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# On some aspects of the Bohl transformation for Hamiltonian and symplectic systems ☆

Ondřej Došlý

*Department of Mathematics and Statistics, Masaryk University, Kotlářská 2, CZ-611 37 Brno, Czechia*

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ABSTRACT

The classical Bohl transformation [4] from 1906 concerns the second order linear differential equations and states, roughly speaking, that a pair of linearly independent solutions of a second order differential equation can be expressed via the sine and cosine functions. Since that time, this transformation has been extended in various directions and became e.g. the theoretical basis for the deeply developed transformation theory of second order linear differential equations [8]. In our paper we discuss this transformation for linear Hamiltonian differential systems and discrete symplectic systems. We provide an alternative proofs to some know results and these new proofs enable to give a new insight into the topics. We also formulate some open problems associated with the discrete Bohl transformation.


## 1. Introduction

Consider the second order linear differential equation in the Jacobi form

$$x'' + p(t)x = 0, \tag{1}$$

where $p$ is a continuous function in an interval under consideration. Let $x_1, x_2$ be a pair of linearly independent solutions of (1) with the wronskian $w = x_1x_2' + x_1x_2' = 1$. Then by a direct computation (see [4]) one can verify that the function $h = \sqrt{x_1^2 + x_2^2}$ is a solution of the second order nonlinear differential equation

$$h'' + p(t)h = \frac{w^2}{h^3}. \tag{2}$$

The previous result easily extends to the more general Sturm–Liouville equation

$$\big(r(t)x'\big)' + p(t)x = 0 \tag{3}$$

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*E-mail address:* dosly@math.muni.cz.

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and (2) then reads as

$$\big(r(t)h'\big)' + p(t)h = \frac{w^2}{r(t)h^3}, \quad w = r(x_1x_2' - x_1'x_2) \tag{4}$$

The previous equation can be seen from the transformation point of view as follows. Consider the transformation of (3) $x = f(t)y$, where $f$ is a sufficiently smooth function. Then we have the identity (which can be verified by an easy computation, see e.g. [1])

$$f(t)\big[(r(t)x')' + p(t)x\big] = (R(t)y')' + P(t)y$$

with

$$R(t) = f^2(t)r(t), \quad P(t) = f(t)\big[(r(t)f'(t))' + p(t)f(t)\big].$$

In particular, if we take $f(t) = h(t) = \sqrt{x_1^2 + x_2^2}$ with the wronskian $w = 1$ of $x_1, x_2$ and we denote $q = \frac{1}{rh^2}$, then $y$ is a solution of the equation

$$\left(\frac{1}{q(t)}y'\right)' + q(t)y = 0 \tag{5}$$

which is explicitly solvable with the pair of linearly independent solutions

$$y_1(t) = \sin\int^t q(s)\,ds, \quad y_2(t) = \cos\int^t q(s)\,ds.$$

If we allow solutions of (3) to be also complex-valued and we put $x_1 = u + iv$, $x_2 = u - iv$, $u, v$, being the real solutions of (3) with $r(u'v - uv') = \frac{1}{2}$, equation (4) takes the form

$$\big(rh'\big)' + ph = -\frac{1}{rh^3}$$

since $w = r(x_1x_2' - x_1'x_2) = i$, i.e. $w^2 = -1$ in this case, and equation (5) has the form

$$\left(\frac{1}{q(t)}y'\right)' - q(t)y = 0.$$

The setup of the paper is as follows. In the next section we recall the concept of trigonometric Hamiltonian differential system and discrete trigonometric system together with basic properties of their solutions. In Section 3 we provide alternative proofs of the Bohl transformations for Hamiltonian and symplectic systems (comparing with those given in [6,10]). We also show how these new proofs are related to the so-called hyperbolic Hamiltonian and symplectic transformation when complex solutions are also considered. The last section is devoted to some open problems concerning geometric oscillation theory associated with Hamiltonian and symplectic systems.

## 2. Preliminaries

The principal concern in our paper are the linear Hamiltonian differential system

$$z' = \mathcal{J}H(t)z, \quad H = \begin{pmatrix} -C & A^T \\ A & B \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{6}$$

with $\binom{x}{u} \in \mathbb{R}^{2n}$, $A, B, C \in \mathbb{R}^{n\times n}$ continuous, $B, C$, symmetric, i.e., $B = B^T$, $C = C^T$ (then, of course, $H^T = H$), and its discrete counterpart, then symplectic difference system

$$z_{k+1} = \mathcal{S}_k z_k, \quad \mathcal{S}_k^T \mathcal{J} \mathcal{S}_k = \mathcal{J}, \quad \mathcal{S}_k = \begin{pmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{pmatrix} \tag{7}$$

with $z = \binom{x}{u} \in \mathbb{R}^{2n}$, $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \in \mathbb{R}^{n\times n}$.

The common feature of (6) and (7) is that they are the more general first order linear systems with the property that their fundamental matrix is symplectic whenever it has this property at an initial condition.

Sometimes, we will need to write (6) and (7) in entries and then we will write these systems in the form

$$x' = A(t)x + B(t)u, \quad u' = C(t)x - A^T(t)u,$$

resp.

$$x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k, \quad u_{k+1} = \mathcal{C}_k x_k + \mathcal{D}_k u_k.$$

The concept of trigonometric (Hamiltonian) differential system was introduced by Barrett and Reid [3,18] in the fifties of the last century in the connection with the Prüfer transformation for (6). A trigonometric differential system is a special system (6) where $A = 0$, $B = -C =: Q$, i.e., written in the matrix form for $\binom{S}{C} \in \mathbb{R}^{2n\times n}$ it is the system

$$S' = Q(t)C, \quad C' = -Q(t)S. \tag{8}$$

Note that the letter $C$ here has nothing to do with the $C$ in the coefficient matrix of (6). The letter $C$ in (8) prompts the cosine function which is a solution of (8) in the scalar case, see later.

