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A new general filter regularization method for Cauchy problems for elliptic equations with a locally Lipschitz nonlinear source

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A R T I C L E I N F O

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A B S T R A C T

Up to now, studies on the semi-linear Cauchy problem for elliptic partial differential equations needed to assume that the source term present in the governing equation is a global Lipschitz function. The current paper is the first investigation to not only the more general but also the more practical case of interest when the source term is only a **local** Lipschitz function. In such a situation, the methods of solution from the previous studies with a global Lipschitz source term are not directly applicable and therefore, novel ideas and techniques need to be developed to tackle the local Lipschitz nonlinearity. This locally Lipschitz source arises in many applications of great physical interest governed by, for example, the sine-Gordon, Lane–Emden, Allen–Cahn and Liouville equations. The inverse problem is severely ill-posed in the sense of Hadamard by violating the continuous dependence upon the input Cauchy data. Therefore, in order to obtain a stable solution we consider theoretical aspects of regularization of the problem by a new generalized filter method. Under some priori assumptions on the exact solution, we prove and obtain rigorously convergence estimates.

1. Introduction

Let H be a Hilbert space with the inner product $\langle .,. \rangle$ and the norm $\|.\|$, and let $A : D(A) \subset H \to H$ be a linear, positive-definite, self-adjoint operator with compact inverse on H. For $L > 0$, consider the inverse problem of finding the function $u : [0, L] \to H$ from the equation

$$\frac{d^2 u(z)}{dz^2} = Au(z) + G(z, u(z)), \quad z \in (0, L), \tag{1.1}$$

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with the (initial) Cauchy conditions

$$\begin{cases} u(0) = f, \\ \dfrac{du}{dz}(0) = h, \end{cases} \tag{1.2}$$

where (f, h) are given data in $H \times H$ and the source function $G : [0, L] \times H \to H$ will be defined later. In practice, the data $(f, h) \in H \times H$ is noisy and is represented by the perturbed data $(f^\delta, h^\delta) \in H \times H$ satisfying

$$\|f^\delta - f\|_H + \|h^\delta - h\|_H \le \delta, \tag{1.3}$$

where the constant $\delta > 0$ represents a known upper bound of the measurement error.

It is well-known that in general the Cauchy problem for elliptic equations is severely ill-posed in Hadamard's sense, i.e. a small perturbation in the given Cauchy data (1.2) may cause a very large error in the output solution $u(z)$ for $z \in (0, L]$. Moreover, the instability increases with increasing the distance z from the boundary $z = 0$. Therefore, it is very difficult to solve the problem by using classical numerical methods of inversion [22]. In order to overcome this instability, regularization methods are naturally required.

Equation (1.1) is an abstract version which generalizes many well-known equations. For a simple example, if $A = -\Delta$ (Laplace operator) and $G(z, u(z)) = -k^2 u(z)$ with k real or purely imaginary, then the equation (1.1) becomes the Helmholtz or modified Helmholtz equation, respectively, which arises in many engineering applications related to propagating waves in different environments or heat transfer in fins. More generally, for $A = -\Delta$ and G a nonlinear function of u, equation (1.1) becomes the nonlinear Poisson equation which is encountered in numerous applications in heat and mass transfer, chemical reactions, gas dynamics and fluid flow in porous media [2,3,19].

In the past, there have been many studies on the homogeneous problem given by equation (1.1) with $G = 0$ and (1.2). For instance, Elden and Berntsson [14] used the logarithmic convexity method to obtain a stability result of Hölder type. Alessandrini et al. [1] provided optimal stability results under minimal assumptions, whilst Reginska and Tautenhahn [33] presented some stability estimates and a regularization method for a Cauchy problem for Helmholtz equation. Many methods have been proposed to solve the Cauchy problem for linear homogeneous elliptic equations, such as the method of successive iterations [10], the alternating method [26], the conjugate gradient method [12,23], the iterative regularization method [15], the quasi-reversibility method [25,30], the fourth-order modified method [29], the Fourier truncation regularized (or spectral regularized) method [18,34], etc. Nevertheless, the literature devoted to the Cauchy problem for linear homogeneous elliptic equations is very rich, see e.g. [4,5,7,9,11,13,16,21,25,28,32,34] and the references therein. Recently, a linear inhomogeneous version of Helmholtz equation (i.e. $G(z, u(z)) = G(z)$ in equation (1.1)) has been considered in [35].

Although there are many works on the linear case, the literature on the nonlinear case is quite scarce. We mention here a nonlinear elliptic problem of [37], where the authors approximated (1.1) and (1.2) by a truncation method. However, their results are only given for globally Lipschitz source terms.

In practice, the applications of nonlinear problem requires the extended Lipschitz source term. For example, if $G(z, u) = \sin u$, then the equation (1.1) is called the elliptic-sine Gordon equation which occurs in several areas of mathematical physics including the theory of Josephson effects, superconductors and spin waves in ferromagnets, see e.g. [17,20]. Furthermore, the Lane–Emden equation $\Delta u = -u^p$, implying $G(z, u) = -u^p$, $p > 1$, plays a vital role in describing the structure of the polytropic stars, where p is called the polytropic index [8]. Also, the reaction–diffusion equation $\Delta u = \Phi^2 u^p$, implying $G(z, u) = \Phi^2 u^p$, governs kinetic and diffusional phenomena in chemical reaction engineering. In this equation, p is the order of the reaction and Φ^2 is called the Thiele parameter representing the ratio of kinetic to transport resistances in the domain, see [3] where other physical models such as thermal explosion $G(z, u) = -\exp(u - 1)$, and substrate inhibition $G(z,u) = \Phi^2 u/(1+\alpha u+\beta u^2)$, are also considered. Finally, for $G(z,u) = u - u^3$, we have the Allen–Cahn equation originally formulated in the description of bi-phase separation in fluids. From this wide range of physical examples one can observe that, except for the sine-Gordon equation in which the sine-nonlinearity is a global Lipschitz function, the other examples present a nonlinear function G which is only locally Lipschitz, i.e. for all $B > 0$, there exists $k(B) > 0$ such that

$$\|G(z,u) - G(z,v)\|_H \le k(B)\,\|u-v\|_H \quad \forall z \in [0,L], \quad \text{if } \max\{\|u\|_H, \|v\|_H\} \le B. \tag{1.4}$$

To the best of our knowledge, the Cauchy problem (1.1) and (1.2) for nonlinear elliptic equations with a locally Lipschitz source term is yet to be investigated. Therefore, in the present paper, we propose a new general filter function method to regularize the problem (1.1) and (1.2) in the case that G is locally Lipschitzian with respect to u. Remark that it is impossible to solve the problem only with the assumption (1.4) by applying directly the method of [37]. To overcome this technical difficulty, in this paper, we propose a new idea in which the locally Lipschitz source function G is approximated by a sequence G_δ of globally Lipschitzian functions. Furthermore, assuming that the function k given in (1.4) is increasing on $[0,+\infty)$, we then choose a positive sequence $\{B_\delta\}_{\delta>0}$ satisfying $\lim\limits_{\delta\to 0^+} B_\delta = +\infty$ on which $k(B_\delta)$ satisfies certain constraints. We then define the function G_δ from G as

$$G_\delta(z,v) = G\left(z, \min\left\{\frac{B_\delta}{\|v\|_H}, 1\right\} v\right), \quad \forall (z,v) \in [0,L]\times H. \tag{1.5}$$

In particular, $G_\delta(z,0) = G(z,0)$. In fact, since $\lim_{\delta\to 0} B_\delta = +\infty$, for δ small enough we have that $\sup_{z\in[0,L]} \|u(z)\|_H \le B_\delta$. From (1.5) this implies that

$$G_\delta(z,u(z)) = G(z,u(z)), \quad \forall z \in [0,L], \quad \text{for } \delta \text{ small enough.} \tag{1.6}$$

We also have the following lemma giving the Lipschitz constant for the function G_δ.

