6. Local uniform rotundity of Lorentz spaces $\Gamma_{p,w}$

Theorem 6.1. *The following conditions are equivalent.*

- (i) $1 < p < \infty$ and $\mu((a, b) \cap \text{supp}(w)) > 0$ for any interval $(a, b) \subset (0, \alpha)$ with $a < b$, $W(\infty) = \infty$ whenever $\alpha = \infty$.

- (ii) $\Gamma_{p,w}$ is locally uniformly rotund.
- (iii) $\Gamma_{p,w}$ is midpoint locally uniformly rotund.
- (iv) $\Gamma_{p,w}$ is rotund.

Proof. Immediately, by the definition of rotundity properties we get (ii) \Rightarrow (iii) \Rightarrow (iv). Moreover, by Theorem 2.3 in [8] we have (iv) \Rightarrow (i). Now we prove (i) \Rightarrow (ii). Let (i) be satisfied and let $x \in S_{\Gamma_{p,w}}$ and $(x_n) \subset B_{\Gamma_{p,w}}$ satisfy $\|x_n\|_{\Gamma_{p,w}} \rightarrow 1$ and $\|x_n + x\|_{\Gamma_{p,w}} \rightarrow 2$. By assumption $W(\infty) = \infty$ and by Proposition 1.4 in [23] it follows that $\Gamma_{p,w}$ is order continuous, whence and by Theorems 5.1, 5.3 and 5.5 we may assume that $x = x^*$ and $x_n = x_n^*$ for any $n \in \mathbb{N}$. Let $\epsilon > 0$, $\delta > 0$. Clearly, for all $n \in \mathbb{N}$,

$$x_n^{**}(t)\phi_{\Gamma}(t) \leq \|x_n\|_{\Gamma_{p,w}} \leq 1,$$

where ϕ_{Γ} is the fundamental function of $\Gamma_{p,w}$ (see [1]). By the above inequality there is $t_1 > 0$ such that for any $t \geq t_1$ and $n \in \mathbb{N}$ we have

$$\max\{x_n^{**}(t), x^{**}(t)\} \leq \epsilon.$$

Let $A = [0, a) \subset [0, \alpha)$ with $t_1 \leq a < \infty$. Define for any $n \in \mathbb{N}$,

$$A_n = \{t \in A : |x_n^{**}(t) - x^{**}(t)| \geq \delta\}, \quad F_n = \left\{t \in A : x_n^{**}(t) > \left(\frac{2}{W(\epsilon/4)}\right)^{1/p}\right\}$$

and

$$G = \left\{t \in A : x^{**}(t) > \left(\frac{2}{W(\epsilon/4)}\right)^{1/p}\right\}.$$

Denote $B_n = F_n \cup G$ for every $n \in \mathbb{N}$. Then, we have

$$\begin{aligned} 2 &\geq \|x_n\|_{\Gamma_{p,w}}^p + \|x\|_{\Gamma_{p,w}}^p \geq \int_0^{\mu(F_n)} x_n^{**p}(t)w(t)dt + \int_0^{\mu(G)} x^{**p}(t)w(t)dt \\ &\geq \int_0^{\mu(F_n)} \frac{2}{W(\epsilon/4)}w(t)dt + \int_0^{\mu(G)} \frac{2}{W(\epsilon/4)}w(t)dt \\ &= \frac{2W(\mu(F_n)) + 2W(\mu(G))}{W(\epsilon/4)} \end{aligned}$$

for any $n \in \mathbb{N}$. Hence, $W(\epsilon/4) \geq W(\mu(F_n)) + W(\mu(G))$ which concludes

$$\mu(B_n) \leq \frac{\epsilon}{2}, \tag{33}$$

for every $n \in \mathbb{N}$. Denote $C_k = \{t \in A : w(t) > 1/k\}$. Clearly, $C_k \subset C_{k+1}$ and $A = \bigcup_{k=1}^{\infty} C_k$. Consequently, $\mu(A \setminus C_k) \rightarrow 0$ as $k \rightarrow \infty$ and so

