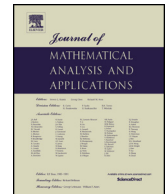




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Logarithmic stabilization of the Euler–Bernoulli transmission plate equation with locally distributed Kelvin–Voigt damping

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ABSTRACT

In this work we consider a transmission problem for a plate equation where one small part of the domain is made of a viscoelastic material with Kelvin–Voigt constitutive relation. We apply the general results due to Burq's [10] in the study of asymptotic behavior of solutions and prove that the semigroup associated to the system is logarithmically stable. The main ingredient of the proof is the Carleman estimates. The method consist to use Carleman estimates to obtain information on the resolvent for high frequency.

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1. Introduction

Recent advances in material science have provided new means to engineers for the suppression of vibration of elastic structures. One such approach is to bond patches made of special materials to the underlying structures as passive or active controllers. For example, engineers at the Ford Motor Company designed a constrained-layer damping patch (an aluminum foil layer, a viscoelastic layer, and an adhesive layer) which was attached to an elastic plate. They compared the natural frequencies and mode shapes of the plate with and without the patch to ascertain the effect of the patch [29]. There is an abundance of literature on the study of laminated/viscoelastic layers. We refer the readers to [3] and the references therein. Another example is the increasing success of using piezoceramic patches as sensors and actuators (see [13,6] and the references quoted there). Due to the presence of the patches, the material properties of the structure, such as the elasticity moduli, damping coefficient, and Poisson ratio, are changed. In particular, jump discontinuity at the location of the edges of the patches is usually introduced to these properties. General systems describing the interactions between patches and elastic structures consisting of a cylindrical shell, plate, and beam have been derived by Banks, Smith, and Wang [6].

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In the last three decades, owing to the large applications of smart materials in modern technology, there has been an increasing research on elastic systems with viscoelastic damping [4,5]. When the smart materials are added into the elastic structures, the Young's modulus, the mass density and the damping coefficients are changed accordingly. This passive method, on the one hand, makes the distributed control practically applicable but on the other hand, brings some new mathematical challenges which attract an increasing research interests [11,25]. For the controllability study of this kind of systems, we refer to [15,25]. The results of exponential stability by bounded viscous damping can be found in [11,9,35]. The exponential stability of one-dimensional wave and Euler–Bernoulli beam equations with Kelvin–Voigt damping was discussed in [26,27,12]. Among these viscoelastic systems, two types of unbounded viscoelastic damping, Kelvin–Voigt damping and Boltzmann damping, are specially important. The Kelvin–Voigt damping, which we are interested in this paper, models of linear viscoelasticity assume that the instantaneous stress depends on the instantaneous strain and the strain rate linearly.

In recent years, there has been much interest in the stability problems for elastic systems with locally distributed damping. Most of the works were devoted to the viscous damping, i.e., the damping is proportional to the velocity (see for instance [11] and [35]). Structures with local viscoelasticity arise from use of smart material or passive stabilization of structures. However, very little is known about exponential stability for elastic systems with local viscoelastic damping, although there is a fairly deep understanding when the damping is distributed over the entire domain but only for one dimension (see [27]). To our knowledge, the first paper in this direction was published in 1998 by Liu and Liu [26] where they obtained exponential stability for the Euler–Bernoulli beam equation with local Kelvin–Voigt damping. Recently, the same result was proven for the transmission case in [16]. Next, two new papers were appeared for a problem of wave/plate transmission equation for the one-dimensional case in [18] and multidimensional case in [17] where a polynomial and an exponential result was proven respectively.

2. Model and statement of results

Consider a clamped elastic domain in \mathbb{R}^n , ($n \geq 2$) which is made of a viscoelastic material with Kelvin–Voigt constitutive relation in which a transmission effect has been established in such a way that the damping is locally effective in only one side of the interface. By the Kirchhoff hypothesis, neglecting the rotatory inertia, the transversal vibration (see [12] for the modeling problem) the model can be described as follows: Let Ω and Ω_1 be two open, bounded and connected domains with smooth boundary (with class C^∞) respectively Γ and S such that $\Omega_1 \subset \Omega$ and $\overline{S} \cap \overline{\Gamma} = \emptyset$. We set also $\Omega_2 = \Omega \setminus \overline{\Omega}_1$ which is an open connected domain with boundary $\partial\Omega_2 = \Gamma \cup S$ (see Fig. 1). The main purpose of this paper is to study the long time behavior of the following coupled system

$$\left\{ \begin{array}{ll} \partial_t^2 u_1 + c_1 \Delta (c_1 \Delta u_1 + a \cdot \Delta \partial_t u_1) = 0 & \text{in } \Omega_1 \times]0, +\infty[, \\ \partial_t^2 u_2 + c_2 \Delta (c_2 \Delta u_2) = 0 & \text{in } \Omega_2 \times]0, +\infty[, \\ u_1 = u_2 & \text{on } S \times]0, +\infty[, \\ \partial_\nu u_1 = \partial_\nu u_2 & \text{on } S \times]0, +\infty[, \\ c_1 \Delta u_1 = c_2 \Delta u_2 & \text{on } S \times]0, +\infty[, \\ \partial_\nu (c_1 \Delta u_1) = \partial_\nu (c_2 \Delta u_2) & \text{on } S \times]0, +\infty[, \\ u_2 = 0 & \text{on } \Gamma \times]0, +\infty[, \\ \Delta u_2 = 0 & \text{on } \Gamma \times]0, +\infty[, \\ u_1(x, 0) = u_1^0(x), \partial_t u_1(x, 0) = u_1^1(x) & \text{in } \Omega_1, \\ u_2(x, 0) = u_2^0(x), \partial_t u_2(x, 0) = u_2^1(x) & \text{in } \Omega_2. \end{array} \right. \quad (2.1)$$

Where ∂_ν denotes the unit outward normal vector of Ω_1 and Ω respectively in S and Γ , $c_1 := c_1(x)$, $c_2 := c_2(x)$ are smooth and positive functions such that for some constants $c_0, c'_0 > 0$, $c_0 \leq c_k \leq c'_0$, $k = 1, 2$

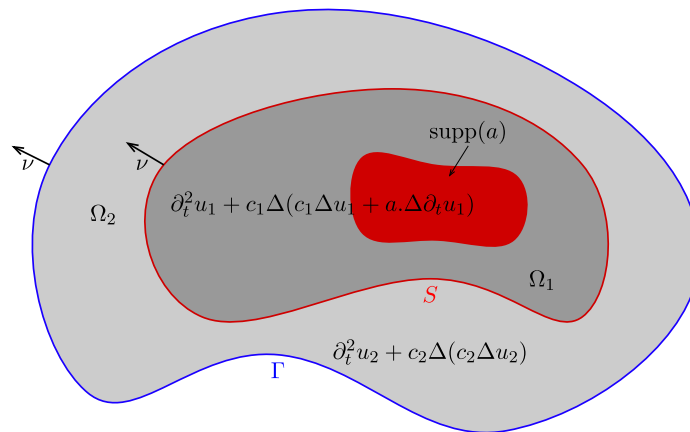


Fig. 1. The domains Ω_1 and Ω_2 and the plate operators.

and $a := a(x)$ are non-negative bounded functions in Ω_1 and we suppose that a vanishing near the boundary S such that there exists a non-empty open domain $\omega \subset \Omega_1$ in such a way $a \geq a_0$ in $\bar{\omega}$ for some strictly positive constant a_0 .

The energy of a solution of (2.1) at time $t \geq 0$ is defined by

$$E(t) = \frac{1}{2} \int_{\Omega_1} \left(|\partial_t u_1(x, t)|^2 + c_1^2 |\Delta u_1(x, t)|^2 \right) c_1^{-1} dx + \frac{1}{2} \int_{\Omega_2} \left(|\partial_t u_2(x, t)|^2 + c_2^2 |\Delta u_2(x, t)|^2 \right) c_2^{-1} dx.$$

By Green's formula we can prove that for all $t_1, t_2 > 0$ we have

$$E(t_2) - E(t_1) = - \int_{t_1}^{t_2} \int_{\Omega_1} a |\Delta \partial_t u_1(x, t)|^2 dx dt,$$

and this mean that the energy is decreasing over the time.