Lemma 1.1. *For $\delta > 0$, $z \in [0,L]$ and v_1, $v_2 \in H$, we have*

$$\|G_\delta(z,v_1) - G_\delta(z,v_2)\|_H \le 2k(B_\delta)\,\|v_1 - v_2\|_H\,. \tag{1.7}$$

Proof. Due to the continuity, it is enough to prove the lemma for non-zero elements v_1 and v_2 in H. We can assume that $\|v_1\| \ge \|v_2\| > 0$. Using the local Lipschitz property (1.4) of G and the definition (1.5) of G_δ, we have

$$\begin{aligned}\|G_\delta(z,v_1) - G_\delta(z,v_2)\|_H &= \left\|G\left(z, \min\left\{\frac{B_\delta}{\|v_1\|_H},1\right\}v_1\right) - G\left(z, \min\left\{\frac{B_\delta}{\|v_2\|_H},1\right\}v_2\right)\right\|_H \\ &\le k(B_\delta)\left\|\min\left\{\frac{B_\delta}{\|v_1\|_H},1\right\}v_1 - \min\left\{\frac{B_\delta}{\|v_2\|_H},1\right\}v_2\right\|_H, \quad \forall z\in[0,L].\end{aligned}$$

It remains to show that

$$\left\|\min\left\{\frac{B_\delta}{\|v_1\|_H},1\right\}v_1 - \min\left\{\frac{B_\delta}{\|v_2\|_H},1\right\}v_2\right\|_H \le 2\,\|v_1 - v_2\|_H\,.$$

This inequality is trivial if $B_\delta \ge \|v_1\|_H \ge \|v_2\|_H$. In the case $\|v_1\|_H \ge \|v_2\|_H \ge B_\delta$, we have

$$\begin{aligned}\left\|\frac{B_\delta}{\|v_1\|_H}v_1-\frac{B_\delta}{\|v_2\|_H}v_2\right\|_H &= B_\delta\left\|\frac{v_1-v_2}{\|v_1\|_H}+\frac{\|v_2\|_H-\|v_1\|_H}{\|v_1\|_H\cdot\|v_2\|_H}v_2\right\|_H\\ &\le B_\delta\left(\left\|\frac{v_1-v_2}{\|v_1\|_H}\right\|_H+\left\|\frac{\|v_2\|_H-\|v_1\|_H}{\|v_1\|_H\cdot\|v_2\|_H}v_2\right\|_H\right)\\ &= \frac{B_\delta}{\|v_1\|_H}\left(\|v_1-v_2\|_H+\Big|\|v_2\|_H-\|v_1\|_H\Big|\right)\le 2\|v_1-v_2\|_H\,.\end{aligned}$$

Finally, if $\|v_1\|_H \ge B_\delta \ge \|v_2\|_H$ then

$$\begin{aligned}\left\|\frac{B_\delta}{\|v_1\|_H}v_1-v_2\right\|_H &= \left\|\frac{B_\delta-\|v_1\|_H}{\|v_1\|_H}v_1+v_1-v_2\right\|_H\\ &\le \left\|\frac{B_\delta-\|v_1\|_H}{\|v_1\|_H}v_1\right\|_H+\|v_1-v_2\|_H\\ &= \Big|B_\delta-\|v_1\|_H\Big|+\|v_1-v_2\|_H\le 2\|v_1-v_2\|_H\,.\end{aligned}$$

This implies the desired result (1.7). □

2. Cauchy problem for elliptic equations

From now on, suppose that $A: D(A)\subset H\to H$ is a linear, positive-definite, self-adjoint operator with compact inverse on H. As a consequence, the operator A admits an orthonormal eigenbasis $\{\phi_n\}_{n\ge1}$ in H, associated with the eigenvalues

$$0<\lambda_1\le\lambda_2\le\lambda_3\le\cdots\le\lim_{n\to\infty}\lambda_n=\infty.$$

We can divide the Cauchy problem for elliptic equations into three cases: homogeneous linear problem, inhomogeneous linear problem and nonlinear problem.

2.1. Homogeneous linear problem

We first consider the homogeneous problem, i.e. $G=0$, of finding a function $u:[0,L]\to H$ satisfying

$$\frac{d^2u(z)}{dz^2}=Au,\quad z\in(0,L) \tag{2.8}$$

subject to the Cauchy conditions (1.2). Let

$$u(z)=\sum_{n=1}^{\infty}\langle u(z),\phi_n\rangle\,\phi_n \tag{2.9}$$

be the Fourier series of u in the Hilbert space H. From (2.8), we obtain the following homogeneous second-order differential equation:

$$\frac{d^2}{dz^2}\langle u(z),\phi_n\rangle-\lambda_n\langle u(z),\phi_n\rangle=0.$$

Solving this equation, we obtain

$$\langle u(z),\phi_n\rangle=A_ne^{\sqrt{\lambda_n}z}+B_ne^{-\sqrt{\lambda_n}z}.$$

It follows from (1.2) that $\langle u(0), \phi_n \rangle = \langle f, \phi_n \rangle$ and $\frac{d}{dz} \langle u(0), \phi_n \rangle = \langle h, \phi_n \rangle$. The obtained results for A_n and B_n imply that

$$u(z) = \sum_{n=1}^{\infty} \left[\cosh\left(\sqrt{\lambda_n} z\right) \langle f, \phi_n \rangle + \frac{\sinh\left(\sqrt{\lambda_n} z\right)}{\sqrt{\lambda_n}} \langle h, \phi_n \rangle \right] \phi_n,$$

leading us to define the linear operators $P(z)$, $S(z) : H \to H$,

$$P(z)f = \sum_{n=1}^{\infty} \cosh\left(\sqrt{\lambda_n} z\right) \left\langle f, \phi_n \right\rangle \phi_n, \tag{2.10}$$

$$S(z)f = \sum_{n=1}^{\infty} \frac{\sinh\left(\sqrt{\lambda_n} z\right)}{\sqrt{\lambda_n}} \left\langle f, \phi_n \right\rangle \phi_n \tag{2.11}$$

for $z \in [0, L]$ and $f \in H$. The solution of the homogeneous problem (1.2) and (2.8) is then given by

$$u(z) = P(z)f + S(z)h, \quad z \in [0, L]. \tag{2.12}$$

2.2. Inhomogeneous linear problem and nonlinear problem

(i) We consider first the linear inhomogeneous problem of finding a function $u : [0, L] \to H$ satisfying

$$\frac{d^2 u(z)}{dz^2} = Au + G(z), \quad 0 \le z \le L \tag{2.13}$$

subject to the Cauchy conditions (1.2). The solution u in this case has the Fourier series expansion (2.9), where $\left\langle u(z), \phi_n \right\rangle$ satisfies inhomogeneous second-order differential equation

$$\frac{d^2}{dz^2} \left\langle u(z), \phi_n \right\rangle - \lambda_n \left\langle u(z), \phi_n \right\rangle = \left\langle G(z), \phi_n \right\rangle.$$

Solving this equation and using (1.2), we obtain the exact solution u to problem (1.2) and (2.13) given by

$$\begin{aligned} u(z) = \sum_{n=1}^{\infty} \Bigg[&\cosh\left(\sqrt{\lambda_n} z\right) \langle f, \phi_n \rangle + \frac{\sinh\left(\sqrt{\lambda_n} z\right)}{\sqrt{\lambda_n}} \langle h, \phi_n \rangle \\ &+ \int_0^z \frac{\sinh\left(\sqrt{\lambda_n}(z-y)\right)}{\sqrt{\lambda_n}} \left\langle G(y), \phi_n \right\rangle dy \Bigg] \phi_n. \end{aligned} \tag{2.14}$$

With the definitions (2.10) and (2.11), equation (2.14) can be rewritten as

$$u(z) = P(z)f + S(z)h + \int_0^z S(z-y)G(y)dy. \tag{2.15}$$

Recently, Tuan et al. [35] regularized a simpler version of the equation (2.14) by truncation and quasi-boundary value methods.