The concept of trigonometric (symplectic) difference system was introduced by D. Anderson [2] as a special symplectic difference system (written for $\binom{S}{C} \in \mathbb{R}^{2n\times n}$ in entries)

$$S_{k+1} = \mathcal{P}_k S_k + \mathcal{Q}_k C_k, \quad C_{k+1} = -\mathcal{Q}_k S_k + \mathcal{P}_k C_k, \tag{9}$$

with the coefficient matrices satisfying

$$\mathcal{P}_k^T \mathcal{P}_k + \mathcal{Q}_k^T \mathcal{Q}_k = I, \quad \mathcal{P}_k^T \mathcal{Q}_k = \mathcal{Q}_k^T \mathcal{P}_k. \tag{10}$$

The adjective "trigonometric" system for (8) and (9) is motivated by the fact that if $n = 1$, i.e., the previous matrix quantities reduce to scalars, then

$$S(t) = \sin \int^t Q(s)\, ds, \quad C(t) = \cos \int^t Q(s)\, ds$$

is a solution of (8), while a solution of (9) is

$$S_k = \sin \left( \sum^{k-1} \varphi_j \right), \quad C_k = \cos \left( \sum^{k-1} \varphi_j \right),$$

where $\varphi$ are given by the formula

$$\sin \varphi_j = \mathcal{Q}_j, \quad \cos \varphi_j = \mathcal{P}_j \tag{11}$$

Concerning oscillatory properties of trigonometric differential systems (8) in the general case, if $Q(t) \geq 0$ (nonnegative definite) and $\operatorname{Tr} Q(t) > 0$ (Tr denotes the trace, i.e., the sum of diagonal entries), then (8) is nonoscillatory (i.e., there exists a $2n \times n$ matrix solution $\binom{S}{C}$ with $S^T C = C^T S$ such that $\det S(t) \neq 0$ for large $t$) if and only if (see [15])

$$\int^{\infty} \operatorname{Tr} Q(t)\, dt < \infty. \tag{12}$$

As for the discrete counterpart of (12), if the matrices $\mathcal{Q}_k > 0$ for large $k$ (positive definiteness), then (9) is nonoscillatory (i.e., there exists a $2n \times n$ solution $\binom{S_k}{C_k}$ with

$$\operatorname{Ker} S_{k+1} \subseteq \operatorname{Ker} S_k, \quad S_k S_{k+1}^{\dagger} \mathcal{Q}_k \geq 0 \tag{13}$$

for large $k$, here Ker denotes the kernel of a matrix and $^\dagger$ stands for the Moore–Penrose generalized inverse) if and only if (see [6])

$$\sum_{k=1}^{\infty} \operatorname{arccot} \lambda_1 \left(\mathcal{Q}_k^{-1} \mathcal{P}_k\right) < \infty, \tag{14}$$

where $\lambda_1$ is the smallest eigenvalue of the matrix indicated.

At the end of this section let us mention another common feature of trigonometric systems, namely, in addition to symplecticity of the fundamental matrix this matrix is also orthogonal whenever it has this property at an initial condition, i.e. $Z^T(t)Z(t) = I$. In the continuous case it follows from the fact that the coefficient matrix in (8) is antisymmetric, while in the discrete case the coefficient matrix in (9) is orthogonal and orthogonal matrices form the group with respect to the matrix multiplication. Note that the condition of positive definiteness of $\mathcal{Q}$ in (14) was recently relaxed to nonnegative definiteness in the recent paper [11], but we will not need this more general statement in our paper.

## 3. Bohl transformation

An alternative terminology for the Bohl transformation is the trigonometric transformation. Here we prefer the terminology Bohl transformation since our approach is closer to the original one of Bohl form 1906 (and also to that of [16]) and better illustrates the main idea of this section. Main results of this section show that every linear Hamiltonian differential system resp. symplectic difference system can be transformed into trigonometric system without changing oscillatory nature of transformed systems. In a modified form, these statements can be found in [6,10] but the proofs presented here are completely different from those given in the above mentioned papers and enable to clarify the relationship of the considered transformation to the so-called hyperbolic transformation.

**Theorem 1.** *Consider system (6). There exist a nonsingular differentiable matrix $H(t) \in \mathbb{R}^{n\times n}$ and a differentiable matrix $G(t) \in \mathbb{R}^{n\times n}$ such that the matrix*

$$\mathcal{R}(t) = \begin{pmatrix} H(t) & 0 \\ G(t) & H^{T-1}(t) \end{pmatrix} \in \mathbb{R}^{2n\times 2n} \tag{15}$$

*is symplectic and the transformation*

$$\begin{pmatrix} x \\ u \end{pmatrix} = \mathcal{R}(t) \begin{pmatrix} s \\ c \end{pmatrix}$$

*transforms* (6) *into a trigonometric system* (8) *with the matrix $Q$ given by the formula*

$$Q(t) = H^{-1}(t)B(t)H^{T-1}(t).$$

**Proof.** The transformation

$$\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} H & 0 \\ G & H^{T-1} \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{u} \end{pmatrix}$$

with $H, G$ satisfying $H^T G = G^T H$ transforms (6) into the system of the same form

$$\tilde{x}' = \widehat{A}(t)\tilde{x} + \widehat{B}(t)\tilde{u}, \quad \tilde{u}' = \widehat{C}(t)\tilde{x} - \widehat{A}^T(t)\tilde{u}, \tag{16}$$

with the matrices $\widehat{\mathcal{A}}, \widehat{\mathcal{B}}, \widehat{\mathcal{C}}$ given by the formulas

$$\begin{aligned} \widehat{\mathcal{A}} &= H^{-1}(-H' + AH + BG), \\ \widehat{\mathcal{B}} &= H^{-1}BH^{T-1}, \\ \widehat{\mathcal{C}} &= -G^T(-H' + AH + BG) + H^T(-G' + CH - A^T G). \end{aligned} \tag{17}$$

