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Complex structures on real vector lattices ☆

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ABSTRACT

The main result of this paper states that a universally complete real vector lattice admits a band preserving complex structure if and only if it contains no locally one-dimensional bands. A description of order bounded disjointness preserving complex structures on Archimedean semiprime f-algebras is also given.

1. Introduction

A real vector space X is said to admit a *complex structure* if there exists a linear operator S on X with $S \circ S = -I_X$ or, equivalently, with S invertible and $S^{-1} = -S$. Such operator S (often called itself a complex structure) enables one to define on X a structure of vector space over the complexes $\mathbb{C}$ by putting $zx = (\alpha + i\beta)x := \alpha x + \beta S(x)$ for all $z = \alpha + i\beta \in \mathbb{C}$ and $x \in X$. Moreover, if X is a Banach space and S is bounded, then one can define a complex norm $|x| := \sup\{\|e^{i\theta}x\| : \theta \in [0, 2\pi]\}$ on X which is equivalent to the original one. Denote by X_S the resulting complex vector (Banach) space. A finite-dimensional vector space admits complex structures if and only if the dimension of the space is even. In the infinite-dimensional setting, there are real Banach spaces admitting no complex structure. This is the case of the James' space, as it was shown by Dieudonné [7]. More examples of this kind have been constructed over the years: uniformly convex examples due to Szarek [25], indecomposable space of Gowers and Maurey [12] or, more generally, any space such that every operator on it is a strictly singular perturbation of a multiple of the identity, etc. (see also [11,13]). We refer to the recent papers by Ferenczi [8] and Ferenczi and Galego [9] for further discussion.

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This line of development may be continued in two opposite directions: to look either for real Banach spaces (lattices) without complex structure but having some additional metric properties, or for complex structures on vector spaces (lattices) with some additional order properties. For example, there is a strong way for a Banach space to lack complex structure: A real Banach space X is said to be *extremely non-complex* if $\|I_X + T^2\| = 1 + \|T^2\|$ holds for every bounded linear operator T on X. There are infinitely many nonisomorphic extremely non-complex Banach spaces (and even Banach lattices), see Koszmider, Martín, and Merí [16] and [17]. At the same time it is an interesting question whether or not a real vector lattice admits a complex structure which is band preserving? To the best of the author's knowledge no research in this direction has been attempted.

The paper is organized as follows. In Section 2 it is proved that there is no order bounded band preserving complex structure on an Archimedean vector lattice and an explicit description of order bounded disjointness preserving complex structures on Archimedean semiprime f-algebras is given. Section 3 collects necessary information about d-bases in vector lattices. Section 4 contains the main result of this paper stating that every real non-locally one-dimensional universally complete vector lattice admits a band preserving complex structure (which is not order bounded according to Proposition 2.1) and is isomorphic to the complexification of some real vector lattice. Some additional remarks are presented in Section 5.

2. Disjointness preserving complex structures

Recall that a linear operator on E is said to be *band preserving* if it leaves every band invariant. If E is a vector lattice with a projection property, then a linear operator in E is band preserving if and only if it commutes with all band projections [3, Theorem 8.3]. For the theory of vector lattices and positive operators we refer to the book [3].

Proposition 2.1. *If E is an Archimedean vector lattice then there is no order bounded band preserving complex structure on E.*

Proof. An order bounded band preserving complex structure in E is an orthomorphism by definition. For every Archimedean vector lattice E, the space of all orthomorphisms $\mathrm{Orth}(E)$ under composition is an Archimedean f-algebra, having the identity operator I as its multiplicative unit, see [3, Theorem 8.24]. But if an f-algebra squares are positive, so the relations $S \in \mathrm{Orth}(E)$ and $S^2 = -I$ are not compatible. □

Proposition 2.1 is no longer true beyond the class of disjointness preserving operators. A linear operator S on E is said to be *disjointness preserving* if $|x| \wedge |y| = 0$ implies $|Tx| \wedge |Ty| = 0$ for all $x, y \in E$. Evidently, a band preserving operator is disjointness preserving. The following simple example shows that there exists an order bounded disjointness preserving complex structure.