(ii) For the nonlinear problem (1.1) and (1.2), its solution u satisfies the following integral equation:

$$u(z) = \sum_{n=1}^{\infty} \Bigg[\cosh\left(\sqrt{\lambda_n} z\right) \left\langle f, \phi_n \right\rangle + \frac{\sinh\left(\sqrt{\lambda_n} z\right)}{\sqrt{\lambda_n}} \left\langle h, \phi_n \right\rangle + \int_0^z \frac{\sinh\left(\sqrt{\lambda_n}(z-y)\right)}{\sqrt{\lambda_n}} \left\langle G(y, u(y)), \phi_n \right\rangle dy \Bigg] \phi_n. \tag{2.16}$$

This integral equation can be rewritten as

$$u(z) = P(z)f + S(z)h + \int_0^z S(z-y)G(y, u(y))dy. \tag{2.17}$$

The transformation of (1.1) and (1.2) into (2.17) is easily proved by the separation of variables method, as above. Prior to this study, a filter regularization method was applied for solving a backward heat conduction problem [31] and for the Cauchy problem of the Helmholtz equation [36]. In the next section, we introduce a new general filter regularization method to stabilize the integral equation (2.17).

3. A general filter regularization method for the nonlinear problem

In this section, we present a new general filter regularization method and establish convergence rates and error estimates.

First, let us remark that $P(z)$ and $S(z)$ given by equations (2.10) and (2.11), respectively, are unbounded linear operators. This means that the solution u of (2.17) is not stable. To approximate u, we introduce a regularized solution u_α^δ obtained by replacing $P(z)$, $S(z)$ by bounded linear operators $P_\alpha^\delta(z)$, $S_\alpha^\delta(z)$, respectively, as follows:

$$u_\alpha^\delta(z) = P_\alpha^\delta(z)f^\delta + S_\alpha^\delta(z)h^\delta + \int_0^z S_\alpha^\delta(z-y)G_\delta(y, u_\alpha^\delta(y))dy, \tag{3.18}$$

where G_δ is defined in (1.5) and $0 < \alpha = \alpha(\delta)$ plays the role of the regularization parameter to be chosen depending on the amount of noise δ in (1.3). Here, $P_\alpha^\delta(z)$ and $S_\alpha^\delta(z)$ are defined by

$$P_\alpha^\delta(z)f = \sum_{n=1}^{\infty} \frac{Q(\alpha, \lambda_n)e^{\sqrt{\lambda_n}z} + R(\alpha, \lambda_n)e^{-\sqrt{\lambda_n}z}}{2} \left\langle f, \phi_n \right\rangle \phi_n, \tag{3.19}$$

$$S_\alpha^\delta(z)f = \sum_{n=1}^{\infty} \frac{Q(\alpha, \lambda_n)e^{\sqrt{\lambda_n}z} - R(\alpha, \lambda_n)e^{-\sqrt{\lambda_n}z}}{2\sqrt{\lambda_n}} \left\langle f, \phi_n \right\rangle \phi_n \tag{3.20}$$

for $z \in [0, L]$ and $f = \sum_{n=1}^{\infty} \left\langle f, \phi_n \right\rangle \phi_n$. In these expressions, $Q(\alpha, \lambda_n)$ and $R(\alpha, \lambda_n)$ are called *"regularizing filter functions"*. For more details on regularizing filter functions, we refer the reader to the book of Kirsch [24]. With a regularization strategy $\alpha = \alpha(\delta)$ for the regularization parameter satisfying

$$\lim_{\delta \to 0} \alpha(\delta) = \lim_{\delta \to 0} \|u_\alpha^\delta(z) - u(z)\|_H = 0, \quad \forall z \in [0, L],$$

then we obtain a so-called *"filter regularization method"*. Now consider the general regularizing filter Q satisfying

$$\begin{cases} 0 \le Q(\alpha, \lambda_n) e^{\sqrt{\lambda_n} z} \le M(\alpha)^{-z/L}, \\ |Q(\alpha, \lambda_n) - 1| e^{\sqrt{\lambda_n}(z-L)} \le M(\alpha)^{1-z/L}, \end{cases} \quad \forall n \in \mathbb{N}^*, \tag{3.21}$$

where $M(\alpha)$ is some positive function satisfying $\lim_{\delta\to 0} M(\alpha) = 0$. We also take the filter $R(\alpha, \lambda_n)$ to be the function

$$R(\alpha, \lambda_n) = Q(\alpha, \lambda_n) \quad \text{or,} \quad R(\alpha, \lambda_n) = 1. \tag{3.22}$$

In this case, we only consider examples for Q, and from this, the filter R is directly defined by (3.22). For more illustration, we give a couple of examples for Q which satisfy (3.21).

Example 1. Let R_1, Q_1 be as

$$R_1(\alpha, \lambda_n) = Q_1(\alpha, \lambda_n) = \frac{e^{-\sqrt{\lambda_n} L}}{\alpha + e^{-\sqrt{\lambda_n} L}}. \tag{3.23}$$

First, we can deduce the following inequality:

$$\begin{aligned} 0 \le Q_1(\alpha, \lambda_n) e^{\sqrt{\lambda_n} z} = \frac{e^{-\sqrt{\lambda_n}(L-z)}}{\alpha + e^{-\sqrt{\lambda_n} L}} &= \frac{e^{-\sqrt{\lambda_n}(L-z)}}{\left(\alpha + e^{-\sqrt{\lambda_n} L}\right)^{1-z/L} \left(\alpha + e^{-\sqrt{\lambda_n} L}\right)^{z/L}} \\ &\le \left(\alpha + e^{-\sqrt{\lambda_n} L}\right)^{-z/L} \le \alpha^{-z/L}. \end{aligned} \tag{3.24}$$

By a similar technique, we get

$$|Q_1(\alpha, \lambda_n) - 1| e^{\sqrt{\lambda_n}(z-L)} = \frac{\alpha e^{-\sqrt{\lambda_n}(L-z)}}{\alpha + e^{-\sqrt{\lambda_n} L}} \le \alpha^{1-z/L}. \tag{3.25}$$

Therefore, Q_1 given in (3.23) satisfies (3.21) with $M(\alpha) = \alpha = \delta$.