Let $\begin{pmatrix} X & \tilde{X} \\ U & \tilde{U} \end{pmatrix} \in \mathbb{R}^{2n \times 2n}$ be a symplectic fundamental matrix of (6), i.e., $Z^T \mathcal{J} Z = \mathcal{J} = Z \mathcal{J} Z^T$, which in entries of $Z$ reads as

$$\begin{gathered} X^T \tilde{U} - U^T \tilde{X} = I = X\tilde{U}^T - \tilde{X}U^T, \\ X^T U = U^T X, \ \tilde{X}^T \tilde{U} = \tilde{U}^T \tilde{X}, \ X\tilde{X}^T = \tilde{X}X^T, \ U\tilde{U}^T = \tilde{U}U^T. \end{gathered} \tag{18}$$

Let $\tilde{H}$ be any matrix for which $\tilde{H}\tilde{H}^T = XX^T + \tilde{X}\tilde{X}^T$ and $G = (UX^T + \tilde{U}\tilde{X}^T)\tilde{H}^{T-1}$. We have using (18)

$$\begin{aligned} G^T\tilde{H} - \tilde{H}^T G &= \tilde{H}^{-1}(XU^T + \tilde{X}\tilde{U}^T)\tilde{H} - \tilde{H}^T(UX^T + \tilde{U}\tilde{X}^T)\tilde{H}^{T-1} \\ &= \tilde{H}^{-1}\big[(XU^T + \tilde{X}\tilde{U}^T)(XX^T + \tilde{X}\tilde{X}^T) \\ &\quad - (XX^T + \tilde{X}\tilde{X}^T)(UX^T + \tilde{U}\tilde{X}^T)\big]\tilde{H}^{T-1} \\ &= \tilde{H}^{-1}\big[XU^T XX^T + XU^T\tilde{X}\tilde{X}^T + \tilde{X}\tilde{U}^T XX^T + \tilde{X}\tilde{U}^T\tilde{X}\tilde{X}^T \\ &\quad - XX^T UX^T - XX^T\tilde{U}\tilde{X}^T - \tilde{X}\tilde{X}^T UX^T - \tilde{X}\tilde{X}^T\tilde{U}\tilde{X}^T\big]\tilde{H}^{T-1} \\ &= \tilde{H}^{-1}\big[X(U^T\tilde{X} - X^T\tilde{U})\tilde{X}^T + \tilde{X}(\tilde{U}^T X - \tilde{X}^T U)X^T\big]\tilde{H}^{T-1} \\ &= \tilde{H}^{-1}(-X\tilde{X}^T + \tilde{X}X^T)\tilde{H}^{T-1} = 0, \end{aligned}$$

i.e., the matrix $\mathcal{R}$ is symplectic, and

$$\begin{aligned} \widehat{A} &= \tilde{H}^{-1}(-\tilde{H}' + A\tilde{H} + BG) = \tilde{H}^{-1}(-\tilde{H}'\tilde{H}^T + A\tilde{H}\tilde{H}^T + BUX^T + B\tilde{U}\tilde{X}^T)H^{T-1} \\ &= \tilde{H}^{-1}(-\tilde{H}'\tilde{H}^T + A(XX^T + \tilde{X}\tilde{X}^T) + BUX^T + B\tilde{U}\tilde{X}^T)\tilde{H}^{T-1} \\ &= \tilde{H}^{-1}(-\tilde{H}'\tilde{H}^T + X'X^T + \tilde{X}'\tilde{X}^T)\tilde{H}^{T-1}. \end{aligned}$$

Then

$$\widehat{A} + \widehat{A}^T = \tilde{H}^{-1}\big[-\tilde{H}'\tilde{H}^T - \tilde{H}\tilde{H}^{T\prime} + X'X^T + XX^{T\prime} + \tilde{X}'\tilde{X}^T + \tilde{X}\tilde{X}^{T\prime}\big] = 0,$$

i.e., the matrix $\widehat{\mathcal{A}}$ is antisymmetric. Denote

$$C_1 = -G^T(-\tilde{H}' + A\tilde{H} + BG), \quad C_2 = \tilde{H}^T(-G' + C\tilde{H} - A^T G).$$

We have

$$\begin{aligned} C_1 &= -G^T(-\tilde{H}'\tilde{H}^T + A(XX^T + \tilde{X}\tilde{X}^T) + B(UX^T + \tilde{U}\tilde{X}^T))\tilde{H}^{T-1} \\ &= G^T(\tilde{H}'\tilde{H}^T - X'X^T - \tilde{X}'\tilde{X}^T)\tilde{H}^{T-1}. \end{aligned}$$