We define the operator \mathcal{A} by

$$\mathcal{A} \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} = (v_1, v_2, -c_1 \Delta(c_1 \Delta u_1 + a \Delta v_1), -c_2 \Delta(c_2 \Delta u_2))$$

in the Hilbert space $\mathcal{H} = X \times H$ where $H = H_1 \times H_2 = L^2(\Omega_1, c_1^{-1} dx) \times L^2(\Omega_2, c_2^{-1} dx)$ and

$$X = \{(u_1, u_2) \in H : u_1 \in H^2(\Omega_1), u_2 \in H^2(\Omega_2), u_2|_{\Gamma} = 0, u_1|_S = u_2|_S, \partial_\nu u_1|_S = \partial_\nu u_2|_S\}, \quad (2.2)$$

with domain

$$\mathcal{D}(\mathcal{A}) = \{(u_1, u_2, v_1, v_2) \in \mathcal{H} : (v_1, v_2, -c_1 \Delta(c_1 \Delta u_1 + a \Delta v_1), -c_2 \Delta(c_2 \Delta u_2)) \in \mathcal{H}, \\ \Delta u_2|_{\Gamma} = 0, c_1 \Delta u_1|_S = c_2 \Delta u_2|_S, \partial_\nu(c_1 \Delta u_1)|_S = \partial_\nu(c_2 \Delta u_2)|_S\}.$$

Now we are able to state our main results.

Theorem 2.1. *There exists $C > 0$ such that for every $\mu \in \mathbb{R}$ with $|\mu|$ large, we have*

$$\|(\mathcal{A} - i\mu \text{Id})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq Ce^{C|\mu|}. \quad (2.3)$$

As an immediate consequence of the previous theorem (see [10] and more recently [7]), we get the following rate of decrease of energy

Theorem 2.2. *For any $k \in \mathbb{N}$, there exists a constant $C > 0$ such that for any initial data $(u_1^0, u_2^0, u_1^1, u_2^1) \in \mathcal{D}(\mathcal{A}^k)$, the energy $E(t)$ of the system (2.1) whose solution $u(x, t)$ is starting from $(u_1^0, u_2^0, u_1^1, u_2^1)$ satisfy*

$$E(t) \leq \frac{C}{(\ln(2+t))^{2k}} \|(u_1^0, u_2^0, u_1^1, u_2^1)\|_{\mathcal{D}(\mathcal{A}^k)}^2, \quad \forall t > 0.$$

The exponential stability of an elastic system with damping distributed either over the entire region or along the boundary of the region has been studied extensively during the past, however, only a relatively small amount of attention was paid to the stability of the system with damping distributed locally inside the domain. It is known that, when viscous damping is only distributed on a subinterval of the domain, exponential stability holds for the Euler–Bernoulli beam equation (longitudinal and transversal motion) with both constant coefficients [11] and variable coefficients [20,21]. For a higher dimensional domain that will depends on the geometric properties of the subregion, where viscous damping is applied. We refer to [11] for a two-dimensional Schrödinger equation on a disk and a rectangle, to [25] for a class of n -dimensional conservative partial differential equations (PDEs), and to [35] for an n -dimensional semilinear wave equation. It should be pointed out that the operator corresponding to the viscous damping is bounded on the underlying space while the one corresponding to the Kelvin–Voigt damping is unbounded, and is not a lower-order perturbation of the elastic operator. The effect of such locally distributed damping on the energy decay is unknown. Since the work of Chen et al. [12] we were able to show that locally distributed Kelvin–Voigt damping ensures the asymptotic stability of a general second-order elastic system.

The study of the stabilization problem for transmission systems has attracted a lot of attention in recent years e.g. [1,2,14,18,19,8,30]. The systems discussed in those paper involve thermoelastic systems, fluid-structure interaction systems, and coupled wave–wave, plate–plate, or plate–wave equations. Recently in [16] transmission problem of transversal vibrations of a clamped elastic beam was studied as problems with locally distributed damping. It was shown, for the one-dimensional case, that when viscoelastic damping is distributed only on a subinterval in the interior of the domain and in one side of the transmission point, the exponential stability holds. In our knowledge the only work made in this direction for the multidimensional case was recently appeared in [17] where a logarithmic rate was proved for a transmission Euler–Bernoulli plate and wave equation with a Kelvin–Voigt damping which arising from the plate equation. In this paper, we will show that when the Kelvin–Voigt feedback is locally distributed on a one side of the interface, a logarithmic stability holds. Our approach is the frequency domain method. The main ingredient of this method is the so-called Carleman estimate. This method has been applied successfully to several models with locally distributed viscous damping (see [24,10,8,14]). However, due to the unboundedness of the Kelvin–Voigt damping operator and the high order of the operator, the arguments used in [24,10,8,14] are not valid here. To overcome this difficulty, we reduces the original system of two elliptic equations and carry out a special analysis of the resolvent.

The outline of this paper is as follow. In section 3 we prove the well-Posedness of the problem (2.1), in section 4 we give a global Carleman estimate and we construct a suitable phases functions and in section 5 we prove the resolvent estimate given by Theorem 2.1.

3. Existence and uniqueness

This section is devoted to show the existence, uniqueness and regularity of solutions to system (2.1).

Throughout this paper, we denote the inner product in the space $H = H_1 \times H_2$ by

$$\left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle_H = \int_{\Omega_1} u_1(x) \overline{v_1(x)} c_1^{-1}(x) dx + \int_{\Omega_2} u_2(x) \overline{v_2(x)} c_2^{-1}(x) dx.$$

The Cauchy problem corresponding to the system (2.1) is written in the following form

$$\begin{cases} \partial_t \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} (t) = \mathcal{A} \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} (t) & t \in]0, +\infty[, \\ \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} (0) = \begin{pmatrix} u_1^0 \\ u_2^0 \\ u_1^1 \\ u_2^1 \end{pmatrix}. \end{cases}$$

Now we have to specify the functional space and the domain of the operator \mathcal{A} . In the space H we define the operator G by

$$G \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (-c_1 \Delta u_1, -c_2 \Delta u_2) \quad \forall (u_1, u_2) \in \mathcal{D}(G)$$

with domain $\mathcal{D}(G) = X$ defined in (2.2). The space X is equipped with the norm

$$\|(u_1, u_2)\|_X = \|G(u_1, u_2)\|_H$$

and we defined the graph norm of G by

$$\|(u_1, u_2)\|_{gr(G)}^2 = \|(u_1, u_2)\|_H^2 + \|G(u_1, u_2)\|_H^2.$$

Proposition 3.1. $(X, \|\cdot\|_X)$ is a Hilbert space with a norm equivalent to the graph norm of G .

Proof. First, we note that by setting $u = \mathbb{1}_{\Omega_1} \cdot u_1 + \mathbb{1}_{\Omega_2} \cdot u_2$, the two following statements are equivalent $(u_1, u_2) \in X$, and $u \in H^2(\Omega)$ with $u|_{\Gamma} = 0$. It is well known that if G is a closed operator then $(X, \|\cdot\|_{gr(G)})$ is a Hilbert space. Thus to prove the proposition it suffices to show that G is closed and both norms are equivalent.

By Green's formula and Poincaré inequality it is easy to show that (see note above) there exists $C > 0$ such that

$$\left\langle G \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\rangle_H = \|\nabla u_1\|_{L^2(\Omega_1)}^2 + \|\nabla u_2\|_{L^2(\Omega_2)}^2 \geq C \|(u_1, u_2)\|_H^2 \quad \forall (u_1, u_2) \in X. \quad (3.1)$$

Then G is a strictly positive operator and we have

$$\|G(u_1, u_2)\|_H \cdot \|(u_1, u_2)\|_H \geq \left\langle G \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\rangle_H \geq C \|(u_1, u_2)\|_H^2 \quad \forall (u_1, u_2) \in X,$$

which prove the equivalence of the two norms.

To prove that the operator G is closed it suffices to show that G is self-adjoint. Since G is symmetric by (3.1), according to [33, Proposition 3.2.4] we have only to show that G is onto. Since the Laplacian Dirichlet operator is onto (see [33]) then by using the remark at the beginning of the proof it follows that G is closed. And this conclude the proof. \square

We set $\mathcal{H} = X \times H$ the Hilbert space with the norm

$$\|(u_1, u_2, v_1, v_2)\|^2 = \|(u_1, u_2)\|_X^2 + \|(v_1, v_2)\|_H^2 \quad \forall (u_1, u_2, v_1, v_2) \in \mathcal{H},$$

and we recall that the domain of the operator \mathcal{A} is defined by

$$\mathcal{D}(\mathcal{A}) = \{(u_1, u_2, v_1, v_2) \in \mathcal{H} : (v_1, v_2, -c_1\Delta(c_1\Delta u_1 + a\Delta v_1), -c_2\Delta(c_2\Delta u_2)) \in \mathcal{H}, \\ \Delta u_2|_{\Gamma} = 0, c_1\Delta u_1|_S = c_2\Delta u_2|_S, \partial_\nu(c_1\Delta u_1)|_S = \partial_\nu(c_2\Delta u_2)|_S\}.$$

Theorem 3.1. *Under the above assumptions, the operator \mathcal{A} generates a strongly semigroup of contractions in \mathcal{H} .*

Proof. According to Lumer–Phillips theorem (see for example [33, p. 103]) we have only to show that \mathcal{A} is m-dissipative.

