

Accepted Manuscript

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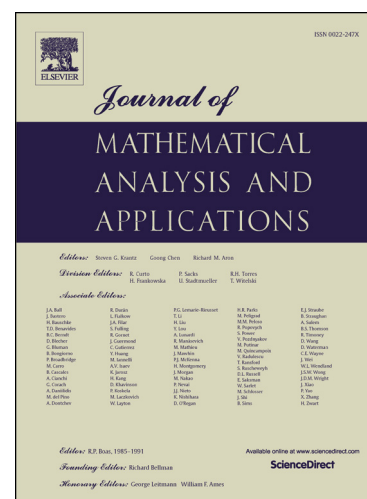
PII: S0022-247X(16)30832-0
DOI: <http://dx.doi.org/10.1016/j.jmaa.2016.12.036>
Reference: YJMAA 20976

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 7 May 2016

Please cite this article in press as: Y. Jia et al., Effect of predator cannibalism and prey growth on the dynamic behavior for a predator-stage structured population model with diffusion, *J. Math. Anal. Appl.* (2017), <http://dx.doi.org/10.1016/j.jmaa.2016.12.036>

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Effect of predator cannibalism and prey growth on the dynamic behavior for a predator-stage structured population model with diffusion *

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Abstract In this paper, a predator-prey model with predator-stage structured and diffusion is concerned. We deal with the system by endowing it with the homogeneous Neumann boundary conditions. We first give the a priori estimates of positive solutions for the reduced reaction diffusion system. Secondly, we discuss the effects of predator cannibalism and prey growth on the stability of nonnegative constant steady states of the model in detail. Thirdly, we investigate the nonexistence and existence of nonconstant positive solutions. Finally, we discuss the Hopf bifurcation created by diffusion.

Keywords Predator-prey model; stage structure; steady states; stability; fixed point index; Hopf bifurcation.

2000 Mathematics subject classifications 92D25; 93C20; 35K57

1 Introduction

In population dynamics, in order to model populations which go through distinct stages, such as laboratory insect populations, it is necessary to take the age structure of the populations into account. The age and development stage always effect the existence and extinction of different species which inhabit in a same living environment. Population growth models with age or stage structure often predict complex population dynamics. Due to these evidences, models with stage structured populations are important in different kinds of ecosystems, see, for example [3, 7, 9, 10, 15, 19, 20, 21, 22, 23, 27, 33, 34]. Generally speaking, population growth models that include stage structure predict more complex population dynamics than those without taking these factors into account.

For stage structured predator-prey systems, the phenomenon of adults cannibalize juveniles within the same species is a common natural phenomenon in some species [4, 5,

*The work was supported in part by the National Science Foundation of China (11271236, 11671243, 61672021), by the Shaanxi New-star Plan of Science and Technology (2015KJXX-21), and also by the Natural Science Basic Research Plan in Shaanxi Province of China (2015JQ1023).

14], especially in a variety of fish species, such as North Atlantic cod [1], salmon [6], perch [8] and striped bass [12] and the references contained therein. Cannibalism is an interesting and important mechanism in population dynamics, as the large literatures on the subject demonstrate. The role in stabilizing or destabilizing population cycles for these models has been deeply studied in recent years. Biologically, such phenomenon can support the idea that cannibalism is used as a means of population control to prevent them from overbreeding. This intraspecific preying behavior has been a subject in mathematics and ecology, and numerous different modelling approaches have been proposed [1, 4, 11, 13, 14].

The predator cannibalism and prey growth both have certain influences on the dynamics of the model, sometimes even a considerable result will occur. In [2], Kohlmeier and Ebenhöf introduced a two species predator-prey model with logistic prey growth and showed that cannibalism could affect the stability of the system. They found that a high cannibalism rate may make the unique equilibrium with positive components change from being unstable to stable. The very similar results has also been obtained in [15] for an age structured predator-prey system with delay. There are some other literatures show that cannibalism may have a few disadvantages practicing on populations, such as cannibalism wastes high cost energy needed to breed, see for example [16, 17]. Furthermore, cannibalism also has unstable effect on some predator-prey systems [18].

On the other hand, the prey growth may destabilize the solutions of a structured predator-prey model [19, 20]. In [19], using the standard method of determining equilibria and locating the roots of the corresponding characteristic equation, the authors analyzed the effect of growth rate on the stability about an equilibrium of a stage structured competition population system. In [20], by linearization method, for a predator-prey system with stage structured for predator, the authors considered the asymptotic stability of the positive equilibrium of the system.

In [27], Du et al. proposed a predator-prey model with stage structure for predator and with diffusion and cross diffusion. The ODE version of this model is

$$\begin{cases} x'(t) = Ax - Bx^2 - \varepsilon Cxy_1 - Cxy_2, & t > 0, \\ y_1'(t) = Kxy_2 - Dy_1 - My_1, & t > 0, \\ y_2'(t) = Dy_1 - Py_2, & t > 0, \end{cases}$$

where x, y_1 and y_2 are densities of the prey, immature predator and mature predator, respectively. The model also assumes that the immature predator consumes the prey. The authors discussed the stability of nonnegative constant steady states for the ODE system and the corresponding reaction diffusion system with homogeneous Neumann boundary conditions, Hopf bifurcation for the ODE system, Turing Pattern created by cross diffusion.

Recently, Buonomo et al. [24, 25] considered the following ODE model.

$$\begin{cases} A'(\tau) = mJ - d_A A, & \tau > 0, \\ J'(\tau) = \eta_1 \delta AP - (1 - \eta_2) \sigma AJ - (m + d_J)J, & \tau > 0, \\ P'(\tau) = \gamma_1 P - \gamma_2 P^2 - \delta AP, & \tau > 0, \end{cases} \quad (1.1)$$

where $A(\tau)$ and $J(\tau)$ denote the number of adults and juveniles of the predator populations and $P(\tau)$ denotes the number of individuals of a non-structured prey population at time

τ , respectively. Other parameters: m is the constant maturation rate from juveniles to adults for predator; d_A and d_J are the death rates of A and J , and assumed to be different; η_1 is the rate of preys in converting themselves into new juveniles of predators; δ is the interspecific competition rate; η_2 is the rate of juvenile predators in converting themselves into new juveniles and is assumed smaller than 1; σ is the cannibalism attack rate; γ_1 and γ_2 are the logistic coefficients. All parameters are positive constants. It is assumed in system (1.1) that the predator is stage structured and has an instantaneous maturation rate. The adults of predator not only consume the prey but also juveniles. For more details of system (1.1), the readers can refer to references [24, 25].

By scaling

$$u = \frac{\delta}{d_A}A, \quad v = \frac{m\delta}{d_A^2}J, \quad w = \frac{\gamma_2}{d_A}P$$

and setting following non-dimensional variables and parameters:

$$t = d_A\tau, \quad a = \frac{m\delta\eta_1}{d_A\gamma_2}, \quad b = \frac{\sigma(1-\eta_2)}{\delta}, \quad c = \frac{m+d_J}{d_A}, \quad d = \frac{\gamma_1}{d_A},$$

then system (1.1) transforms into following form

$$\begin{cases} u'(t) = v - u, & t > 0, \\ v'(t) = auw - buv - cv, & t > 0, \\ w'(t) = dw - w^2 - uw, & t > 0. \end{cases} \quad (1.2)$$

Thus, in system (1.2), u, v and w represent the adults, the juveniles of the predator and the non-structured prey populations, respectively. We may regard b and d as the predator cannibalism attack rate and the prey growth rate respectively in this new system.

What is of interest in a ecological system is whether the various species can coexist. Sometimes, the species coexist in a steady state. In the case where the species are homogeneously distributed, this phenomenon may be reflected by a constant positive solution to the mathematical model. While in the spatially inhomogeneous case, the existence of a nonconstant time-independent positive solution, also called stationary solution, is an indication of the dynamics of system. Many authors have established the existence of stationary states in various population dynamics models in the presence of diffusion, some of which have been cited in the present paper. Among many classical predator-prey mathematical ecology models, there reflects only population changes due to predation in a situation where predator and prey densities are not spatially dependent. It takes into account neither the fact that population is usually not homogeneously distributed, nor the fact that predators and preys naturally develop strategies for survival. Both of these considerations involve diffusion processes which can be quite intricate as different concentration levels of preys and predators cause different population movements [10].

Motivated by their works and ideas in [24, 25, 27], to take into account the inhomogeneous distribution of the predator and prey in different spatial locations within a fixed bounded domain at any given time, and the natural tendency of each species to diffuse to areas of smaller population concentration, we should consider the corresponding system of partial differential equations. When the species are confined to a fixed and bounded living

space, we are led to the following partial differential equations system of reaction-diffusion type

$$\begin{cases} u_t - d_1 \Delta u = v - u, & x \in \Omega, \quad t > 0, \\ v_t - d_2 \Delta v = auw - buv - cv, & x \in \Omega, \quad t > 0, \\ w_t - d_3 \Delta w = dw - w^2 - uw, & x \in \Omega, \quad t > 0, \\ \partial_n u = \partial_n v = \partial_n w = 0, & x \in \partial\Omega, \quad t > 0, \\ u = u_0 \geq, \neq 0, v = v_0 \geq, \neq 0, w = w_0 \geq, \neq 0, & x \in \Omega, \quad t = 0, \end{cases} \quad (1.3)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, ∂_n is the outer normal directional derivative on $\partial\Omega$. The homogeneous Neumann boundary condition indicates that the predator-prey system is self-contained with zero population flux across the boundary. The constants d_1, d_2 and d_3 are the diffusion coefficients of the species and assumed to be positive. The initial data u_0, v_0 and w_0 are nonnegative continuous functions in Ω .

The main aim of this paper is to study the effects of the system parameters on the nonnegative steady states of system (1.3), namely, the solutions of the following elliptic system

$$\begin{cases} -d_1 \Delta u = v - u, & x \in \Omega, \\ -d_2 \Delta v = auw - buv - cv, & x \in \Omega, \\ -d_3 \Delta w = dw - w^2 - uw, & x \in \Omega, \\ \partial_n u = \partial_n v = \partial_n w = 0, & x \in \partial\Omega. \end{cases} \quad (1.4)$$

The opposite effect on the model dynamics by prey growth and predator cannibalism rate, respectively, drives us to a deeper investigation about the model dynamics when both these parameters are above their critical values. This is one of the question that this paper deals with.

The paper is organized as follows. First, in Section 2, we establish the a priori upper and lower bounds for the positive solutions of (1.4), which is necessary and will be used in subsequent sections. In Section 3, we study the stability of nonnegative constant solutions of (1.4) in detail by applying the Routh-Hurwitz criterion. In Section 4, we give the nonexistence and existence results of nonconstant positive solution for the model by considering the diffusion and the fixed point index respectively. Finally, we discuss the Hopf bifurcation solution which emanates from the unique positive constant solution, this is done by using the Hopf bifurcation theory.