$$\begin{aligned} C_2 &= \tilde{H}^T \big\{ \big[-(CX - A^T U)X^T - U(X^T A^T + U^T B) \\ &\quad - (C\tilde{X} - A^T\tilde{U})\tilde{X}^T - \tilde{U}(\tilde{X}^T A^T + \tilde{U}^T B)\big]\tilde{H}^{T-1} \\ &\quad + (UX^T + \tilde{U}\tilde{X}^T)\tilde{H}^{T-1}\tilde{H}^{T\prime}\tilde{H}^{T-1} + C\tilde{H} - A^T(UX^T + \tilde{U}\tilde{X}^T)\tilde{H}^{T-1}\big\} \\ &= \tilde{H}^T \big\{ -CXX^T + A^T UX^T - UX^T A^T - UU^T B - C\tilde{X}\tilde{X}^T + A^T\tilde{U}\tilde{X}^T \\ &\quad - \tilde{U}\tilde{X}^T A^T - \tilde{U}\tilde{U}^T B + G\tilde{H}^{T\prime} + C\tilde{H}\tilde{H}^T - A^T UX^T - A^T\tilde{U}\tilde{X}^T \big\} H^{T-1} \\ &= \tilde{H}^T \big\{ -UX^T A^T - UU^T B - \tilde{U}\tilde{X}^T A^T - \tilde{U}\tilde{U}^T B + G\tilde{H}^{T\prime} \big\} \tilde{H}^{T-1} \\ &= \tilde{H}^{-1} \big\{ (XX^T + \tilde{X}\tilde{X}^T)(-UX^T - \tilde{U}\tilde{X}^T)A^T \\ &\quad - (XX^T + \tilde{X}\tilde{X}^T)(UU^T + \tilde{U}\tilde{U}^T)B + \tilde{H}\tilde{H}^T G\tilde{H}^{T\prime} \big\} \tilde{H}^{T-1} \\ &= \tilde{H}^{-1} \big\{ -XX^T UX^T A^T - XX^T\tilde{U}\tilde{X}^T A^T - \tilde{X}\tilde{X}^T UX^T A^T - \tilde{X}\tilde{X}^T\tilde{U}\tilde{X}^T A^T \\ &\quad - XX^T UU^T B - XX^T\tilde{U}\tilde{U}^T B - \tilde{X}\tilde{X}^T UU^T B - \tilde{X}\tilde{X}^T\tilde{U}\tilde{U}^T B \\ &\quad + (XU^T + \tilde{X}\tilde{U}^T)\tilde{H}\tilde{H}^{T\prime} \big\} \tilde{H}^{T-1} \\ &= \tilde{H}^{-1} \big\{ (-(XX^T UX^T + XX^T\tilde{U}\tilde{X}^T + \tilde{X}\tilde{X}^T UX^T + \tilde{X}\tilde{X}^T\tilde{U}\tilde{X}^T)A^T \\ &\quad - (XX^T UU^T + XX^T\tilde{U}\tilde{U}^T + \tilde{X}\tilde{X}^T UU^T + \tilde{X}\tilde{X}^T\tilde{U}\tilde{U}^T)B \\ &\quad + \tilde{H}G^T\tilde{H}\tilde{H}^{T\prime} + \tilde{H}G^T(XX^{T\prime} + \tilde{X}\tilde{X}^{T\prime}) - \tilde{H}G^T(XX^{T\prime} + \tilde{X}\tilde{X}^{T\prime}) \big\} \tilde{H}^{T-1} \\ &= \tilde{H}^{-1} \big\{ -(XX^T UX^T + XX^T\tilde{U}\tilde{X}^T + \tilde{X}\tilde{X}^T UX^T + \tilde{X}\tilde{X}^T\tilde{U}\tilde{X}^T)A^T \\ &\quad - (XX^T UU^T + XX^T\tilde{U}\tilde{U}^T + \tilde{X}\tilde{X}^T UU^T + \tilde{X}\tilde{X}^T\tilde{U}\tilde{U}^T)B \\ &\quad + (XU^T + \tilde{X}\tilde{U}^T)[X(X^T A^T + U^T B) + \tilde{X}(\tilde{X}^T A^T + \tilde{U}^T B)] \big\} \tilde{H}^{T-1} \\ &\quad + \tilde{H}^{-1}\big[\tilde{H}G^T\tilde{H}\tilde{H}^{T\prime} - \tilde{H}G^T(XX^{T\prime} + \tilde{X}\tilde{X}^{T\prime})\big]\tilde{H}^{T-1} \\ &= \tilde{H}^{-1} \big\{ \big[X(-X^T U + U^T X)X^T + X(-X^T\tilde{U} + U^T\tilde{X})\tilde{X}^T \\ &\quad + \tilde{X}(-\tilde{X}^T U + \tilde{U}^T X)X^T + \tilde{X}(-\tilde{X}^T\tilde{U} + \tilde{U}^T\tilde{X})\tilde{X}^T\big]A^T \\ &\quad + \big[X(-X^T U + U^T X)U^T + X(-X^T\tilde{U} + U^T\tilde{X})\tilde{U}^T \\ &\quad + \tilde{X}(-\tilde{X}^T U + \tilde{U}^T X)U^T + \tilde{X}(-\tilde{X}^T\tilde{U} + \tilde{U}^T\tilde{X})\tilde{U}^T\big]B \big\} \tilde{H}^{T-1} \\ &\quad + G^T(\tilde{H}\tilde{H}^{T\prime} - XX^{T\prime} - \tilde{X}\tilde{X}^{T\prime})H^{T-1} \\ &= \tilde{H}^{-1} \big\{ (-X\tilde{X}^T + \tilde{X}X^T)A^T + (-X\tilde{U}^T + \tilde{X}U^T)B \big\} \tilde{H}^{T-1} \\ &\quad + G^T(\tilde{H}\tilde{H}^{T\prime} - XX^{T\prime} - \tilde{X}\tilde{X}^{T\prime})\tilde{H}^{T-1} \\ &= -\tilde{H}^{-1}B\tilde{H}^{T-1} + G^T(\tilde{H}\tilde{H}^{T\prime} - XX^{T\prime} - \tilde{X}\tilde{X}^{T\prime})\tilde{H}^{T-1}. \end{aligned}$$