$$\mu((A_n \setminus B_n) \setminus C_k) < \frac{\epsilon}{4} \tag{34}$$

for some $k \in \mathbb{N}$ which does not depend on n . By uniform convexity of a power function u^p for $p > 1$ there exists $\xi \in (0, 1)$ such that for any $u, v \in [0, \infty)$, $|u - v| \geq \delta(W(\frac{\epsilon}{4})/2)^{1/p} \max\{u, v\}$ we have

$$\left(\frac{u+v}{2}\right)^p \leq \frac{1-\xi}{2} (u^p + v^p).$$

Therefore,

$$\left(\frac{x_n^{**}(t) + x^{**}(t)}{2}\right)^p \leq \frac{1-\xi}{2} (x_n^{**p}(t) + x^{**p}(t))$$

for any $t \in A_n \setminus B_n$ and $n \in \mathbb{N}$. Moreover, for every $t > 0$ and $n \in \mathbb{N}$ we get

$$\left(\frac{x_n^{**}(t) + x^{**}(t)}{2}\right)^p \leq \frac{x_n^{**p}(t) + x^{**p}(t)}{2}.$$

Hence,

$$\left(\frac{x_n^{**}(t) + x^{**}(t)}{2}\right)^p \leq \frac{x_n^{**p}(t) + x^{**p}(t)}{2} - \frac{\xi}{2} (x_n^{**p}(t) + x^{**p}(t)) \chi_{A_n \setminus B_n}(t)$$

for any $t \in (0, \alpha)$ and $n \in \mathbb{N}$. Consequently,

$$\begin{aligned} \left\| \frac{x_n + x}{2} \right\|_{\Gamma_{p,w}}^p &\leq \int_0^\alpha \left(\frac{x_n^{**}(t) + x^{**}(t)}{2}\right)^p w(t) dt \\ &\leq \int_0^\alpha \frac{x_n^{**p}(t) + x^{**p}(t)}{2} w(t) dt - \frac{\xi}{2} \int_{A_n \setminus B_n} (x_n^{**p}(t) + x^{**p}(t)) w(t) dt \\ &\leq 1 - \xi \int_{A_n \setminus B_n} \left(\frac{x_n^{**}(t) + x^{**}(t)}{2}\right)^p w(t) dt \\ &\leq 1 - \xi \int_{A_n \setminus B_n} \left| \frac{x_n^{**}(t) - x^{**}(t)}{2} \right|^p w(t) dt \leq 1 - \xi \left(\frac{\delta}{2}\right)^p \int_{A_n \setminus B_n} w(t) dt \\ &\leq 1 - \xi \left(\frac{\delta}{2}\right)^p \int_{(A_n \setminus B_n) \cap C_k} w(t) dt \leq 1 - \frac{\delta^p \xi}{2^p k} \mu((A_n \setminus B_n) \cap C_k) \end{aligned}$$

for any $k, n \in \mathbb{N}$. Therefore, by assumption $\frac{1}{2} \|x_n + x\|_{\Gamma_{p,w}} \rightarrow 1$ we obtain

$$\mu((A_n \setminus B_n) \cap C_k) \leq \frac{2^p k}{\delta^p \xi} \left(1 - \frac{1}{2^p} \|x_n + x\|_{\Gamma_{p,w}}^p\right) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

which implies

$$\mu((A_n \setminus B_n) \cap C_k) < \frac{\epsilon}{4}$$

for large enough $n \in \mathbb{N}$ and for all $k \in \mathbb{N}$. Hence, by conditions (33) and (34) it follows that

$$\begin{aligned} \mu(A_n) &= \mu(A_n \setminus B_n) + \mu(A_n \cap B_n) \\ &\leq \mu(B_n) + \mu((A_n \setminus B_n) \cap C_k) + \mu((A_n \setminus B_n) \setminus C_k) < \epsilon, \end{aligned}$$

for large enough $n \in \mathbb{N}$, which concludes that (x_n^{**}) converges to x^{**} in measure as $n \rightarrow \infty$. Now, according to assumption (i) and by Proposition 2.5 in [11] we get strict K -monotonicity of $\Gamma_{p,w}$, which means that x is a UKM point. Finally, since $x_n^{**} \rightarrow x^{**}$ in measure and $\|x_n\|_{\Gamma_{p,w}} \rightarrow \|x\|_{\Gamma_{p,w}}$, by Theorem 4.1 in [12] and by Theorem 3.8 we complete the proof. \square