2 A priori estimates of positive solutions

The main purpose of this section is to give the a priori upper and lower bounds for positive solutions of (1.4). To this aim, we first introduce a simple but very useful result due to Lou and Ni [26] by virtue of the maximum principle.

Lemma 2.1. [26] *Suppose that $g \in C(\bar{\Omega} \times \mathbb{R}^1)$.*

(i) *If $z \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies*

$$\Delta z(x) + g(x, z(x)) \geq 0, \quad \text{in } \Omega, \quad \partial_n z \leq 0 \quad \text{on } \partial\Omega,$$

and $z(x_0) = \max_{\bar{\Omega}} z$, then $g(x_0, z(x_0)) \geq 0$;

(ii) If $z \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies

$$\Delta z(x) + g(x, z(x)) \leq 0, \quad \text{in } \Omega, \quad \partial_n z \geq 0 \quad \text{on } \partial\Omega,$$

and $z(x_0) = \min_{\overline{\Omega}} z$, then $g(x_0, z(x_0)) \leq 0$.

Theorem 2.2. Suppose that $b > a$ and $a^2 d < b(ad - c) < d(a^2 + b^2)$. Then any positive solution (u, v, w) of (1.4) satisfies

$$\frac{b(ad - c) - a^2 d}{b^2} \leq u, v \leq \frac{ad}{b}, \quad \frac{d(b - a)}{b} \leq w \leq \frac{d(a^2 + b^2) - b(ad - c)}{b^2}.$$

Proof. Let (u, v, w) be a positive solution of system (1.4) and

$$\begin{aligned} u(x_{11}) &= \max_{\overline{\Omega}} u(x), & v(x_{12}) &= \max_{\overline{\Omega}} v(x), & w(x_{13}) &= \max_{\overline{\Omega}} w(x), \\ u(x_{21}) &= \min_{\overline{\Omega}} u(x), & v(x_{22}) &= \min_{\overline{\Omega}} v(x), & w(x_{23}) &= \min_{\overline{\Omega}} w(x). \end{aligned}$$

Applying Lemma 2.1 to the three equations of (1.4), in turn, we have

$$\begin{cases} v(x_{11}) - u(x_{11}) \geq 0, \\ v(x_{21}) - u(x_{21}) \leq 0, \end{cases} \quad (2.1)$$

$$\begin{cases} au(x_{12})w(x_{12}) - bu(x_{12})v(x_{12}) - cv(x_{12}) \geq 0, \\ au(x_{22})w(x_{22}) - bu(x_{22})v(x_{22}) - cv(x_{22}) \leq 0, \end{cases} \quad (2.2)$$

and

$$\begin{cases} d - w(x_{13}) - u(x_{13}) \geq 0, \\ d - w(x_{23}) - u(x_{23}) \leq 0. \end{cases} \quad (2.3)$$

From equalities (2.1), we know that

$$v(x_{22}) \leq v(x_{21}) \leq u(x_{21}) \leq u(x_{11}) \leq v(x_{11}) \leq v(x_{12}).$$

Thus, once we get the lower and upper bounds of v , considering (2.3), we can obtain the estimates of u, w .

Obviously, the third equation of (1.4) implies that $\max_{\overline{\Omega}} w \leq d$. By the first inequality of (2.2), we have

$$v(x_{12}) \leq \frac{au(x_{12})w(x_{12})}{bu(x_{12}) + c} \leq \frac{au(x_{12})w(x_{12})}{bu(x_{12})} \leq \frac{ad}{b}.$$

Set $f(u) = \frac{auw}{bu+c}$. Then $f(u)$ is increasing in u . Using the monotonicity of $f(u)$ and the second inequalities of (2.2) and (2.3), we have

$$\begin{aligned} v(x_{22}) &\geq \frac{au(x_{22})w(x_{22})}{bu(x_{22}) + c} \geq \frac{au(x_{21})w(x_{23})}{bu(x_{21}) + c} \geq \frac{au(x_{21})(d - u(x_{23}))}{bu(x_{21}) + c} \\ &\geq \frac{au(x_{21})(d - u(x_{11}))}{bu(x_{21}) + c} \geq \frac{au(x_{21})(d - v(x_{12}))}{bu(x_{21}) + c} \geq \frac{a(d - \frac{ad}{b})u(x_{21})}{bu(x_{21}) + c} \\ &= \frac{ad(b - a)u(x_{21})}{b(bu(x_{21}) + c)} \geq \frac{ad(b - a)v(x_{22})}{b(bv(x_{22}) + c)}. \end{aligned}$$

Therefore,

$$v(x_{22}) \geq \frac{b(ad - c) - a^2d}{b^2}.$$

Further, u satisfies

$$\frac{b(ad - c) - a^2d}{b^2} \leq u(x) \leq \frac{ad}{b}.$$

From (2.3) we get

$$\begin{aligned} w(x_{13}) &\leq d - u(x_{13}) \leq d - u(x_{21}) \\ &\leq d - \frac{b(ad - c) - a^2d}{b^2} = \frac{d(a^2 + b^2) - b(ad - c)}{b^2}, \\ w(x_{23}) &\geq d - u(x_{23}) \geq d - u(x_{11}) \geq d - \frac{ad}{b} = \frac{d(b - a)}{b}. \end{aligned}$$

This completes the proof.

Remark 2.3. It is easy to verify that

$$\frac{b(ad - c) - a^2d}{b^2} < \frac{ad}{b} \quad \text{and} \quad \frac{d(b - a)}{b} < \frac{d(a^2 + b^2) - b(ad - c)}{b^2}$$

with our assumptions.

Remark 2.4. From the third equation of (1.4), we see that $w \leq d$ whenever (u, v, w) is a positive solution of (1.4). In fact, one can easily check that the upper bound of w in Theorem 2.2 is less than d .

3 Stability of constant solutions

In this section, using the well-known results in [28], we discuss the effects of the prey growth and predator cannibalism on the stability of constant solutions of system (1.4). The possible constant solutions are $U_0 = (0, 0, 0)$, $U_1 = (0, 0, d)$ and $U_2 = \left(\frac{ad-c}{a+b}, \frac{ad-c}{a+b}, \frac{bd+c}{a+b}\right)$ when $ad > c$, which are also the possible constant solutions of system (1.2). It can be observed from U_2 that increasing the cannibalistic parameter b results in an increase of the prey steady state w and a decrease of adult u and juvenile v predator steady states, irrespective of parameter values. This feature accords with the analysis presented in [15] for an age-structured model including delayed juvenile maturation.

In [24, 25], the stability and unstability of the three possible constant solutions of (1.2) has been discussed. In the case that if the solution is unstable for the ODE system (1.2), then it is also unstable for the PDE system (1.4). Here, for discussing the Hopf bifurcation and for the convenience to reader, we give the necessary details. Our discussion shows that U_1 and U_2 have the same stability properties as in [24, 25].

Let $0 = \mu_0 < \mu_1 < \mu_2 < \mu_3 < \dots$ be the eigenvalues of $-\Delta$ in Ω with the homogeneous Neumann boundary condition. For convenience in use, we set

$$X = \{(u, v, w) \in (C^1(\bar{\Omega}))^3 \mid \partial_n u = \partial_n v = \partial_n w = 0, x \in \partial\Omega\}.$$

Again, for each $i = 0, 1, 2, 3, \dots$, let E_i be the eigenspace corresponding to μ_i in $C^1(\overline{\Omega})$ and $\{\phi_{ij} : j = 1, \dots, \dim E_i\}$ is an orthonormal basis of E_i and $X_{ij} = \{l\phi_{ij} | l \in \mathbb{R}^3\}$. Then

$$X = \oplus_{i=0}^{\infty} X_i, \quad X_i = \oplus_{j=1}^{\dim E_i} X_{ij}.$$

Theorem 3.1. *The solution U_0 is unstable, U_1 is locally asymptotically stable if $d < \frac{c}{a}$ and unstable if $ad > c$. Therefore the stability properties of U_0 and U_1 do not depend on the parameter b , which is in turn proportional to the cannibalism attack rate σ . This means that the cannibalism per se can not lead to predator extinction.*

Proof. The linearized operator of (1.4) at U_0 is

$$\begin{pmatrix} d_1\Delta - 1 & 1 & 0 \\ 0 & d_2\Delta - c & 0 \\ 0 & 0 & d_3\Delta + d \end{pmatrix} =: L_0.$$

Since all eigenvalues of L_0 are $\sigma\{d_1\Delta - 1\} \cup \sigma\{d_2\Delta - c\} \cup \sigma\{d_3\Delta + d\}$, here we use $\sigma\{L\}$ to denote the spectrum set of operator L , we see that L_0 has at least a positive eigenvalue d , and U_0 is unstable.

The corresponding Jacobian matrix of (1.4) at U_1 is

$$\begin{pmatrix} d_1\Delta - 1 & 1 & 0 \\ ad & d_2\Delta - c & 0 \\ -d & 0 & d_3\Delta - d \end{pmatrix} =: L_1.$$

For each $i \geq 0$, X_i is L_1 -invariant, and μ is an eigenvalue of L_1 on X_i if and only if μ is an eigenvalue of the matrix

$$\begin{pmatrix} -d_1\mu_i - 1 & 1 & 0 \\ ad & -d_2\mu_i - c & 0 \\ -d & 0 & -d_3\mu_i - d \end{pmatrix} =: A.$$

The characteristic polynomial of A is given by

$$f_i(\mu) = \mu^3 + A_{1i}\mu^2 + A_{2i}\mu + A_{3i},$$

where

$$\begin{aligned} A_{1i} &= (d_1 + d_2 + d_3)\mu_i + c + d + 1, \\ A_{2i} &= (d_1d_2 + d_2d_3 + d_1d_3)\mu_i^2 + (d_1(c + d) + d_2(1 + d) + d_3(1 + c))\mu_i + c + d + cd - ad, \\ A_{3i} &= d_1d_2d_3\mu_i^3 + (dd_1d_2 + d_2d_3 + cd_1d_3)\mu_i^2 + (d(cd_1 + d_2) + d_3(c - ad))\mu_i + d(c - ad). \end{aligned}$$

A complicated computation gives

$$A_{1i}A_{2i} - A_{3i} = P_3\mu_i^3 + P_2\mu_i^2 + P_1\mu_i + P_0$$

with

$$\begin{aligned}
 P_3 &= (d_1 + d_2 + d_3)(d_1 d_2 + d_2 d_3 + d_1 d_3) - d_1 d_2 d_3, \\
 P_2 &= (d_1 + d_2 + d_3)(d_1(c + d) + d_2(d + 1) + d_3(c + 1)) \\
 &\quad + (c + d + 1)(d_1 d_2 + d_2 d_3 + d_1 d_3) - (d d_1 d_2 + d_2 d_3 + c d_1 d_3) \\
 &= d_1(d_1(c + d) + d_2(d + 1) + d_3) + d_2(c d_1 + d_2(d + 1) + d_3(c + 1)) \\
 &\quad + d_3(d_1(c + d) + d d_2 + d_3(c + 1)), \\
 P_1 &= (d_1 + d_2 + d_3)(c + d + c d - a d) + (c + d + 1)(d_1(c + d) + d_2(d + 1) + d_3(c + 1)) \\
 &\quad - (d(c d_1 + d_2) + d_3(c - a d)) \\
 &= (d_1 + d_2)((c + d)(d + 1) - a d) + d_3(c + d + c d) + c(d_1(c + d) + d_2(d + 1) + c d_3) \\
 &\quad + d(d_2 + d_3(c + 1)) + d_1(c + d) + d_2 + d_3(c + 1), \\
 P_0 &= c(c(d + 1) - a d) + (d + 1)(c + d + c d) - a d = (c + 1)((c + d)(d + 1) - a d).
 \end{aligned}$$

It is easily seen that $A_{1i}, P_3, P_2 > 0$. If $d < \frac{c}{a}$ then we must have $A_{2i}, A_{3i}, P_1, P_0 > 0$. Thus, for all $i \geq 0$, we conclude that $A_{1i}A_{2i} - A_{3i} > 0$ provided that $d < \frac{c}{a}$. It thus follows from the Routh-Hurwitz stability criterion [28] that, for each $i \geq 0$, the three roots of characteristic equation $f_i(\mu) = 0$ all have negative real parts, and therefore U_1 is stable. Conversely, if $ad > c$, then there exists at least some $i \geq 0$ such that $f_i(\mu) = 0$ has at least a positive real root or a pair of conjugate complex roots with positive real parts and thus U_1 is unstable. The proof is finished.