Let $(u_1, u_2, v_1, v_2) \in \mathcal{D}(\mathcal{A})$ then by Green's formula we have

$$\begin{aligned} \operatorname{Re} \left\langle \mathcal{A} \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} \right\rangle_{\mathcal{H}} &= \operatorname{Re} \left\langle \begin{pmatrix} v_1 \\ v_2 \\ -c_1\Delta(c_1\Delta u_1 + a\Delta v_1) \\ -c_2\Delta(c_2\Delta u_2) \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} \right\rangle_{\mathcal{H}} \\ &= -\|a^{\frac{1}{2}}\Delta v_1\|_{L^2(\Omega_1)}^2 \leq 0. \end{aligned}$$

This shows that \mathcal{A} is dissipative.

Let $(f_1, f_2, g_1, g_2) \in \mathcal{H}$ we shall find $(u_1, u_2, v_1, v_2) \in \mathcal{D}(\mathcal{A})$ such that

$$\mathcal{A} \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ -c_1\Delta(c_1\Delta u_1 + a\Delta v_1) \\ -c_2\Delta(c_2\Delta u_2) \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ g_1 \\ g_2 \end{pmatrix},$$

which means that we have to find $(u_1, u_2, v_1, v_2) \in \mathcal{D}(\mathcal{A})$ such that

$$\begin{cases} v_1 = f_1 \\ v_2 = f_2 \\ -c_1\Delta(c_1\Delta u_1 + a\Delta f_1) = g_1 \\ -c_2\Delta(c_2\Delta u_2) = g_2. \end{cases}$$

In fact, by Riesz representation theorem, there exists a unique $(u_1, u_2) \in X = \mathcal{D}(G)$ such that for all $(\varphi_1, \varphi_2) \in X$ we have

$$\begin{aligned} -\langle c_1^{-1}g_1, \varphi_1 \rangle_{L^2(\Omega_1)} - \langle c_2^{-1}g_2, \varphi_2 \rangle_{L^2(\Omega_2)} - \langle a\Delta f_1, \Delta \varphi_1 \rangle_{L^2(\Omega_1)} \\ = \langle c_1\Delta u_1, \Delta \varphi_1 \rangle_{L^2(\Omega_1)} + \langle c_2\Delta u_2, \Delta \varphi_2 \rangle_{L^2(\Omega_2)}. \end{aligned} \quad (3.2)$$

In particular, for all $(\varphi_1, \varphi_2) \in \mathcal{C}_c^\infty(\Omega_1) \times \mathcal{C}_c^\infty(\Omega_2)$ the expression (3.2) yields

$$\begin{aligned}\langle \Delta(c_1 \Delta u_1 + a \Delta f_1) + c_1^{-1} g_1, \varphi_1 \rangle_{D'(\Omega_1)} &= 0, \\ \langle \Delta(c_2 \Delta u_2) + c_2^{-1} g_2, \varphi_2 \rangle_{D'(\Omega_2)} &= 0,\end{aligned}$$

then we obtain

$$\begin{aligned}-c_1 \Delta(c_1 \Delta u_1 + a \Delta f_1) &= g_1 \quad \text{in } L^2(\Omega_1), \\ -c_2 \Delta(c_2 \Delta u_2) &= g_2 \quad \text{in } L^2(\Omega_2).\end{aligned}\tag{3.3}$$

Back again to (3.2) then by Green's formula we can write

$$\begin{aligned}\langle \Delta(c_1 \Delta u_1 + a \Delta f_1) + c_1^{-1} g_1, \varphi_1 \rangle_{L^2(\Omega_1)} + \langle \Delta(c_2 \Delta u_2) + c_2^{-1} g_2, \varphi_2 \rangle_{L^2(\Omega_2)} \\ = \langle c_2 \Delta u_2, \partial_\nu \varphi_2 \rangle_{L^2(\Gamma)} + \langle c_1 \Delta u_1, \partial_\nu \varphi_1 \rangle_{L^2(S)} - \langle c_2 \Delta u_2, \partial_\nu \varphi_2 \rangle_{L^2(S)} \\ + \langle \partial_\nu(c_1 \Delta u_1), \varphi_2 \rangle_{L^2(S)} - \langle \partial_\nu(c_2 \Delta u_2), \varphi_2 \rangle_{L^2(S)},\end{aligned}$$

then using (3.3) we get for all $(\varphi_1, \varphi_2) \in X$ that

$$\langle \partial_\nu(c_1 \Delta u_1) - \partial_\nu(c_2 \Delta u_2), \varphi_1 \rangle_{L^2(S)} + \langle c_1 \Delta u_1 - c_2 \Delta u_2, \partial_\nu \varphi_1 \rangle_{L^2(S)} + \langle c_2 \Delta u_2, \partial_\nu \varphi_2 \rangle_{L^2(\Gamma)} = 0,$$

which yields to the following equalities

$$c_1 \Delta u_1|_S = c_2 \Delta u_2|_S, \quad \partial_\nu(c_1 \Delta u_1)|_S = \partial_\nu(c_2 \Delta u_2)|_S, \quad \Delta u_2|_\Gamma = 0.$$

And this concludes the proof. \square

One consequence of this last result is that, if we assume that $(u_1^0, u_2^0, u_1^1, u_2^1) \in \mathcal{D}(\mathcal{A})$, there exists a unique solution of (2.1) which can be expressed by means of a semigroup on \mathcal{H} as follows

$$\begin{pmatrix} u_1 \\ u_2 \\ \partial_t u_1 \\ \partial_t u_2 \end{pmatrix} = e^{t\mathcal{A}} \begin{pmatrix} u_1^0 \\ u_2^0 \\ u_1^1 \\ u_2^1 \end{pmatrix},\tag{3.4}$$

where $e^{t\mathcal{A}}$ is the C_0 -semigroup of contractions generated by the operator \mathcal{A} . And we have the following regularity of the solution

$$\begin{pmatrix} u_1 \\ u_2 \\ \partial_t u_1 \\ \partial_t u_2 \end{pmatrix} \in C([0, +\infty[, \mathcal{D}(\mathcal{A})) \cap C^1([0, +\infty[, \mathcal{H}).$$

Besides, if $(u_1^0, u_2^0, u_1^1, u_2^1) \in \mathcal{H}$, the function $(u_1(t), u_2(t))$ given by (3.4) is the mild solution of (2.1) and it lives in $C([0, +\infty[, \mathcal{H})$.

4. Carleman estimate and construction of the weight functions

4.1. Carleman estimate

We consider two open and disjoint domains \mathcal{O}_1 and \mathcal{O}_2 in which we define respectively the second order elliptic semi-classical operators $P_1 = -h^2\Delta - \alpha_1 h$ and $P_2 = -h^2\Delta - \alpha_2 h$ with principal symbol $p(x, \xi) = |\xi|^2$ where h is a very small semi-classical parameter and $\alpha_1 := \alpha_1(x)$, $\alpha_2 := \alpha_2(x)$ are two smooth functions, and we suppose that $\partial\mathcal{O}_1 = \gamma \cup \gamma_1$, $\partial\mathcal{O}_2 = \gamma \cup \gamma_2$ and $\overline{\gamma}_1 \cap \overline{\gamma} = \overline{\gamma}_2 \cap \overline{\gamma} = \emptyset$.

Let $\varphi_1 \in \mathcal{C}^\infty(\overline{\mathcal{O}}_1)$ and $\varphi_2 \in \mathcal{C}^\infty(\overline{\mathcal{O}}_2)$ be two real value functions. We define the two adjoint operators $P_{\varphi_1} = e^{\varphi_1/h} P_1 e^{\varphi_1/h}$ and $P_{\varphi_2} = e^{\varphi_2/h} P_2 e^{\varphi_2/h}$ of principal symbol respectively $p_1(x, \xi) = p(x, \xi + i\nabla\varphi_1)$ and $p_2(x, \xi) = p(x, \xi + i\nabla\varphi_2)$.