Consequently,

$$\begin{aligned}\widehat{C} = C_1 + C_2 &= -G^T(-\tilde{H}'\tilde{H}^T + X'X^T + \tilde{X}'\tilde{X})\tilde{H}^{T-1} - \tilde{H}^{-1}B\tilde{H}^{T-1} \\ &\quad + G^T(\tilde{H}\tilde{H}^{T\prime} - XX^{T\prime} - \tilde{X}\tilde{X}^{T\prime})\tilde{H}^{T-1} \\ &= -\tilde{H}^{-1}B\tilde{H}^{T-1} + G^T(\tilde{H}'\tilde{H}^T - X'X^T - \tilde{X}'\tilde{X}^T + \tilde{H}\tilde{H}^{T\prime} - XX^{T\prime} - \tilde{X}\tilde{X}^{T\prime})\tilde{H}^{T-1} \\ &= -\tilde{H}^{-1}B\tilde{H}^{T-1} = -\widehat{B}.\end{aligned}$$

Now, let $R$ be the fundamental matrix of the system $R' = \widehat{A}(t)R$. Since the matrix $\widehat{A}$ is antisymmetric ($\widehat{A} + \widehat{A}^T = 0$), the matrix $R$ is orthogonal, i.e., $R^TR = I$. Let us apply the transformation $\tilde{x} = R(t)s$, $\tilde{u} = R^{T-1}c$ to system (16). According to (17), this transformation transforms (16) into the system

$$s' = R^{-1}\widehat{B}R^{T-1}c, \quad c' = -R^T\widehat{C}Rs.$$

Since $R^T = R^{-1}$, denoting $Q = -R^T\widehat{C}R = R^{-1}\widehat{B}R^{T-1} = H^{-1}BH^{T-1}$ with $H = \tilde{H}R$ we see that the last system is the trigonometric Hamiltonian system. □

**Theorem 2.** *Consider the discrete symplectic system (7). There exist nonsingular matrices $H_k \in \mathbb{R}^{n\times n}$ and $G_k \in \mathbb{R}^{n\times n}$ such that the matrix*

$$\mathcal{R}_k = \begin{pmatrix} H_k & 0 \\ G_k & H_k^{T-1}\end{pmatrix} \in \mathbb{R}^{2n\times 2n}$$

*is symplectic and the transformation*

$$\begin{pmatrix} x_k \\ u_k \end{pmatrix} = \mathcal{R}_k \begin{pmatrix} s_k \\ c_k \end{pmatrix}$$

*transforms (7) into (9) with the matrices $\mathcal{P}, \mathcal{Q}$ given by the formulas*

$$\mathcal{P}_k = H_{k+1}^{-1}\big(\mathcal{A}_kH_k + \mathcal{B}_kG_k\big), \quad \mathcal{Q}_k = H_{k+1}^{-1}\mathcal{B}_kH_k^{T-1}. \tag{19}$$

*Moreover, the matrices $H_k$ can be chosen in such a way that the matrix $\mathcal{Q}$ is symmetric and $\mathcal{Q}_k \geq 0$.*

**Proof.** We have the transformation formulas for the transformed system (no index by a matrix means index $k$), see e.g. [5],

$$\begin{aligned}\widehat{\mathcal{A}} &= H_{k+1}^{-1}(\mathcal{A}H + \mathcal{B}G), \quad \widehat{\mathcal{B}} = H_{k+1}^{-1}\mathcal{B}H^{T-1} \\ \widehat{\mathcal{C}} &= -G_{k+1}^T(\mathcal{A}H + \mathcal{B}G) + H_{k+1}^T(\mathcal{C}H + \mathcal{D}G), \\ \widehat{\mathcal{D}} &= -G_{k+1}^T\mathcal{B}H^{T-1} + H_{k+1}^T\mathcal{D}H^{T-1}.\end{aligned}$$

Let $\binom{X}{U}$, $\binom{Y}{Z}$ be the conjoined bases of (7) satisfying (18) with $Y, Z$ instead of $\tilde{X}, \tilde{U}$, $H$ be any matrix satisfying $HH^T = XX^T + YY^T$, and $G = (UX^T + ZY^T)H^{T-1}$. Then using the fact that $\binom{X}{U}$ and $\binom{Y}{Z}$ are also solutions of the reversed system

$$X_k = \mathcal{D}_k^TX_{k+1} - \mathcal{B}_k^TU_{k+1}, \quad U_k = -\mathcal{C}_k^TX_{k+1} + \mathcal{A}_k^tU_{k+1} \tag{20}$$

$$\begin{aligned}\widehat{\mathcal{C}} &= -G_{k+1}^T(\mathcal{A}H + \mathcal{B}G) + H_{k+1}^T(\mathcal{C}H + \mathcal{D}G) \\ &= \big\{-G_{k+1}^T(\mathcal{A}HH^T + \mathcal{B}GH^T) + H_{k+1}^T(\mathcal{C}HH^T + \mathcal{D}GH^T\big\}\, H^{T-1} \\ &= \big\{-G_{k+1}^T(\mathcal{A}XX^T + \mathcal{A}YY^T) + \mathcal{B}(UX^T + ZY^T\big)\end{aligned}$$