Example 2. Let us choose R_2 and Q_2 as follows:

$$R_2(\alpha, \lambda_n) = Q_2(\alpha, \lambda_n) = \begin{cases} 1, & \text{if } \lambda_n \le N_\alpha, \\ 0, & \text{if } \lambda_n > N_\alpha, \end{cases} \tag{3.26}$$

where N_α is some positive number satisfying $\lim_{\delta\to 0} N_{\alpha(\delta)} = +\infty$. It follows from

$$Q_2(\alpha, \lambda_n) e^{\sqrt{\lambda_n} z} = \begin{cases} e^{\sqrt{\lambda_n} z}, & \text{if } \lambda_n \le N_\alpha, \\ 0, & \text{if } \lambda_n > N_\alpha, \end{cases}$$

that

$$Q_2(\alpha, \lambda_n) e^{\sqrt{\lambda_n} z} \le e^{\sqrt{N_\alpha} z}$$

and

$$|Q_2(\alpha, \lambda_n) - 1| e^{\sqrt{\lambda_n}(z-L)} \le e^{\sqrt{N_\alpha}(z-L)}.$$

Therefore, Q_2 given in (3.26) satisfies (3.21) with $M(\alpha) = e^{-L\sqrt{N_\alpha}} = \delta$. The solution of (3.18) with filters (3.26) is called a *"truncation solution"*, and it has recently been studied in [37].

At this stage, let us introduce the Gevrey-type space [6,27],

$$\mathrm{Ge}_\beta = D\left(e^{\beta\sqrt{-A}}\right) = \left\{\zeta \in H; \sum_{n=1}^{\infty} e^{2\beta\sqrt{\lambda_n}} |\langle \zeta, \phi_n\rangle|^2 < \infty\right\}, \tag{3.27}$$

for some $\beta > 0$. This is a Hilbert space with the norm

$$\|\zeta\|_{\mathrm{Ge}_\beta} = \|\zeta\|_{D\left(e^{\beta\sqrt{-A}}\right)} = \sqrt{\sum_{n=1}^{\infty} e^{2\beta\sqrt{\lambda_n}} |\langle \zeta, \phi_n\rangle|^2}.$$

Clearly, if $\zeta \in D\left(e^{\beta\sqrt{-A}}\right)$, then the Fourier coefficients of ζ must decay exponentially, as $n \to \infty$.

The next theorem states the main result of the paper.

Theorem 3.1 *(General regularization filters). Assume that the problem (1.1) and (1.2) has a solution $u \in C([0, L]; H)$. Choose $M(\alpha)$ such that $\delta/M(\alpha)$ is bounded and choose B_δ such that*

$$\lim_{\delta\to 0} M(\alpha)^{1-z/L} \exp\left(\frac{\sqrt{2}k(B_\delta)z}{\sqrt{\lambda_1}}\right) = 0, \quad z \in [0, L). \tag{3.28}$$

Suppose that there exist positive constants I_1 or I_2 such that

$$\|f\|_{D\left(e^{L\sqrt{-A}}\right)} + \frac{1}{\sqrt{\lambda_1}}\|h\|_{D\left(e^{L\sqrt{-A}}\right)} + \frac{1}{\sqrt{\lambda_1}}\int_0^L \left\|G(z, u(z))\right\|_{D\left(e^{L\sqrt{-A}}\right)} dz \le I_1, \tag{3.29}$$

or,

$$\sup_{0\le z\le L} \left\{\left\|u(z)\right\|_{D\left(e^{(L-z)\sqrt{-A}}\right)}, \left\|\frac{du(z)}{dz}\right\|_{D\left(e^{(L-z)\sqrt{-A}}\right)}\right\} \le I_2. \tag{3.30}$$

Then, for δ small enough, the solution u_α^δ of (3.18) satisfies the following estimates:

$$\left\|u_\alpha^\delta(z) - u(z)\right\|_H \le \begin{cases} P_1 \exp\left(\frac{\sqrt{2}k(B_\delta)z}{\sqrt{\lambda_1}}\right) M(\alpha)^{1-z/L}, & \text{if (3.29) holds} \\ P_2 \exp\left(\frac{\sqrt{2}k(B_\delta)z}{\sqrt{\lambda_1}}\right) M(\alpha)^{1-z/L}, & \text{if (3.30) holds} \end{cases} =: \Theta, \tag{3.31}$$

where

$$P_1 = I_1 + \left(1 + \frac{1}{\sqrt{2\lambda_1}}\right)\frac{\delta}{M(\alpha)}, \quad P_2 = 2I_2 \max\left\{1, \frac{1}{\sqrt{\lambda_1}}\right\} + \left(1 + \frac{1}{\sqrt{2\lambda_1}}\right)\frac{\delta}{M(\alpha)}. \tag{3.32}$$

Moreover, there exists $z_\delta \in [0, L]$ satisfying $\lim_{\delta\to 0} z_\delta = 0$ such that

$$\|u_\alpha^\delta(z_\delta) - u(L)\|_H \le \sup_{0\le z\le L}\left\|\frac{du(z)}{dz}\right\|_H \sqrt{\frac{L}{\ln(\frac{1}{M(\alpha)})}} + \Theta. \tag{3.33}$$

Remark 3.1. If in Theorem 3.1 we choose the regularization parameter α from $M(\alpha) = \delta$ then, by taking B_δ such that

$$k(B_\delta) \le \frac{\gamma\sqrt{\lambda_1}}{\sqrt{2}L} \ln\left(\ln(\delta^{-1})\right)$$

for some $\gamma > 0$, we can conclude that (3.28) holds. Indeed, it is easy to see that

$$\lim_{\delta\to 0} M(\alpha)^{1-z/L} \exp\left(\frac{\sqrt{2}k(B_\delta)z}{\sqrt{\lambda_1}}\right) \le \lim_{\delta\to 0} \delta^{1-z/L} \ln^\gamma(\delta^{-1}) = 0, \quad \forall z \in [0, L).$$

For proving Theorem 3.1 the following lemmas are needed.

Lemma 3.1. *The operators $P_\alpha^\delta(z)$ and $S_\alpha^\delta(z)$ defined by (3.19) and (3.20), respectively, are bounded and linear, and their norms satisfy*

$$\left\|P_\alpha^\delta(z)\right\|_{\mathbb{L}(H)} \le M(\alpha)^{-z/L}, \quad \left\|S_\alpha^\delta(z)\right\|_{\mathbb{L}(H)} \le \frac{M(\alpha)^{-z/L}}{\sqrt{2\lambda_1}}, \quad z \in [0, L], \tag{3.34}$$

where $\|\cdot\|_{\mathbb{L}(H)}$ stands for the operator norm on the space of bounded linear operators from H onto itself.

Proof. Let $f \in H$ be arbitrary and represented as $f = \sum_{n=1}^{\infty} \left\langle f, \phi_n \right\rangle \phi_n$. Then, from (3.19), (3.21) and noting that $R(\alpha, \lambda_n)e^{-\sqrt{\lambda_n}z} \le \max\{Q(\alpha, \lambda_n), 1\}e^{-\sqrt{\lambda_n}z} \le \max\{M(\alpha)^{-z/L}, 1\} = M(\alpha)^{-z/L}$, we have

$$\begin{aligned}\|P_\alpha^\delta(z)f\|_H^2 &= \sum_{n=1}^{\infty} \Big[\frac{Q(\alpha, \lambda_n)e^{\sqrt{\lambda_n}z} + R(\alpha, \lambda_n)e^{-\sqrt{\lambda_n}z}}{2}\Big]^2 \left|\left\langle f, \phi_n \right\rangle\right|^2 \\ &\le M(\alpha)^{-2z/L} \sum_{n=1}^{\infty} \left|\left\langle f, \phi_n \right\rangle\right|^2 = M(\alpha)^{-2z/L}\|f\|_H^2.\end{aligned}$$

This latter estimate implies that

$$\left\|P_\alpha^\delta(z)\right\|_{\mathbb{L}(H)} \le M(\alpha)^{-z/L}, \quad z \in [0, L].$$