By denoting ∂_ν the unit outward normal vector of \mathcal{O}_1 and \mathcal{O}_2 respectively in $\gamma \cup \gamma_1$ and γ_2 we assume that the weight functions φ_1 and φ_2 satisfy

- 1) $|\nabla\varphi_1|(x) > 0$, $\forall x \in \overline{\mathcal{O}}_1$ and $|\nabla\varphi_2|(x) > 0$, $\forall x \in \overline{\mathcal{O}}_2$,
- 2) $\partial_\nu\varphi_1|_{\gamma_1} \neq 0$ and $\partial_\nu\varphi_2|_{\gamma_2} < 0$,
- 3) $\varphi_1|_\gamma = \varphi_2|_\gamma$,
- 4) $(\partial_\nu\varphi_1)|_\gamma < 0$, $(\partial_\nu\varphi_2)|_\gamma < 0$ and $(\partial_\nu\varphi_1)|_\gamma^2 - (\partial_\nu\varphi_2)|_\gamma^2 > 0$,
- 5) the sub-ellipticity condition respectively in $\overline{\mathcal{O}}_1$ and $\overline{\mathcal{O}}_2$

$$\forall (x, \xi) \in \overline{\mathcal{O}}_1 \times \mathbb{R}^n; p_{\varphi_1}(x, \xi) = 0 \implies \{\operatorname{Re}(p_{\varphi_1}), \operatorname{Im}(p_{\varphi_1})\}(x, \xi) > 0,$$

$$\forall (x, \xi) \in \overline{\mathcal{O}}_2 \times \mathbb{R}^n; p_{\varphi_2}(x, \xi) = 0 \implies \{\operatorname{Re}(p_{\varphi_2}), \operatorname{Im}(p_{\varphi_2})\}(x, \xi) > 0.$$

We consider the following transmission boundary value problem

$$\begin{cases} -\Delta w_1 - \frac{\alpha_1}{h} w_1 = f_1 & \text{in } \mathcal{O}_1 \\ -\Delta w_2 - \frac{\alpha_2}{h} w_2 = f_2 & \text{in } \mathcal{O}_2 \\ w_1 = w_2 + e_1 & \text{on } \gamma \\ \partial_\nu w_1 = \partial_\nu w_2 + e_2 & \text{on } \gamma \\ w_2 = 0 & \text{on } \gamma_2 \end{cases} \quad (4.1)$$

to which we correspond a Carleman estimate given by

Theorem 4.1. *Under the above assumptions on the weight functions φ_1 and φ_2 , there exists $h_0 > 0$ and $C > 0$ such that*

$$\begin{aligned} & h\|e^{\varphi_1/h} w_1\|_{L^2(\mathcal{O}_1)}^2 + h^3\|e^{\varphi_1/h} \nabla w_1\|_{L^2(\mathcal{O}_1)}^2 + h\|e^{\varphi_1/h} w_1\|_{L^2(\gamma)}^2 + h^3\|e^{\varphi_1/h} \nabla w_1\|_{L^2(\gamma)}^2 + \\ & h^3\|e^{\varphi_1/h} \partial_\nu w_1\|_{L^2(\gamma)}^2 + h\|e^{\varphi_2/h} w_2\|_{L^2(\mathcal{O}_2)}^2 + h^3\|e^{\varphi_2/h} \nabla w_2\|_{L^2(\mathcal{O}_2)}^2 + h\|e^{\varphi_2/h} w_2\|_{L^2(\gamma)}^2 + \\ & h^3\|e^{\varphi_2/h} \nabla w_2\|_{L^2(\gamma)}^2 + h^3\|e^{\varphi_2/h} \partial_\nu w_2\|_{L^2(\gamma)}^2 \leq C(h^4\|e^{\varphi_1/h} f_1\|_{L^2(\mathcal{O}_1)}^2 + h^4\|e^{\varphi_2/h} f_2\|_{L^2(\mathcal{O}_2)}^2 + \\ & h\|e^{\varphi_1/h} w_1\|_{L^2(\gamma_1)}^2 + h^3\|e^{\varphi_2/h} \partial_\nu w_1\|_{L^2(\gamma_1)}^2 + h\|e^{\varphi_1/h} e_1\|_{L^2(\gamma)}^2 + h^3\|e^{\varphi_1/h} \nabla e_1\|_{L^2(\gamma)}^2 + h^3\|e^{\varphi_1/h} e_2\|_{L^2(\gamma)}^2) \end{aligned} \quad (4.2)$$

for all $w_1 \in \mathcal{C}^\infty(\overline{\mathcal{O}}_1)$ and $w_2 \in \mathcal{C}^\infty(\overline{\mathcal{O}}_2)$ satisfying the system (4.1) and $h \in]0, h_0]$.

Remarks 4.1.

- 1) The previous theorem follows from some local Carleman estimates. In fact, the boundary terms are provided by [23, Proposition 2] and [24, Proposition 1] while the interface's terms are provided from [32, Theorem 2.2] or [8, Theorem 2.3]. Indeed, one first chooses the partition of unity (θ_i) on some neighborhood of $\partial\mathcal{O}_1$ and $\partial\mathcal{O}_2$ such that any element of this partition θ lies in one of the following cases:
 - i) $\text{supp}(\theta) \cap \gamma_1 \neq \emptyset$, $\text{supp}(\theta) \cap \gamma_2 = \emptyset$ and $\text{supp}(\theta) \cap \gamma = \emptyset$,
 - ii) $\text{supp}(\theta) \cap \gamma_2 \neq \emptyset$, $\text{supp}(\theta) \cap \gamma_1 = \emptyset$ and $\text{supp}(\theta) \cap \gamma = \emptyset$,
 - iii) $\text{supp}(\theta) \cap \gamma \neq \emptyset$, $\text{supp}(\theta) \cap \gamma_1 = \emptyset$ and $\text{supp}(\theta) \cap \gamma_2 = \emptyset$.
 Next, $\text{supp}(\theta)$ chosen sufficiently small, one defines $v = \theta.u$. Working in local coordinates, we may apply to function v
 - in case i) [24, Proposition 1] where especially we need the assumption $\partial_\nu \varphi_1|_{\gamma_1} \neq 0$,
 - in case ii) [23, Proposition 2] since $\partial_\nu \varphi_2|_{\gamma_2} < 0$,
 - in case iii) [32, Theorem 2.2] where assumptions 3) and 4) are needed here,
 and, summing up these inequalities, we directly get the estimate (4.2).
- 2) If the function w_1 is supported away from γ_1 the estimate (4.2) is allows true even if we don't assume that $(\partial_\nu \varphi_1)|_{\gamma_1} \neq 0$, while the proof of Theorem 4.1 is local.
- 3) We can not assume that $(\partial_\nu \varphi_1)|_{\gamma_1} < 0$ (it means $\partial_\nu \varphi_1 < 0$ in whole $\partial\mathcal{O}_1$), otherwise the weight function attain his global maximum in \mathcal{O}_1 and in particular we have $\nabla \varphi_1 = 0$ in some points of \mathcal{O}_1 .

4.2. Construction of the weight functions

In this section we will try to find two phases that satisfy the Hörmander's condition except in a finite number of ball where one of them do not satisfies this condition the second does and is strictly greater. Note that this result is similar to the Burq's one [10, Proposition 3.2], but here we give a new proof due to F. Laudenbach. Then, we will adapt this result to our case to construct a suitable weight functions that will be needed in the next section. The main ingredient of this section is the following.

Proposition 4.1. *Let \mathcal{O} be a bounded open subset with boundary $\Gamma_1 \cup \Gamma_2$ where $\bar{\Gamma}_1 \cap \bar{\Gamma}_2 = \emptyset$, then there exist two real functions $\psi_1, \psi_2 \in \mathcal{C}^\infty(\mathcal{O})$ and continuous on $\bar{\mathcal{O}}$ satisfying for $k = 1, 2$ that $(\partial_\nu \psi_k)|_{\Gamma_1} > 0$ and $(\partial_\nu \psi_k)|_{\Gamma_2} < 0$ having only degenerate critical points (of finite number) such that when $\nabla \psi_k = 0$ then $\nabla \psi_{k+1} \neq 0$ and $\psi_{k+1} > \psi_k$ where we have assumed that $k+1 = 2$ if $k = 1$ and $k+1 = 1$ if $k = 2$.*

Remarks 4.2.