Theorem 3.2. *Let $ad > c$. Then there is a unique positive constant b_0 such that U_2 is locally asymptotically stable for $b > b_0$ and unstable for $b < b_0$. In particular, in the case that cannibalism is absent, U_2 is locally asymptotically stable for $d < \frac{1}{a^2}((a + 1)c^2 + (3a + 1)c + a)$ and unstable for $d > \frac{1}{a^2}((a + 1)c^2 + (3a + 1)c + a)$.*

Proof. We always suppose that $ad > c$ holds. The Jacobian matrix of (1.4) at U_2 is

$$\begin{pmatrix} d_1\Delta + a_{11} & a_{12} & a_{13} \\ a_{21} & d_2\Delta + a_{22} & a_{23} \\ a_{31} & a_{32} & d_3\Delta + a_{33} \end{pmatrix} =: L_2$$

with

$$\begin{cases} a_{11} = -1, & a_{12} = 1, & a_{13} = 0, \\ a_{21} = c, & a_{22} = -\frac{a(c+bd)}{a+b}, & a_{23} = \frac{a(ad-c)}{a+b}, \\ a_{31} = -\frac{c+bd}{a+b}, & a_{32} = 0, & a_{33} = -\frac{c+bd}{a+b}. \end{cases}$$

As the same as previously described, for each $i \geq 0$, X_i is invariant under L_2 , and μ is an eigenvalue of L_2 on X_i if and only if μ is an eigenvalue of the matrix

$$\begin{pmatrix} -d_1\mu_i + a_{11} & a_{12} & a_{13} \\ a_{21} & -d_2\mu_i + a_{22} & a_{23} \\ a_{31} & a_{32} & -d_3\mu_i + a_{33} \end{pmatrix} =: B.$$

The characteristic polynomial of B is

$$g_i(\mu) = \mu^3 + B_{1i}\mu^2 + B_{2i}\mu + B_{3i},$$

where

$$\begin{aligned} B_{1i} &= (d_1 + d_2 + d_3)\mu_i - a_{11} - a_{22} - a_{33}, \\ B_{2i} &= (d_1d_2 + d_2d_3 + d_1d_3)\mu_i^2 - (d_1(a_{22} + a_{33}) + d_2(a_{11} + a_{33}) + d_3(a_{11} + a_{22}))\mu_i \\ &\quad + a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{13}a_{31} - a_{23}a_{32} - a_{12}a_{21}, \\ B_{3i} &= d_1d_2d_3\mu_i^3 - (d_1d_2a_{33} + d_2d_3a_{11} + d_1d_3a_{22})\mu_i^2 \\ &\quad + (d_1(a_{22}a_{33} - a_{23}a_{32}) + d_2(a_{11}a_{33} - a_{13}a_{31}) + d_3(a_{11}a_{22} - a_{12}a_{21}))\mu_i \\ &\quad - a_{11}a_{22}a_{33} - a_{13}a_{21}a_{32} - a_{12}a_{23}a_{31} + a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33} + a_{11}a_{23}a_{32}. \end{aligned}$$

Considering the signs of $a_{ij}, i, j = 1, 2, 3$, we see that $B_{1i} > 0$. Furthermore, in the expression of B_{2i} ,

$$a_{11}a_{22} - a_{12}a_{21} = \frac{b(ad - c)}{a + b},$$

and in B_{3i} ,

$$-a_{11}a_{22}a_{33} - a_{12}a_{23}a_{31} + a_{12}a_{21}a_{33} = \frac{(a + b)(ad - c)}{(a + b)^2}.$$

So, $B_{2i}, B_{3i} > 0$.

A series of calculation yields

$$B_{1i}B_{2i} - B_{3i} = Q_3\mu_i^3 + Q_2\mu_i^2 + Q_1\mu_i + Q_0$$

with

$$\begin{aligned} Q_3 &= (d_1 + d_2 + d_3)(d_1d_2 + d_2d_3 + d_1d_3) - d_1d_2d_3, \\ Q_2 &= -(a_{11} + a_{22})(d_1d_2 + d_3(d_1 + d_2 + d_3)) - (a_{22} + a_{33})(d_2d_3 + d_1(d_1 + d_2 + d_3)) \\ &\quad - (a_{11} + a_{33})(d_1d_3 + d_2(d_1 + d_2 + d_3)), \\ Q_1 &= d_1((a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{13}a_{31} - a_{23}a_{32} - a_{12}a_{21}) \\ &\quad + a_{23}a_{32} + a_{22}(a_{11} + a_{22} + a_{33}) + a_{33}(a_{11} + a_{33})) \\ &\quad + d_2((a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{13}a_{31} - a_{23}a_{32} - a_{12}a_{21}) \\ &\quad + a_{13}a_{31} + a_{33}(a_{11} + a_{22} + a_{33}) + a_{11}(a_{11} + a_{22})) \\ &\quad + d_3((a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{13}a_{31} - a_{23}a_{32} - a_{12}a_{21}) \\ &\quad + a_{12}a_{21} + a_{22}(a_{11} + a_{22} + a_{33}) + a_{11}(a_{11} + a_{33})), \\ Q_0 &= -(a_{11} + a_{22} + a_{33})(a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{13}a_{31} - a_{23}a_{32} - a_{12}a_{21}) \\ &\quad - (-a_{11}a_{22}a_{33} - a_{13}a_{21}a_{32} - a_{12}a_{23}a_{31} + a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33} + a_{11}a_{23}a_{32}). \end{aligned}$$

It is easily seen that $Q_3, Q_2 > 0$. Furthermore,

$$a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{13}a_{31} - a_{23}a_{32} - a_{12}a_{21} = \frac{b(ad - c) + c + bd}{a + b} + \frac{a(c + bd)^2}{(a + b)^2},$$

thus, $Q_1 > 0$, and therefore $B_{1i}B_{2i} - B_{3i} \geq Q_0$. Now, we consider Q_0 . Substituting the values of a_{ij} , $i, j = 1, 2, 3$ into Q_0 and sorting them, we obtain

$$\begin{aligned} Q_0 &= a(a+1) \left(\frac{c+bd}{a+b} \right)^3 + (a^2+3a+1) \left(\frac{c+bd}{a+b} \right)^2 \\ &\quad + (1+a-ac-ad) \left(\frac{c+bd}{a+b} \right) - c \\ &=: a(a+1)y^3(b) + (a^2+3a+1)y^2(b) + (1+a-ac-ad)y(b) - c =: Q_0(y). \end{aligned}$$

We see that

$$\lim_{y \rightarrow 0^+} Q_0(y) = -c < 0, \quad \lim_{y \rightarrow +\infty} Q_0(y) = +\infty,$$

so there at least exists a $y_0 = y(b_0) > 0$ such that $Q_0(y_0) = 0$. Meanwhile, the derivative of Q_0 on y is

$$Q'_0(y) = 3a(a+1)y^2(b) + 2(a^2+3a+1)y(b) + (1+a-ac-ad).$$

Since the coefficients of the first and second order of $Q'_0(y)$ are positive, and

$$\begin{aligned} &(2(a^2+3a+1))^2 - 4 \cdot 3a(a+1)(1+a-ac-ad) \\ &= 4(a^4+3(c+d+1)a^3+(3c+3d+5)a^2+3a+1) > 0, \end{aligned}$$

considering the positivity of y , we know that the equation $Q'_0(y) = 0$ has at most a root in $(0, \infty)$. If it has no root in $(0, \infty)$, then $Q_0(y)$ is increasing in $(0, \infty)$, and y_0 is unique, $Q_0(y) > 0$ for $y > y_0$ (c.f. Figure 1). If there is a root in $(0, \infty)$, say, \bar{y} , then $Q_0(y)$ is decreasing in $(0, \bar{y})$ and increasing in (\bar{y}, ∞) , y_0 is also unique, $Q_0(y) > 0$ for $y > y_0$ (c.f. Figure 2). These facts show that there exists a unique y_0 such that $Q_0(y) > 0$ when a, b, c, d satisfy $\frac{c+bd}{a+b} > y_0 = \frac{c+b_0d}{a+b_0}$. Considering the monotonicity of $y(b)$ on b , it induces that there is a unique $b_0 > 0$ such that $Q_0(y) > 0$ provided that $b > b_0$.

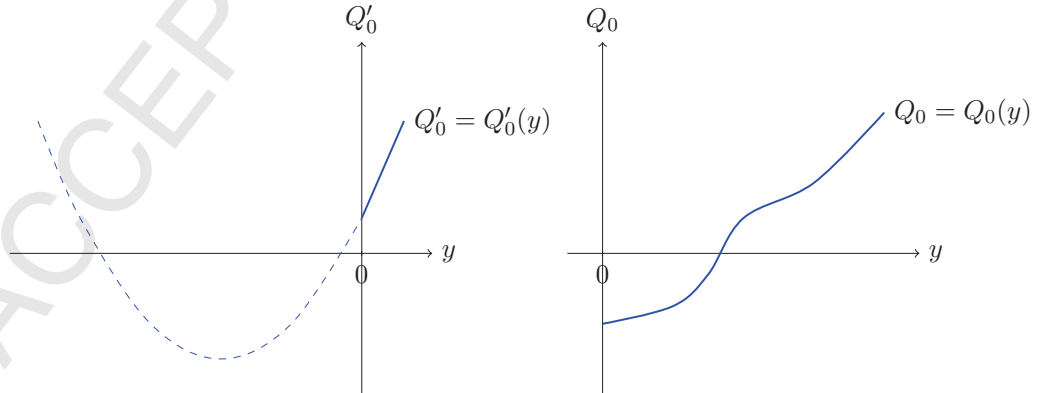


Fig. 1. Equation $Q'_0(y) = 0$ has no root in $(0, \infty)$ (left panel), the intersection of curve $Q_0(y)$ and y -axis, say y_0 , is unique (right panel).

