



Navier–Stokes equations with external forces in Lorentz spaces and its application to the self-similar solutions



Hideo Kozono^{a,*}, Senjo Shimizu^b

^a Department of Mathematics, Waseda University, 169-8555 Tokyo, Japan

^b Graduate School of Human and Environmental Studies, Kyoto University, 606-8501 Kyoto, Japan

ARTICLE INFO

Article history:

Received 18 July 2017

Available online 19 October 2017

Submitted by Y. Yamada

Keywords:

Navier–Stokes equations

Self-similar solutions

Lorentz space

Maximal regularity theorem

Global solutions

Implicit function theorem

ABSTRACT

We show existence theorem of global mild solutions with small initial data and external forces in Lorentz spaces with scaling invariant norms. If the initial data have more regularity in another scaling invariant class, then our mild solution is actually the strong solution. The result on local existence of solutions for large data is also discussed. Our method is based on the maximal regularity theorem on the Stokes equations in Lorentz spaces. Then we apply our theorem to prove existence of self-similar solutions provided both initial data and external forces are homogeneous functions. Since we construct the global solution by means of the implicit function theorem, as a byproduct, its stability with respect to the given data is necessarily obtained.

© 2017 Elsevier Inc. All rights reserved.

0. Introduction

Let us consider the Cauchy problem of the Navier–Stokes equations in \mathbb{R}^n , $n \geq 2$;

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = f & \text{in } \mathbb{R}^n \times (0, \infty), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u|_{t=0} = a & \text{in } \mathbb{R}^n, \end{cases} \quad (\text{N-S})$$

where $u = u(x, t) = (u_1(x, t), \dots, u_n(x, t))$ and $p = p(x, t)$ denote the unknown velocity vector and the unknown pressure at the point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and the time $t \in (0, \infty)$, while $a = a(x) = (a_1(x), \dots, a_n(x))$ and $f = f(x, t) = (f_1(x, t), \dots, f_n(x, t))$ are the given initial data of velocity and the given external force, respectively. In this paper, we prove the existence of global mild and strong solutions to (N-S) for small initial data $a \in L^{n, \infty}(\mathbb{R}^n)$ and small external force $f \in L^{s, \infty}(0, \infty; L^{q, \infty}(\mathbb{R}^n))$ with

* Corresponding author.

E-mail addresses: kozono@waseda.jp (H. Kozono), shimizu.senjo.5s@kyoto-u.ac.jp (S. Shimizu).

$2/s + n/3 = 3$, $n/3 < q < \infty$, where $L^{r,\infty}$ denotes the Lorentz space. It is well-known that (N–S) is invariant under such a change of scaling as $u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t)$ and $p_\lambda(x, t) = \lambda^2 p(\lambda x, \lambda^2 t)$ for all $\lambda > 0$. The Banach space \mathcal{Y} of functions with the space and time variables with the norm $\|\cdot\|_{\mathcal{Y}}$ is called *scaling invariant* to (N–S) if it holds that $\|u_\lambda\|_{\mathcal{Y}} = \|u\|_{\mathcal{Y}}$ for all $\lambda > 0$. For instance, in the usual Lebesgue spaces L^q , we see that the scaling invariant space \mathcal{Y} to (N–S) is the Serrin class $L^{s_0}(0, \infty; L^{q_0}(\mathbb{R}^n))$ for $2/s_0 + n/q_0 = 1$ with $n \leq q_0 \leq \infty$. Since the corresponding scaling law to the initial data is like $a_\lambda(x) = \lambda a(\lambda x)$, the suitable Banach space X with the norm $\|\cdot\|_X$ for the initial data should have the property that $\|a_\lambda\|_X = \|a\|_X$ for all $\lambda > 0$.

Since the pioneer work of Fujita–Kato [7], many efforts have been made to find such a space X as large as possible. Indeed, Kato [11] and Giga–Miyakawa [8] succeed to find the space $X = L^n(\mathbb{R}^n)$. Later on, Kozono–Yamazaki [13], [14] and Cannone–Planchon [4] extended to $X = L^{n,\infty}(\mathbb{R}^n)$, $\dot{B}_{q,\infty}^{-1+n/q}(\mathbb{R}^n)$ with $n < q < \infty$, where $\dot{B}_{q,r}^s(\mathbb{R}^n)$ denotes the homogeneous Besov space. Introducing the space \mathcal{PM}^k of the pseudo measure defined by $\mathcal{PM}^k = \{a \in \mathcal{S}' ; \sup_{\xi \in \mathbb{R}^n} |\xi|^k |\hat{a}(\xi)| < \infty\}$, Cannone–Karch [3] proved that $X = \mathcal{PM}^2$ is a suitable space for $n = 3$. The largest space of X was obtained by Koch–Tataru [12] who proved local well-posedness of (N–S) for $a \in X = BMO^{-1} = \dot{B}_{\infty,2}^{-1}(\mathbb{R}^n)$, where $\dot{B}_{q,r}^s(\mathbb{R}^n)$ denotes the homogeneous Triebel–Lizorkin space. Their result [12] seems to be optimal in the sense that continuous dependence of solutions with respect to the initial data breaks down in $X = \dot{B}_{\infty,r}^{-1}(\mathbb{R}^n)$ for $2 < r \leq \infty$, which was proved by Bourgain–Pavlović [2], Yoneda [23] and Wang [21]. Iwabuchi [10] introduced the modulation space $M_{q,r}^s(\mathbb{R}^n)$ and proved similar ill-posedness to those of [2] and [23] in $X = M_{2,r}^s$ for $s < -1$ and $1 \leq r < \infty$.

Concerning the external force f , the corresponding scaling law is like $f_\lambda(x, t) = \lambda^3 f(\lambda x, \lambda^2 t)$. However, in comparison with a number of papers on well-posedness with respect to the initial data, there is a little literature for investigating the suitable space Y of external forces satisfying $\|f_\lambda\|_Y = \|f\|_Y$ for all $\lambda > 0$. A typical Lebesgue space of Y can be chosen as $Y = L^s(0, \infty; L^q(\mathbb{R}^n))$ for $2/s + n/q = 3$. In this direction, Cannone–Planchon [5] treated $f = \operatorname{div} F$ with $F \in L^s(0, \infty; L^q(\mathbb{R}^3))$ for $2/s + 3/q = 2$ with $2/3 < q < \infty$. Then Cannone–Karch [3] showed that $Y = C_w(0, \infty; \mathcal{PM}^0)$ is a suitable space for $n = 3$. (C_w denotes the class of weakly continuous functions.) Their space may be regarded as the variant of $L^\infty(0, \infty; L^{n/3,\infty}(\mathbb{R}^n))$.