Remark 6.2. Now we present a class of symmetric spaces obtained by the K -method of interpolation which are locally uniformly rotund. Let ϕ_0 and ϕ_1 be increasing concave functions on $[0, \infty)$ with $\phi_0(0^+) = \phi_1(0^+) = 0$ whose quotient $\lambda = \phi_0/\phi_1$ is increasing and vanishes continuously at zero and also

$$\lim_{s \rightarrow \infty} \min\{\phi_0(s), \phi_1(s)\} = \lim_{s \rightarrow \infty} \lambda(s) = \infty.$$

By condition 6 in [24], for simplicity we may use the following notations Λ_{ϕ_0} and Λ_{ϕ_1} for Lorentz spaces Λ_{1,ϕ'_0} and Λ_{1,ϕ'_1} , respectively. Considering $(\Lambda_{\phi_0}, \Lambda_{\phi_1})$, a Banach couple of two Lorentz spaces, by Lemma 3.2 in [16] we obtain the formula for the K -functional

$$\begin{aligned} K(t, x, \Lambda_{\phi_0}, \Lambda_{\phi_1}) &= \inf\{\|x_0\|_{\Lambda_{\phi_0}} + t\|x_1\|_{\Lambda_{\phi_1}} : x = x_0 + x_1, x_0 \in \Lambda_{\phi_0}, x_1 \in \Lambda_{\phi_1}\} \\ &= \int_0^{s(t)} x^*(\tau) d\phi_0(\tau) + t \int_{s(t)}^{\infty} x^*(\tau) d\phi_1(\tau) \end{aligned}$$

for any $t \in (0, \infty)$ and $x \in \Lambda_{\phi_0} + \Lambda_{\phi_1}$, where $s(t)$ is uniquely determined by the condition $\lambda(s(t)) = t$. Let ρ be a strictly positive weight function on $[0, \infty)$ and $1 < p < \infty$ and let w be a weight function such that $\mu((a, b) \cap \text{supp}(w)) > 0$ for any interval $(a, b) \subset (0, \alpha)$ with $a < b$, $W(\infty) = \infty$ whenever $\alpha = \infty$. Assuming

$$(\Lambda_{\phi_0}, \Lambda_{\phi_1})_{\rho, \Gamma_{p,w}} = \{x \in \Lambda_{\phi_0} + \Lambda_{\phi_1} : \rho(\cdot)K(\cdot, x, \Lambda_{\phi_0}, \Lambda_{\phi_1}) \in \Gamma_{p,w}\}$$

by Corollary 3.7 and Remark (see page 339 in [16]) and by Theorem 6.1 we conclude that the Banach space $(\Lambda_{\phi_0}, \Lambda_{\phi_1})_{\rho, \Gamma_{p,w}}$ equipped with the norm $\|x\|_{\rho, \Gamma_{p,w}} = \|\rho(\cdot)K(\cdot, x, \Lambda_{\phi_0}, \Lambda_{\phi_1})\|_{\Gamma_{p,w}}$ is locally uniformly rotund.

Theorem 6.3. Let $1 < p < \infty$, w be a weight function such that $W(t) = \int_0^t w$ is strictly increasing on $[0, \alpha) \setminus (a, b)$, $W(\infty) = \infty$ and $w = 0$ on (a, b) and let $x \in \Gamma_{p,w} \setminus \{0\}$. If x^* is an LUR point of $\Gamma_{p,w}$, then one of the following conditions is satisfied:

- (i) $\mu(t \in (a, b) : x^*(t) = x^*(a^-)) = b - a$.
- (ii) $\mu(t \in (a, b) : x^*(t) = x^*(b)) = b - a$.