$$
\begin{aligned}
&\quad + H_{k+1}^T[\mathcal{C}XX^T + \mathcal{C}YY^T + \mathcal{D}(UX^T + ZY^T)]\big\} H^{T-1} \\
&= -H_{k+1}^{-1}(X_{k+1}U_{k+1}^T + Y_{k+1}Z_{k+1}^T)(\mathcal{A}HH^T + \mathcal{B}(UX^T + ZY^T))H^{T-1} \\
&\quad + H_{k+1}^T(\mathcal{C}XX^T + \mathcal{C}YY^T + \mathcal{D}UX^T + \mathcal{D}ZY^T)H^{T-1} \\
&= H_{k+1}^{-1}\big\{-(X_{k+1}U_{k+1}^T + Y_{k+1}Z_{k+1}^T)\mathcal{A}(XX^T + YY^T) \\
&\quad - (X_{k+1}U_{k+1}^T + Y_{k+1}Z_{k+1}^T)\mathcal{B}(UX^T + ZY^T) \\
&\quad + H_{k+1}H_{k+1}^T\mathcal{C}(XX^T + YY^T) + H_{k+1}H_{k+1}^T\mathcal{D}(UX^T + ZY^T)\big\} H^{T-1} \\
&= H_{k+1}^{-1}\big\{-(X_{k+1}U_{k+1}^T + Y_{k+1}Z_{k+1}^T)(X_{k+1}X^T + Y_{k+1}Y^T) \\
&\quad + (X_{k+1}X_{k+1}^T + Y_{k+1}Y_{k+1}^T)(U_{k+1}X^T + Z_{k+1}Y^T)\big\} H^{T-1} \\
&= H_{k+1}^{-1}\big\{-(X_{k+1}U_{k+1}^T + Y_{k+1}Z_{k+1}^T)X_{k+1}(X_{k+1}^T\mathcal{D} - U_{k+1}^T\mathcal{B}) \\
&\quad - (X_{k+1}U_{k+1}^T + Y_{k+1}Z_{k+1}^T)Y_{k+1}(Y_{k+1}^T\mathcal{D} - Z_{k+1}^T\mathcal{B}) \\
&\quad + (X_{k+1}X_{k+1}^T + Y_{k+1}Y_{k+1}^T)U_{k+1}(X_{k+1}^T\mathcal{D} - U_{k+1}^T\mathcal{B}) \\
&\quad + (X_{k+1}X_{k+1}^T + Y_{k+1}Y_{k+1}^T)Z_{k+1}(Y_{k+1}^T\mathcal{D} - Z_{k+1}^T\mathcal{B}\big\} H^{T-1}.
\end{aligned}
$$

In the continuation of this computation we change the notation, no index by a matrix means now the index $k+1$. Using (18)

$$
\begin{aligned}
\widehat{\mathcal{C}}_k &= H^{-1}\big\{XU^TXX^T\mathcal{D}_k + XU^TXU^T\mathcal{B}_k - YZ^TXX^T\mathcal{D}_k + YZ^TXU^T\mathcal{B}_k \\
&\quad - XU^TYY^T\mathcal{D}_k + XU^TYZ^T\mathcal{B}_k - YZ^TYY^T\mathcal{D}_k + YZ^TYZ^T\mathcal{B}_k \\
&\quad + XX^TUX^T\mathcal{D}_k - XX^TUU^T\mathcal{B}_k + YY^TUX^T\mathcal{D}_k - YY^TUU^T\mathcal{B}_k \\
&\quad + XX^TZY^T\mathcal{D}_k - XX^TZZ^T\mathcal{B}_k + YY^TZY^T\mathcal{D}_k - YY^TZZ^T\mathcal{B}_k\big\} H_k^{T-1} \\
&= H^{-1}\big\{Y(-Z^TX + Y^TU)X^T\mathcal{D}_k + X(-U^TY + X^TZ)Y^T\mathcal{D}_k + Y(Z^TX - Y^TU)U^TB_k \\
&\quad + X(U^TY - X^TZ)Z^T\mathcal{B}_k + X(-U^TX + X^TU)X^T\mathcal{D}_k + X(U^TX - X^TU)U^T\mathcal{B}_k \\
&\quad + Y(-Z^TY + Y^TZ)Y^T\mathcal{D}_k + Y(Z^TY - Y^TZ)Z^T\mathcal{B}_k\big\} H_k^{T-1} \\
&= H^{-1}\big\{(YX^T - XY^T)\mathcal{D}_k + (YU^T - XZ^T)\mathcal{B}_k\big\} H_k^{T-1} = -H^{-1}\mathcal{B}_kH_k^{T-1} = -\widehat{\mathcal{B}}_k
\end{aligned}
$$

and again with no index meaning the index $k$, using (20)

$$
\begin{aligned}
\widehat{D} &= (-G_{k+1}^T\mathcal{B} + H_{k+1}^T\mathcal{D})H^{T-1} \\
&= -H_{k+1}^{-1}(X_{k+1}U_{k+1}^T + Y_{k+1}Z_{k+1}^T)\mathcal{B} + H_{k+1}^T\mathcal{D}H^{T-1} \\
&= H_{k+1}^{-1}\left[-(X_{k+1}U_{k+1}^T + Y_{k+1}Z_{k+1}^T)\mathcal{B} + (X_{k+1}X_{k+1}^T + Y_{k+1}Y_{k+1}^T)\mathcal{D}\right] H^{T-1} \\
&= H_{k+1}^{-1}[X_{k+1}X^T + Y_{k+1}Y^T]H^{T-1} \\
&= H_{k+1}^{-1}[(\mathcal{A}X + \mathcal{B}U)X^T + (\mathcal{A}Y + \mathcal{B}Z)Y^T]H^{T-1} \\
&= H_{k+1}^{-1}[\mathcal{A}(XX^T + YY^T) + \mathcal{B}(UX^T + ZY^T)]H^{T-1} \\
&= H_{k+1}^{-1}[\mathcal{A}HH^T + \mathcal{B}(UX^T + ZY^T)]H^{T-1} = H_{k+1}^{-1}(\mathcal{A}H + \mathcal{B}G) = \widehat{A}.
\end{aligned}
$$