In this paper, for the global well-posedness of (N–S), we take such a space Y as the Lorentz space $L^{s,\infty}(0, \infty; L^{q,\infty}(\mathbb{R}^n))$ for $2/s + n/q = 3$ with $3/n < q < \infty$ as well as $X = L^{n,\infty}(\mathbb{R}^n)$. Our method is based on the maximal regularity theorem on the Stokes operator $A = -P\Delta$ in $L^{s,\infty}(0, \infty; L^{q,\infty}(\mathbb{R}^n))$ with P denoting by the projection onto the space of solenoidal vector fields, while they [3] made use of analyticity and decay properties of the Stokes semigroup $\{e^{-tA}\}_{t>0}$ in \mathcal{PM}^k . Since the maximal regularity theorem enables us to choose $1 < s, q < \infty$ arbitrarily, we may take larger spaces than those of [3]. Furthermore, since the homogeneous functions can be handled in the class of Lorentz spaces $L^{q,\infty}(\mathbb{R}^n)$, we show the existence of self-similar solutions to (N–S). It should be also emphasized that our construction of solutions relies on the implicit function theorem which yields necessarily continuous dependence $(a, f) \rightarrow u$ from $X \times Y$ to \mathcal{Y} . Hence, we prove global well-posedness of (N–S) with small data a and f in our Lorentz spaces. Concerning the existence of local strong solutions to (N–S) for large given data a and f , because of lack of strong continuity of the semi-group $\{e^{-tA}\}_{t>0}$ in $L^{q,\infty}(\mathbb{R}^n)$, we have a certain restriction on a and f . However, our condition may handle arbitrary data $a \in L^n(\mathbb{R}^n)$ and $f \in L^s(0, T; L^{q,\infty}(\mathbb{R}^n))$ for $2/s + n/q = 3$ with $3/n < q < \infty$.

1. Results

Before stating our result, let us first recall the Lorentz space $L^{q,\infty}$ on \mathbb{R}^n defined by

$$L^{q,\infty} = \{f : \mathbb{R}^n \rightarrow \mathbb{R}^n; \|f\|_{L^{q,\infty}} = \sup_{R>0} R [\mu\{x \in \mathbb{R}^n : |f(x)| > R\}]^{\frac{1}{q}} < \infty\}, \quad (1.1)$$

where μ denotes the Lebesgue measure in \mathbb{R}^n . It should be noted that the norm $\|f\|_{L^{q,\infty}}$ defined by (1.1) is actually a quasi norm. However, there is a norm of $L^{q,\infty}$ which is equivalent to the above $\|\cdot\|_{L^{q,\infty}}$. So, we may regard $L^{q,\infty}$ as the Banach space. For the detail, see, e.g., Stein–Weiss [20].

Let us introduce some function spaces and the Stokes operator in $L^{q,\infty}_\sigma$. We denote by P the Helmholtz projection from $L^q(\mathbb{R}^n)$, $1 < q < \infty$ onto the subspace of solenoidal vector fields. It is well-known that P is expressed as

$$P = (P_{ij})_{1 \leq i,j \leq n}, \quad P_{ij} = \delta_{ij} + R_i R_j, \quad i, j = 1, \dots, n,$$

where $\{\delta_{ij}\}_{1 \leq i,j \leq n}$ is the Kronecker symbol and $R_i = \frac{\partial}{\partial x_i}(-\Delta)^{-\frac{1}{2}}$, $i = 1, \dots, n$ is the Riesz transform. Hence we may regard P as the projection operator also in $L^{q,\infty}$ for all $1 < q < \infty$. The Stokes operator A on $L^{q,\infty}_\sigma \equiv PL^{q,\infty}$ is defined by $A = -P\Delta$ with the domain $D(A) = \{u \in L^{q,\infty}_\sigma; \nabla^2 u \in L^{q,\infty}\}$, $1 < q < \infty$. Then the original equations (N–S) can be written to the abstract evolution equation:

$$\begin{cases} \frac{du}{dt} + Au + P(u \cdot \nabla u) = Pf & \text{on } (0, \infty), \\ u(0) = a. \end{cases} \quad (1.2)$$

The solution u of (1.2) is called a *strong* solution of (N–S). Our first purpose is to solve (1.2) in the sense of *mild* solution with the form

$$u(t) = e^{-tA}a + \int_0^t e^{-(t-\tau)A}Pf(\tau) d\tau - \int_0^t e^{-(t-\tau)A}P(u \cdot \nabla u)(\tau) d\tau, \quad 0 < t < \infty. \quad (1.3)$$

Our main result now reads

Theorem 1.1. *Let $1 < s < 2$ and $\max\{1, \frac{n}{3}\} < q < \infty$. We assume that $2/s + n/q = 3$.*

(i) *Suppose that $n < q_0 < \infty$, $2 < s_0 < \infty$, $n \leq q_1 < \infty$, $1 < s_1 \leq 2$ satisfy*

$$q < q_0, \quad s < s_0, \quad \frac{2}{s_0} + \frac{n}{q_0} = 1, \quad (1.4)$$

$$q \leq q_1, \quad s < s_1, \quad \frac{2}{s_1} + \frac{n}{q_1} = 2, \quad (1.5)$$

$$1/s_0 + 1/s_1 < 1. \quad (1.6)$$

There exists a constant $\delta = \delta(n, q, q_0, q_1) > 0$ such that if $a \in L^{n,\infty}_\sigma$ and $f \in L^{s,\infty}(0, \infty; L^{q,\infty})$ satisfy

$$\|a\|_{L^{n,\infty}} + \|f\|_{L^{s,\infty}(0,\infty;L^{q,\infty})} \leq \delta, \quad (1.7)$$

then there exists a mild solution u of (N–S) on $(0, \infty)$ with the property that

$$u \in L^{s_0,\infty}(0, \infty; L^{q_0,\infty}_\sigma), \quad \nabla u \in L^{s_1,\infty}(0, \infty; L^{q_1,\infty}). \quad (1.8)$$

(ii) *Concerning uniqueness, there is a constant $M = M(n, q_0) > 0$ such that if u_1 and u_2 are two mild solutions of (N–S) in the class (1.8) with*

$$\|u_1\|_{L^{s_0,\infty}(0,\infty;L^{q_0,\infty})} \leq M, \quad \|u_2\|_{L^{s_0,\infty}(0,\infty;L^{q_0,\infty})} \leq M$$

then we have

$$u_1 \equiv u_2 \quad \text{on } \mathbb{R}^n \times [0, \infty).$$

(iii) Furthermore, we have continuous dependence and stability in the following sense. For every $\varepsilon > 0$, there is a constant $\eta(n, q, q_0, q_1; \varepsilon)$ such that if $a_i \in L_\sigma^{n, \infty}$ and $f_i \in L^{s, \infty}(0, \infty; L^{q, \infty})$ satisfy (1.7) for both $i = 1$ and $i = 2$, and if it holds that

$$\|a_1 - a_2\|_{L^{n, \infty}} + \|f_1 - f_2\|_{L^{s, \infty}(0, \infty; L^{q, \infty})} < \eta, \quad (1.9)$$

then the mild solutions u_i of (N-S) with $a = a_i$ and $f = f_i$ for $i = 1, 2$ given by the above (i) are subject to the estimate

$$\|u_1 - u_2\|_{L^{s_0, \infty}(0, \infty; L^{q_0, \infty})} + \|\nabla u_1 - \nabla u_2\|_{L^{s_1, \infty}(0, \infty; L^{q_1, \infty})} < \varepsilon. \quad (1.10)$$