Proof. Although the following proof is partially similar as the proof of Proposition 4.1 we will present the details of this proof for the sake of convenience. Let $x \in S_{\Gamma_{p,w}}$ and x^* be an LUR point of $\Gamma_{p,w}$. Suppose for a contrary that (i) and (ii) are not fulfilled. Then either there are $\gamma, \beta \in (a, b)$ such that $x^*(\gamma) > x^*(\beta)$ or there exists $c \in (x^*(b), x^*(a^-))$ such that $x^*(t) = c$ for all $t \in (a, b)$. Now we consider two cases.

Case 1. Assume that there exist $\gamma, \beta \in (a, b)$ such that $x^*(\gamma) > x^*(\beta)$. Since $a < \gamma < \beta < b$ we have $x^*(b) \leq x^*(\beta) < x^*(\gamma) \leq x^*(a)$. By the right-continuity of the decreasing rearrangement x^* there is $\xi \in (\gamma, \beta)$ such that $x^*(\beta) < x^*(\xi) \leq x^*(\gamma)$. Now we consider case 1 into two parts.

Part 1. Assume that $x^*(\beta) < x^*(\xi) < x^*(\gamma)$. Denote

$$L = \int_{\gamma}^{\xi} (x^* - x^*(\xi)), \quad P = \int_{\xi}^{\beta} (x^*(\xi) - x^*).$$

By the right-continuity of x^* we get $L > 0$ and since $\beta < b$ we may assume without loss of generality that $P > 0$. If $P \geq L$ then, by monotonicity and continuity of the maximal function x^{**} , we are able to find $\eta \in (\xi, \beta]$ such that

$$L = \int_{\xi}^{\eta} (x^*(\xi) - x^*). \quad (35)$$

Define

$$y = x^* \chi_{[0, \alpha) \setminus (\gamma, \eta)} + x^*(\xi) \chi_{(\gamma, \eta)}.$$

We claim that $\|x^* + y\|_{\Gamma_{p,w}} = 2$ and $\|x^* - y\|_{\Gamma_{p,w}} > 0$. First observe that y is decreasing and

$$x^* + y = 2x^* + (x^*(\xi) - x^*) \chi_{(\gamma, \eta)}.$$

Considering $t \in (0, \gamma]$ we have

$$(x^* + y)^{**}(t) = \frac{1}{t} \int_0^t (2x^* + (x^*(\xi) - x^*) \chi_{(\gamma, \eta)}) = 2x^{**}(t).$$

If $t \in [\eta, \alpha)$, then by condition (35) we obtain

$$\begin{aligned} (x^* + y)^{**}(t) &= \frac{2}{t} \int_0^t x^* + \frac{1}{t} \int_{\gamma}^{\xi} (x^*(\xi) - x^*) + \frac{1}{t} \int_{\xi}^{\eta} (x^*(\xi) - x^*) \\ &= 2x^{**}(t) - \frac{L}{t} + \frac{1}{t} \int_{\xi}^{\eta} (x^*(\xi) - x^*) = 2x^{**}(t). \end{aligned}$$

Consequently, since $\|x\|_{\Gamma_{p,w}} = 1$ and by assumption $w = 0$ on (a, b) and $(\gamma, \eta) \subset (a, b)$ it follows that

$$\begin{aligned} \|x^* + y\|_{\Gamma_{p,w}}^p &= \int_0^a (x^* + y)^{**p} w + \int_b^{\alpha} (x^* + y)^{**p} w \\ &= \int_0^a 2^p x^{**p} w + \int_b^{\alpha} 2^p x^{**p} w = 2^p. \end{aligned}$$