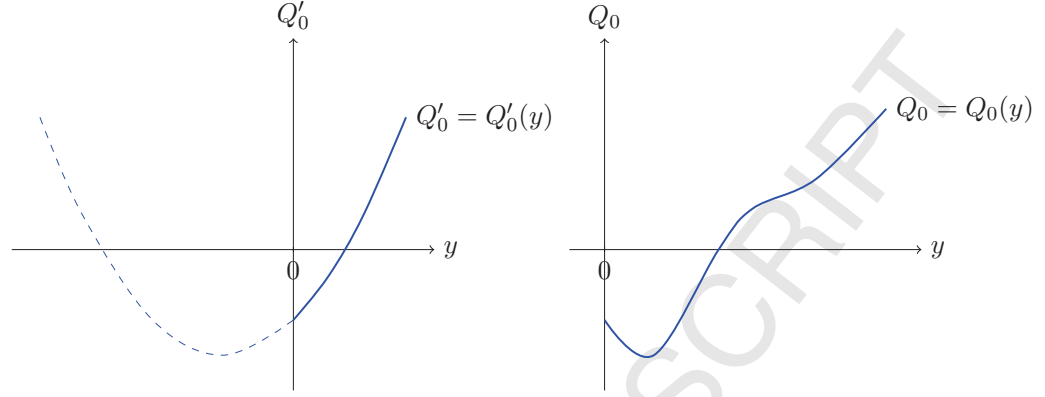


Fig. 2. Equation $Q'_0(y) = 0$ has a root \bar{y} in $(0, \infty)$ (left panel), the intersection y_0 of curve $Q_0(y)$ and y -axis is also unique (right panel).

Therefore, in the case $b > b_0$, we must have $B_{1i}B_{2i} - B_{3i} > 0$ and the Routh-Hurwitz criterion implies that for each $i \geq 0$, the three roots of characteristic equation $g_i(\mu) = 0$ all have negative real parts, and therefore U_2 is stable. Whereas, if $b < b_0$, then $\frac{c+bd}{a+b} < y_0$, $Q_0 < 0$, and there exists at least some $i \geq 0$ such that $g_i(\mu) = 0$ has at least a positive real root or a pair of conjugate complex roots with positive real parts and thus U_2 is unstable.

If the predator cannibalism is absent, namely, $b = 0$, then we still have $Q_3, Q_2, Q_1 > 0$, and

$$Q_0 = a(a+1) \left(\frac{c}{a}\right)^3 + (a^2 + 3a + 1) \left(\frac{c}{a}\right)^2 + \frac{c}{a}(1 + a - ac - ad) - c.$$

Simple computation shows that $Q_0 > 0$ if $d < \frac{1}{a^2}((a+1)c^2 + (3a+1)c + a)$ and $Q_0 < 0$ if $d > \frac{1}{a^2}((a+1)c^2 + (3a+1)c + a)$. Then the stable properties can be obtained similarly as the above, and the proof is complete.

Remark 3.3. In fact, for fixed parameters, the unique positive root y_0 of $Q_0(y) = 0$ can be calculated. According to the theory of roots for cubic equation, we have

$$y_0 = \frac{-(a^2 + 3a + 1) - 2\sqrt{m_1}\cos\frac{m_2}{3}}{3a(a+1)},$$

or

$$y_0 = \frac{-(a^2 + 3a + 1) + \sqrt{m_1}\cos\frac{m_2}{3} + \sqrt{3}\sin\frac{m_2}{3}}{3a(a+1)},$$

where

$$\begin{aligned} m_1 &= (a^2 + 3a + 1)^2 - 3a(a+1)(1 + a - ac - ad) > 0, \\ m_2 &= \frac{\arccos(2m_1(a^2 + 3a + 1) - 3a(a+1)m_3)}{2\sqrt{m_1^3}} \end{aligned}$$

with $m_3 = (a^2 + 3a + 1)(1 + a - ac - ad) + 9ac(a+1)$.

Remark 3.4. Theorem 3.1 indicates that there exists some constant $\delta_1 > 0$ such that the spectrum of L_1 , which consists of eigenvalues, lies in the half plane $\{\text{Re} \leq -\delta_1\}$ when $d < \frac{c}{a}$. In the case $ad > c$, Theorem 3.2 shows that there is some constant $\delta_2 > 0$ such that

the spectrum of L_2 , which also consists of eigenvalues, lies in the half plane $\{\text{Re} \leq -\delta_2\}$ when a, b, c, d satisfy $\frac{c+bd}{a+b} > \frac{c+b_0d}{a+b_0}$ for some $b_0 > 0$.

4 Nonexistence and existence of nonconstant positive solutions

In this section, we discuss the nonexistence and existence of nonconstant positive solutions to system (1.4) by mainly taking into account the diffusion rates of the species u, v, w and the fixed point indexes of the related operators, respectively. It should be noted that the boundedness (which we will use frequently) of positive solutions that we obtained in Section 2 plays an important role in the arguments.

First, we consider the nonexistence.

Theorem 4.1. *If $ad \leq c$, then (1.4) does not admit any non-constant positive solution.*

Proof. By the existence condition of constant positive solution $\left(\frac{ad-c}{a+b}, \frac{ad-c}{a+b}, \frac{bd+c}{a+b}\right)$, we know that (1.4) has no constant positive solution when $ad \leq c$. Hence, it needs to show that (1.4) has no non-constant positive solution when $ad \leq c$.

Suppose on the contrary that (1.4) has a non-constant positive solution (u, v, w) when $ad \leq c$. Then u must be nonconstant, otherwise, it is easily seen that v , and further, w must be constant from the first and second equations of (1.4). Using the fact that $w \leq, \neq d$, integrating the first and second equations of (1.4) on Ω , and considering the positivity of u, v, w , we have

$$0 = -d_1 \int_{\Omega} \Delta u dx = \int_{\Omega} (v - u) dx, \quad (4.1)$$

and

$$0 = -d_2 \int_{\Omega} \Delta v dx = \int_{\Omega} (auw - buv - cv) dx < \int_{\Omega} (adu - cv) dx. \quad (4.2)$$

The equality (4.2) implies that $adu \geq, \neq cv$. Combine with our assumption, it leads to $u > v$. This is an obvious contradiction with (4.1). Therefore, if (1.4) has a positive solution, it is necessary that $ad > c$.

In order to state our another non-existence result of positive solutions, for convenience, we give the following known inequality which will be useful.

For any $p, q \in \mathbb{R}$ and $s > 0$, using the Cauchy inequality we can get

$$pq \leq \frac{1}{4s} p^2 + sq^2. \quad (4.3)$$

Especially, if we take $s = \frac{1}{\mu_1}$, then $pq \leq \frac{\mu_1}{4} p^2 + \frac{1}{\mu_1} q^2$.

Theorem 4.2. *There exist positive constants $\tilde{d}_1 = \tilde{d}_1(a, d, \mu_1)$, $\tilde{d}_2 = \tilde{d}_2(a, b, d, \mu_1)$, $\tilde{d}_3 = \tilde{d}_3(a, b, d, \mu_1)$, such that problem (1.4) has no nonconstant positive solution when $d_1 \geq \tilde{d}_1, d_2 \geq \tilde{d}_2, d_3 \geq \tilde{d}_3$.*

Proof. For $\varphi \in L^1(\Omega)$, let $\bar{\varphi} = \frac{1}{|\Omega|} \int_{\Omega} \varphi(x) dx$. Suppose on the contrary that (u, v, w) is a nonconstant positive solution to (1.4) when $d_1 > \tilde{d}_1, d_2 > \tilde{d}_2, d_3 > \tilde{d}_3$ for some

positive constants $\tilde{d}_1, \tilde{d}_2, \tilde{d}_3$. Multiplying the three equalities of (1.4) by $u - \bar{u}, v - \bar{v}, w - \bar{w}$ respectively, and then integrating over Ω , by the boundedness of u, v, w in Theorem 2.2, the inequality (4.3) and the fact that $\int_{\Omega}(\varphi - \bar{\varphi})dx = 0$, we get

$$\begin{aligned} d_1 \int_{\Omega} |\nabla u|^2 dx &= \int_{\Omega} (v - u)(u - \bar{u}) dx = \int_{\Omega} (v - u - (\bar{v} - \bar{u}))(u - \bar{u}) dx \\ &< \int_{\Omega} (u - \bar{u})^2 dx + \int_{\Omega} |(v - \bar{v})(u - \bar{u})| dx \\ &\leq \int_{\Omega} (u - \bar{u})^2 dx + \frac{\mu_1}{4} \int_{\Omega} (u - \bar{u})^2 dx + \frac{1}{\mu_1} \int_{\Omega} (v - \bar{v})^2 dx, \end{aligned}$$

$$\begin{aligned} d_2 \int_{\Omega} |\nabla v|^2 dx &= \int_{\Omega} (auw - buv - cv)(v - \bar{v}) dx \\ &= \int_{\Omega} (auw - buv - cv - (a\bar{u}\bar{w} - b\bar{u}\bar{v} - c\bar{v}))(v - \bar{v}) dx \\ &= a \int_{\Omega} (uw - \bar{u}\bar{w})(v - \bar{v}) dx - b \int_{\Omega} (uv - \bar{u}\bar{v})(v - \bar{v}) dx - c \int_{\Omega} (v - \bar{v})^2 dx \\ &= a \int_{\Omega} ((uw - u\bar{w}) + (u\bar{w} - \bar{u}\bar{w}))(v - \bar{v}) dx \\ &\quad - b \int_{\Omega} ((uv - u\bar{v}) + (u\bar{v} - \bar{u}\bar{v}))(v - \bar{v}) dx - c \int_{\Omega} (v - \bar{v})^2 dx \\ &= a \int_{\Omega} u(w - \bar{w})(v - \bar{v}) dx + a\bar{w} \int_{\Omega} (u - \bar{u})(v - \bar{v}) dx \\ &\quad - b \int_{\Omega} u(v - \bar{v})^2 dx - b\bar{v} \int_{\Omega} (u - \bar{u})(v - \bar{v}) dx - c \int_{\Omega} (v - \bar{v})^2 dx \\ &< \frac{a^2 d}{b} \int_{\Omega} |w - \bar{w}| |v - \bar{v}| dx + ad \int_{\Omega} |u - \bar{u}| |v - \bar{v}| dx \\ &\quad + b \cdot \frac{ad}{b} \int_{\Omega} |u - \bar{u}| |v - \bar{v}| dx \\ &\leq \frac{a^2 d}{b} \left(\frac{\mu_1}{4} \int_{\Omega} (w - \bar{w})^2 dx + \frac{1}{\mu_1} \int_{\Omega} (v - \bar{v})^2 dx \right) \\ &\quad + 2ad \left(\frac{\mu_1}{4} \int_{\Omega} (u - \bar{u})^2 dx + \frac{1}{\mu_1} \int_{\Omega} (v - \bar{v})^2 dx \right), \end{aligned}$$