(iv) In addition to (1.4) and (1.5), we assume that $n < q_0 < \infty$, $2 < s_0 < \infty$, $n \leq q_1 < \infty$, $1 < s_1 \leq 2$ satisfy

$$\frac{1}{q} = \frac{1}{q_0} + \frac{1}{q_1}, \quad \frac{1}{s} = \frac{1}{s_0} + \frac{1}{s_1}. \quad (1.11)$$

If $a \in B_{q, \infty}^{2(1-1/s)} \cap L_\sigma^{n, \infty}$ and $f \in L^{s, \infty}(0, \infty; L^{q, \infty})$ fulfill (1.7), then the mild solution u given by the above (i) has an additional regularity such that

$$\partial_t u, Au \in L^{s, \infty}(0, \infty; L_\sigma^{q, \infty}) \quad (1.12)$$

and satisfies (1.2).

Since the Lorenz spaces $L^{q, \infty}$ include homogeneous functions, the above theorem enables us to show the existence of self-similar solutions of (N-S). Here, we say that u is a self-similar mild (resp. strong) solution of (N-S) if u satisfies (1.3) (resp. (1.2)) with the property that $u_\lambda(x, t) \equiv \lambda u(\lambda x, \lambda^2 t) = u(x, t)$ for all $(x, t) \in \mathbb{R}^n \times (0, \infty)$ and all $\lambda > 0$.

Our result on the self-similar solutions reads as follows.

Corollary 1.1. Let q, s, q_0, s_0, q_1, s_1 be as in Theorem 1.1. Suppose that $a \in L_\sigma^{n, \infty}$ and $f \in L^{s, \infty}(0, \infty; L^{q, \infty})$ have such properties as

$$a(\lambda x) = \lambda^{-1} a(x), \quad f(\lambda x, \lambda^2 t) = \lambda^{-3} f(x, t)$$

for all $(x, t) \in \mathbb{R}^n \times (0, \infty)$ and all $\lambda > 0$. If a and f satisfy (1.7), then the mild and the strong solutions u given by Theorem 1.1 (i) and (iv) are self-similar.

Remarks. (i) Since $a \in L_\sigma^{n, \infty}$, we need to take q_1 so that $n \leq q_1 < \infty$. Therefore, the restriction $1 < s < s_1 \leq 2$ is due to q_1 which is required to satisfy $n \leq q_1$ with (1.5). More precisely, to find the mild solution u in the class $\nabla u \in L^{s_1, \infty}(0, \infty; L^{q_1, \infty})$ for the initial data $a \in L_\sigma^{n, \infty}$, we need such restriction such as $1 < s < 2$ on the external force $f \in L^{s, \infty}(0, \infty; L^{q, \infty})$. In other words, it is necessary to assume a rapid decay of $f(\cdot, t)$ at $t = \infty$ in such a way that $\|f(\cdot)\|_{L^{q, \infty}} \in L^{s, \infty}(0, \infty)$ for $1 < s < 2$. Notice that the smaller s is assumed, the more rapid decay of $f(t)$ as $t \rightarrow \infty$ is required. However, if we rewrite (1.3) as

$$u(t) = e^{-tA}a + \int_0^t e^{-(t-\tau)A} P f(\tau) d\tau - \int_0^t P \nabla \cdot e^{-(t-\tau)A} (u \otimes u)(\tau) d\tau, \quad 0 < t < \infty, \quad (1.13)$$

and if we find the solution u of (1.13) in a larger class $u \in L^{s_0, \infty}(0, \infty; L^{q_0, \infty})$ for $n < q_0 < \infty$ and $2 < s_0 < \infty$ satisfying (1.4), then we may choose $1 < s < \infty$.

(ii) For given $1 < s < 2$ and $\max\{1, \frac{n}{3}\} < q < \infty$, we can find $n < q_0 < \infty$, $2 < s_0 < \infty$, $n \leq q_1 < \infty$, $1 < s_1 \leq 2$ so that (1.4), (1.5) and (1.6) are fulfilled. It should be noted that the conditions (1.4), (1.5) and (1.6) are equivalent to

$$q < q_0, \quad \frac{n}{2} \left(\frac{1}{q} - \frac{1}{q_0} \right) < 1, \quad \frac{2}{s_0} + \frac{n}{q_0} = 1, \quad (1.14)$$

$$q \leq q_1, \quad \frac{n}{2} \left(\frac{1}{q} - \frac{1}{q_1} \right) < \frac{1}{2}, \quad \frac{2}{s_1} + \frac{n}{q_1} = 2, \quad (1.15)$$

$$1 < n \left(\frac{1}{q_0} + \frac{1}{q_1} \right), \quad (1.16)$$

respectively. The space (1.8) is scaling invariant to (N–S).

(iii) Based on the space \mathcal{PM}^k of the pseudo measure, Cannone–Karch [3] proved similar results to Theorem 1.1 (i), (ii) and Corollary 1.1 for small $\{a, f\} \in \mathcal{PM}^2 \times C_w(0, \infty; \mathcal{PM}^0)$ in \mathbb{R}^3 . Since $\mathcal{PM}^k \subset L^{\frac{n}{n-k}, \infty}$ for $0 \leq k < n$, our result includes that of [3].

(iv) For uniqueness of mild solutions of (N–S) we need to assume that both of two solutions are small in $L^{s_0, \infty}(0, \infty; L^{q_0, \infty})$. This is a typical difference of uniqueness assertion between the Lorentz space and the Lebesgue space due to Serrin [19]. On the other hand, in the usual Lebesgue space such as the class $C([0, T]; L^n_\sigma)$ uniqueness holds without any restriction on smallness (see, e.g. Lions–Masmoudi [17]). Because of lack of strong continuity of the semigroup $\{e^{-tA}\}_{t>0}$ in the Lorentz space $L^{n, \infty}_\sigma$, it seems to be difficult to prove such an unconditional uniqueness in our case. However, in comparison with the stationary problem like [15] and Nakatsuka [18], it is an open question whether uniqueness does hold provided one solution is small, while another solution is large, but has better regularity than the scaling invariant class.

(v) For existence of strong solutions in the class (1.12), we assume an additional regularity condition on the initial data a such as $a \in B^{2(1-1/s)}_{q, \infty} \cap L^{n, \infty}_\sigma$. However, we do not need any restriction on the size of a in $B^{2(1-1/s)}_{q, \infty}$. On the other hand, such regularity seems to be a *necessary condition* for existence of solutions of (N–S) in the class (1.12), because $B^{2(1-1/s)}_{q, \infty} = B^{-1+n/q}_{q, \infty}$ coincides with the critical scaling invariant class of initial data obtained by [13], [4] and Farwig–Sohr–Varnhorn [6].