Moreover, by the definition of y we get $\|y\|_{\Gamma_{p,w}} = 1$. Now we show $\|x^* - y\|_{\Gamma_{p,w}} > 0$. Observe that

$$\begin{aligned} (x^* - y)^{**}(t) &= \frac{1}{t} \int_0^t (x^* - x^* \chi_{[0, \alpha) \setminus (\gamma, \eta)} - x^*(\xi) \chi_{(\gamma, \eta)})^* \\ &= \frac{1}{t} \int_0^t ((x^* - x^*(\xi)) \chi_{(\gamma, \xi)} + (x^* - x^*(\xi)) \chi_{(\xi, \eta)})^* \\ &\geq \frac{1}{t} \int_0^t (x^* - x^*(\xi)) \chi_{(\gamma, \xi)} + \frac{1}{t} \int_0^t (x^*(\xi) - x^*) \chi_{(\xi, \eta)} = \psi(t) \end{aligned}$$

for any $t \in (0, \alpha)$. Clearly, if $t \in (0, \gamma]$, then $\psi(t) = 0$. Moreover, if $t \in [\eta, \alpha)$, then $\psi(t) = \frac{2L}{t}$. Therefore, by assumption $W(\infty) = \infty$ we conclude

$$\begin{aligned} \|x^* - y\|_{\Gamma_{p,w}}^p &= \int_0^\alpha (x^* - y)^{**p} w \geq \int_0^\alpha \psi^p w = \int_0^a \psi^p w + \int_b^\alpha \psi^p w \\ &= \left(\frac{2L}{b}\right)^p W_p(b) > 0, \end{aligned}$$

which completes the claim. Now considering $P < L$ we may choose $\eta \in (\gamma, \xi)$ such that

$$P = \int_\eta^\xi (x^* - x^*(\xi)). \quad (36)$$

Next assuming

$$y = x^* \chi_{[0,\alpha) \setminus (\eta,\beta)} + x^*(\xi) \chi_{(\eta,\beta)},$$

it is easy to show $\|x^* + y\|_{\Gamma_{p,w}} = 2$ and $\|y\|_{\Gamma_{p,w}} = 1$. Furthermore,

$$(x^* - y)^{**}(t) \geq \frac{1}{t} \int_0^t (x^* - x^*(\xi)) \chi_{(\eta,\xi)} + \frac{1}{t} \int_0^t (x^*(\xi) - x^*) \chi_{(\xi,\beta)},$$

whence

$$\|x^* - y\|_{\Gamma_{p,w}}^p \geq \left(\frac{2P}{b}\right)^p W_p(b) > 0.$$

Now combining in part 1 both cases when $P \geq L$ and $P < L$ we obtain a contradiction with assumption that x^* is an *LUR* point.

Part 2. Assume that for all $t \in (a, b)$ we have either $x^*(\gamma) = x^*(t)$ or $x^*(\beta) = x^*(t)$. Then, we may assume that $x^*(\beta) = x^*(\beta^-)$ and find $\xi \in (\gamma, \beta)$ such that $x^*(\xi^-) > x^*(\xi)$. Let $\epsilon = \frac{1}{2} \min\{\beta - \xi, \xi - \gamma\}$ and $\delta = \frac{1}{2}(x^*(\gamma) + x^*(\beta))$. Define

$$y = x^* \chi_{[0,\alpha) \setminus [\xi-\epsilon, \xi+\epsilon)} + \delta \chi_{[\xi-\epsilon, \xi+\epsilon)}.$$