Similarly, we can easily show the second estimate of the lemma. Indeed, since $\lambda_n \ge \lambda_1$ for $n \ge 1$, and using (3.20), (3.21) and that $\left(\frac{a-b}{2}\right)^2 \le \frac{a^2+b^2}{4}$ for a, $b \ge 0$, we have

$$\begin{aligned}\|S_\alpha^\delta(z)f\|_H^2 &= \sum_{n=1}^{\infty} \Big[\frac{Q(\alpha, \lambda_n)e^{\sqrt{\lambda_n}z} - R(\alpha, \lambda_n)e^{-\sqrt{\lambda_n}z}}{2\sqrt{\lambda_n}}\Big]^2 \left|\left\langle f, \phi_n \right\rangle\right|^2 \\ &\le \frac{M(\alpha)^{-2z/L}}{2\lambda_1} \sum_{n=1}^{\infty} \left|\left\langle f, \phi_n \right\rangle\right|^2 = \frac{M(\alpha)^{-2z/L}}{2\lambda_1}\|f\|_H^2.\end{aligned}$$

This latter inequality implies the second estimate of the lemma. □

Lemma 3.2. *For each $(f^\delta, h^\delta) \in H \times H$, the integral equation (3.18) has a unique solution $u_\alpha^\delta \in C([0, L]; H)$.*

Proof. For each $w \in C([0, L]; H)$, we define

$$F(w)(z) = P_\alpha^\delta(z)f^\delta + S_\alpha^\delta(z)h^\delta + \int_0^z S_\alpha^\delta(z - y)G_\delta(y, w(y))dy.$$

It is sufficient to show that F has a unique fixed point in $C([0, L]; H)$. This fact will be proved by the contraction principle.

We claim by mathematical induction with respect to $m = 1, 2, \ldots$ that, for all $w,\, v \in C([0, L]; H)$,

$$\|F^m(w)(z) - F^m(v)(z)\|_H \le \frac{(\mathrm{C}\, z/M(\alpha))^m}{m!} |||w - v|||, \quad \forall z \in [0, L], \tag{3.35}$$

where $|||.|||$ is the sup norm in $C([0, L]; H)$ and $\mathrm{C} = \mathrm{C}(\lambda_1, k(B_\delta))$ is given by

$$\mathrm{C} = \frac{\sqrt{2}k(B_\delta)}{\sqrt{\lambda_1}}. \tag{3.36}$$

For $m = 1$, using (1.7), (3.34) and (3.36) we have

$$\begin{aligned}
\|F(w)(z) - F(v)(z)\|_H &= \left\| \int_0^z S_\alpha^\delta(z-y) \Big[G_\delta(y, w(y)) - G_\delta(y, v(y)) \Big] dy \right\|_H \\
&\le \int_0^z \left\| S_\alpha^\delta(z-y) \right\|_{\mathbb{L}(H)} \left\| G_\delta(y, w(y)) - G_\delta(y, v(y)) \right\|_H dy \\
&\le 2k(B_\delta) \int_0^z \frac{M(\alpha)^{-(z-y)/L}}{\sqrt{2\lambda_1}} \left\| w(y) - v(y) \right\|_H dy \\
&\le \frac{\mathrm{C}}{M(\alpha)} \int_0^z \left\| w(y) - v(y) \right\|_H dy \le \frac{\mathrm{C}\, z}{M(\alpha)} |||w - v|||.
\end{aligned}$$

Suppose that (3.35) holds for $m = j$. We prove that (3.35) holds for $m = j + 1$. Indeed, we have

$$\begin{aligned}
\left\|F^{j+1}(w)(z) - F^{j+1}(v)(z)\right\|_H &= \left\|F(F^j(w))(z) - F(F^j(v))(z)\right\|_H \\
&\le \frac{\mathrm{C}}{M(\alpha)} \int_0^z \left\|F^j(w)(y) - F^j(v)(y)\right\|_H dy \\
&\le \frac{\mathrm{C}}{M(\alpha)} |||w - v||| \int_0^z \left(\frac{\mathrm{C}}{M(\alpha)} \right)^j \frac{y^j}{j!} dy \\
&= \left(\frac{\mathrm{C}}{M(\alpha)} \right)^{j+1} \frac{z^{j+1}}{(j+1)!} |||w - v|||.
\end{aligned}$$

Therefore, the inequality (3.35) holds for all $m = 1, 2, \ldots$ by the induction principle. In particular, one has

$$|||F^m(w)(z) - F^m(v)(z)||| \le \frac{(\mathrm{C}\, L/M(\alpha))^m}{m!} |||w - v|||.$$

Since

$$\lim_{m \to +\infty} \frac{(\mathrm{C}\, L/M(\alpha))^m}{m!} = 0,$$

there exists a positive integer number m_0 such that F^{m_0} is a contraction mapping. It follows that F^{m_0} has a unique fixed point u_α^δ in $C([0, L]; H)$. Since $F^{m_0}(F(u_\alpha^\delta)) = F(F^{m_0}(u_\alpha^\delta)) = F(u_\alpha^\delta)$, we obtain $F(u_\alpha^\delta) = u_\alpha^\delta$ due to the uniqueness of the fixed point of F^{m_0}. The uniqueness of the fixed point of F also follows from the uniqueness of the fixed point of F^{m_0}. The unique fixed point u_α^δ of F is the solution of (3.18). $\square$

Lemma 3.3. *The integral equation*

$$w(z) = P_\alpha^\delta(z)f + S_\alpha^\delta(z)h + \int_0^z S_\alpha^\delta(z-y)G_\delta(y,w(y))dy \tag{3.37}$$

has a unique solution $v_\alpha^\delta \in C([0,L];H)$. Furthermore, we have the following estimate:

$$\|u_\alpha^\delta(z) - v_\alpha^\delta(z)\|_H \le \left(1 + \frac{1}{\sqrt{2\lambda_1}}\right) M(\alpha)^{-z/L} \exp(\mathrm{C}\, z)\delta, \quad z \in [0,L]. \tag{3.38}$$

Proof. Using Lemma 3.2, we conclude that the integral equation (3.37) has a unique solution $v_\alpha^\delta \in C([0,L];H)$. Using (1.7), (3.18) and (3.34), we have

$$\begin{aligned}
\|u_\alpha^\delta(z) - v_\alpha^\delta(z)\|_H &\le \left\|P_\alpha^\delta(z)(f^\delta - f)\right\|_H + \left\|S_\alpha^\delta(z)(h^\delta - h)\right\|_H \\
&\quad + \left\| \int_0^z S_\alpha^\delta(z-y)\Big[G_\delta(y,u_\alpha^\delta(y)) - G_\delta(y,v_\alpha^\delta(y))\Big]dy\right\|_H \\
&\le \|P_\alpha^\delta(z)\|_{\mathbb{L}(H)}\|f^\delta - f\|_H + \|S_\alpha^\delta(z)\|_{\mathbb{L}(H)}\|h^\delta - h\|_H \\
&\quad + \int_0^z \|S_\alpha^\delta(z-y)\|_{\mathbb{L}(H)}\|G_\delta(y,u_\alpha^\delta(y)) - G_\delta(y,v_\alpha^\delta(y))\|_H dy \\
&\le M(\alpha)^{-z/L}\delta + \frac{M(\alpha)^{-z/L}}{\sqrt{2\lambda_1}}\delta + k(B_\delta)\int_0^z \frac{\sqrt{2}M(\alpha)^{\frac{y-z}{L}}}{\sqrt{\lambda_1}}\|u_\alpha^\delta(y) - v_\alpha^\delta(y)\|_H dy.
\end{aligned} \tag{3.39}$$

Multiplying both sides of (3.39) by $M(\alpha)^{z/L}$, it yields

$$M(\alpha)^{z/L}\|u_\alpha^\delta(z) - v_\alpha^\delta(z)\|_H \le \left(1 + \frac{1}{\sqrt{2\lambda_1}}\right)\delta + \mathrm{C}\int_0^z M(\alpha)^{y/L}\|u_\alpha^\delta(y) - v_\alpha^\delta(y)\|_H dy.$$

Applying Gronwall's inequality, we obtain

$$M(\alpha)^{z/L}\|u_\alpha^\delta(z) - v_\alpha^\delta(z)\|_H \le \left(1 + \frac{1}{\sqrt{2\lambda_1}}\right)\exp(\mathrm{C}\, z)\delta.$$

Dividing both sides the latter estimate by $M(\alpha)^{z/L}$, we conclude that (3.38) holds. This completes the proof of the lemma. □

Now, we present some estimates in the Gevrey space (3.27).