The proof of the fact that the matrices $H_k$ can be chosen in such a way that the matrices $\mathcal{Q}_k$ are nonnegative definite is the same as presented in [6, Remark 3.2], so we omit it here. □

**Remark 1.** Similarly as in the scalar case, if we take into consideration complex-valued solutions with the wronskian $X^T\tilde{U} - U^T\tilde{X} = i$ resp. $X^T Z - U^T Y = i$, a closer examination of the proofs of Theorems 1, 2, show that the resulting systems upon the considered transformations are

$$s' = Q(t)c, \quad c' = Q(t)s$$

resp.

$$s_{k+1} = \mathcal{P}_k s_k + \mathcal{Q}_k c_k, \quad c_{k+1} = \mathcal{Q}_k s_k + \mathcal{P}_k c_k. \tag{21}$$

These systems are called in [9,12] hyperbolic systems (since in the scalar case $n = 1$ their solutions are expressed via the hyperbolic sine and cosine functions/sequences). For example, concerning (21) with $n = 1$, a solution $\binom{s}{c}$ of this system is $s_k = \sinh \sum^{k-1} \varphi_j$, $c_k = \cosh \sum^{k-1} \varphi_j$, where $\sinh \varphi_k = \mathcal{Q}_k$, i.e., $\varphi_k = \log(\mathcal{Q}_k + \sqrt{1 + \mathcal{Q}_k^2})$. This implies that scalar system (21) has also solutions

$$\begin{pmatrix} s_k \\ c_k \end{pmatrix} = \begin{pmatrix} e^{\sum^{k-1} \varphi_j} \\ e^{\sum^{k-1} \varphi_j} \end{pmatrix} = \begin{pmatrix} \prod^{k-1} e^{\varphi_j} \\ \prod^{k-1} e^{\varphi_j} \end{pmatrix},$$
$$\begin{pmatrix} s_k \\ c_k \end{pmatrix} = \begin{pmatrix} e^{-\sum^{k-1} \varphi_j} \\ e^{-\sum^{k-1} \varphi_j} \end{pmatrix} = \begin{pmatrix} \prod^{k-1} e^{-\varphi_j} \\ \prod^{k-1} e^{-\varphi_j} \end{pmatrix}.$$

The last formulas are essentially formulas (3.6) in [16]. The new proofs of Theorems 1, 2 from the previous part of this section enable to obtain transformation of nonoscillatory systems (6), (7) to hyperbolic systems as a special case of the Bohl transformation and not as "separate" results of [9,12] (which were proved there via rather tedious computations).

## 4. Open problems

The continuous Bohl transformation provides a nice geometrical view into the oscillation theory of linear Hamiltonian differential systems. To show this, consider a solution $\binom{S}{C}$ of (8) for which (such a solution always exists)

$$S^T S + C^T C = I = SS^T + CC^T, \quad S^T C = C^T S, \quad SC^T = CS^T, \tag{22}$$

and let

$$G(t) := C(t)C^T(t) - S(t)S^T(t) + 2iS(t)C^T(t),$$

then using (22) it is not difficult to show that the matrix $G$ is unitary ($G^*(t)G(t) = I$). Observe also that $G(t) = \cos 2\int^t Q + i \sin 2\int^t Q$ in the scalar case. Etgen [14,15] showed that if the matrix $Q(t)$ is nonnegative definite, then the eigenvalues of $G$ move around the unit circle in the complex plane in the positive direction when $t$ increases. Moreover, $\lambda = 1$ is an eigenvalue of $G(t)$ if and only if the matrix $S(t)$ is singular and $\lambda = -1$ is an eigenvalue of $G(t)$ if and only if the matrix $C(t)$ is singular.

Since by the Bohl transformation, the first entries $X$, $\tilde{X}$ of a pair of conjoined bases $\binom{X}{U}$, $\binom{\tilde{X}}{\tilde{U}}$ of (6) satisfying $X^T\tilde{U} - U^T\tilde{X} = I$ can be expressed via the matrices $S$, $C$ (observe that if $\binom{S}{C}$ is a solution of (8) then $\binom{C}{-S}$ is a solution as well, i.e., we can express $X = H(t)S$, $\tilde{X} = H(t)C$) the above idea gives a nice geometrical proof of the statement that the numbers of focal points (i.e., the algebraic singularities of the first $X$-component) of conjoined bases $\binom{X}{U}$, $\binom{\tilde{X}}{\tilde{U}}$ in a given interval differ by at most $n$.

Now let us turn our attention to the discrete cases. Consider a conjoined basis $\binom{X}{U}$ of (7), i.e.,

$$\operatorname{rank}\begin{pmatrix} X_k \\ U_k \end{pmatrix} = n, \quad X_k^T U = U_k^T X.$$

Following [5], an interval $(k, k+1]$ contains no focal point of $\binom{X}{U}$ if

$$\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k, \quad X_k X_{k+1}^{\dagger} \mathcal{B}_k \geq 0, \tag{23}$$

see [5]. It took relatively a great effort to define the concept of *multiplicity* of a focal point when one of the conditions in (23) is violated. This problem was resolved in [17] in 2003 as follows. Define the matrices

$$M_k = (I - X_{k+1} X_{k+1}^{\dagger}) \mathcal{B}_k, \quad P_k = T_k X_k X_{k+1}^{\dagger} \mathcal{B}_k T_k, \tag{24}$$

where $T_k = I - M_k^{\dagger} M_k$. It was shown in [17] that if one of the conditions (23) is violated then

$$\operatorname{rank} M_k + \operatorname{ind} P_k > 0$$

(ind denotes the index, i.e., the number of negative eigenvalues of a matrix) and this quantity defines the multiplicity of a focal point of $\binom{X}{U}$ in $(k, k+1]$. More precisely, the quantity $m_1(k) = \operatorname{rank} M_k$ is called the multiplicity of a focal point at $k+1$, while the quantity $m_2(k) = \operatorname{ind} P_k$ is the called the multiplicity of a focal point between $k$ and $k+1$. Since discrete trigonometric system (9) is a special case of (7), these definitions apply to this system as well. Now, based on the scalar case $n = 1$ we conjecture that a similar geometrical oscillation theory as for Hamiltonian trigonometric differential systems holds also in the discrete case.