Example 2.2. There exist order bounded disjointness preserving complex structures on $C(K)$ space. Take $K := [-2, -1] \cup [1, 2]$ and a continuous function $b : [1, 2] \to \mathbb{R} \setminus \{0\}$. Let $a \in C(K)$ be such that a coincides with b on $[1, 2]$ and $a(t) = -1/b(t)$ for $t \in [-2, -1]$. Define a linear operator S in $C(K)$ by $(Sx)(t) = a(t)x(\sigma(t))$, $(t \in K)$ with the continuous function $\sigma : K \to K$ given by $\sigma(t) = -t$. Then S is an order bounded disjointness preserving complex structures on $C(K)$.

Next we describe all order bounded disjointness preserving complex structures on Archimedean semiprime f-algebras. To do this we need the following result by Hart [15].

Theorem 2.3. *Let A and B be Archimedean semiprime f-algebras and h be a bijective order bounded disjointness preserving operator from A onto B. Then there exists a unique f-algebra isomorphism $\tilde{h}$ from* $\mathrm{Orth}(A)$ *onto* $\mathrm{Orth}(B)$ *such that $h(xy) = \tilde{h}(x)h(y)$ for all $x, y \in A$.*

We need one more f-algebra concept. An f-algebra A is called *n-root closed* if for every $x \in A_+$ there exists $\sqrt[n]{x} \in A_+$ with $(\sqrt[n]{x})^n = x$.

Proposition 2.4. *Let A be an Archimedean semiprime f-algebra which is n-root closed for some $1 < n \in \mathbb{N}$. An order bounded linear operator $S : A \to A$ is a disjointness preserving complex structure if and only if there exists an invertible orthomorphism $w \in \mathrm{Orth}(A)$ and a unique f-algebra isomorphism $h : A \to A$ such that*

$$h \circ h = I_A, \quad w \circ \tilde{h}(w) = -I_A, \quad S = w \circ h.$$

Proof. Assume that A satisfies the stated requirements. It was proved by Boulabiar, Buskes, and Triki in [6, Corollary 6.2] that an order bounded disjointness preserving operator $S : A \to A$ is representable as $S = w \circ h$, whenever S is bijective, $w \in \mathrm{Orth}(A)$ is invertible and $h : A \to A$ is an f-algebra isomorphism. If S is a complex structure then applying this representation and Theorem 2.3 we deduce

$$-x = S(Sx) = wh(wh(x)) = w\tilde{h}(w)h(h(x)),$$

so that $\pi \circ h^2 = -I_A$ where $\pi := w\tilde{h}(w)$. It follows that

$$-xy = \pi(h^2(xy)) = h^2(x)\pi(h^2(y)) = -h^2(x)y$$

for all $x, y \in A$ and hence $h^2 = I_A$. Moreover, $\pi = \pi \circ h^2 = -I_A$. The converse is straightforward. $\square$

Corollary 2.5. *Let A be an Archimedean f-algebra with unit element $\mathbb{1}$. An order bounded linear operator $S : A \to A$ is a disjointness preserving complex structure if and only if there exists a unique f-algebra isomorphism $h : A \to A$ such that the representation $S(x) = S(\mathbb{1})h(x)$ holds for all $x \in A$ with $h \circ h = I_A$ and $S(\mathbb{1})h(S(\mathbb{1})) = -\mathbb{1}$.*

Proof. This can be deduced from Proposition 2.4 or, alternatively, proved making use of the following result due to Boulabiar, Buskes and Henriksen [5]: An order bounded disjointness preserving operator $S : A \to A$ is representable as $Sx = S(\mathbb{1})h(x)$, $(x \in A)$, where $h : A \to A$ is an f-algebra isomorphism, provided that S^{-1} is also band preserving. $\square$

Corollary 2.6. *Let K be a compact Hausdorff space. An order bounded linear operator $S : C(K) \to C(K)$ is a disjointness preserving complex structure if and only if there exists a unique homeomorphism $\sigma : K \to K$ such that*

$$\sigma \circ \sigma = I_K, \quad a(a \circ \sigma) = -\mathbb{1},$$
$$Sx = a(x \circ \sigma) \quad (x \in C(K)),$$

with $a = S(\mathbb{1})$ and $\mathbb{1}$ being the function identically equal to one on K.