- 1) One consequence of Proposition 4.1 is that for $k = 1, 2$ we can find a finite number of points x_{kj} where $j = 1, \dots, N_k$ and $\epsilon > 0$ such that $\overline{B(x_{kj}, 2\epsilon)} \subset \mathcal{O}$ and $B(x_{1j_1}, 2\epsilon) \cap B(x_{2j_2}, 2\epsilon) = \emptyset$, for all $k = 1, 2$ and $j_k = 1, \dots, N_k$ and in $B(x_{kj}, 2\epsilon)$ we have $\psi_{k+1} > \psi_k$ for all $j = 1, \dots, N_k$ (see Fig. 2).
- 2) For $\lambda > 0$ large enough the weight functions $\varphi_k = e^{\lambda \psi_k}$ satisfy the Hörmander's condition in $U_k = \mathcal{O} \cap \left(\bigcup_{j=1}^{N_k} B(x_{kj}, \epsilon) \right)^c$. Indeed, we have only to prove that for an open bounded subset $U \subset \mathbb{R}^n$, $\psi \in \mathcal{C}^\infty(\bar{U})$ satisfying $|\nabla \psi| \geq C$ in \bar{U} and $\varphi = e^{\lambda \psi}$ we have $\{\text{Re}(p_\varphi), \text{Im}(p_\varphi)\}(x, \xi) \geq C'$ in $\bar{U} \times \mathbb{R}^n$ for $\lambda > 0$ large enough. We have

$$\begin{cases} \nabla \varphi = \lambda e^{\lambda \psi} \nabla \psi & \text{and} \quad \varphi'' = e^{\lambda \psi} (\lambda \nabla \psi \cdot {}^t \nabla \psi + \lambda \psi'') \\ p_\varphi(x, \xi) = 0 \implies \langle \xi, \nabla \varphi \rangle = 0 \text{ and } |\xi|^2 = |\nabla \varphi|^2 \end{cases}$$

then we obtain

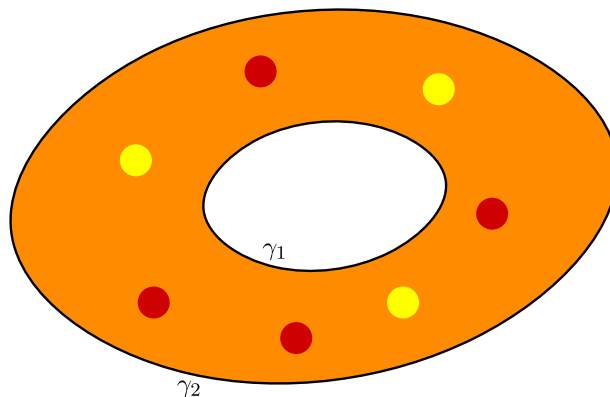


Fig. 2. The functions φ_1 and ψ_1 (resp. φ_2 and ψ_2) have not critical points in yellow and orange (resp. in red and orange) domains. However, $\varphi_1 > \varphi_2$ in the yellow balls and $\varphi_2 > \varphi_1$ in the red balls. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$\begin{aligned} \{\operatorname{Re}(p_\varphi), \operatorname{Im}(p_\varphi)\}(x, \xi) &= 4\lambda e^{\lambda\psi} {}^t\xi \cdot \psi'' \cdot \xi + 4e^{3\lambda\psi} (\lambda^4 |\nabla\psi|^2 + \lambda^3 {}^t\nabla\psi \cdot \psi'' \cdot \nabla\psi) \\ &= 4e^{3\lambda\psi} (\lambda^4 |\nabla\psi|^2 + O(\lambda^3)). \end{aligned}$$

Which conclude the result.

- 3) In general, [Proposition 4.1](#) is also true for any smooth manifold with boundary of disjoint union of two open and closed submanifolds.

Proof. Since the Morse functions are dense (for the \mathcal{C}^∞ topology) in the set of \mathcal{C}^∞ functions then we can find ψ_1 a Morse function such that $(\partial_\nu \psi_1)|_{\Gamma_1} < 0$ and $(\partial_\nu \psi_1)|_{\Gamma_2} > 0$. We can suppose that ψ_1 have no local maximum in \mathcal{O} (The proceeding of the elimination of the maximum has been done by Burq in [\[10, Appendix A\]](#), we can see also [\[28, Theorem 8.1\]](#) and [\[22, Lemma 2.6\]](#)).

Let c be a critical point of ψ_1 since its index is different from n then we can find a \mathcal{C}^∞ arc $\gamma_c : [-1, 1] \rightarrow \Omega$ such that $\gamma_c(0) = c$ and $\psi_1(\gamma_c(1)) = \psi_1(\gamma_c(-1)) > \psi_1(c)$. We do this construction for all the critical points of ψ_1 so that all the arcs are mutually disjoint. This allows us to find a vector field X in \mathcal{O} , vanishing near the boundary of \mathcal{O} such that for all critical points c of ψ_1 we have

$$X(\gamma_c(t)) = \dot{\gamma}_c(t),$$

where $\dot{\gamma}$ stand for the time derivative.

We denote ϕ_t its flow that satisfies

$$\dot{\phi}_t(x) = X(\phi_t(x)),$$

and we set $\psi_2 = \psi_1 \circ \phi_1$, thus ψ_1 and ψ_2 satisfy the required properties. Indeed, since $X \equiv 0$ near the boundary Γ_1 and Γ_2 which mean that $\phi_t(x) = x$ near Γ_1 and Γ_2 then $\partial_\nu \psi_1|_{\Gamma_1} = \partial_\nu \psi_2|_{\Gamma_1}$ and $\partial_\nu \psi_1|_{\Gamma_2} = \partial_\nu \psi_2|_{\Gamma_2}$. If c is a critical point of ψ_1 then we have $\psi_2(c) = \psi_1(\gamma_c(1)) > \psi_1(c)$, and if c' is a critical point of ψ_2 then $c' = \phi_{-1}(c)$ where c is a critical point of ψ_1 and we have $\psi_2(c') = \psi_1(\phi_1 \circ \phi_{-1}(c)) = \psi_1(c) < \psi_1(\phi_{-1}(c)) = \psi_1(c')$, and this achieve the proof. \square

Now we return to our geometric baseline as described in the introduction of this paper. We note by $\tilde{\Omega}_1 = \Omega_1 \setminus \overline{B}_r$ where B_r is an open ball of Ω_1 with radius $r > 0$ such that $\overline{B}_r \subset \Omega_1$. Applying [Proposition 4.1](#) and the first and second items of [Remarks 4.2](#) respectively on the domains $\tilde{\Omega}_1$ and Ω_2 we can find four phases

$\varphi_{1,1}$, $\varphi_{1,2}$, $\varphi_{2,1}$ and $\varphi_{2,2}$ verifying the Hörmander's condition respectively in $U_{1,1} = \tilde{\Omega}_1 \cap \left(\bigcup_{j=1}^{N_{11}} B(x_{11}^j, \epsilon) \right)^c$, $U_{1,2} = \tilde{\Omega}_1 \cap \left(\bigcup_{j=1}^{N_{12}} B(x_{12}^j, \epsilon) \right)^c$, $U_{2,1} = \Omega_2 \cap \left(\bigcup_{j=1}^{N_{21}} B(x_{21}^j, \epsilon) \right)^c$ and $U_{2,2} = \Omega_2 \cap \left(\bigcup_{j=1}^{N_{22}} B(x_{22}^j, \epsilon) \right)^c$ such that $|\nabla \varphi_{1,1}| > 0$ in $\overline{U}_{1,1}$, $|\nabla \varphi_{1,2}| > 0$ in $\overline{U}_{1,2}$, $|\nabla \varphi_{2,1}| > 0$ in $\overline{U}_{2,1}$ and $|\nabla \varphi_{2,2}| > 0$ in $\overline{U}_{2,2}$, moreover $\varphi_{1,k} < \varphi_{1,k+1}$ in $B(x_{1k}^j, 2\epsilon)$ for all $j = 1, \dots, N_{1,k}$ and $\varphi_{2,k} < \varphi_{2,k+1}$ in $B(x_{2k}^j, 2\epsilon)$ for all $j = 1, \dots, N_{2,k}$. Furthermore, for all $k = 1, 2$ we have

$$(\partial_\nu \varphi_{1,k})|_S < 0, \quad (\partial_\nu \varphi_{2,k})|_S < 0 \text{ and } (\partial_\nu \varphi_{2,k})|_\Gamma < 0.$$

Besides, we can suppose that $\varphi_{1,k}|_S = \varphi_{2,k}|_S$, and by argument of density (see [8]) we can suppose also that

$$(\partial_\nu \varphi_{1,k})|_S^2 - (\partial_\nu \varphi_{2,k})|_S^2 > 0.$$

And this concludes the construction of weight functions that will be used in next section. For more details of that construction of the weight functions we refer the reader to [8] and [17].

5. Resolvent estimate

This section is devoted to prove a resolvent estimate, precisely we will show that for some constant $C > 0$ we have

$$\|(\mathcal{A} - i\mu \text{Id})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C e^{C|\mu|},$$

for every $\mu \in \mathbb{R}$ large enough (i.e. $|\mu| \geq 1$).