To specify our idea in more details, consider a conjoined basis $\binom{S}{C}$ of (9) with $\mathcal{Q}_k \geq 0$ (what we may assume without loss of generality) satisfying (22) and let

$$G_k = C_k C_k^T - S_k S_k^T + 2i S_k C_k^T,$$

this matrix is unitary and $G_k = \cos 2\left(\sum^{k-1} \varphi_j\right) + i \sin 2\left(\sum^{k-1} \varphi_j\right)$ in the scalar case $n = 1$ with $\varphi_j$ given by (11).

Our conjecture is as follows. Suppose that the interval $(k, k+1]$ contains a focal point of $\binom{S}{C}$ of multiplicity

$$m_1(k) + m_2(k) = \operatorname{rank} M_k + \operatorname{ind} P_k$$

with

$$M_k = (I - S_{k+1} S_{k+1}^{\dagger}) \mathcal{Q}_k, \quad P_k = T_k S_k S_{k+1}^{\dagger} \mathcal{Q}_k T_k,$$

where $T_k = I - M_k^{\dagger} M_k$. We conjecture, in terms of behavior of the eigenvalues of the unitary matrix $G_k$, that $m_1(k)$ eigenvalues of $G_k$ which are different from $\lambda = 1$ (the point $[1, 0]$ on the unit circle) "skip" to $\lambda = 1$ at $k+1$ (the focal point of multiplicity $m_1(k)$ at $k+1$) and $m_2(k)$ eigenvalues of $G_k$ skip over the point $\lambda = 1$ on the unit circle in the complex plane between $k$ and $k+1$. The formulation "skip over the point $\lambda = 1$" we understand in the following sense. Let $e^{i\varphi_k^{[1]}}, \dots, e^{i\varphi_k^{[n]}}$ be the eigenvalues of $G_k$ such that $\varphi_k^{[1]} \leq \varphi_k^{[2]} \leq \dots \leq \varphi_k^{[n]}$ and let $e^{i\varphi_{k+1}^{[1]}}, \dots, e^{i\varphi_{k+1}^{[n]}}$ with $\varphi_{k+1}^{[1]} \leq \varphi_{k+1}^{[2]} \leq \dots \leq \varphi_{k+1}^{[n]}$ be the eigenvalues of $G_{k+1}$. We say that an eigenvalue $e^{i\varphi^{[j]}}$ skips over the point $\lambda = 1$ on the unit circle between $k$ and $k+1$ if the positively oriented arc of the unit circle between $e^{i\varphi_k^{[j]}}$ and $e^{i\varphi_{k+1}^{[j]}}$ intersects the point $[1, 0]$ on the unit circle.

A positive answer to this conjecture would open a new view on the discrete oscillation theory. Note that the Sturmian separation theorem holds for (7) similarly as in the continuous case. The geometrical discrete oscillation theory would provide a nice alternative proof of the basic statement of this theory which says that the numbers of focal points of two conjoined bases of (7) in a given discrete interval differ by at most $n$, see [7,13].

Another open problem associated with the Bohl transformation is related to the Sturm comparison theorem for Hamiltonian and symplectic systems. Consider a pair of Hamiltonian differential systems with the coefficient matrices $H(t)$ and $\tilde{H}(t)$ such that $H(t) \geq \tilde{H}(t)$ for large $t$. Sturmian comparison theorem states that if the minorant system with $\tilde{H}$ is oscillatory, then the majorant system with $H$ is oscillatory as well. The Bohl transformation from Theorem 1 shows that there is no "ideal" comparison system in the following sense. Given an oscillatory Hamiltonian system with the coefficient matrix $H$, there exists its minorant system with

$$\tilde{H}(t) \leq H(t), \quad \tilde{H}(t) \not\equiv H(t) \tag{25}$$

such that the minorant system is still oscillatory. The proof of this statement is based on the Bohl transformation. We transform the original oscillatory Hamiltonian system to the trigonometric system with the matrix $Q$ satisfying $\operatorname{Tr} Q(t)\, dt = \infty$. Then we take the trigonometric system with the matrix $\mu Q(t)$ with $\mu \in (0, 1)$ and transform it "back" to general Hamiltonian system using the same transformation matrices as in transformation of the system with $H(t)$ to trigonometric system. As a result we obtain a Hamiltonian system which is also oscillatory and whose coefficient matrix $\tilde{H}$ satisfies (25).

A completely open problem is what is the situation in the discrete cases. Even in the most simple case of the second order oscillatory difference equation (note that Sturmian comparison theorem holds for this equation similarly as in the continuous case)

$$\Delta^2 x_k + p_k x_{k+1} = 0$$

it is not clear how to construct a sequence $\tilde{p}_k < p_k$ for large $k$ in such a way that the resulting difference equation

$$\Delta^2 x_k + \tilde{p}_k x_{k+1} = 0$$

is still oscillatory.

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