Proof. This can be deduced from Corollary 2.5 or, alternatively, proved making use of following result due to Arendt [4]: An order bounded disjointness preserving operator $S : C(K) \to C(K)$ is a weighted composition operator, i.e. there exists a mapping $\sigma : K \to K$ such that

$$(Sx)(t) = a(t)x(\sigma(t)) \quad (x \in C(K),\, t \in K)),$$

with $a = S(\mathbb{1})$ and σ uniquely defined and continuous on $\{t \in K : a(t) \neq 0\}$. $\square$

3. d-bases

With the use of a Hamel basis, it can be proved that there exist nontrivial additive involutions on $\mathbb{R}$, that is, nonzero functions $f : \mathbb{R} \to \mathbb{R}$ with $f \circ f = I_{\mathbb{R}}$, see Kuczma [18, Theorem 12.5.2]. We carry out similar constructions making use of a d-basis instead of a Hamel basis for the proof of our main result. Before launching into details we state some needed properties of d-bases from Abramovich and Kitover [1].

Recall that a vector lattice is *universally complete* if it is Dedekind complete and laterally complete. Let E be a universally complete vector lattice. A subset $\mathcal{E} \subset E$ is called *d-independent*, if for each band projection ρ on E the set $\{\rho e : \rho e \neq 0,\ e \in \mathcal{E}\}$ is linearly independent, that is, the collection of all non-zero members of the set $\rho\mathcal{E}$ is linearly independent. Any maximal (by inclusion) set of d-independent vectors is called a *d-basis*. A universally complete vector lattice E with weak order unit $\mathbb{1}$ is called *locally one-dimensional* if $\{\mathbb{1}\}$ is a d-basis in E, [1, Definition 6.1]. For the notions of d-independence and d-basis for an arbitrary vector lattice see Abramovich and Kitover [2].

Lemma 3.1. *Let $\mathcal{E}$ be a fixed d-basis in a universally complete vector lattice E. Then for each $x \in E$ there exists a collection $(\rho_\xi)_{\xi\in\Xi}$ of pairwise disjoint band projections (depending on x) such that $\sum_{\xi\in\Xi} \rho_\xi = I_E$ and the following representation holds:*

$$x = \sum_{\xi\in\Xi} \sum_{e\in\mathcal{E}} \alpha_{\xi,e} \rho_\xi e, \tag{1}$$

where $\alpha_{\xi,e}$ are some scalars (depending on x), such that for each $\xi \in \Xi$ only a finite number of coefficients $\alpha_{\xi,e}$ may be nonzero.

Proof. See [1, p. 33] and [19, Proposition 5.1.1 (3)]. □

The expression (1) is called a *d-expansion* of x with respect to d-basis $\mathcal{E}$. A d-expansion is not unique, as we always can subdivide any projection band E_i into the direct sum of two or more complementary projection bands.

Theorem 3.2. *If E is a universally complete vector lattice, then for each non-zero band B in E there is a non-zero band $B_0 \subseteq B$ such that there exists a d-basis in B_0 consisting of weak order units in B_0.*

Proof. See [1, Theorem 6.4] and [20, Theorem 4.6.9]. □

Theorem 3.3. *Let $\mathcal{E}$ be a d-basis in a universally complete vector lattice consisting of weak units. Then either $\mathcal{E}$ is a singleton, or $\mathcal{E}$ is of infinite cardinality.*

Proof. See [1, Theorem 6.8]. □

The following result was obtained by Abramovich and Kitover in [1, Theorem 14.9] and by McPolin and Wickstead in [22, Theorem 3.2].

Theorem 3.4. *A universally complete vector lattice is locally one-dimensional if and only if every band preserving linear operator in it is order bounded.*

Proof. The proof is based on d-basis technique (see Abramovich and Kitover [1, Theorem 14.9]), while the Boolean valued proof is presented by Kusraev and Kutateladze in [20, Theorem 4.6.4]. □

We need one more tool. Say that a Dedekind complete vector lattice E is *principally universally complete* if each principal band in E is universally complete, see [1, Definition 14.1].

Theorem 3.5. *Let E be a Dedekind complete real vector lattice and $S : E \to E^\delta$ be a band preserving linear operator. Then there exists a maximal band E_0 in E such that the restriction of S to E_0 is order bounded, and hence the restriction of S to any nonzero ideal in $E_0^\perp$ is not order bounded. Moreover, for every $x \in E_+$ and for every $\lambda \in \mathbb{R}_+$ there exists an element $u \in E$ such that $0 \le u \le x$ and $|Su| \ge \lambda x$. In particular, $E_0^\perp$ is principally universally complete.*

Proof. See [1, Theorem 14.8]. □

4. Main results

We will also need the following auxiliary fact.