We suppose that the resolvent estimate (2.3) is false. Then by the continuity of the resolvent and the resonance theorem there exist $K_m > 0$, $\mu_m \in \mathbb{R}$ and two sequences $(u_{1,m}, u_{2,m}, v_{1,m}, v_{2,m}) \in \mathcal{D}(\mathcal{A})$ and $(f_{1,m}, f_{2,m}, g_{1,m}, g_{2,m}) \in \mathcal{H}$, $m = 1, 2, \dots$ such that

$$|\mu_m| \longrightarrow +\infty, \quad K_m \longrightarrow +\infty, \quad \|(u_{1,m}, u_{2,m}, v_{1,m}, v_{2,m})\|_{\mathcal{H}} = 1, \quad (5.1)$$

and

$$e^{K_m|\mu_m|}(\mathcal{A} - i\mu_m) \begin{pmatrix} u_{1,m} \\ u_{2,m} \\ v_{1,m} \\ v_{2,m} \end{pmatrix} = \begin{pmatrix} f_{1,m} \\ f_{2,m} \\ g_{1,m} \\ g_{2,m} \end{pmatrix} \longrightarrow 0 \text{ in } \mathcal{H}. \quad (5.2)$$

This imply that

$$e^{K_m|\mu_m|}(v_{1,m} - i\mu_m u_{1,m}) = f_{1,m} \longrightarrow 0 \text{ in } H^2(\Omega_1), \quad (5.3)$$

$$e^{K_m|\mu_m|}(v_{2,m} - i\mu_m u_{2,m}) = f_{2,m} \longrightarrow 0 \text{ in } H^2(\Omega_2), \quad (5.4)$$

$$e^{K_m|\mu_m|}(-c_1 \Delta(c_1 \Delta u_{1,m} + a \Delta v_{1,m}) - i\mu_m v_{1,m}) = g_{1,m} \longrightarrow 0 \text{ in } L^2(\Omega_1), \quad (5.5)$$

$$e^{K_m|\mu_m|}(-c_2 \Delta(c_2 \Delta u_{2,m}) - i\mu_m v_{2,m}) = g_{2,m} \longrightarrow 0 \text{ in } L^2(\Omega_2). \quad (5.6)$$

From (5.1) and (5.2), we get

$$\operatorname{Re} \left\langle \begin{pmatrix} f_{1,m} \\ f_{2,m} \\ g_{1,m} \\ g_{2,m} \end{pmatrix}, \begin{pmatrix} u_{1,m} \\ u_{2,m} \\ v_{1,m} \\ v_{2,m} \end{pmatrix} \right\rangle_{\mathcal{H}} = -e^{K_m |\mu_m|} \int_{\Omega_1} a |\Delta v_{1,m}|^2 dx \rightarrow 0. \quad (5.7)$$

Then by (5.3) and (5.7), we obtain

$$|\mu_m|^2 e^{\frac{K_m}{2} |\mu_m|} \int_{\omega} |\Delta u_{1,m}|^2 dx \rightarrow 0. \quad (5.8)$$

Hence from (5.7) and (5.8) we obtain

$$e^{\frac{K_m}{2} |\mu_m|} \left(\int_{\omega} |\Delta u_{1,m}|^2 dx + \int_{\omega} |\Delta v_{1,m}|^2 dx \right) \rightarrow 0. \quad (5.9)$$

And by (5.3) we have

$$\frac{1}{|\mu_m|^2} \|\Delta(\psi \cdot v_{1,m})\|_{L^2(\Omega_1)}^2 = O(1), \quad \forall \psi \in \mathcal{C}^\infty(\Omega_1). \quad (5.10)$$

Then by multiplying (5.5) by $\mu_m^{-1} \psi \cdot \bar{v}_{1,m}$ where $\psi \in \mathcal{C}^\infty(\Omega_1)$ and $\operatorname{supp}(\psi) \subset \omega$ we obtain by (5.9) and (5.10) that

$$e^{\frac{K_m}{4} |\mu_m|} \int_{\omega} |v_{1,m}|^2 \psi dx \rightarrow 0.$$

In particular, since $B_{4r} \subset \omega$ it follows that

$$e^{\frac{K_m}{4} |\mu_m|} \int_{B_{4r}} |v_{1,m}|^2 dx \rightarrow 0.$$

We set B_{4r} a ball of radius $4r > 0$, such that $B_{4r} \subset \omega$ then by using (5.3) we find that

$$e^{\frac{K_m}{4} |\mu_m|} \int_{B_{4r}} |u_{1,m}|^2 dx \rightarrow 0. \quad (5.11)$$

Let's consider now the following transmission problem

$$\begin{cases} v_1 - i\mu u_1 = f_1 & \text{in } \Omega_1 \\ v_2 - i\mu u_2 = f_2 & \text{in } \Omega_2 \\ -c_1 \Delta(c_1 \Delta u_1 + a \Delta v_1) - i\mu v_1 = g_1 & \text{in } \Omega_1 \\ -c_2 \Delta(c_2 \Delta u_2) - i\mu v_2 = g_2 & \text{in } \Omega_2 \\ u_1 = u_2, \quad \partial_\nu u_1 = \partial_\nu u_2 & \text{on } S \\ c_1 \Delta u_1 = c_2 \Delta u_2, \quad \partial_\nu(c_1 \Delta u_1) = \partial_\nu(c_2 \Delta u_2) & \text{on } S \\ u_2 = 0, \quad \Delta u_2 = 0 & \text{on } \Gamma. \end{cases} \quad (5.12)$$

Then the solution (u_1, u_2, v_1, v_2) of (5.12) satisfies

$$\begin{cases} v_1 = i\mu u_1 + f_1 & \text{in } \Omega_1 \\ v_2 = i\mu u_2 + f_2 & \text{in } \Omega_2 \\ \mu^2 u_1 - c_1 \Delta(c_1 \Delta u_1 + a \Delta v_1) = g_1 + i\mu f_1 & \text{in } \Omega_1 \\ \mu^2 u_2 - c_2 \Delta(c_2 \Delta u_2) = g_2 + i\mu f_2 & \text{in } \Omega_2 \\ u_1 = u_2, \quad \partial_\nu u_1 = \partial_\nu u_2 & \text{on } S \\ c_1 \Delta u_1 = c_2 \Delta u_2, \quad \partial_\nu(c_1 \Delta u_1) = \partial_\nu(c_2 \Delta u_2) & \text{on } S \\ u_2 = 0, \quad \Delta u_2 = 0 & \text{on } \Gamma. \end{cases} \quad (5.13)$$

This can be rewritten as follows

$$\begin{cases} v_1 = i\mu u_1 + f_1 & \text{in } \Omega_1 \\ v_2 = i\mu u_2 + f_2 & \text{in } \Omega_2 \\ (-\Delta - \frac{|\mu|}{c_1})(c_1 \Delta u_1 + a \Delta v_1 - |\mu|u_1) = \Phi_1 = \frac{1}{c_1}g_1 + i\frac{\mu}{c_1}f_1 - a\frac{|\mu|}{c_1}\Delta v_1 & \text{in } \Omega_1 \\ (-\Delta - \frac{|\mu|}{c_2})(c_2 \Delta u_2 - |\mu|u_2) = \Phi_2 = \frac{1}{c_2}g_2 + i\frac{\mu}{c_2}f_2 & \text{in } \Omega_2 \\ u_1 = u_2, \quad \partial_\nu u_1 = \partial_\nu u_2 & \text{on } S \\ c_1 \Delta u_1 = c_2 \Delta u_2, \quad \partial_\nu(c_1 \Delta u_1) = \partial_\nu(c_2 \Delta u_2) & \text{on } S \\ u_2 = 0, \quad \Delta u_2 = 0 & \text{on } \Gamma. \end{cases} \quad (5.14)$$

We set now

$$w_1 = c_1 \Delta u_1 - |\mu|u_1 + a \Delta v_1 \quad \text{and} \quad w_2 = c_2 \Delta u_2 - |\mu|u_2, \quad (5.15)$$

then it is easy to show that w_1 and w_2 satisfy the following simple transmission problem

$$\begin{cases} -\Delta w_1 - \frac{|\mu|}{c_1}w_1 = \Phi_1 & \text{in } \Omega_1 \\ -\Delta w_2 - \frac{|\mu|}{c_2}w_2 = \Phi_2 & \text{in } \Omega_2 \\ w_1 = w_2, \quad \partial_\nu w_1 = \partial_\nu w_2 & \text{on } S \\ w_2 = 0 & \text{on } \Gamma. \end{cases} \quad (5.16)$$

We recall the notation given in the end of the previous section $\tilde{\Omega}_1 = \Omega_1 \setminus \overline{B}_r$. The main ingredient of the resolvent estimate is the following lemma which is essentially a consequence of the Carleman estimate.