$$\begin{aligned}
 d_3 \int_{\Omega} |\nabla w|^2 dx &= \int_{\Omega} (dw - w^2 - uw)(w - \bar{w}) dx \\
 &= \int_{\Omega} (dw - w^2 - uw - (d\bar{w} - \bar{w}^2 - \bar{u}\bar{w}))(w - \bar{w}) dx \\
 &= d \int_{\Omega} (w - \bar{w})^2 dx - \int_{\Omega} (w + \bar{w})(w - \bar{w})^2 dx - \int_{\Omega} (uw - \bar{u}\bar{w})(w - \bar{w}) dx \\
 &= d \int_{\Omega} (w - \bar{w})^2 dx - \int_{\Omega} (w + \bar{w})(w - \bar{w})^2 dx \\
 &\quad - \int_{\Omega} ((uw - u\bar{w}) + (u\bar{w} - \bar{u}\bar{w}))(w - \bar{w}) dx \\
 &< d \int_{\Omega} (w - \bar{w})^2 dx - \int_{\Omega} u(w - \bar{w})^2 dx - \bar{w} \int_{\Omega} (u - \bar{u})(w - \bar{w}) dx \\
 &< d \int_{\Omega} (w - \bar{w})^2 dx + d \int_{\Omega} |u - \bar{u}| |w - \bar{w}| dx \\
 &\leq d \int_{\Omega} (w - \bar{w})^2 dx + d \left(\frac{\mu_1}{4} \int_{\Omega} (u - \bar{u})^2 dx + \frac{1}{\mu_1} \int_{\Omega} (w - \bar{w})^2 dx \right).
 \end{aligned}$$

Add up these three inequalities to yield

$$\begin{aligned}
 d_1 \int_{\Omega} |\nabla u|^2 dx + d_2 \int_{\Omega} |\nabla v|^2 dx + d_3 \int_{\Omega} |\nabla w|^2 dx &< \frac{4 + \mu_1 + d\mu_1 + 2ad\mu_1}{4} \int_{\Omega} (u - \bar{u})^2 dx \\
 &\quad + \frac{b + a^2d + 2abd}{b\mu_1} \int_{\Omega} (v - \bar{v})^2 dx \\
 &\quad + \frac{d(a^2\mu_1^2 + 4b\mu_1 + 4b)}{4b\mu_1} \int_{\Omega} (w - \bar{w})^2 dx.
 \end{aligned}$$

By the Poincaré inequality $\mu_1 \int_{\Omega} (\varphi - \bar{\varphi})^2 dx \leq \int_{\Omega} |\nabla \varphi|^2 dx$, it follows that

$$\begin{aligned}
 &d_1\mu_1 \int_{\Omega} (u - \bar{u})^2 dx + d_2\mu_1 \int_{\Omega} (v - \bar{v})^2 dx + d_3\mu_1 \int_{\Omega} (w - \bar{w})^2 dx \\
 &\leq d_1 \int_{\Omega} |\nabla u|^2 dx + d_2 \int_{\Omega} |\nabla v|^2 dx + d_3 \int_{\Omega} |\nabla w|^2 dx.
 \end{aligned}$$

Set

$$\tilde{d}_1 = \frac{4 + \mu_1 + d\mu_1 + 2ad\mu_1}{4\mu_1}, \quad \tilde{d}_2 = \frac{b + a^2d + 2abd}{b\mu_1^2}, \quad \tilde{d}_3 = \frac{d(a^2\mu_1^2 + 4b\mu_1 + 4b)}{4b\mu_1^2}.$$

Then we get

$$\begin{aligned}
 &d_1 \int_{\Omega} (u - \bar{u})^2 dx + d_2 \int_{\Omega} (v - \bar{v})^2 dx + d_3 \int_{\Omega} (w - \bar{w})^2 dx \\
 &< \tilde{d}_1 \int_{\Omega} (u - \bar{u})^2 dx + \tilde{d}_2 \int_{\Omega} (v - \bar{v})^2 dx + \tilde{d}_3 \int_{\Omega} (w - \bar{w})^2 dx.
 \end{aligned}$$

Thus, it leads to a contradiction in view of $d_1 \geq \tilde{d}_1, d_2 \geq \tilde{d}_2, d_3 \geq \tilde{d}_3$. This supports our conclusion.

Now, we focus our attention on investigating the existence of nonconstant positive solutions to system (1.4) via the fixed point index theory. We need to calculate some fixed point indexes. For the existence, considering Theorem 3.2, we know that it is impossible to expect nonconstant positive solutions of (1.4) near U_2 when $b > b_0$. So, from now on, we always suppose that $ad > c$ and $b \leq b_0$.

First, we introduce some preliminaries (due to Dancer [29] and Li [30]) which serve as the basic tools in the sequel. The main tool is a theorem concerning the fixed point index of compact maps on positive cones. We use the notations introduced in [29, 30].

Let E be a real Banach space and $W \subset E$ a closed convex set. For all real number $\alpha \geq 0$, if $\alpha W \subset W$, then W is called a wedge in E . A wedge W is said to be a cone if $W \cap \{-W\} = \{0\}$. For $y \in W$, we define

$$W_y = \{x \in E | y + \nu x \in W, \nu > 0\}, \quad S_y = \{x \in \overline{W}_y | -x \in \overline{W}_y\},$$

then \overline{W}_y is a wedge and S_y is closed in E . Furthermore, both \overline{W}_y and S_y is convex.

Let T be a compact linear map on E which is invariant on \overline{W}_y . If there exist $t \in (0, 1)$ and $w \in \overline{W}_y \setminus S_y$ such that $w - tTw \in S_y$, then we say that T has property α . Suppose W is a wedge in E and $G : W \rightarrow W$ is a compact map with fixed point $y_0 \in W$. Let $L = G'(y_0)$ be the Fréchet derivative of G at y_0 . Then L maps \overline{W}_{y_0} into itself. Now parts of the Dancer's and Li's fixed point index result can be stated as follows.

Lemma 4.3 (i) Assume $I - L$ is invertible on E . If L has property α on \overline{W}_{y_0} , then $\text{index}(G, y_0) = 0$;

(ii) Assume $I - L$ is not invertible on E but is on $\overline{W}_{y_0} \setminus \{0\}$. If $I - L : \overline{W}_{y_0} \rightarrow \overline{W}_{y_0}$ is not surjective, then $\text{index}(G, y_0) = 0$.

By $\text{index}(G, y_0)$ we mean the Leray-Schauder degree $\deg(I - G, N(y_0), 0)$, where $N(y_0)$ is a neighborhood of y_0 in W .

If (u, v, w) is a positive solution of (1.4), then by Theorem 2.2, there exist constants $C_1, C_2 > 0$ such that $C_1 < u, v, w < C_2$. Let X be as in Section 3 and define

$$\begin{aligned} X^+ &= \{(u, v, w) \in X | u, v, w \geq 0, x \in \overline{\Omega}, \partial_n u = \partial_n v = \partial_n w = 0, x \in \partial\Omega\}, \\ \mathfrak{S} &= \{(u, v, w) \in X^+ | C_1 < u, v, w < C_2, x \in \overline{\Omega}\}, \\ K &= \{u \in C_0(\overline{\Omega}) | u \geq 0, x \in \overline{\Omega}, \partial_n u = 0, x \in \partial\Omega\}, \\ W &= K \oplus K \oplus K. \end{aligned}$$

Then K is a cone in $C_0(\overline{\Omega})$.

Since (u, v, w) is bounded, we can find a large constant $M > 0$, such that

$$\max\{\max_{\overline{\Omega}}\{|v - u|\}, \max_{\overline{\Omega}}\{|auw - buv - cv|\}, \max_{\overline{\Omega}}\{|dw - w^2 - uw|\}\} < M.$$

In order to calculate the $\text{index}(G, U_0)$ and $\text{index}(G, U_1)$, we set

$$G(u, v, w) = \begin{pmatrix} (-d_1\Delta + M)^{-1}(v - u + Mu) \\ (-d_2\Delta + M)^{-1}(auw - buv - cv + Mv) \\ (-d_3\Delta + M)^{-1}(dw - w^2 - uw + Mw) \end{pmatrix}.$$

By the strong maximum principle we know that $(-d_k\Delta + M)^{-1}, k = 1, 2, 3$, are positive compact linear operators, so G is the direct sum of positive compact linear operators. Clearly, (u, v, w) is the solution of (1.4) if and only if (u, v, w) is the fixed point of G (This is independent of the choice of M).

Consider the system

$$\begin{cases} -d_1\Delta u = \theta v - u, & x \in \Omega, \\ -d_2\Delta v = a\theta uw - buv - cv, & x \in \Omega, \\ -d_3\Delta w = d\theta w - w^2 - uw, & x \in \Omega, \\ \partial_n u = \partial_n v = \partial_n w = 0, & x \in \partial\Omega, \end{cases} \quad (4.4)$$

where $\theta \in [0, 1]$ is parameter. Then it similarly can be proved that if $(u_\theta, v_\theta, \omega_\theta)$ is a solution of (4.4) then $(u_\theta, v_\theta, \omega_\theta)$ is also bounded, we might as well suppose that $C_1 < u_\theta, v_\theta, \omega_\theta < C_2$.