Lemma 4.1. *Let E be a universally complete vector lattice and $(E_\xi)_{\xi\in\Xi}$ be a collection of pairwise disjoint bands with $(\bigcup_{\xi\in\Xi} E_\xi)^\perp = \{0\}$. If $S_\xi : E_\xi \to E_\xi$ is a band preserving linear operator for all $\xi \in \Xi$, then there exists a unique band preserving linear operator $S : E \to E$ such that $S|_{E_\xi} = S_\xi$. Moreover, S is a complex structure on E if and only if S_ξ is a complex structure on X_ξ for all $\xi \in \Xi$.*

Proof. Define an operator S on E by putting

$$Sx := \sum_\xi S_\xi x_\xi \quad (x \in E),$$

where $x_\xi \in E_\xi$, $\pi_\xi x = \pi_\xi x_\xi$ ($\xi \in \Xi$), and π_ξ is a band projection corresponding to E_ξ. Obviously, S is the operator sought. □

Now we are ready to state and prove the main result.

Theorem 4.2. *A universally complete real vector lattice E admits a band preserving complex structure if and only if there is no locally one-dimensional band in E.*

Proof. *Necessity.* If E is locally one-dimensional, then every band preserving linear operator is order bounded by Theorem 3.4. Thus, there is no band preserving complex structure on E in accordance with Proposition 2.1.

Sufficiency. Assume that E has no locally one-dimensional band. According to Theorem 3.2 there exists a family of pairwise disjoint non-zero bands (B_ξ) in E such that $(\bigcup_{\xi\in\Xi} B_\xi)^\perp = \{0\}$ and each of them has a d-basis consisting of weak order units. In a view of Lemma 4.1 there is no loss of generality in assuming that there exists a d-basis $\mathcal{E}$ in E, consisting of weak units.

According to our assumption $\mathcal{E}$ is not a singleton and hence $\mathcal{E}$ is of infinite cardinality by Theorem 3.3. Therefore, there exists a decomposition $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$, with $\mathcal{E}_1 \cap \mathcal{E}_2 = \varnothing$ and $\mathcal{E}_1$ and $\mathcal{E}_2$ having the same (infinite) cardinality. Hence there exists a one-to-one mapping g from $\mathcal{E}_1$ onto $\mathcal{E}_2$. Thus the function g^{-1} is defined on $\mathcal{E}_2$ and maps $\mathcal{E}_2$ onto $\mathcal{E}_1$. Now we define an operator $S : \mathcal{E} \to \mathcal{E}$ as follows:

$$S(e) = \begin{cases} -g(e), & \text{for } e \in \mathcal{E}_1, \\ g^{-1}(e), & \text{for } e \in \mathcal{E}_2. \end{cases}$$

Given the d-expansion (1) of $x \in E$ we define Sx by

$$Sx = \sum_{\xi\in\Xi} \Big(- \sum_{e\in\mathcal{E}_1} \alpha_{\xi,e}\rho_\xi g(e) + \sum_{e\in\mathcal{E}_2} \alpha_{\xi,e}\rho_\xi g^{-1}(e) \Big). \tag{2}$$

In particular, $S(\pi e) = \pi Se$ for all $\pi \in \mathbb{P}(E)$ and $e \in \mathcal{E}$, since $S(\pi e) = -\pi g(e)$ $(e \in \mathcal{E}_1)$ and $S(\pi e) = \pi g^{-1}(e)$ $(e \in \mathcal{E}_2)$ by definition. The definition of S is sound. Indeed, if x has two distinct d-expansions of the form (1) with two different partitions of unity (π_η) and (ρ_ξ), then the two values of Sx defined by (2) using these partitions of unity, coincide with the value of Sx defined by (2) using the common refinement $(\pi_\eta \rho_\xi)$ of given partitions of unity as can be easily seen changing the order of summation. In a similar way one can prove that S is linear.