Lemma 3.4. *Assume that $f \in D\left(e^{L\sqrt{-A}}\right)$. Then, we have the following estimates:*

$$\|P_\alpha^\delta(z)f - P(z)f\|_H \le M(\alpha)^{1-\frac{z}{L}}\|f\|_{D\left(e^{L\sqrt{-A}}\right)}, \tag{3.40}$$

and

$$\|S_\alpha^\delta(z)f - S(z)f\|_H \le \frac{M(\alpha)^{1-\frac{z}{L}}}{\sqrt{\lambda_1}}\|f\|_{D\left(e^{L\sqrt{-A}}\right)}. \tag{3.41}$$

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Proof. Since $f = \sum_{n=1}^{\infty} \left\langle f, \phi_n \right\rangle \phi_n$, noting that $[R(\alpha, \lambda_n) - 1]^2 \leq [Q(\alpha, \lambda_n) - 1]^2$ and using (2.10), (2.11), (3.19)–(3.21), we have

$$\begin{aligned}
\|P_\alpha^\delta(z)f - P(z)f\|_H^2 &\leq \sum_{n=1}^{\infty} \frac{\left[Q(\alpha, \lambda_n) - 1\right]^2 e^{2\sqrt{\lambda_n}z} + \left[R(\alpha, \lambda_n) - 1\right]^2 e^{-2\sqrt{\lambda_n}z}}{2} \left|\left\langle f, \phi_n \right\rangle\right|^2 \\
&\leq \sum_{n=1}^{\infty} \left[Q(\alpha, \lambda_n) - 1\right]^2 e^{2\sqrt{\lambda_n}(z-L)} e^{2\sqrt{\lambda_n}L} \left|\left\langle f, \phi_n \right\rangle\right|^2 \\
&\leq M(\alpha)^{2-\frac{2z}{L}} \sum_{n=1}^{\infty} e^{2\sqrt{\lambda_n}L} \left|\left\langle f, \phi_n \right\rangle\right|^2 = M(\alpha)^{2-\frac{2z}{L}} \|f\|^2_{D\left(e^{L\sqrt{-A}}\right)}
\end{aligned}$$

and

$$\begin{aligned}
\|S_\alpha^\delta(z)f - S(z)f\|_H^2 &\leq \sum_{n=1}^{\infty} \frac{\left[Q(\alpha, \lambda_n) - 1\right]^2 e^{2\sqrt{\lambda_n}z}}{\lambda_n} \left|\left\langle f, \phi_n \right\rangle\right|^2 \\
&\leq \sum_{n=1}^{\infty} \frac{\left[Q(\alpha, \lambda_n) - 1\right]^2 e^{2\sqrt{\lambda_n}(z-L)}}{\lambda_1} e^{2\sqrt{\lambda_n}L} \left|\left\langle f, \phi_n \right\rangle\right|^2 \\
&\leq \frac{M(\alpha)^{2-\frac{2z}{L}}}{\lambda_1} \sum_{n=1}^{\infty} e^{2\sqrt{\lambda_n}L} \left|\left\langle f, \phi_n \right\rangle\right|^2 = \frac{M(\alpha)^{2-\frac{2z}{L}}}{\lambda_1} \|f\|^2_{D\left(e^{L\sqrt{-A}}\right)}.
\end{aligned}$$

Taking the square roots in these expressions we obtain the estimates (3.40) and (3.41). □

Lemma 3.5. *For δ small enough, we have:*

(i) *If the assumption (3.29) holds, then*

$$\|v_\alpha^\delta(z) - u(z)\|_H \leq I_1 \exp(\mathrm{C}\, z) M(\alpha)^{1-z/L}, \quad z \in [0, L]. \tag{3.42}$$

(ii) *If the assumption (3.30) holds and $R(\alpha, \lambda_n) = 1$, then*

$$\|v_\alpha^\delta(z) - u(z)\|_H \leq 2I_2 \max\left\{1, \frac{1}{\sqrt{\lambda_1}}\right\} \exp(\mathrm{C}\, z) M(\alpha)^{1-z/L}, \quad z \in [0, L]. \tag{3.43}$$

(iii) *If the assumption (3.30) holds and $R(\alpha, \lambda_n) = Q(\alpha, \lambda_n)$, then*

$$\|v_\alpha^\delta(z) - u(z)\|_H \leq I_2 \exp(\mathrm{C}\, z) M(\alpha)^{1-z/L}, \quad z \in [0, L]. \tag{3.44}$$

Proof. We assume that δ is small enough such that (1.6) holds. We divide the proof into two parts.

Part A. Assume that (3.29) holds. From (2.17) and Lemma 3.3, we have

$$\begin{aligned}
v_\alpha^\delta(z) - u(z) &= \left[P_\alpha^\delta(z)f - P(z)f\right] + \left[S_\alpha^\delta(z)h - S(z)h\right] + \int_0^z S_\alpha^\delta(z-y)\left[G_\delta(y, v_\alpha^\delta(y)) - G_\delta(y, u(y))\right]dy \\
&\quad + \int_0^z \left[S_\alpha^\delta(z-y)G(y, u(y)) - S(z-y)G(y, u(y))\right]dy.
\end{aligned}$$

By taking the norms in H on both sides and using (1.6), (3.34), (3.40) and (3.41), we obtain

$$\begin{aligned}
\|v_\alpha^\delta(z) - u(z)\|_H &\le \|P_\alpha^\delta(z)f - P(z)f\|_H + \|S_\alpha^\delta(z)h - S(z)h\|_H \\
&\quad + \int_0^z \|S_\alpha^\delta(z-y)\|_{\mathbb{L}(H)} \Big\|G_\delta(y, v_\alpha^\delta(y)) - G_\delta(y,u(y))\Big\|_H dy \\
&\quad + \int_0^z \Big\|S_\alpha^\delta(z-y)G(y,u(y)) - S(z-y)G(y,u(y))\Big\|_H dy \\
&\le M(\alpha)^{1-z/L}\|f\|_{D\left(e^{L\sqrt{-A}}\right)} + \frac{M(\alpha)^{1-z/L}}{\sqrt{\lambda_1}}\|h\|_{D\left(e^{L\sqrt{-A}}\right)} \\
&\quad + \mathrm{C}\int_0^z M(\alpha)^{\frac{y-z}{L}} \Big\|v_\alpha^\delta(y) - u(y)\Big\|_H dy + \frac{1}{\sqrt{\lambda_1}}\int_0^z M(\alpha)^{1-\frac{z-y}{L}}\Big\|G(y,u(y))\Big\|_{D\left(e^{L\sqrt{-A}}\right)} dy.
\end{aligned}$$