Define

$$G_\theta(u, v, w) = \begin{pmatrix} (-d_1\Delta + M)^{-1}(\theta v - u + Mu) \\ (-d_2\Delta + M)^{-1}(a\theta uw - buv - cv + Mw) \\ (-d_3\Delta + M)^{-1}(d\theta w - w^2 - uw + Mw) \end{pmatrix}.$$

By the definition of \mathfrak{S} , there is no fixed point of G_θ on $\partial\mathfrak{S}$. Using the property of homotopy invariance of degree, we know that the fixed index $\text{index}(G_\theta, \text{int}\mathfrak{S}, W)$ is independent of θ . Moreover, if (u, v, ω) is a fixed point of G_θ , then

$$\deg(I - G_\theta, N(u, v, \omega), (0, 0, 0)) = \text{index}(G'_\theta(u, v, \omega), (0, 0, 0), W) = (-1)^\gamma,$$

where $N(u, v, \omega)$ is a neighborhood of (u, v, ω) , γ is the sum of algebraic multiplicities of all eigenvalues of $G'_\theta(u, v, \omega)$ which are greater than 1. Thus, if the spectral radius $r(G'_\theta(u, v, \omega)) \leq 1$, then we must have

$$\text{index}(G'_\theta(u, v, \omega), (0, 0, 0), W) = 1.$$

Now, take $\theta = 0$. Then (4.4) has a unique nonnegative solution $(0, 0, 0)$, and then, $(0, 0, 0)$ is the unique fixed point of G_θ on W . So

$$\text{index}(G_\theta, \text{int}\mathfrak{S}, W) = \deg(I - G_\theta, N(0, 0, 0), (0, 0, 0)).$$

Direct calculation yields

$$\begin{aligned} & G'_\theta(0, 0, 0) \\ &= \begin{pmatrix} (-d_1\Delta + M)^{-1}(-1 + M) & (-d_1\Delta + M)^{-1}\theta & 0 \\ 0 & (-d_2\Delta + M)^{-1}(-c + M) & 0 \\ 0 & 0 & (-d_3\Delta + M)^{-1}(d\theta + M) \end{pmatrix}. \end{aligned}$$

Obviously, $G'_\theta(0, 0, 0)$ has no eigenvalues which are greater than 1 when $\theta = 0$. Therefore, $r(G'_\theta(u, v, \omega)) \leq 1$ and

$$\begin{aligned} \text{index}(G_\theta, \text{int}\mathfrak{S}, W) &= \deg(I - G_\theta, N(0, 0, 0), (0, 0, 0)) \\ &= \text{index}(G'_\theta(0, 0, 0), (0, 0, 0), W) \\ &= 1. \end{aligned} \quad (4.5)$$

(4.5) holds for all $\theta \in [0, 1]$, therefore, for $G_1 := G$ we have

$$\text{index}(G, \text{int}\mathfrak{S}, W) = 1.$$

Later, we will investigate the indexes of G at U_0, U_1, U_2 . However, to derive the index of G at U_2 , we need some other preliminaries. Set $U = (u, v, w)^T$ and

$$D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}, \quad F(U) = \begin{pmatrix} v - u \\ auw - buv - cv \\ dw - w^2 - uw \end{pmatrix}, \quad C = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

with a_{ij} given as in the proof of Theorem 3.2. Then system (1.4) can be rewritten as

$$-D\Delta U = F(U), \quad x \in \Omega; \quad \partial_n U = 0, \quad x \in \partial\Omega. \quad (4.6)$$

Thus, U is a solution of (1.4) if and only if U is a solution of (4.6) if and only if U is a solution of the problem

$$H(U) := U - (I - \Delta)^{-1}(D^{-1}F(U) + U) = 0, \quad x \in \Omega; \quad \partial_n U = 0, \quad x \in \partial\Omega$$

in X^+ , where I represents the identity operator on X . Clearly, the operator H is a compact perturbation of I . By the definition of \mathfrak{S} , we know that the Leray-Schauder degree $\deg(H, \mathfrak{S}, 0)$ is well-defined provided that $H(U) \neq 0$ on $\partial\Omega$. Moreover, it is easy to see that U is a fixed point of G if and only if U is a zero point of H . In the following, we will seek for the index of G at U_2 by H .

The main result of this section reads as follows.

Theorem 4.4. *Suppose that $I - G'(U_1)$ is invertible on $(C_0(\bar{\Omega}))^3$. If the determinant $\det(C + D) < 0$, then the system (1.4) has at least a nonconstant positive solution.*

Proof. We first calculate the $\text{index}(G, U_0)$ and $\text{index}(G, U_1)$.

For the solution U_0 , it is easy to see that $\bar{W}_{U_0} = W$, $S_{U_0} = \{(0, 0, 0)\}$. The linearized operator of G at U_0 is

$$G'(U_0) = \begin{pmatrix} (-d_1\Delta + M)^{-1} \begin{pmatrix} -1 + M & 1 & 0 \end{pmatrix} \\ (-d_2\Delta + M)^{-1} \begin{pmatrix} 0 & -c + M & 0 \end{pmatrix} \\ (-d_3\Delta + M)^{-1} \begin{pmatrix} 0 & 0 & d + M \end{pmatrix} \end{pmatrix}.$$

Take $(\varphi, \psi, \omega) \in (C_0(\bar{\Omega}))^3$. If $(I - G'(U_0))(\varphi, \psi, \omega)^T = 0$, then we have

$$\begin{cases} -d_1\Delta\varphi = -\varphi + \psi, \\ -d_2\Delta\psi = -c\psi, \\ -d_3\Delta\omega = d\omega. \end{cases} \quad (4.7)$$

Now we consider the invertibility of $I - G'(U_0)$ on $(C_0(\bar{\Omega}))^3$.

The second equation of (4.7) implies that $\psi \equiv 0$, further, the first equation induces that $\varphi \equiv 0$. For the third equation, we need to consider the following two cases.

Case 1. $\frac{d}{d_3} \notin \sigma(-\Delta)$, where $\sigma(-\Delta)$ is the spectrum set of $-\Delta$ with homogenous Neumann boundary conditions. Then $\omega \equiv 0$. This shows that $I - G'(U_0)$ is invertible on

$(C_0(\bar{\Omega}))^3$. Let $t = \frac{M}{d+M}$. Then $t \in (0, 1)$. Moreover, if φ_0 is the principal eigenfunction of $-\Delta$, then $(0, 0, \varphi_0) \in \bar{W}_{U_0} \setminus S_{U_0}$. So we have

$$\begin{aligned} (I - tG'(U_0)) \begin{pmatrix} 0 \\ 0 \\ \varphi_0 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ \varphi_0 - t(-d_3\Delta + M)^{-1}(d+M)\varphi_0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in S_{U_0}. \end{aligned}$$

This shows that $G'(U_0)$ has property α on \bar{W}_{U_0} . By Lemma 4.3, $\text{index}(G, U_0) = 0$.

Case 2. $\frac{d}{d_3} \in \sigma(-\Delta)$. Then there exist $\varphi, \psi, \omega \in C_0(\bar{\Omega})$, $\omega \neq 0$, such that $(I - G'(U_0))(\varphi, \psi, \omega)^T = 0$ (In fact, $\varphi = \psi \equiv 0$ by (4.7)). Therefore, $I - G'(U_0)$ is not invertible on $(C_0(\bar{\Omega}))^3$. Let ω_d be the eigenfunction of $I - G'(U_0)$ corresponding to $\frac{d}{d_3}$. Then ω_d must change sign in Ω (This implies that $I - G'(U_0)$ is invertible on \bar{W}_{U_0}). Moreover, the nodes of ω_d divide Ω into finite subdomains. And then, there exists a positive function, say, $\tilde{\omega}$, $\partial_n \tilde{\omega}|_{\partial\Omega} = 0$ on Ω such that $\int_{\Omega} \omega_d \tilde{\omega} dx \neq 0$. We claim that $I - G'(U_0)$ is not surjective on \bar{W}_{U_0} .

$\forall \varphi, \psi \in K$, suppose there exist $\varphi_0, \psi_0, \omega_0 \in K$ such that $(I - G'(U_0))(\varphi_0, \psi_0, \omega_0)^T = (\varphi, \psi, \tilde{\omega})^T$. Then we have

$$\begin{cases} \varphi_0 - (-1 + M)(-d_1\Delta + M)^{-1}\varphi_0 + (-d_1\Delta + M)^{-1}\psi_0 = \varphi, \\ \psi_0 - (-c + M)(-d_2\Delta + M)^{-1}\psi_0 = \psi, \\ \omega_0 - (d + M)(-d_3\Delta + M)^{-1}\omega_0 = \tilde{\omega}. \end{cases} \quad (4.8)$$

Multiply the third equation of (4.8) by ω_d and then integrate over Ω we get

$$\int_{\Omega} (-d_3\Delta - d)\omega_d\omega_0 dx = \int_{\Omega} (-d_3\Delta + M)\omega_d\tilde{\omega} dx. \quad (4.9)$$

The left-hand side of (4.9) is equivalent to $\int_{\Omega} (-d_3\Delta\omega_d - d\omega_d)\omega_0 dx = 0$, while the righthand side is $(d + M) \int_{\Omega} \omega_d\tilde{\omega} dx \neq 0$. This shows that there do not exist $\varphi_0, \psi_0, \omega_0 \in K$ such that $(I - G'(U_0))(\varphi_0, \psi_0, \omega_0)^T = (\varphi, \psi, \tilde{\omega})^T$ holds. Hence $(I - G'(U_0))$ is not surjective on \bar{W}_{U_0} . Then we still have $\text{index}(G, U_0) = 0$ by Lemma 4.3.

Now we consider the solution U_1 . For U_1 , $\bar{W}_{U_1} = K \oplus K \oplus C_0(\bar{\Omega})$, $S_{U_1} = \{0\} \oplus \{0\} \oplus C_0(\bar{\Omega})$. The linearized operator of G at U_1 is

$$G'(U_1) = \begin{pmatrix} (-d_1\Delta + M)^{-1} \begin{pmatrix} -1 + M & 1 & 0 \end{pmatrix} \\ (-d_2\Delta + M)^{-1} \begin{pmatrix} ad & -c + M & 0 \end{pmatrix} \\ (-d_3\Delta + M)^{-1} \begin{pmatrix} -d & 0 & -d + M \end{pmatrix} \end{pmatrix}.$$

$\forall \varepsilon > 0$, let $t = \frac{-(d+\varepsilon)+M}{-d+M}$. Then $t \in (0, 1)$ since M is large. We still use φ_0 to denote the principal eigenfunction of $-\Delta$ with homogenous Neumann boundary conditions, then

$(0, 0, \varphi_0) \in \overline{W}_{U_1} \setminus S_{U_1}$ and we have

$$\begin{aligned} (I - tG'(U_1)) \begin{pmatrix} 0 \\ 0 \\ \varphi_0 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ \varphi_0 - t(-d_3\Delta + M)^{-1}(-d + M)\varphi_0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \varphi_0 - (-d_3\Delta + M)^{-1}(-(d + \varepsilon) + M)\varphi_0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \frac{d+\varepsilon}{M}\varphi_0 \end{pmatrix} \in S_{U_1}. \end{aligned}$$

This shows that $G'(U_1)$ has property α on \overline{W}_{U_1} . By the assumption and Lemma 4.3, $\text{index}(G, U_1) = 0$.

In order to calculate the index of G at U_2 , we should change our idea.

We note that the Fréchet derivative of F at U_2 is exactly the matrix C , so

$$I - H'(U_2) = (I - \Delta)^{-1}(D^{-1}C + I).$$

It follows from the definition of fixed point index that if the operator $I - H'(U_2)$ is invertible then the fixed point index of $I - H$ at U_2 is defined by $\text{index}(I - H, U_2) = (-1)^\sigma$, here σ is the number of negative eigenvalues of $I - H'(U_2)$.

As a matter of course, to calculate the $\text{index}(I - H, U_2)$, in the following, we consider the eigenvalues of $I - H'(U_2)$.