To see that the operator S constructed is band preserving note first that the equation $\pi x = \alpha_0 \pi^\perp e_0 + \sum_{\xi\in\Xi}\sum_{e\in\mathcal{E}} \alpha_{\xi,e}\pi\rho_\xi e$ with $e_0 \in \mathcal{E}$ and $\alpha_0 = 0$ implies by definition

$$S(\pi x) = \sum_{\xi\in\Xi}\Big(-\sum_{e\in\mathcal{E}_1}\alpha_{\xi,e}\rho_\xi\pi g(e) + \sum_{e\in\mathcal{E}_2}\alpha_{\xi,e}\rho_\xi\pi g^{-1}(e)\Big).$$

Now, taking into account that a linear operator is band preserving if and only if it commutes with all band projections, it is sufficient to observe

$$\begin{aligned} S(\pi x) &= \sum_{\xi\in\Xi}\Big(-\sum_{e\in\mathcal{E}_1}\alpha_{\xi,e}\rho_\xi\pi g(e) + \sum_{e\in\mathcal{E}_2}\alpha_{\xi,e}\rho_\xi\pi g^{-1}(e)\Big) = \\ &= \pi\sum_{\xi\in\Xi}\rho_\xi\Big(-\sum_{e\in\mathcal{E}_1}\alpha_{\xi,e}\pi e + \sum_{e\in\mathcal{E}_2}\alpha_{\xi,e}\pi e\Big) = \pi Sx. \end{aligned}$$

It remains to show that $S^2 = -I_E$. For every fixed ξ we have by (2) that

$$\begin{aligned} \rho_\xi S^2 x = (\rho_\xi S)^2 x &= \rho_\xi S\Big(-\sum_{e\in\mathcal{E}_1}\alpha_{\xi,e}\rho_\xi g(e) + \sum_{e\in\mathcal{E}_2}\alpha_{\xi,e}\rho_\xi g^{-1}(e)\Big) = \\ &= -\sum_{e\in\mathcal{E}_1}\alpha_{\xi,e}\rho_\xi Sg(e) + \sum_{e\in\mathcal{E}_2}\alpha_{\xi,e}\rho_\xi Sg^{-1}(e) = -\rho_\xi x \end{aligned}$$

holds for all ξ. The proof is complete. □

Given a complex structure S on a real vector lattice E, denote by E_S the complex vector space with the underlying additive group E and the complex multiplication defined as $zu = (\alpha + i\beta)u := \alpha u + \beta S(u)$ for all $z = \alpha + i\beta \in \mathbb{C}$ and $u \in E$. If S is band preserving, then the corresponding complex multiplication is also band preserving, i.e. $u \perp v$ implies $zu \perp v$ for all $z \in \mathbb{C}$ and $u, v \in E$. Thus, we have the following easy corollary.

Corollary 4.3. *A universally complete real vector lattice without locally one-dimensional bands admits a structure of a complex vector space with a band preserving complex multiplication.*

Remark 4.4. A *complex vector lattice* is the complexification $X_{\mathbb{C}} = X + iX$ of a real vector lattice X with modulus defined by

$$|x + iy| := \sup_{0\le s<2\pi} |x\cos s + y\sin s| \quad (x, y \in X).$$

The right-hand supremum exists whenever X is uniformly complete and in this case we have the modulus mapping $|\cdot| : X_{\mathbb{C}} \to X$, see [23]. (An axiomatic approach see in Mittelmeyer and Wolff [24].) Observe that E_S is a uniformly complete real vector lattice and a complex vector space simultaneously, but E_S is not a complex vector lattice. Indeed, if E_S were a complex vector lattice, i.e. $E_S = X_{\mathbb{C}}$ for some real vector lattice X, we would have $|Sx| = |ix| = |x|$ for all $x \in X$, hence S would be order bounded. But S cannot be order bounded by Proposition 2.1.

Theorem 4.5. *Assume that a universally complete real vector lattice E contains no locally one-dimensional band and fix a band preserving complex structure S on E. Then there exists an order dense vector sublattice $X \subset E$ such that $E_S \simeq X_{\mathbb{C}}$, i.e. E_S and $X_{\mathbb{C}}$ are isomorphic as complex vector spaces.*

Proof. Take a non-locally one-dimensional universally complete real vector lattice E and denote by $*$ the complex multiplication on E associated with a fixed band preserving complex structure S on E. Let X consist of all elements $x \in E$ admitting the representation (1) with $\mathcal{E} := \mathcal{E}_1$. The proof of the fact that X is a real subspace of E is similar to the proof of soundness and linearity of S given above in Theorem 4.2. If $x \in X$ and $\pi \in \mathbb{P}(E)$ is the band projection onto the band $\{x^+\}^{\perp\perp}$, then $x^+ = \pi x = \sum_{\xi\in\Xi}\sum_{e\in\mathcal{E}_1}\alpha_{\xi,e}\rho_\xi\pi e$ and hence $x^+ \in X$. It follows that X is a vector sublattice of E. If $0 < u \in E$, then for an arbitrary $e \in \mathcal{E}_1$ there exists a band projection π on E and a positive real λ such that $0 < \lambda\pi e \leq x$. This means that X is order dense in E, since $\lambda\pi e \in X$ by definition.