Lemma 5.1. *There exists a constant $C > 0$ such that for any $(u_1, u_2, v_1, v_2) \in \mathcal{D}(\mathcal{A})$ solution of (5.12) the following result holds*

$$\begin{aligned} \|\Delta u_1\|_{L^2(\Omega_1)}^2 + \|\Delta u_2\|_{L^2(\Omega_2)}^2 + \|v_1\|_{L^2(\Omega_1)}^2 + \|v_2\|_{L^2(\Omega_2)}^2 &\leq C e^{C|\mu|} \left(\|\Delta f_1\|_{L^2(\Omega_1)}^2 \right. \\ &\quad \left. + \|\Delta f_2\|_{L^2(\Omega_2)}^2 + \|g_1\|_{L^2(\Omega_1)}^2 + \|g_2\|_{L^2(\Omega_2)}^2 + \int_{\Omega_1} a |\Delta v_1|^2 dx + \int_{B_{4r}} |u_1|^2 dx \right), \end{aligned} \quad (5.17)$$

for all $\mu \in \mathbb{R}$ large enough.

Proof. We introduce the cut-off function $\chi \in \mathcal{C}^\infty(\Omega_1)$ by setting

$$\chi(x) = \begin{cases} 1 & \text{in } B_{3r}^c \\ 0 & \text{in } B_{2r} \end{cases}$$

Next, denote $\tilde{w}_1 = \chi w_1$. And by (5.16), one sees that

$$-\Delta \tilde{w}_1 - \frac{|\mu|}{c_1} \tilde{w}_1 = \tilde{\Phi}_1 = \chi \Phi_1 - [\Delta, \chi] w_1. \quad (5.18)$$

Now keeping the same notations as the previous section and let $\varphi_{1,1}$, $\varphi_{1,2}$, $\varphi_{2,1}$ and $\varphi_{2,2}$ four weight functions that satisfy the conclusion of the section 4. Let $\chi_{1,1}$, $\chi_{1,2}$, $\chi_{2,1}$ and $\chi_{2,2}$ four cut-off functions equal to one respectively in $\left(\bigcup_{j=1}^{N_{11}} B(x_{1j}^1, 2\epsilon)\right)^c$, $\left(\bigcup_{j=1}^{N_{12}} B(x_{1j}^2, 2\epsilon)\right)^c$, $\left(\bigcup_{j=1}^{N_{21}} B(x_{2j}^1, 2\epsilon)\right)^c$ and $\left(\bigcup_{j=1}^{N_{22}} B(x_{2j}^2, 2\epsilon)\right)^c$ and supported respectively in $\left(\bigcup_{j=1}^{N_{11}} B(x_{1j}^1, \epsilon)\right)^c$, $\left(\bigcup_{j=1}^{N_{12}} B(x_{1j}^2, \epsilon)\right)^c$, $\left(\bigcup_{j=1}^{N_{21}} B(x_{2j}^1, \epsilon)\right)^c$ and $\left(\bigcup_{j=1}^{N_{22}} B(x_{2j}^2, \epsilon)\right)^c$ (in order to eliminate the critical points of the phases functions $\varphi_{1,1}$, $\varphi_{1,2}$, $\varphi_{2,1}$ and $\varphi_{2,2}$ (see Fig. 2)). We set now $w_{1,1} = \chi_{1,1} \tilde{w}_1$, $w_{1,2} = \chi_{1,2} \tilde{w}_1$, $w_{2,1} = \chi_{2,1} w_2$ and $w_{2,2} = \chi_{2,2} w_2$. Then from the system (5.16) for $k = 1, 2$ we obtain

$$\begin{cases} -\Delta w_{1,k} - \frac{|\mu|}{c_1} w_{1,k} = \Psi_{1,k} & \text{in } \Omega_1 \\ -\Delta w_{2,k} - \frac{|\mu|}{c_2} w_{2,k} = \Psi_{2,k} & \text{in } \Omega_2 \\ w_{1,k} = w_{2,k}, \quad \partial_\nu w_{1,k} = \partial_\nu w_{2,k} & \text{on } S \\ w_{2,k} = 0 & \text{on } \Gamma, \end{cases} \quad (5.19)$$

where

$$\begin{cases} \Psi_{1,k} = \chi_{1,k} \tilde{\Phi}_1 - [\Delta, \chi_{1,k}] \tilde{w}_1 \\ \Psi_{2,k} = \chi_{2,k} \Phi_2 - [\Delta, \chi_{2,k}] w_2. \end{cases} \quad (5.20)$$

Applying now the Carleman estimate as given in the previous section (Theorem 4.1) to the system (5.19) for $h = \frac{1}{|\mu|}$ then for $k = 1, 2$ we obtain

$$\begin{aligned} & h \|e^{\varphi_{1,k}/h} w_{1,k}\|_{L^2(U_{1,k})}^2 + h^3 \|e^{\varphi_{1,k}/h} \nabla w_{1,k}\|_{L^2(U_{1,k})}^2 + h \|e^{\varphi_{2,k}/h} w_{2,k}\|_{L^2(U_{2,k})}^2 + \\ & h^3 \|e^{\varphi_{2,k}/h} \nabla w_{2,k}\|_{L^2(U_{2,k})}^2 \leq Ch^4 (\|e^{\varphi_{1,k}/h} \Psi_{1,k}\|_{L^2(U_{1,k})}^2 + \|e^{\varphi_{2,k}/h} \Psi_{2,k}\|_{L^2(U_{2,k})}^2). \end{aligned}$$

Relations (5.18) and (5.20) yield

$$\begin{aligned} & h \|e^{\varphi_{1,k}/h} w_{1,k}\|_{L^2(U_{1,k})}^2 + h^3 \|e^{\varphi_{1,k}/h} \nabla w_{1,k}\|_{L^2(U_{1,k})}^2 + h \|e^{\varphi_{2,k}/h} w_{2,k}\|_{L^2(U_{2,k})}^2 + \\ & h^3 \|e^{\varphi_{2,k}/h} \nabla w_{2,k}\|_{L^2(U_{2,k})}^2 \leq Ch^4 (\|e^{\varphi_{1,k}/h} \Phi_1\|_{L^2(U_{1,k})}^2 + \|e^{\varphi_{2,k}/h} \Phi_2\|_{L^2(U_{2,k})}^2 + \\ & \|e^{\varphi_{1,k}/h} [\Delta, \chi] w_1\|_{L^2(U_{1,k})}^2 + \|e^{\varphi_{1,k}/h} [\Delta, \chi_{1,k}] \tilde{w}_1\|_{L^2(U_{1,k})}^2 + \|e^{\varphi_{2,k}/h} [\Delta, \chi_{2,k}] w_2\|_{L^2(U_{2,k})}^2). \end{aligned} \quad (5.21)$$

We addition the two last estimates for $k = 1, 2$ and using the properties of phases $\varphi_{1,k} < \varphi_{1,\sigma(k)}$ in $\left(\bigcup_{j=1}^{N_{1k}} B(x_{1k}^j, 2\epsilon)\right)$ and $\varphi_{2,k} < \varphi_{2,\sigma(k)}$ in $\left(\bigcup_{j=1}^{N_{2k}} B(x_{2k}^j, 2\epsilon)\right)$ then we can absorb the terms $[\Delta, \chi_{1,k}]\tilde{w}_1$ and $[\Delta, \chi_{2,k}]w_2$ at the right hand side of (5.21) into the left hand side for $h > 0$ small. More precisely we obtain

$$\begin{aligned} & h \int_{\tilde{\Omega}_1} (e^{2\varphi_{1,1}/h} + e^{2\varphi_{1,2}/h}) |\tilde{w}_1|^2 dx + h \int_{\Omega_2} (e^{2\varphi_{2,1}/h} + e^{2\varphi_{2,2}/h}) |w_2|^2 dx \leq \\ & Ch^4 \left(\int_{\Omega_1} (e^{2\varphi_{1,1}/h} + e^{2\varphi_{1,2}/h}) |\Phi_1|^2 dx + \int_{\Omega_2} (e^{2\varphi_{2,1}/h} + e^{2\varphi_{2,2}/h}) |\Phi_2|^2 dx \right. \\ & \quad \left. + \int_{\tilde{\Omega}_1} (e^{2\varphi_{1,1}/h} + e^{2\varphi_{1,2}/h}) |[\Delta, \chi]w_1|^2 dx \right). \end{aligned}$$

Consequently, by using that $\Omega_1 = \tilde{\Omega}_1 \cup B_{2r}$ and the expressions of Φ_1 and Φ_2 in (5.14) we see that

$$\begin{aligned} \int_{\Omega_1} |w_1|^2 dx + \int_{\Omega_2} |w_2|^2 dx & \leq Ce^{C/h} \left(\int_{\tilde{\Omega}_1} |f_1|^2 dx + \int_{\Omega_1} |g_1|^2 dx + \int_{\Omega_2} |f_2|^2 dx \right. \\ & \quad \left. + \int_{\Omega_2} |g_2|^2 dx + \int_{\Omega_1} a |\Delta v_1|^2 dx + \int_{B_{2r}} |w_1|^2 dx + \int_{\tilde{\Omega}_1} |[\Delta, \chi]w_1|^2 dx \right). \end{aligned} \quad (5.22)$$