Now we prove $\|x^* + y\|_{\Gamma_{p,w}} = 2$, $\|x^* - y\|_{\Gamma_{p,w}} > 0$ and $\|y\|_{\Gamma_{p,w}} = 1$. First observe

$$\begin{aligned} (x^* + y)^{**}(t) &= \frac{1}{t} \int_0^t (2x^* + (\delta - x^*) \chi_{(\xi-\epsilon, \xi+\epsilon)})^* \\ &= 2x^{**}(t) + \frac{1}{t} \int_0^t (\delta - x^*(\gamma)) \chi_{(\xi-\epsilon, \xi)} + \frac{1}{t} \int_0^t (\delta - x^*(\beta)) \chi_{(\xi, \xi+\epsilon)} \\ &= 2x^{**}(t) + \frac{1}{2t} \int_0^t (x^*(\beta) - x^*(\gamma)) \chi_{(\xi-\epsilon, \xi)} + \frac{1}{2t} \int_0^t (x^*(\gamma) - x^*(\beta)) \chi_{(\xi, \xi+\epsilon)} \end{aligned}$$

for all $t > 0$. Thus, if $t \in (0, \xi - \epsilon) \cup [\xi + \epsilon, \alpha)$, then $(x^* + y)^{**}(t) = 2x^{**}(t)$. Consequently, since $w = 0$ on (a, b) and $(\xi - \epsilon, \xi + \epsilon) \subset (a, b)$, we get $\|x^* + y\|_{\Gamma_{p,w}} = 2$. Furthermore, for any $t > 0$,

$$\begin{aligned} (x^* - y)^{**}(t) &= \frac{1}{t} \int_0^t ((x^* - \delta)\chi_{(\xi-\epsilon, \xi+\epsilon)})^* \\ &= \frac{1}{2t} \int_0^t ((x^*(\gamma) - x^*(\beta))\chi_{(\xi-\epsilon, \xi)} + (x^*(\beta) - x^*(\gamma))\chi_{(\xi, \xi+\epsilon)})^* \\ &= \frac{x^*(\gamma) - x^*(\beta)}{2} (\chi_{[0, 2\epsilon]})^{**}(t), \end{aligned}$$

whence

$$\|x^* - y\|_{\Gamma_{p,w}} = \frac{x^*(\gamma) - x^*(\beta)}{2} \|\chi_{[0, 2\epsilon]}\|_{\Gamma_{p,w}} > 0.$$

Clearly $y^* = y$, and so for all $t \in (0, \alpha)$ we have

$$\begin{aligned} y^{**}(t) &= \frac{1}{t} \int_0^t (x^* + (\delta - x^*)\chi_{(\xi-\epsilon, \xi+\epsilon)}) \\ &= x^{**}(t) + \frac{1}{t} \int_0^t (\delta - x^*(\gamma))\chi_{(\xi-\epsilon, \xi)} + \frac{1}{t} \int_0^t (\delta - x^*(\beta))\chi_{(\xi, \xi+\epsilon)} \\ &= x^{**}(t) + \frac{1}{2t} \int_0^t (x^*(\beta) - x^*(\gamma))\chi_{(\xi-\epsilon, \xi)} + \frac{1}{2t} \int_0^t (x^*(\gamma) - x^*(\beta))\chi_{(\xi, \xi+\epsilon)}. \end{aligned}$$

Thus, if $t \in [0, \xi - \epsilon] \cup [\xi + \epsilon, \alpha)$, then $y^{**}(t) = x^{**}(t)$ and by assumption $w = 0$ a.e. on (a, b) it follows that $\|y\|_{\Gamma_{p,w}} = \|x\|_{\Gamma_{p,w}} = 1$. Finally, we obtain a contradiction in view of assumption that x^* is an *LUR* point.

Case 2. Assume that $x^* = c$ on (a, b) for some $c \in (x^*(b), x^*(a^-))$. Taking

$$\delta = \frac{1}{2} \min\{c - x^*(b), x^*(a^-) - c\},$$

and

$$y = x^* \chi_{[0, \alpha] \setminus (a, b)} + (c + \delta) \chi_{(a, \frac{a+b}{2})} + (c - \delta) \chi_{(\frac{a+b}{2}, b)}$$

we may easily prove that $\|y\|_{\Gamma_{p,w}} = 1$, $\|x^* + y\|_{\Gamma_{p,w}} = 2$ and $\|x^* - y\|_{\Gamma_{p,w}} > 0$, which contradicts with assumption that x^* is an *LUR* point. \square

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