(vi) In the case $f = 0$, the corresponding result to Corollary 1.1 was obtained by Barraza [1, Theorem 2]. Another aspect of the advantage in using Lorentz spaces was seen in Yamazaki [22] who proved existence of periodic solutions in 3D exterior domains. Recently, Lemarie-Rieusset [16] exhibited a systematic treatment of various function spaces which are useful to (N–S).

Finally, we show local existence theorem of mild solutions for given data in Lorentz spaces under some restriction.

Theorem 1.2. *Let q, s, q_0, s_0, q_1, s_1 be the same numbers as in Theorem 1.1. In addition, we assume that $n < q_1 < \infty$. There exists $\epsilon_0 = \epsilon_0(n, q, q_0, q_1) > 0$ such that if $a \in L^{n, \infty}_\sigma$ and $f \in L^{s, \infty}(0, T; L^{q, \infty})$ satisfy*

$$\limsup_{R \rightarrow \infty} R\mu\{x \in \mathbb{R}^n; |a(x)| > R\}^{\frac{1}{n}} + \limsup_{r \rightarrow \infty} r\mu\{t \in (0, T); \|f(t)\|_{L^{q, \infty}} > r\}^{\frac{1}{s}} \leq \epsilon_0, \quad (1.17)$$

then there exists $T_ \leq T$ and a mild solution u of (N–S) on $(0, T_*)$ with the property that*

$$u \in L^{s_0, \infty}(0, T_*; L_\sigma^{q_0, \infty}), \quad \nabla u \in L^{s_1, \infty}(0, T_*; L^{q_1, \infty}). \quad (1.18)$$

Remark. The condition (1.17) implies that the local singularities of a in \mathbb{R}^n and $\|f(t)\|_{L^{q, \infty}}$ in $t \in (0, T)$ behave like L^n and L^s , respectively. It is easy to see that if $a \in L^r$ for $n \leq r \leq \infty$ and if $f \in L^\alpha(0, T; L^{q, \infty})$ for $s \leq \alpha \leq \infty$, then we have that

$$\limsup_{R \rightarrow \infty} R\mu\{x \in \mathbb{R}^n; |a(x)| > R\}^{\frac{1}{n}} + \limsup_{r \rightarrow \infty} r\mu\{t \in (0, T); \|f(t)\|_{L^{q, \infty}} > r\}^{\frac{1}{s}} = 0,$$

and hence the hypothesis (1.17) necessarily holds.

2. Preliminaries

We first recall elementary properties of $L^p - L^q$ estimates of the Stokes semigroup $\{e^{-tA}\}_{t>0}$. For $1 < p \leq q < \infty$, there is a constant $C = C(n, p, q)$ such that

$$\|e^{-tA}a\|_{L^{q, \infty}} \leq Ct^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{q})} \|a\|_{L^{p, \infty}}, \quad (2.1)$$

$$\|\nabla e^{-tA}a\|_{L^{q, \infty}} \leq Ct^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{q}) - \frac{1}{2}} \|a\|_{L^{p, \infty}} \quad (2.2)$$

hold for all $a \in L_\sigma^{p, \infty}$ and all $0 < t < \infty$. The above estimates are well-known in usual Lebesgue spaces L^p and L^q so that the Hunt interpolation theorem yields (2.1) and (2.2). The following lemma is the $L^{s, \infty}$ -bounds in time for the Stokes flow $e^{-tA}a$.

Lemma 2.1. *Let $a \in L_\sigma^{p, \infty}$ for $1 < p < \infty$.*

(i) *For $1 < q_0, s_0 < \infty$ satisfying $p \leq q_0 < \infty$, $1/p - 1/q_0 < 2/n$ and $2/s_0 + n/q_0 = n/p$, we have $e^{-tA}a \in L^{s_0, \infty}(0, \infty; L_\sigma^{q_0, \infty})$ with the estimate*

$$\|e^{-tA}a\|_{L^{s_0, \infty}(0, \infty; L^{q_0, \infty})} \leq C\|a\|_{L^{p, \infty}}, \quad (2.3)$$

where $C = C(n, p, q_0)$.

(ii) *For $1 < q_1, s_1 < \infty$ satisfying $p \leq q_1 < \infty$, $1/p - 1/q_1 < 1/n$ and $2/s_1 + n/q_1 = n/p + 1$, we have $\nabla e^{-tA}a \in L^{s_1, \infty}(0, \infty; L^{q_1, \infty})$ with the estimate*

$$\|\nabla e^{-tA}a\|_{L^{s_1, \infty}(0, \infty; L^{q_1, \infty})} \leq C\|a\|_{L^{p, \infty}}, \quad (2.4)$$

where $C = C(n, p, q_1)$.

Proof. (i) Since $p \leq q_0 < \infty$, we have by (2.1) that

$$\|e^{-tA}a\|_{L^{q_0, \infty}} \leq Ct^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{q_0})} \|a\|_{L^{p, \infty}}, \quad 0 < t < \infty \quad (2.5)$$

with $C = C(n, p, q_0)$. By the assumption it holds that $0 < 1/s_0 = n(1/p - 1/q_0)/2 < 1$, and we have $t^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{q_0})} \in L^{s_0, \infty}(0, \infty)$. Hence, the desired estimate (2.3) follows from (2.5).

(ii) Similarly, since $p \leq q_1 < \infty$, we have by (2.2) that

$$\|\nabla e^{-tA}a\|_{L^{q_1, \infty}} \leq Ct^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{q_1}) - \frac{1}{2}} \|a\|_{L^{p, \infty}}, \quad 0 < t < \infty \quad (2.6)$$

with $C = C(n, p, q_1)$. By the assumption it holds that $1/2 \leq 1/s_1 = n(1/p - 1/q_1)/2 + 1/2 < 1$, and we have $t^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{q_1}) - \frac{1}{2}} \in L^{s_1, \infty}(0, \infty)$. Hence, the desired estimate (2.4) follows from (2.6). This proves Lemma 2.1. \square

We next investigate the inhomogeneous term $\int_0^t e^{-(t-\tau)A} Pf(\tau) d\tau$.

Lemma 2.2. *Let $f \in L^{s,\infty}(0, \infty; L^{q,\infty})$ for $2/s + n/q = 3$ with $\max\{1, n/3\} < q < \infty$ and $1 < s < \infty$. We set*

$$Ff(t) = \int_0^t e^{-(t-\tau)A} Pf(\tau) d\tau, \quad 0 < t < \infty.$$

(i) *For $1 < q_0, s_0 < \infty$ satisfying*

$$n < q_0, \quad q < q_0, \quad \frac{n}{2} \left(\frac{1}{q} - \frac{1}{q_0} \right) < 1, \quad \frac{2}{s_0} + \frac{n}{q_0} = 1 \quad (2.7)$$

we have that $Ff \in L^{s_0,\infty}(0, \infty; L^{q_0,\infty})$ with the estimate

$$\|Ff\|_{L^{s_0,\infty}(0,\infty;L^{q_0,\infty})} \leq C \|f\|_{L^{s,\infty}(0,\infty;L^{q,\infty})}, \quad (2.8)$$

where $C = C(n, q, q_0)$.