For each $i \geq 0$ and $1 \leq j \leq \dim E_i$, X_{ij} is invariant under $I - H'(U_2)$, and it is easy to check that μ is an eigenvalue of $I - H'(U_2)$ on X_{ij} if and only if μ is an eigenvalue of the matrix operator $\frac{1}{1+\mu_i}(D^{-1}C + I)$ or $I - \frac{1}{1+\mu_i}(\mu_i I - D^{-1}C)$. This shows that $I - H'(U_2)$ is invertible if and only if the matrix $D^{-1}C + I$ is non-degenerate. Therefore, we only need to consider the determinant of $D^{-1}C + I$. Since

$$\det(D^{-1}C + I) = \frac{1}{d_1 d_2 d_3} \det(C + D),$$

if $\det(C + D) \neq 0$, then $\det(D^{-1}C + I) < 0$ if and only if $I - H'(U_2)$ is invertible and the number of negative eigenvalues of $I - H'(U_2)$ is odd. And then our assumption implies that

$$\text{index}(I - H, U_2) = -1,$$

which induces that $\text{index}(G, U_2) = -1$. Considering the indexes of G in \mathfrak{S} and at U_0, U_1 , we know that the result follows and the proof is completed.

Remark 4.5. On the assumption “ $\det(C + D) < 0$ ”.

We consider $\det(C + D)$ when we calculate $\text{index}(G, U_2)$. Directly calculating gives

$$\det(C + D) = R_3 + R_2 + R_1 + R_0$$

with

$$\begin{aligned}
 R_3 &= d_1 d_2 d_3, \quad R_2 = d_1 d_2 a_{33} + d_2 d_3 a_{11} + d_1 d_3 a_{22}, \\
 R_1 &= d_1(a_{22}a_{33} - a_{23}a_{32}) + d_2(a_{11}a_{33} - a_{13}a_{31}) + d_3(a_{11}a_{22} - a_{12}a_{21}) \\
 &= d_1 a_{22} a_{33} + d_2 a_{11} a_{33} + d_3(a_{11}a_{22} - a_{12}a_{21}) \\
 R_0 &= a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} \\
 &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33}.
 \end{aligned}$$

An easily check indicates that $R_3, R_1 > 0$, $R_2, R_0 < 0$ ($R_0 = -B_{30}$), so our assumption is reasonable. Conversely, if R_1, R_2, R_3 and R_4 are all positive, then the assumption is invalid.

It is also worth pointing out that the condition “ $\det(C + D) < 0$ ” is independent of μ_i for all $i \geq 0$. This simplifies the existence conditions for positive solutions and the analysis process indeed.

Remark 4.6. If we assume “ $I - G'(U_2)$ is invertible on $(C_0(\bar{\Omega}))^3$ ” directly as the assumption we add on “ $I - G'(U_1)$ ” in Theorem 4.4, then we can not show whether $G'(U_2)$ has property α on \bar{W}_{U_2} or not. In addition, we also can not show whether $I - G'(U_2)$ is surjective on \bar{W}_{U_2} or not even we assume $I - G'(U_2)$ is not invertible on $(C_0(\bar{\Omega}))^3$ but is on $\bar{W}_{U_2} \setminus \{0\}$. In both cases, $\text{index}(G, U_2)$ can not be given. So, we obtain the index of G at U_2 through H .

5 Hopf bifurcation solution emanating from U_2 for PDE system

In this section, using Hopf bifurcation theorem [31], we take b as a parameter to discuss the bifurcation solution of (1.4), which bifurcates from U_2 . Considering Theorem 3.2, we know that it is impossible to expect nonconstant positive solutions of (1.4) near U_2 when $b > b_0$. So, throughout this section, we still suppose that $ad > c$ and $b \leq b_0$.

The following two results correspond to the cases μ_0 and $\mu_i, i \geq 1$ respectively.

Theorem 5.1. *There is a Hopf bifurcation at $b = b_0$ for system (1.4), where b_0 is given in Theorem 3.2.*

Proof. Originally, according to Hopf bifurcation theory, we should work out the three roots of the characteristic equation $g_0(\mu) = 0$ and examine whether they satisfy the conditions of Hopf bifurcation. However, the expressions of the roots are very complicated for a cubic equation. So, we will consider this by some other means, that is, we will not seek for the three roots of $g_0(\mu) = 0$ directly.

By Hopf bifurcation theory, the characteristic equation $g_0(\mu) = 0$ must have a pair of conjugate pure imaginary roots at the Hopf bifurcation point, we might as well denote them by $\pm i\beta, \beta \neq 0$, where i is the imaginary unit. Since μ_0 is simple, $i\beta$ is a simple eigenvalue of the matrix operator B (see Section 3). Let $-\eta$ be the unique real root of $g_0(\mu) = 0$. Then we have

$$(\mu + \eta)(\mu^2 + \beta^2) = 0,$$

namely,

$$\mu^3 + \eta\mu^2 + \beta^2\mu + \eta\beta^2 = 0,$$

Thus, by contrasting the coefficients of $g_0(\mu)$, at the Hopf bifurcation point, we must have

$$\eta = B_{10}, \quad \beta^2 = B_{20}, \quad \eta\beta^2 = B_{30}$$

with B_{10}, B_{20}, B_{30} given in $g_0(\mu)$. It is obvious that $B_{10}B_{20} = B_{30}$.

On the other hand, at $b = b_0$, it follows from Section 3 that $Q_0 = 0$, so

$$0 = Q_3\mu_0^3 + Q_2\mu_0^2 + Q_1\mu_0 + Q_0 = B_{10}B_{20} - B_{30}$$

in view of $\mu_0 = 0$. It is just $B_{10}B_{20} = B_{30}$, therefore b_0 is a Hopf bifurcation point.

In [24], Proposition 2.4 shows that U_2 is unstable for small b for the ODE system (1.2). In fact, in that case, there is no Hopf bifurcation from U_2 for system (1.2), that is because the linearized operator of (1.2) at U_2 has a negative root and a pair of complex roots with positive real parts. In the following we prove that, in this case, the Hopf bifurcation from U_2 occurs for the PDE system (1.4). Of course, such bifurcation solutions are spatial-dependent. Different from the ODE cases, we may call this phenomenon as bifurcation created by the diffusion exactly. To this aim and for convenience, we introduce a few notations. Set

$$y = y(b) = \frac{c + bd}{a + b}, \quad H_i(y) = B_{1i}B_{2i} - B_{3i}.$$

Substituting the values of $a_{ij}, i, j = 1, 2, 3$ into $B_{ki}, k = 1, 2, 3$, we obtain

$$\begin{aligned} H_i(y) = & a(a+1)y^3 + ((2a(d_1 + d_2 + d_3) + (a^2 + 1)d_1 + d_2 + a^2d_3)\mu_i + a^3 + 3a + 1)y^2 \\ & + ((ad_1d_2 + (a+1)d_2d_3 + d_1d_3 + (d_1 + d_2 + d_3)((a+1)d_1 + d_2 + ad_3))\mu_i^2 \\ & + 2(a+1)(d_1 + d_2 + d_3)\mu_i + a - a(c+d) + 1)y \\ & + (d_2 + d_3)(d_1(d_1 + d_2 + d_3) + d_2d_3)\mu_i^3 + (d_2 + d_3)(2d_1 + d_2 + d_3)\mu_i^2 \\ & + (d_2 + d_3 - c(d_1 + d_2))\mu_i - c. \end{aligned} \quad (5.1)$$

Now we have the following result.

Theorem 5.2. *Let $b < b_0$ with b_0 given in Theorem 3.2. Assume that, for some $j \geq 1$, μ_j is a simple eigenvalue of the operator $-\Delta$ with homogeneous Neumann boundary condition. If the following conditions hold:*

- (i) $1 + \frac{1}{a} > c + d$;
- (ii) *for enough small $y > 0$, d_1, d_2, d_3 satisfy $H_j(y) < 0$,*

then there is a $b_1 > 0$ such that a branch of solution bifurcates from U_2 at $b = b_1$.

Proof. First, the condition $b < b_0$ tells us that the equilibrium U_2 is unstable.

By condition (ii), we see that $\lim_{y \rightarrow 0+} H_j(y) < 0$. Meanwhile, for large y , it is obvious that $H_j(y) > 0$. Hence, there exists a $y_1 > 0$ such that $H_j(y_1) = 0$, furthermore, near y_1 , we have $H_j(y) < 0$ for $y < y_1$ and $H_j(y) > 0$ for $y > y_1$. In other words, there exists a $b_1 > 0$ such that $H_j(y_1(b_1)) = 0$, and in any small neighborhood of b_1 , $H_j(y) < 0$ for $b < b_1$ and $H_j(y) > 0$ for $b > b_1$ in view of the monotonicity of y on b .

Since $B_{kj} > 0, k = 1, 2, 3$, the equation $g_j(\mu) = 0$ has no nonnegative root for all $b > 0$. The three roots of $g_j(\mu) = 0$, denoted by $\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3$, are either negative and real

or one is negative and real, say, $\bar{\mu}_1$, and the other two are a pair of conjugate complex roots: $\bar{\mu}_2 = \alpha + \mathbf{i}\beta, \bar{\mu}_3 = \alpha - \mathbf{i}\beta$ with $\beta \neq 0$. We need to eliminate the first case. Suppose that $\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3$ are negative and real. Then there exists a constant $r > 0$ such that $|\bar{\mu}_k| > r, k = 1, 2, 3$. Basing on the theory of the roots of a univariate cubic equation, we have

$$\bar{\mu}_1 + \bar{\mu}_2 + \bar{\mu}_3 = -B_{1j}, \quad \bar{\mu}_1\bar{\mu}_2 + \bar{\mu}_2\bar{\mu}_3 + \bar{\mu}_1\bar{\mu}_3 = B_{2j}, \quad \bar{\mu}_1\bar{\mu}_2\bar{\mu}_3 = -B_{3j}.$$

And therefore

$$\begin{aligned} H_j(y) &= -(\bar{\mu}_1 + \bar{\mu}_2 + \bar{\mu}_3)(\bar{\mu}_1\bar{\mu}_2 + \bar{\mu}_2\bar{\mu}_3 + \bar{\mu}_1\bar{\mu}_3) + \bar{\mu}_1\bar{\mu}_2\bar{\mu}_3 \\ &= -(\bar{\mu}_1^2(\bar{\mu}_2 + \bar{\mu}_3) + \bar{\mu}_2^2(\bar{\mu}_1 + \bar{\mu}_3) + \bar{\mu}_3^2(\bar{\mu}_1 + \bar{\mu}_2) + 2\bar{\mu}_1\bar{\mu}_2\bar{\mu}_3) \\ &\geq 8r^3 \end{aligned} \tag{5.2}$$

for all $y > 0$. This contradicts with $H_j(y_1) = 0$. Thus, the equation $g_j(\mu) = 0$ must have a negative real root and a pair of conjugate complex roots $\bar{\mu}_2 = \alpha + \mathbf{i}\beta, \bar{\mu}_3 = \alpha - \mathbf{i}\beta, \beta \neq 0$.