Multiplying by $M(\alpha)^{z/L}$ both sides, we have

$$\begin{aligned}
&M(\alpha)^{z/L}\|v_\alpha^\delta(z) - u(z)\|_H \\
&\le M(\alpha)\left[\|f\|_{D\left(e^{L\sqrt{-A}}\right)} + \frac{1}{\sqrt{\lambda_1}}\|h\|_{D\left(e^{L\sqrt{-A}}\right)} + \frac{1}{\sqrt{\lambda_1}}\int_0^z M(\alpha)^{y/L}\Big\|G(y,u(y))\Big\|_{D\left(e^{L\sqrt{-A}}\right)} dy\right] \\
&\quad + \mathrm{C}\int_0^z M(\alpha)^{y/L}\Big\|v_\alpha^\delta(y) - u(y)\Big\|_H dy.
\end{aligned}$$

This together with (3.29) implies that

$$M(\alpha)^{z/L}\|v_\alpha^\delta(z) - u(z)\|_H \le M(\alpha) I_1 + \mathrm{C}\int_0^z M(\alpha)^{y/L}\Big\|v_\alpha^\delta(y) - u(y)\Big\|_H dy.$$

Then Gronwall's inequality yields

$$M(\alpha)^{z/L}\|v_\alpha^\delta(z) - u(z)\|_H \le M(\alpha) I_1 \exp(\mathrm{C}\, z).$$

From this we obtain

$$\|v_\alpha^\delta(z) - u(z)\|_H \le I_1 M(\alpha)^{1-z/L}\exp(\mathrm{C}\, z), \quad z \in [0, L],$$

which is the desired estimate (3.42).

Part B. Assume that (3.30) holds.

For the proof of this part, we consider two cases, as follows.

Case 1: The filter $R(\alpha, \lambda_n) = 1$.

Taking the inner product of $u(z)$ and its derivative from (2.16), and adding the results give

$$\Big\langle u(z), \phi_n\Big\rangle + \frac{\Big\langle \frac{du(z)}{dz}, \phi_n\Big\rangle}{\sqrt{\lambda_n}} = e^{\sqrt{\lambda_n}z}\Big\langle f, \phi_n\Big\rangle + \frac{e^{\sqrt{\lambda_n}z}}{\sqrt{\lambda_n}}\Big\langle h, \phi_n\Big\rangle + \int_0^z \frac{e^{\sqrt{\lambda_n}(z-y)}}{\sqrt{\lambda_n}}\Big\langle G(y,u(y)), \phi_n\Big\rangle dy.$$

From (2.16), Lemma 3.3, and (3.19), (3.20) with $R(\alpha, \lambda_n) = 1$, we obtain

$$v_\alpha^\delta(z) - u(z) = \sum_{n=1}^{\infty} \frac{(Q(\alpha,\lambda_n) - 1)}{2}\left[\left\langle u(z), \phi_n \right\rangle + \frac{\left\langle \frac{du(z)}{dz}, \phi_n \right\rangle}{\sqrt{\lambda_n}}\right]\phi_n$$
$$+ \sum_{n=1}^{\infty}\left[\int_0^z S_\alpha^\delta(z-y)\Big[G_\delta(y, v_\alpha^\delta(y)) - G(y, u(y))\Big]dy\right]\phi_n.$$

Then the triangle inequality and equations (1.6), (1.7), (3.21) and (3.34) lead to

$$\|v_\alpha^\delta(z) - u(z)\|_H \le \sqrt{\sum_{n=1}^{\infty}\Big(Q(\alpha,\lambda_n) - 1\Big)^2 e^{2\sqrt{\lambda_n}(z-L)} e^{2\sqrt{\lambda_n}(L-z)}\left[\left\langle u(z), \phi_n \right\rangle + \frac{\left\langle \frac{du(z)}{dz}, \phi_n \right\rangle}{\sqrt{\lambda_n}}\right]^2}$$
$$+ \int_0^z \|S_\alpha^\delta(z-y)\|_{\mathbb{L}(H)} \Big\| G_\delta(y, v_\alpha^\delta(y)) - G_\delta(y, u(y))\Big\|_H dy$$
$$\le M(\alpha)^{1-z/L}\sqrt{\sum_{n=1}^{\infty} e^{2\sqrt{\lambda_n}(L-z)}\left(\langle u(z), \phi_n\rangle + \frac{\left\langle \frac{du(z)}{dz}, \phi_n \right\rangle}{\sqrt{\lambda_n}}\right)^2}$$
$$+ \mathrm{C}\int_0^z M(\alpha)^{\frac{y-z}{L}}\Big\|v_\alpha^\delta(y) - u(y)\Big\|_H dy$$
$$\le M(\alpha)^{1-z/L}\sqrt{2\sum_{n=1}^{\infty} e^{2\sqrt{\lambda_n}(L-z)}\Big|\langle u(z), \phi_n\rangle\Big|^2 + \frac{2}{\lambda_1}\sum_{n=1}^{\infty} e^{2\sqrt{\lambda_n}(L-z)}\Big|\langle\frac{du(z)}{dz}, \phi_n\rangle\Big|^2}$$
$$+ \mathrm{C}\int_0^z M(\alpha)^{\frac{y-z}{L}}\Big\|v_\alpha^\delta(y) - u(y)\Big\|_H dy$$
$$\le M(\alpha)^{1-z/L}\sqrt{2\Big\|u(z)\Big\|^2_{D\left(e^{(L-z)\sqrt{-A}}\right)} + \frac{2}{\lambda_1}\Big\|\frac{du(z)}{dz}\Big\|^2_{D\left(e^{(L-z)\sqrt{-A}}\right)}}$$
$$+ \mathrm{C}\int_0^z M(\alpha)^{\frac{y-z}{L}}\Big\|v_\alpha^\delta(y) - u(y)\Big\|_H dy.$$

Multiplying by $M(\alpha)^{z/L}$ both sides, we have

$$M(\alpha)^{z/L}\|v_\alpha^\delta(z) - u(z)\|_H \le 2\max\left\{1, \frac{1}{\sqrt{\lambda_1}}\right\} M(\alpha) I_2 + \mathrm{C}\int_0^z M(\alpha)^{y/L}\Big\|v_\alpha^\delta(y) - u(y)\Big\|_H dy.$$

Then Gronwall's inequality yields

$$M(\alpha)^{z/L}\|v_\alpha^\delta(z) - u(z)\|_H \le 2\max\left\{1, \frac{1}{\sqrt{\lambda_1}}\right\} M(\alpha) I_2 \exp(\mathrm{C}\, z).$$

This implies that

$$\|v_\alpha^\delta(z)-u(z)\|_H \le 2I_2 \max\left\{1, \frac{1}{\sqrt{\lambda_1}}\right\} M(\alpha)^{1-z/L}\exp(\mathrm{C}\, z), \quad z\in[0,L], \tag{3.45}$$

which is the desired estimate (3.43).

Case 2: The filter $R(\delta,\lambda_n)=Q(\delta,\lambda_n)$.