(ii) *For $1 < q_1, s_1 < \infty$ satisfying*

$$\frac{n}{2} < q_1, \quad q \leq q_1, \quad \frac{n}{2} \left(\frac{1}{q} - \frac{1}{q_1} \right) < \frac{1}{2}, \quad \frac{2}{s_1} + \frac{n}{q_1} = 2 \quad (2.9)$$

we have that $\nabla Ff \in L^{s_1,\infty}(0, \infty; L^{q_1,\infty})$ with the estimate

$$\|\nabla Ff\|_{L^{s_1,\infty}(0,\infty;L^{q_1,\infty})} \leq C \|f\|_{L^{s,\infty}(0,\infty;L^{q,\infty})}, \quad (2.10)$$

where $C = C(n, q, q_1)$.

Proof. (i) Since the projection P is a bounded operator in $L^{q,\infty}$, we have by (2.1) that

$$\|Ff(t)\|_{L^{q_0,\infty}} \leq \int_0^t \|e^{-(t-\tau)A} Pf(\tau)\|_{L^{q_0,\infty}} d\tau \leq C \int_0^t (t-\tau)^{-\frac{n}{2} \left(\frac{1}{q} - \frac{1}{q_0} \right)} \|f(\tau)\|_{L^{q,\infty}} d\tau, \quad (2.11)$$

where $C = C(n, q, q_0)$. By (2.7) it holds that

$$0 < \alpha \equiv 1 - \frac{n}{2} \left(\frac{1}{q} - \frac{1}{q_0} \right) < \frac{1}{s}.$$

Since $1/s_0 = 1/s - \alpha$ and since $\|f(\cdot)\|_{L^{q,\infty}} \in L^{s,\infty}(0, \infty)$, it follows from (2.11) and the Hardy–Littlewood–Sobolev inequality in the weak Lebesgue space that $\|Ff(\cdot)\|_{L^{q_0,\infty}} \in L^{s_0,\infty}(0, \infty)$ with the estimate (2.8).

(ii) Similarly by (2.2), we have that

$$\|\nabla Ff(t)\|_{L^{q_1,\infty}} \leq \int_0^t \|\nabla e^{-(t-\tau)A} Pf(\tau)\|_{L^{q_1,\infty}} d\tau \leq C \int_0^t (t-\tau)^{-\frac{n}{2} \left(\frac{1}{q} - \frac{1}{q_1} \right) - \frac{1}{2}} \|f(\tau)\|_{L^{q,\infty}} d\tau, \quad (2.12)$$

where $C = C(n, q, q_1)$. By (2.9) it holds that

$$0 < \beta \equiv \frac{1}{2} - \frac{n}{2} \left(\frac{1}{q} - \frac{1}{q_1} \right) < \frac{1}{s}.$$

Since $1/s_1 = 1/s - \beta$ and since $\|f(\cdot)\|_{L^{q,\infty}} \in L^{s,\infty}(0, \infty)$, it follows from (2.12) and the Hardy–Littlewood–Sobolev inequality in the weak Lebesgue space that $\|\nabla Ff(\cdot)\|_{L^{q_1,\infty}} \in L^{s_1,\infty}(0, \infty)$ with the estimate (2.10). This proves Lemma 2.2. \square

The following lemma on bilinear estimates of

$$N(u, v)(t) \equiv \int_0^t e^{-(t-\tau)A} P(u \cdot \nabla v)(\tau) d\tau, \quad 0 < t < \infty \quad (2.13)$$

plays an important role for the proof of our theorem.

Lemma 2.3. (i) Let $n < q_0 < \infty$ and $2 < s_0 < \infty$ satisfy $2/s_0 + n/q_0 = 1$. For every $u, v \in L^{s_0,\infty}(0, \infty; L^{q_0,\infty})$, we have that $N(u, v) \in L^{s_0,\infty}(0, \infty; L^{q_0,\infty})$ with the estimate

$$\|N(u, v)\|_{L^{s_0,\infty}(0,\infty;L^{q_0,\infty})} \leq C \|u\|_{L^{s_0,\infty}(0,\infty;L^{q_0,\infty})} \|v\|_{L^{s_0,\infty}(0,\infty;L^{q_0,\infty})}, \quad (2.14)$$

where $C = C(n, q_0)$.

(ii) Let $n/2 < q_1 < \infty$ and $1 < s_1 < \infty$ satisfy $2/s_1 + n/q_1 = 2$. Suppose that $\nabla v \in L^{s_1,\infty}(0, \infty; L^{q_1,\infty})$. For every $u \in L^{s_0,\infty}(0, \infty; L^{q_0,\infty})$ with $1 < q_0, s_0 < \infty$ satisfying $1/q_0 < 1 - 1/q_1$ and $1/s_0 < 1 - 1/s_1$, we have that $N(u, v) \in L^{s_0,\infty}(0, \infty; L^{q_0,\infty})$ with the estimate

$$\|N(u, v)\|_{L^{s_0,\infty}(0,\infty;L^{q_0,\infty})} \leq C \|u\|_{L^{s_0,\infty}(0,\infty;L^{q_0,\infty})} \|\nabla v\|_{L^{s_1,\infty}(0,\infty;L^{q_1,\infty})}, \quad (2.15)$$

where $C = C(n, q_1, q_0, s_0)$.

(iii) Let $n < q_0 < \infty$ and $2 < s_0 < \infty$ satisfy $2/s_0 + n/q_0 = 1$. Suppose that $u \in L^{s_0,\infty}(0, \infty; L^{q_0,\infty})$. For every $\nabla v \in L^{s_1,\infty}(0, \infty; L^{q_1,\infty})$ with $1 < q_1, s_1 < \infty$ satisfying $1/q_1 < 1 - 1/q_0$ and $1/s_1 < 1 - 1/s_0$, we have that $\nabla N(u, v) \in L^{s_1,\infty}(0, \infty; L^{q_1,\infty})$ with the estimate

$$\|\nabla N(u, v)\|_{L^{s_1,\infty}(0,\infty;L^{q_1,\infty})} \leq C \|u\|_{L^{s_0,\infty}(0,\infty;L^{q_0,\infty})} \|\nabla v\|_{L^{s_1,\infty}(0,\infty;L^{q_1,\infty})}, \quad (2.16)$$

where $C = C(n, q_0, q_1, s_1)$.

Remark. It should be noted that for the estimate (2.15) the integral exponents q_0 and s_0 for u are arbitrarily chosen so far as the conditions satisfying $1/q_0 < 1 - 1/q_1$ and $1/s_0 < 1 - 1/s_1$ are satisfied, because the other ∇v is in the scaling invariant class. Similarly, for the estimate (2.16), we may take the integral exponents q_1 and s_1 of ∇v arbitrarily so far as the conditions satisfying $1/q_1 < 1 - 1/q_0$ and $1/s_1 < 1 - 1/s_0$ are satisfied, because u is also in the scaling invariant class.