Substitute $\bar{\mu}_2 = \alpha + \mathbf{i}\beta, \bar{\mu}_3 = \alpha - \mathbf{i}\beta$ into the equality (5.2) to get

$$H_j(y) = -2\alpha((\alpha + \bar{\mu}_1)^2 + \beta^2).$$

Regarding $\bar{\mu}_1, \alpha, \beta$ as functions of b , we can rewrite $H_j(y)$ as

$$H_j(y) = -2\alpha(b)((\alpha(b) + \bar{\mu}_1(b))^2 + \beta^2(b)) =: H_j(y(b)).$$

Since $H_j(y_1) = 0$, we have $\alpha(b_1) = 0$. This shows that the linearized operator of (1.4) at U_2 has a pair of conjugate pure imaginary eigenvalues $\pm \mathbf{i}\beta$ when $b = b_1$. It is easily seen that $H_j(y)$ is strictly increasing in μ_i . So $H_k(b_1) \neq 0$ provided that $k \neq j$ for any $k \geq 1$. By our assumption, μ_j is simple, hence $\mathbf{i}\beta$ is a simple eigenvalue of the operator B .

Differentiate $H_j(y)$ on b to yield

$$H'_j(y(b_1)) = -2\alpha'(b_1)(\bar{\mu}_1^2(b_1) + \beta^2(b_1)). \tag{5.3}$$

On the other hand, using (5.1) we also have

$$\begin{aligned} H'_j(y(b_1)) &= 3a(a+1)y^2(b_1) \\ &\quad + 2((2a(d_1 + d_2 + d_3) + (a^2 + 1)d_1 + d_2 + a^2d_3)\mu_j + a^3 + 3a + 1)y(b_1) \\ &\quad + (ad_1d_2 + (a+1)d_2d_3 + d_1d_3 + (d_1 + d_2 + d_3)((a+1)d_1 + d_2 + ad_3))\mu_j^2 \\ &\quad + 2(a+1)(d_1 + d_2 + d_3)\mu_j + a - a(c+d) + 1. \end{aligned}$$

However, the assumption $1 + \frac{1}{a} > c + d$ ensures $H'_j(y(b_1)) \neq 0$, so (5.3) implies that $\alpha'(b_1) \neq 0$. Therefore, the Hopf bifurcation theorem [31] can be applied to conclude that system (1.4) has a branch of solution bifurcating from U_2 . The proof is completed.

Remark 5.3 It has been shown that $Q_0 < 0$ provided that $b < b_0$. On the other hand, by the expressions of Q_3, Q_2, Q_1 , if d_1, d_2, d_3 are small, then we must have Q_3, Q_2, Q_1 are small and further, $H_j(y) = B_{1j}B_{2j} - B_{3j} < 0$.

Remark 5.4 In the case that Ω is one dimensional, all $\mu_i, i \geq 1$ are simple. If $1 + \frac{1}{a} > c + d$ and the constant term of the function $H_i(y)$ is negative, then for each i , there always exists some $b_i > 0$ such that system (1.4) has a branch of solution bifurcating from U_2 . That is, the following conclusion holds.

Suppose that Ω is one dimensional. Let the conditions (i) and (ii) in Theorem 5.2 hold. Then there exist sequences $\{(d_{1i}, d_{2i}, d_{3i})\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ such that system (1.4) with $(d_1, d_2, d_3) = (d_{1i}, d_{2i}, d_{3i})$ has a branch of solution bifurcating from U_2 for each $i \geq 1$.

6 Conclusions

Theorem 3.2 shows that the unique positive constant solution U_2 is locally asymptotically stable with the conditions $ad > c$ and $b > b_0$ for some b_0 . As a consequence, system (1.4) has no nonconstant positive solutions in any neighborhood of U_2 . Since U_2 has the same stability properties for both systems (1.2) and (1.4), applying the implications of Turing instability [32], we see that such unstable phenomenon does not occur for system (1.3) if $ad > c$ and $b > b_0$ hold. However, in Theorem 5.2, we show that when $b < b_0$, if a, c, d satisfy $1 + \frac{1}{a} > c + d$, then, for the suitable ranges of the diffusion rates d_1, d_2, d_3 , system (1.4) has at least a nonconstant positive solution. This shows that diffusive phenomenon may cause the pattern information to occur under unstable conditions. Due to Turing's contributions, it is known that the diffusion can lead to stability and instability of positive solutions, that is, diffusion causes Turing stability or instability occurring. In our present model, we have shown that the stability and instability of equilibrium solutions will remain unaffected regardless diffusion (see Theorems 3.1-3.2), but the diffusive phenomenon leads to the bifurcation phenomenon emerging indeed. We may call such situation as diffusion driven Hopf bifurcation, which is a very interesting topic in ecologically dynamical systems. Undoubtedly, phenomena like this must exist in more dynamical systems, and the corresponding results to different dynamical systems, even more complicated stage structured models, can be obtained by applying the analysis techniques which we used in this paper.

Acknowledgement. The authors thank the referees for their suggestions and comments which improved the presentation of this manuscript.

References

- [1] A. Wikan, A. Eide, An analysis of a nonlinear stage-structured cannibalism model with application to the northeast arctic cod stock, *Bull. Math. Biol.* 66 (2004), 1685-1704.
- [2] C. Kohlmeier, W. Ebenhöf, The stabilizing role of cannibalism in a predator-prey system, *Bull. Math. Biol.* 57 (1995), 401-411.

- [3] S.M. Baer, B.W. Kooi, Y.A. Kuznetsov, H.R. Thieme, Multiparametric bifurcation analysis of a basic two-stage population model, *SIAM J. Appl. Math.* 66 (2006), 1339-1365.
- [4] M.E. Gurtin, D.S. Levine, On populations that cannibalize their young, *SIAM J. Appl. Math.* 42 (1982), 94-108.
- [5] Y. Chowa, S.R.-J. Jang, Cannibalism in discrete-time predator-prey systems, *J. Biol. Dynamics* 6 (2012), 38-62.
- [6] D.H. Cushing, The Problem of Stock and Recruitment, in *Fish Population Dynamics*, John Wiley, New York, 1977.
- [7] S. Liu, E. Beretta, A stage-structured predator-prey model of Beddington-Deangelis type, *SIAM J. Appl. Math.* 66 (2006), 1101-1129.
- [8] W.E. Ricker, The Historical Development, in *Fish Population Dynamics*, John Wiley, New York, 1977.
- [9] A. Wikan, Age or stage structure? A comparison of dynamic outcomes from discrete age- and stage-structured population models, *Bull. Math. Biol.* 74 (2012), 1354-1378.
- [10] M. Wang, Stability and Hopf bifurcation for a prey-predator model with prey-stage structure and diffusion, *Math. Biosci.* 212 (2008), 149-160.
- [11] F.J. Solis, Self-limitation, fishing and cannibalism, *Appl. Math. Comput.* 135 (2003), 39-48.
- [12] H.K. Chadwick, D.E. Stevens, L.W. Miller, Some Factors Regulating the Striped Bass Population in the Sacramento-San Joaquin Estuary, California, in *Assessing the Effects of Power-Plant-Induced Mortality on Fish Populations*, Pergamon, New York, 1977.
- [13] J.M. Cushing, R.F. Costantino, B. Dennis, R.A. Desharnais, S.M. Henson, Nonlinear population dynamics: models, experiments and data, *J. Theor. Biol.* 194 (1998), 1-9.
- [14] D. Claessen, A.M. de Roos, L. Persson, Population dynamic theory of size-dependent cannibalism, *Proc. Biol. Sci.* 271 (2004), 333-340.
- [15] F.V.D. Bosh, W. Gabriel, Cannibalism in an age-structured predator-prey system, *Bull. Math. Biol.* 59 (1997), 551-567.
- [16] M. Mesteron-Gibbons, E. Adams, Animal contests as evolutionary games, *Am. Sci.* 86 (1998), 334-341.
- [17] C. Kaewmanee, I.M. Tang, Cannibalism in an age-structured predator-prey system, *Ecol. Model.* 167 (2003), 213-220.
- [18] K.G. Magnússon, Destabilizing effect of cannibalism on a structured predator-prey system, *Math. Biosci.* 155 (1999), 61-75.

- [19] W. Wang, A stage structured population model, in Mathematical Modelling and Computing in Biology and Medicine, Proceedings of the European Conference, Miriam-Milan, 2003, 326-331.
- [20] W. Wang, L. Chen, A predator-prey system with stage-structure for predator, Comput. Math. Appl. 33 (1997), 83-91.
- [21] Z. Ge, J. Yan, Hopf bifurcation of a predator-prey system with stage structure and harvesting, Nonlinear Anal. 74 (2011), 652-660.
- [22] M. Barfield, R.D. Holt, R. Gomulkiewicz, Evolution in stage-structured populations, Am. Nat. 177 (2011), 397-409.
- [23] S. L. X. Xue, Hopf bifurcation in a three-stage-structured prey-predator system with predator density dependent, Commun. Comput. Inform. Sci. 288 (2012), 740-747.
- [24] B. Buonomo, D. Lacitignola, On the stabilizing effect of cannibalism in stage structured population models, Math. Biosci. Eng. 3 (2006), 717-731.
- [25] B. Buonomo, D. Lacitignola, S. Rionero, Effect of prey growth and predator cannibalism rate on the stability of a structured population model, Nonlinear Anal. RWA 11 (2010), 1170-1181.
- [26] Y. Lou, W.-M. Ni, Diffusion, self-diffusion and cross-diffusion, J. Differential Equations 131 (1996), 79-131.
- [27] Y. Du, P.Y.H. Pang, M. Wang, Qualitative analysis of a prey-predator model with stage structure for the predator, SIAM J. Appl. Math. 69 (2008), 596-620.
- [28] L.J.S. Allen, An Introduction to Mathematical Biology, Pearson, New York, 2007.
- [29] E.N. Dancer, On the indices of fixed points of mappings in cones and applications, J. Math. Anal. Appl. 91 (1983), 131-151.
- [30] L. Li, On positive solutions of a nonlinear equilibrium boundary value problem, J. Math. Anal. Appl. 138 (1989), 537-549.
- [31] B.D. Hassard, N.D. Kazarinoff and Y.-H. Wan, Theory and Applications of Hopf Bifurcation, Cambridge Univ. Press, Cambridge, 1981.
- [32] A.M. Turing, The chemical basis of morphogenesis, Philos. Trans. Roy. Soc. B 237 (1952), 37-72.
- [33] R.M. Nisbet, S.P. Blythe, W.S.C. Gurney, J.A.J. Metz, Stage-structure models of populations with distinct growth and development processes, Math. Med. Biol. 2 (1985), 57-68.
- [34] S.L. Robertson, J.M. Cushing, A bifurcation analysis of stage-structured density dependent integrodifference equations, J. Math. Anal. Appl. 388 (2012), 490-499.