Denote by Y the vector sublattice of E constructed as X but using $\mathcal{E}_2$ instead of $\mathcal{E}_1$. From the relations $\mathcal{E}_1 \cap \mathcal{E}_2 = \{0\}$ and $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ using the representation (1), we deduce easily that $X \cap Y = \{0\}$ and $E = X + Y$. Moreover, the restriction $S_0 := S|_X$ is an $\mathbb{R}$-linear bijection from X onto Y. Now, if P_X and P_Y stand for the canonical projections onto X and Y, respectively, then the mapping $u \mapsto (P_X u, S_0^{-1}P_Y u)$ is an $\mathbb{R}$-linear isomorphism from E onto $X \times X$. It follows that for every $u \in E$ there exists a unique pair $(x, y) \in X \times X$ such that $u = x + Sy$. Clearly the mapping $\varphi : u \mapsto x + iy$ is an $\mathbb{R}$-isomorphism from E to $X_{\mathbb{C}}$. Identifying $x + iy$ with (x, y), it remains to observe that

$$\begin{aligned}\varphi(z * u) &= \alpha\varphi(u) + \beta\varphi(S(u)) = \alpha(x, y) + \beta\varphi(-y + Sx)\\ &= (\alpha x - \beta y, \alpha y + \beta x) = (\alpha + i\beta)(x, -y) = z\varphi(u).\end{aligned}$$

It follows that φ is a $\mathbb{C}$-isomorphism. $\square$

Remark 4.6. The modulus $|u|$ of an element $u \in X_{\mathbb{C}}$ exists and belongs to E. Moreover, the mapping $u \mapsto |u|$ of $X_{\mathbb{C}}$ into E satisfies the properties: $|u| = 0 \iff u = 0$; $|\lambda u| = |\lambda||u|$; $|u + v| \leq |u| + |v|$ for all $\lambda \in \mathbb{C}$ and $u, v \in X$.

Remark 4.7. Evidently, $T := S^{-1}$ is also a complex structure on E provided that S is a complex structure on E. Moreover, T is band preserving provided that so is S, see Abramovich and Kitover [1, Theorem 7.4]. Thus, we can define one more complex multiplication on E by letting $z \star u = (\alpha + i\beta) \star u := \alpha u + \beta T(u)$ for all $z = \alpha + i\beta \in \mathbb{C}$ and $u \in E$. Denote $E_S := (E, *)$ and $E_T := (E, \star)$. Then T is a $\mathbb{C}$-isomorphism of E_S onto itself and S is a $\mathbb{C}$-isomorphism of E_T onto itself. Indeed,

$$\begin{aligned}T(z * u) &= \alpha T(u) + \beta T(Su) = \alpha T(u) + \beta S(Tu) = z * T(u),\\ S(z \star u) &= \alpha S(u) + \beta S(Tu) = \alpha S(u) + \beta T(Su) = z \star S(u).\end{aligned}$$

Moreover, $S(z * u) = (iz) * S(u)$ and $T(z \star u) = (iz) \star T(u)$.

Remark 4.8. The complex vector lattice $Y_{\mathbb{C}}$ shares the properties of $X_{\mathbb{C}}$; in particular, $E_T \simeq Y_{\mathbb{C}}$. Moreover, the mapping $x + iy \mapsto S_0 x + iS_0 y$ with $S_0 := S|_X$ is a $\mathbb{C}$-isomorphism of $X_{\mathbb{C}}$ onto $Y_{\mathbb{C}}$. But this isomorphism is not modulus preserving.