Proof of Lemma 2.3. (i) Since $\operatorname{div} u = 0$, we see that $N(u, v)$ is expressed as

$$N(u, v)(t) = \int_0^t P \nabla \cdot e^{-(t-\tau)A} (u \otimes v)(\tau) d\tau.$$

Hence, it follows from (2.2) and the weak Hölder inequality (see, e.g. [15, Proposition 2.1]) that

$$\|N(u, v)(t)\|_{L^{q_0,\infty}} \leq C \int_0^t (t - \tau)^{-\frac{n}{2}(\frac{2}{q_0} - \frac{1}{q_0}) - \frac{1}{2}} \|u \otimes v(\tau)\|_{L^{\frac{q_0}{2},\infty}} d\tau$$

$$\leq C \int_0^t (t-\tau)^{\frac{1}{2}-\frac{n}{2q_0}-1} \|u(\tau)\|_{L^{q_0,\infty}} \|v(\tau)\|_{L^{q_0,\infty}} d\tau. \quad (2.17)$$

Since

$$\|u(\cdot)\|_{L^{q_0,\infty}} \|v(\cdot)\|_{L^{q_0,\infty}} \in L^{\frac{s_0}{2},\infty}(0,\infty)$$

and since

$$0 < \frac{1}{2} - \frac{n}{2q_0} < \frac{2}{s_0}, \quad \frac{1}{s_0} = \frac{2}{s_0} - \left(\frac{1}{2} - \frac{n}{2q_0} \right),$$

it follows from (2.17) and the Hardy–Littlewood–Sobolev inequality in the weak Lebesgue space that $\|N(u, v)(\cdot)\|_{L^{q_0,\infty}} \in L^{s_0,\infty}(0,\infty)$ with the estimate

$$\begin{aligned} \| \|N(u, v)(\cdot)\|_{L^{q_0,\infty}} \|_{L^{s_0,\infty}(0,\infty)} &\leq C \| \|u(\cdot)\|_{L^{q_0,\infty}} \|v(\cdot)\|_{L^{q_0,\infty}} \|_{L^{\frac{s_0}{2},\infty}(0,\infty)} \\ &\leq C \| \|u(\cdot)\|_{L^{q_0,\infty}} \|_{L^{s_0,\infty}(0,\infty)} \| \|v(\cdot)\|_{L^{q_0,\infty}} \|_{L^{s_0,\infty}(0,\infty)}, \end{aligned}$$

which implies (2.14).

(ii) It follows from (2.1) and the weak Hölder inequality that

$$\begin{aligned} \|N(u, v)(t)\|_{L^{q_0,\infty}} &\leq C \int_0^t (t-\tau)^{-\frac{n}{2}(\frac{1}{q_0}+\frac{1}{q_1}-\frac{1}{q_0})} \|u(\tau)\|_{L^{q_0,\infty}} \|\nabla v(\tau)\|_{L^{q_1,\infty}} d\tau \\ &\leq C \int_0^t (t-\tau)^{1-\frac{n}{2q_1}-1} \|u(\tau)\|_{L^{q_0,\infty}} \|\nabla v(\tau)\|_{L^{q_1,\infty}} d\tau. \end{aligned} \quad (2.18)$$

Since

$$\|u(\cdot)\|_{L^{q_0,\infty}} \|\nabla v(\cdot)\|_{L^{q_1,\infty}} \in L^{\tilde{s},\infty}(0,\infty) \quad \text{with} \quad \frac{1}{\tilde{s}} \equiv \frac{1}{s_0} + \frac{1}{s_1}$$

and since

$$0 < 1 - \frac{n}{2q_1} < \frac{1}{\tilde{s}}, \quad \frac{1}{s_0} = \frac{1}{\tilde{s}} - \left(1 - \frac{n}{2q_1} \right),$$

it follows from (2.18) and the Hardy–Littlewood–Sobolev inequality in the weak Lebesgue space that $\|N(u, v)(\cdot)\|_{L^{q_0,\infty}} \in L^{s_0,\infty}(0,\infty)$ with the estimate

$$\begin{aligned} \| \|N(u, v)(\cdot)\|_{L^{q_0,\infty}} \|_{L^{s_0,\infty}(0,\infty)} &\leq C \| \|u(\cdot)\|_{L^{q_0,\infty}} \|\nabla v(\cdot)\|_{L^{q_1,\infty}} \|_{L^{\tilde{s},\infty}(0,\infty)} \\ &\leq C \| \|u(\cdot)\|_{L^{q_0,\infty}} \|_{L^{s_0,\infty}(0,\infty)} \| \|\nabla v(\cdot)\|_{L^{q_1,\infty}} \|_{L^{s_1,\infty}(0,\infty)}, \end{aligned}$$

which implies (2.15).

(iii) Similarly, it follows from (2.2) and the weak Hölder inequality that

$$\|\nabla N(u, v)(t)\|_{L^{q_1,\infty}} \leq C \int_0^t (t-\tau)^{-\frac{n}{2}(\frac{1}{q_0}+\frac{1}{q_1}-\frac{1}{q_1})-\frac{1}{2}} \|u(\tau)\|_{L^{q_0,\infty}} \|\nabla v(\tau)\|_{L^{q_1,\infty}} d\tau$$

$$\leq C \int_0^t (t-\tau)^{\frac{1}{2}-\frac{n}{2q_0}-1} \|u(\tau)\|_{L^{q_0,\infty}} \|\nabla v(\tau)\|_{L^{q_1,\infty}} d\tau. \quad (2.19)$$

Since

$$\|u(\cdot)\|_{L^{q_0,\infty}} \|\nabla v(\cdot)\|_{L^{q_1,\infty}} \in L^{\tilde{s},\infty}(0,\infty)$$

with the same \tilde{s} as in the above (ii) and since

$$0 < \frac{1}{2} - \frac{n}{2q_0} < \frac{1}{\tilde{s}}, \quad \frac{1}{s_1} = \frac{1}{\tilde{s}} - \left(\frac{1}{2} - \frac{n}{2q_0} \right),$$

it follows from (2.19) and the Hardy–Littlewood–Sobolev inequality in the weak Lebesgue space that $\|\nabla N(u, v)(\cdot)\|_{L^{q_1,\infty}} \in L^{s_1,\infty}(0,\infty)$ with the estimate

$$\begin{aligned} \|\|\nabla N(u, v)(\cdot)\|_{L^{q_1,\infty}}\|_{L^{s_1,\infty}(0,\infty)} &\leq C \| \|u(\cdot)\|_{L^{q_0,\infty}} \|\nabla v(\cdot)\|_{L^{q_1,\infty}} \|_{L^{\tilde{s},\infty}(0,\infty)} \\ &\leq C \| \|u(\cdot)\|_{L^{q_0,\infty}} \|_{L^{s_0,\infty}(0,\infty)} \| \|\nabla v(\cdot)\|_{L^{q_1,\infty}} \|_{L^{s_1,\infty}(0,\infty)}, \end{aligned}$$

which implies (2.16). This proves Lemma 2.3. \square

Finally in this section, we investigate the maximal regularity theorem in the Lorenz space of the Stokes equations:

$$\begin{cases} \frac{du}{dt} + Au = Pf & \text{on } (0, \infty), \\ u(0) = a. \end{cases} \quad (2.20)$$