Let the function w_α^δ be defined by

$$w_\alpha^\delta(z)=\sum_{n=1}^{\infty} Q(\alpha,\lambda_n)\Big\langle u(z),\phi_n\Big\rangle\phi_n. \tag{3.46}$$

From (3.19), (3.20) with $R(\alpha,\lambda_n)=Q(\delta,\lambda_n)$ we have

$$P_\alpha^\delta(z)f=\sum_{n=1}^{\infty} Q(\alpha,\lambda_n)\cosh(\sqrt{\lambda_n}z)\Big\langle f,\phi_n\Big\rangle\phi_n, \quad S_\alpha^\delta(z)h=\sum_{n=1}^{\infty} Q(\alpha,\lambda_n)\frac{\sinh(\sqrt{\lambda_n}z)}{\sqrt{\lambda_n}}\Big\langle h,\phi_n\Big\rangle\phi_n,$$

and it is easy to see that w_α^δ satisfies the following identity:

$$w_\alpha^\delta(z)=P_\alpha^\delta(z)f+S_\alpha^\delta(z)h+\int_0^z S_\alpha^\delta(z-y)G(y,u(y))dy. \tag{3.47}$$

This is equivalent to replacing in (2.16) the eigenfunctions ϕ_n by the filtered ones $\phi_n\sqrt{Q(\alpha,\lambda_n)}$. Combining (3.37) and (3.47), we get

$$\|v_\alpha^\delta(z)-w_\alpha^\delta(z)\|_H=\left\|\int_0^z S_\alpha^\delta(z-y)G_\delta(y,v_\alpha^\delta(y))dy-\int_0^z S_\alpha^\delta(z-y)G(y,u(y))dy\right\|_H.$$

Then (1.6), (1.7) and (3.34) lead to

$$\begin{aligned}\|v_\alpha^\delta(z)-w_\alpha^\delta(z)\|_H &\le \int_0^z \left\|S_\alpha^\delta(z-y)\right\|_{\mathbb{L}(H)}\left\|G_\delta(y,v_\alpha^\delta(y))-G_\delta(y,u(y))\right\|_H dy\\ &\le \mathrm{C}\int_0^z M(\alpha)^{\frac{y-z}{L}}\left\|v_\alpha^\delta(y)-u(y)\right\|_H dy,\end{aligned} \tag{3.48}$$

where we note again equation (1.6) holds for δ small enough. Moreover, from (2.9), (3.21) and (3.46), we deduce that

$$\begin{aligned}\|w_\alpha^\delta(z)-u(z)\|_H &= \sqrt{\sum_{n=1}^{\infty}\left|Q(\alpha,\lambda_n)-1\right|^2 e^{2\sqrt{\lambda_n}(z-L)}e^{2\sqrt{\lambda_n}(L-z)}\left|\Big\langle u(z),\phi_n\Big\rangle\right|^2}\\ &\le M(\alpha)^{1-z/L}\sqrt{\sum_{n=1}^{\infty} e^{2\sqrt{\lambda_n}(L-z)}\left|\Big\langle u(z),\phi_n\Big\rangle\right|^2}\\ &\le M(\alpha)^{1-z/L}\|u(z)\|_{D\left(e^{(L-z)\sqrt{-A}}\right)}\le M(\alpha)^{1-z/L}I_2.\end{aligned} \tag{3.49}$$

Summing up (3.48) and (3.49), and using the triangle inequality we obtain

$$\|v_\alpha^\delta(z)-u(z)\|_H \le M(\alpha)^{1-z/L} I_2 + \mathrm{C}\int_0^z M(\alpha)^{\frac{y-z}{L}} \left\|v_\alpha^\delta(y)-u(y)\right\|_H dy. \tag{3.50}$$

Multiplying by $M(\alpha)^{z/L}$ both sides of (3.50) we obtain

$$M(\alpha)^{z/L}\left\|v_\alpha^\delta(z)-u(z)\right\|_H \le M(\alpha) I_2 + \mathrm{C}\int_0^z M(\alpha)^{y/L}\left\|v_\alpha^\delta(y)-u(y)\right\|_H dy. \tag{3.51}$$

Finally, Gronwall's inequality yields

$$\left\|v_\alpha^\delta(z)-u(z)\right\|_H \le I_2 M(\alpha)^{1-z/L}\exp(\mathrm{C}\, z), \quad z\in[0,L],$$

which is the desired estimate (3.44). □

Now, we shall finish the proof of Theorem 3.1. Applying triangle inequality together with (3.38), we obtain

$$\begin{aligned}\left\|u_\alpha^\delta(z)-u(z)\right\|_H &\le \left\|u_\alpha^\delta(z)-v_\alpha^\delta(z)\right\|_H + \left\|v_\alpha^\delta(z)-u(z)\right\|_H \\ &\le \left(1+\frac{1}{\sqrt{2\lambda_1}}\right) M(\alpha)^{-z/L}\exp(\mathrm{C}\, z)\delta + \left\|v_\alpha^\delta(z)-u(z)\right\|_H.\end{aligned}$$

If assumption (3.29) holds then

$$\left\|u_\alpha^\delta(z)-u(z)\right\|_H \le M(\alpha)^{1-z/L}\exp(\mathrm{C}\, z)\left[I_1+\left(1+\frac{1}{\sqrt{2\lambda_1}}\right)\frac{\delta}{M(\alpha)}\right]. \tag{3.52}$$

If assumption (3.30) holds then

$$\left\|u_\alpha^\delta(z)-u(z)\right\|_H \le M(\alpha)^{1-z/L}\exp(\mathrm{C}\, z)\left[2\max\left\{1,\frac{1}{\sqrt{\lambda_1}}\right\} I_2+\left(1+\frac{1}{\sqrt{2\lambda_1}}\right)\frac{\delta}{M(\alpha)}\right]. \tag{3.53}$$

Hence, the estimates (3.31) hold.

We shall show that for each fixed $0\le z<L$, the function $u_\alpha^\delta(z)$ gives a good approximation to $u(z)$. However, it is difficult to derive its approximation at $z=L$. We therefore need an adjustment in choosing the regularized solution. The main idea is that we first use the continuity of u to approximate the initial value $u(L)$ by $u(z_\delta)$ for some suitable small $z_\delta<L$, and then approximate $u(z_\delta)$ by $u_\alpha^\delta(z_\delta)$. The parameter z_δ will be chosen as follows. For every $\delta>0$, there exists a unique $z_\delta\in(0,L)$ such that

$$(L-z_\delta) = M(\alpha)^{1-\frac{z_\delta}{L}}. \tag{3.54}$$

It implies that $\frac{\ln(L-z_\delta)}{L-z_\delta} = \frac{\ln(M(\alpha))}{L}$. Using the inequality $\ln z > -\frac{1}{z}$ for every $z>0$, we obtain $L-z_\delta < \sqrt{\frac{L}{\ln(\frac{1}{M(\alpha)})}}$. To estimate the error we use the triangle inequality

$$\begin{aligned}\|u_\alpha^\delta(z)-u(L)\|_H &\le \|u(L)-u(z)\|_H + \|u(z)-u_\alpha^\delta(z)\|_H \\ &\le \sup_{0\le z\le L}\left\|\frac{du(z)}{dz}\right\|_H (L-z) + \|u(z)-u_\alpha^\delta(z)\|_H.\end{aligned}$$

The estimate above applied for $z=z_\delta$ together with the estimates (3.31) lead to

$$\|u_\alpha^\delta(z_\delta) - u(L)\|_H \le \sup_{0\le z\le L} \left\|\frac{du(z)}{dz}\right\|_H (L - z_\delta) + \|u(z_\delta) - u_\alpha^\delta(z_\delta)\|_H$$
$$\le \sup_{0\le z\le L} \left\|\frac{du(z)}{dz}\right\|_H \sqrt{\frac{L}{\ln(\frac{1}{M(\alpha)})}} + \Theta,$$

hence (3.33) holds. This ends the proof of the main Theorem 3.1 about general regularization filters for quasilinear Cauchy problems with locally Lipschitz nonlinear source.

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