Theorem 4.9. *A Dedekind complete real vector lattice admitting a band preserving complex structure is principally universally complete.*

Proof. Let E be a Dedekind complete vector lattice and S be a complex structure on E. Then the restriction of S to any nonzero band in E is again a complex structure which is not order bounded by Proposition 2.1. By Theorem 3.5 $E_0 = \{0\}$ and $E = E_0^{\perp}$, where E_0 is a band of regularity of S, and hence E is principally universally complete. □

5. Concluding remarks

The Wickstead problem. Theorem 4.2 enables one to specify the Wickstead problem raised in [26]: *In which vector lattices is each band preserving linear operator automatically order bounded?* An overview of the main ideas and results on the Wickstead problem and its variations may be found in [1,14,20]. Theorem 3.4 says that in the class of universally complete vector lattices this phenomena happens only in locally one-dimensional case. Combining this fact with Theorem 4.2 shows that the existence of band preserving complex structures is intimately related to the absence of locally one-dimensional bands.

Corollary 5.1. *Let E be a nonzero universally complete vector lattice. Then the following assertions are equivalent:*

(1) *There is a locally one-dimensional band in E.*
(2) *There is a nonzero band of E in which all band preserving linear operators are order bounded.*
(3) *There is no band preserving complex structure on E.*

Vector lattice of real measurable functions. Consider an important particular cases of the vector lattice of cosets of (almost everywhere equal) real measurable function. Let (Ω, Σ, μ) be a measure space and let $L^0 := L^0(\Omega, \Sigma, \mu)$ denote the f-algebra of all cosets of real measurable functions on Ω. If (Ω, Σ, μ) is a Maharam (localizable) measure space then L^0 is a universally complete vector lattice, see Fremlin [10]. Moreover, L^0 is locally one-dimensional if and only if the Boolean algebra $\mathbb{B} := \mathbb{B}(\Omega, \Sigma, \mu) := \Sigma/\mu^{-1}(0)$ of measurable sets modulo negligible sets is atomic (and so isomorphic to the boolean $\mathcal{P}(A)$ of a nonempty set A), see Kusraev and Kutateladze [20, Proposition 4.7.11 and Corollary 4.13.8].

Corollary 5.2. *If (Ω, Σ, μ) is an atomless Maharam measure space then the vector lattice $L^0(\Omega, \Sigma, \mu)$ admits a structure of complex vector space with a band preserving complex multiplication.*

Proof. This is immediate from the above remarks and Theorem 4.2. □

Kernel representation. Sections 2 and 4 provide some insight to the existence and description of complex structures on real vector lattices with band preserving or disjointness preserving multiplication. It is an interesting question whether there are similar results for another classes of operators. An easy fact follows.

Proposition 5.3. *Let (Ω, Σ, μ) be a non-atomic σ-finite measure space and X be an order dense ideal in $L^0(\Omega, \Sigma, \mu)$. There does not exist a complex structure on X admitting a kernel representation.*

Proof. Since (Ω, Σ, μ) is non-atomic, there exists an order bounded sequence (x_n) in X which converges to zero in measure but doesn't converge to zero almost everywhere. Assume that a complex structure S on X admits a kernel representation. Then (y_n) with $y_n = Sx_n$ converges to zero almost everywhere and hence (x_n) also converges to zero almost everywhere, since $x_n = -Ty_n$; a contradiction. □

Boolean valued approach. A result similar to Theorem 4.2 can be proved for fields:

Theorem 5.4. *Let $\mathbb{F}$ be a proper subfield of $\mathbb{R}$. Then there exists a discontinuous $\mathbb{F}$-linear function $f : \mathbb{R} \to \mathbb{R}$ such that $f \circ f = -I_{\mathbb{R}}$.*

The proof of this result rely upon Hamel basis and is similar to that of Kuczma [18, Theorem 12.5.2], taking into account the fact that a subfield $\mathbb{F}$ of $\mathbb{R}$ is proper if and only if $\mathbb{R}$ is an infinite dimensional vector space over $\mathbb{F}$, see Lang [21, Ch. VI, Corollary 9.3]. It should be noted that despite of a drastic difference between the concepts of a Hamel basis and a d-basis, they are essentially equivalent: *d-basis is an interpretation of Hamel basis in a Boolean valued model of set theory*, see [20, Theorem 4.5.7].

It is shown by Kusraev and Kutateladze in [20, Theorem 4.13.5] that interpreting Theorem 5.4 in an appropriate Boolean valued model yields that if a universally complete real vector lattice is not locally one-dimensional, then there exist band preserving complex structures on it. This gives another proof of Theorem 4.2.

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