Lemma 2.4. *Let $1 < q, s < \infty$. For every $a \in B_{q,\infty}^{2(1-1/s)}$ with $\operatorname{div} a = 0$ and every $f \in L^{s,\infty}(0, \infty; L^{q,\infty})$, there is a unique solution u of (2.20) in the class*

$$\frac{du}{dt}, Au \in L^{s,\infty}(0, \infty; L_\sigma^{q,\infty}). \quad (2.21)$$

Such a solution u is subject to the estimate

$$\left\| \frac{du}{dt} \right\|_{L^{s,\infty}(0,\infty; L^{q,\infty})} + \|Au\|_{L^{s,\infty}(0,\infty; L^{q,\infty})} \leq C(\|a\|_{B_{q,\infty}^{2(1-1/s)}} + \|f\|_{L^{s,\infty}(0,\infty; L^{q,\infty})}) \quad (2.22)$$

with $C = C(n, s, q)$.

Remark. For the solution u given by Lemma 2.4, we have

$$u, \nabla u \in L^{s,\infty}(0, T; L^{q,\infty}) \quad \text{for all } 0 < T < \infty \text{ and } \nabla^2 u \in L^{s,\infty}(0, \infty; L^{q,\infty}).$$

Proof of Lemma 2.4. The corresponding result in the usual Lebesgue space $L^s(0, \infty; L^q)$ is well-known. See, e.g. Giga–Sohr [9]. Indeed, let $1 < s_0 < s_1 < \infty$ and $1 < q_0 < q < q_1 < \infty$. For every $a \in B_{q_i, s_i}^{2(1-1/s_i)}$ with $\operatorname{div} a = 0$ and every $f \in L^{s_i}(0, \infty; L^{q_i})$, there is a unique solution u of (2.20) such that

$$\frac{du}{dt}, Au \in L^{s_i}(0, \infty; L_\sigma^{q_i}) \quad (2.23)$$

for $i = 0, 1$. Such a solution u is subject to the estimate

$$\left\| \frac{du}{dt} \right\|_{L^{s_i}(0, \infty; L^{q_i})} + \|Au\|_{L^{s_i}(0, \infty; L^{q_i})} \leq C(\|a\|_{B_{q_i, s_i}^{2(1-1/s_i)}} + \|f\|_{L^{s_i}(0, \infty; L^{q_i})}) \quad (2.24)$$

with $C(n, s_i, q_i)$ for $i = 0, 1$. We take spaces X_i , Y_i and \mathcal{Y}_i as

$$X_i = \{a \in B_{q_i, s_i}^{2(1-1/s_i)}; \operatorname{div} a = 0\}, \quad Y_i = L^{s_i}(0, \infty; L^{q_i}),$$

$$\mathcal{Y}_i = \left\{ u \in L_{\operatorname{loc}}^{s_i}(0, \infty; L_{\sigma}^{q_i}); \frac{du}{dt}, Au \in L^{s_i}(0, \infty; L_{\sigma}^{q_i}) \right\}$$

for $i = 0, 1$. We may regard \mathcal{Y}_i as the Banach space with the norm $\|\cdot\|_{\mathcal{Y}_i}$ as

$$\|u\|_{\mathcal{Y}_i} = \left\| \frac{du}{dt} \right\|_{L^{s_i}(0, \infty; L^{q_i})} + \|Au\|_{L^{s_i}(0, \infty; L^{q_i})}$$

for $i = 0, 1$. Then the above solvability of (2.20) in the class (2.23) with the estimate (2.24) enables us to define the bounded linear operator

$$S : (a, f) \in X_i \times Y_i \mapsto u \in \mathcal{Y}_i, \quad i = 0, 1. \quad (2.25)$$

Let us first take $0 < \theta < 1$ so that $1/q = (1 - \theta)/q_0 + \theta/q_1$. Then by the real interpolation theory, S can be extended to the bounded operator as

$$S : (X_0 \times Y_0, X_1 \times Y_1)_{\theta, \infty} \mapsto (\mathcal{Y}_0, \mathcal{Y}_1)_{\theta, \infty}. \quad (2.26)$$

It is easy to see that

$$(X_0 \times Y_0, X_1 \times Y_1)_{\theta, \infty} = \{a \in B_{q, \infty}^{2(1-1/\tilde{s})}; \operatorname{div} a = 0\} \times L^{\tilde{s}, \infty}(0, \infty; L^{q, \infty}) \quad (2.27)$$

$$(\mathcal{Y}_0, \mathcal{Y}_1)_{\theta, \infty} = \left\{ u \in L_{\operatorname{loc}}^{\tilde{s}, \infty}(0, \infty; L_{\sigma}^{q, \infty}); \frac{du}{dt}, Au \in L^{\tilde{s}, \infty}(0, \infty; L_{\sigma}^{q, \infty}) \right\}, \quad (2.28)$$

where \tilde{s} is determined by $1/\tilde{s} = (1 - \theta)/s_0 + \theta/s_1$. Since we have freedom to choose s_0 and s_1 arbitrarily, we see that (2.26), (2.27) and (2.28) hold for all $1 < \tilde{s} < \infty$. Hence, taking $\tilde{s}_0 < s < \tilde{s}_1$ and $0 < \tilde{\theta} < 1$ so that $1/s = (1 - \tilde{\theta})/\tilde{s}_0 + \tilde{\theta}/\tilde{s}_1$, based on (2.26), we obtain the desired solution u of (2.20) with (2.21) and (2.22) again from the real interpolation theory with the exponents $\tilde{\theta}$ and ∞ . This proves Lemma 2.4. \square

3. Proof of theorems

3.1. Global existence of mild and strong solutions; Proof of Theorem 1.1

For the proof of Theorem 1.1, we make use of the *implicit function theorem* for Banach spaces. For that purpose, let us introduce the spaces X , Y and \mathcal{Y} by

$$X \equiv L_{\sigma}^{n, \infty}, \quad Y \equiv L^{s, \infty}(0, \infty; L^{q, \infty}),$$

$$\mathcal{Y} \equiv \{u \in L^{s_0, \infty}(0, \infty; L_{\sigma}^{q_0, \infty}); \nabla u \in L^{s_1, \infty}(0, \infty; L^{q_1, \infty})\}.$$

Equipped with the norm

$$\|u\|_{\mathcal{Y}} = \|u\|_{L^{s_0, \infty}(0, \infty; L^{q_0, \infty})} + \|\nabla u\|_{L^{s_1, \infty}(0, \infty; L^{q_1, \infty})},$$

we may regard \mathcal{Y} as a Banach space. For $(a, f) \in X \times Y$ and $u \in \mathcal{Y}$, we define a map $S(a, f, u)$ by

$$\begin{aligned} S(a, f, u)(t) \\ \equiv u(t) - e^{-tA}a - \int_0^t e^{-(t-\tau)A}Pf(\tau)d\tau + \int_0^t e^{-(t-\tau)A}P(u \cdot \nabla u)(\tau)d\tau, \quad 0 < t < \infty. \end{aligned} \quad (3.1)$$