We estimate now the two last terms in the right hand side of (5.22). We set $\tilde{\chi}$ a cut-off function equal to 1 in a neighborhood of B_{3r} and supported in B_{4r} then we have

$$(-1 + \Delta)(\tilde{\chi}w_1) = [\Delta, \tilde{\chi}]w_1 - \tilde{\chi}w_1 - \frac{|\mu|}{c_1} \tilde{\chi}w_1 - \tilde{\chi}\Phi_1,$$

and hence by elliptic estimates (see [34]) we get

$$\begin{aligned} \|w_1\|_{H^1(B_{3r})}^2 & \leq C(\|(-1 + \Delta)(\tilde{\chi}w_1)\|_{H^{-1}(B_{4r})}^2 + \|w_1\|_{L^2(B_{4r})}^2) \\ & \leq C(\|\Phi_1\|_{L^2(\Omega_1)}^2 + (1 + |\mu|^2)\|w_1\|_{L^2(B_{4r})}^2) \\ & \leq C \left(|\mu|^2 \|f_1\|_{L^2(\Omega_1)}^2 + \|g_1\|_{L^2(\Omega_1)}^2 + (1 + |\mu|^2)\|w_1\|_{L^2(B_{4r})}^2 + |\mu|^2 \int_{\Omega_1} a |\Delta v_1|^2 dx \right). \end{aligned} \quad (5.23)$$

Combining the first line of (5.14) with (5.15), using (5.23) and the fact that $\text{supp}([\Delta, \chi]) \subset B_{3r}$ and $a > 0$ in B_{3r} we deduce

$$\begin{aligned} \int_{B_{2r}} |w_1|^2 dx + \int_{\tilde{\Omega}_1} |[\Delta, \chi]w_1|^2 dx & \leq C\|w_1\|_{H^1(B_{3r})}^2 \\ & \leq C \left(|\mu|^2 \|f_1\|_{L^2(\Omega_1)}^2 + \|g_1\|_{L^2(\Omega_1)}^2 + (1 + |\mu|^2)^2 \|u_1\|_{L^2(B_{4r})}^2 + |\mu|^2 \int_{\Omega_1} a |\Delta v_1|^2 dx \right). \end{aligned} \quad (5.24)$$

On other hand from (5.15) and the transmission conditions we see that

$$\begin{aligned} \|w_1\|_{L^2(\Omega_1)}^2 + \|w_2\|_{L^2(\Omega_2)}^2 &\geq \|c_1\Delta u_1 - |\mu|u_1\|_{L^2(\Omega_1)}^2 + \|c_2\Delta u_2 - |\mu|u_2\|_{L^2(\Omega_2)}^2 - C \int_{\Omega_1} a|\Delta v_1|^2 dx \\ &\geq -C \int_{\Omega_1} a|\Delta v_1|^2 dx + c_1^2 \|\Delta u_1\|_{L^2(\Omega_1)}^2 + c_2^2 \|\Delta u_2\|_{L^2(\Omega_2)}^2 + |\mu|^2 (\|u_1\|_{L^2(\Omega_1)}^2 + \|u_2\|_{L^2(\Omega_2)}^2) \\ &\quad + |\mu|(c_1 \|\nabla u_1\|_{L^2(\Omega_1)}^2 + c_2 \|\nabla u_2\|_{L^2(\Omega_2)}^2) \geq \|c_1\Delta u_1\|_{L^2(\Omega_1)}^2 + \|c_2\Delta u_2\|_{L^2(\Omega_2)}^2 - C \int_{\Omega_1} a|\Delta v_1|^2 dx, \end{aligned} \quad (5.25)$$

and by the expression of v_1 and v_2 in (5.13), one gets

$$\begin{aligned} \|v_1\|_{L^2(\Omega_1)}^2 &\leq \|f_1\|_{L^2(\Omega_1)}^2 + |\mu|^2 \|u_1\|_{L^2(\Omega_1)}^2 \\ \|v_2\|_{L^2(\Omega_2)}^2 &\leq \|f_2\|_{L^2(\Omega_2)}^2 + |\mu|^2 \|u_2\|_{L^2(\Omega_2)}^2. \end{aligned} \quad (5.26)$$

Then by combining Proposition 3.1, and estimates (5.22), (5.24), (5.25) and (5.26) we obtain the results. \square

Now by applying inequality (5.17) to the system (5.3)–(5.6) it follows that

$$\begin{aligned} &\|\Delta u_{1,m}\|_{L^2(\Omega_1)}^2 + \|\Delta u_{2,m}\|_{L^2(\Omega_2)}^2 + \|v_{1,m}\|_{L^2(\Omega_1)}^2 + \|v_{2,m}\|_{L^2(\Omega_2)}^2 \leq \\ &C e^{C|\mu_m|} \left(e^{-2K_m|\mu_m|} \left(\|\Delta f_{1,m}\|_{L^2(\Omega_1)}^2 + \|\Delta f_{2,m}\|_{L^2(\Omega_2)}^2 + \|g_{1,m}\|_{L^2(\Omega_1)}^2 + \|g_{2,m}\|_{L^2(\Omega_2)}^2 \right) \right. \\ &\quad \left. + e^{-\frac{K_m}{4}|\mu_m|} \left(\int_{\Omega_1} a|\Delta v_{1,m}|^2 dx + \int_{B_{4r}} |u_{1,m}|^2 dx \right) e^{\frac{K_m}{4}|\mu_m|} \right). \end{aligned} \quad (5.27)$$

From (5.1)–(5.2) and estimates (5.7) and (5.11), the right hand side of (5.27) go to zero as $m \rightarrow +\infty$ which contradicts (5.1). And this conclude the proof of the resolvent estimate.

Now, follows to [12, Lemma 4.1] it just remains to show that \mathcal{A} has no purely imaginary eigenvalue. Further, $0 \in \rho(\mathcal{A})$, where $\rho(\mathcal{A})$ stands for the resolvent set of \mathcal{A} . Let $\mu \neq 0$ be a real number. Suppose that for some $U = (u_1, u_2, v_1, v_2) \in \mathcal{D}(\mathcal{A})$, one has

$$\mathcal{A}U = i\mu U. \quad (5.28)$$

We shall show that $U = 0$. Taking the inner product with U on both side of (5.28) and taking the real part we immediately find that $v_1 = 0$ in $\text{supp}(a)$. Now (5.28) can be recast as

$$\begin{cases} v_1 = i\mu_1 u_1 & \text{in } \Omega_1, \\ v_2 = i\mu_2 u_2 & \text{in } \Omega_2, \\ -c_1\Delta(c_1\Delta u_1 + a\Delta v_1) - i\mu v_1 = 0 & \text{in } \Omega_1, \\ -c_2\Delta(c_2\Delta u_2) - i\mu v_2 = 0 & \text{in } \Omega_2. \end{cases} \quad (5.29)$$

Since $v_1 = 0$ in $\text{supp}(a)$ and $\mu \neq 0$ the top line of (5.29) yields $u_1 = 0$ in $\text{supp}(a)$. The third line of (5.29) combined with the first one could be written as

$$c_1\Delta z + \mu z = 0 \quad \text{in } \Omega_1 \quad \text{and} \quad z = 0 \quad \text{in } \text{supp}(a),$$

where we denoted by $z = c_1 \Delta u_1 - \mu u_1$. Since $\omega \subset \text{supp}(a)$ then by Calderón's theorem [31, Theorem 4.2] for elliptic operators we find that $z = 0$, this mean that $c_1 \Delta u_1 - \mu u_1 = 0$ which imply for the same argument as previously that $u_1 = 0$ in Ω_1 . Reporting that in the first line of (5.29), we derive $v_1 = 0$ in Ω_1 . The second and fourth line of (5.29) lead to

$$c_2 \Delta(c_2 \Delta u_2) - \mu^2 u_2 = 0 \quad \text{in } \Omega_2 \quad (5.30)$$

where the transmission conditions lead to

$$u_2 = 0, \Delta u_2 = 0 \quad \text{on } \partial\Omega_2 \quad \text{and} \quad \partial_\nu u_2 = 0, \partial_\nu(c_2 \Delta u_2) = 0 \quad \text{on } S.$$

Following to these boundary conditions, we can extend u_2 by zero in whole Ω where (5.30) remains valid on all Ω , then by using the same arguments above as for u_1 one can show also that $u_2 = 0$ in Ω_2 . Using the second line of (5.29), we get $v_2 = 0$ in Ω_2 ; hence we obtain that $U = 0$. Therefore, \mathcal{A} has no purely eigenvalue.

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