Then we have:

Lemma 3.1. *Let $\max\{1, n/3\} < q < \infty$ and $1 < s < 2$ be as $2/s + n/q = 3$. Suppose that $n < q_0 < \infty$, $2 < s_0 < \infty$ and $n \leq q_1 < \infty$, $1 < s_1 \leq 2$ satisfy (1.4), (1.5) and (1.6).*

- (i) *The map S defined by (3.1) is continuous from $X \times Y \times \mathcal{Y}$ into \mathcal{Y} .*
- (ii) *For each $(a, f) \in X \times Y$, the map $S(a, f, \cdot)$ is of class C^1 from \mathcal{Y} into itself and the Fréchet derivative $S_u(a, f, u) \in \mathbb{B}(\mathcal{Y})$ with respect to $u \in \mathcal{Y}$ is given by*

$$S_u(a, f, u)h(t) = h(t) + \int_0^t e^{-(t-\tau)A}P(u \cdot \nabla h + h \cdot \nabla u)(\tau)d\tau, \quad 0 < t < \infty$$

for $h \in \mathcal{Y}$. ($\mathbb{B}(\mathcal{Y})$ denotes the set of bounded linear operators on \mathcal{Y} .)

Proof. (i) By Lemma 2.1 with $p = n$, Lemma 2.2 with (1.14) and (1.15), and Lemma 2.3, it is easy to see that S is a continuous map from $X \times Y \times \mathcal{Y}$ into \mathcal{Y} .

- (ii) For each $u \in \mathcal{Y}$ we define the linear operator L_u by

$$(L_u h)(t) = h(t) + \int_0^t e^{-(t-\tau)A}P(u \cdot \nabla h + h \cdot \nabla u)(\tau)d\tau, \quad 0 < t < \infty$$

for $h \in \mathcal{Y}$. By Lemma 2.3, we have

$$\begin{aligned} \|L_u h\|_{\mathcal{Y}} &\leq \|h\|_{\mathcal{Y}} + \|N(u, h)\|_{\mathcal{Y}} + \|N(h, u)\|_{\mathcal{Y}} \\ &\leq \|h\|_{\mathcal{Y}} + C\|u\|_{\mathcal{Y}}\|h\|_{\mathcal{Y}} \\ &= C(1 + \|u\|_{\mathcal{Y}})\|h\|_{\mathcal{Y}} \end{aligned}$$

for all $u, h \in \mathcal{Y}$ with $C = C(n, q, q_0, q_1)$, which implies that $L_u \in \mathbb{B}(\mathcal{Y})$ with the estimate $\|L_u\|_{\mathbb{B}(\mathcal{Y})} \leq C\|u\|_{\mathcal{Y}}$.

In the similar manner, we obtain that

$$\|L_{u_1} - L_{u_2}\|_{\mathbb{B}(\mathcal{Y})} \leq C\|u_1 - u_2\|_{\mathcal{Y}} \quad (3.2)$$

for all $u_1, u_2 \in \mathcal{Y}$ with $C = C(n, q, q_0, q_1)$, which means that the map $u \in \mathcal{Y} \mapsto L_u \in \mathbb{B}(\mathcal{Y})$ is continuous. Furthermore, we have by Lemma 2.3 that

$$\limsup_{\|h\|_{\mathcal{Y}} \rightarrow 0} \frac{\|S(a, f, u+h) - S(a, f, u) - L_u h\|_{\mathcal{Y}}}{\|h\|_{\mathcal{Y}}} = \limsup_{\|h\|_{\mathcal{Y}} \rightarrow 0} \frac{\|N(h, h)\|_{\mathcal{Y}}}{\|h\|_{\mathcal{Y}}} \leq C \limsup_{\|h\|_{\mathcal{Y}} \rightarrow 0} \|h\|_{\mathcal{Y}} = 0$$

for all $(a, f) \in X \times Y$ and all $u \in \mathcal{Y}$, which means, with the aid of (3.2), that the map $S(a, f, \cdot)$ is of class C^1 from \mathcal{Y} into itself with its Fréchet derivative $S_u(a, f, u) = L_u$. This proves Lemma 3.1. \square

Proof of Theorem 1.1. (i) We make use of Lemma 3.1. Recall the map S defined by (3.1). We may solve the equation $S(a, f, u) = 0$ so that $u \in \mathcal{Y}$ can be expressed by means of $(a, f) \in X \times Y$. Indeed, since we see from Lemma 3.1 that

$$S(0, 0, 0) = 0, \quad S_u(0, 0, 0) = I_{\mathcal{Y}} \text{ (the identity map on } \mathcal{Y} \text{)},$$

it follows from the implicit function theorem on Banach spaces that there exist positive constants δ, δ' and a continuous map u

$$u : U_{\delta} \equiv \{(a, f) \in X \times Y; \|a\|_X + \|f\|_Y \leq \delta\} \mapsto u(a, f) \in V_{\delta'} \equiv \{v \in \mathcal{Y}; \|v\|_{\mathcal{Y}} \leq \delta'\}$$

such that the equation $S(a, f, u) = 0$ can be uniquely expressed as $u = u(a, f)$ for $(a, f) \in U_{\delta}$. Obviously, such a constant δ and the continuous map u are the desired ones for (1.7) and (1.8), respectively.

(ii) Assume that u_1 and u_2 are two mild solutions of (N-S) on $(0, \infty)$ in the class (1.8). Then it follows from (1.3), (2.13) and (2.14) that

$$\begin{aligned} & \|u_1 - u_2\|_{L^{s_0, \infty}(0, \infty; L^{q_0, \infty})} \\ &= \|N(u_1 - u_2, u_1) + N(u_2, u_1 - u_2)\|_{L^{s_0, \infty}(0, \infty; L^{q_0, \infty})} \\ &\leq C(\|u_1\|_{L^{s_0, \infty}(0, \infty; L^{q_0, \infty})} + \|u_2\|_{L^{s_0, \infty}(0, \infty; L^{q_0, \infty})})\|u_1 - u_2\|_{L^{s_0, \infty}(0, \infty; L^{q_0, \infty})} \end{aligned}$$

with $C(n, q_0)$. Hence, we may choose $0 < M < 1/(2C)$.

(iii) The continuous dependence of solutions such as (1.10) under the condition (1.9) is an immediate consequence of continuity of the map $u : U_{\delta} \rightarrow V_{\delta'}$ given by the above (i).

(iv) Let us define v and w by

$$v(t) = e^{-tA}a + \int_0^t e^{-(t-\tau)A}Pf(\tau)d\tau, \quad w(t) = -N(u, u)(t).$$

By Lemma 2.4, v is a unique solution of (2.20) with the property that

$$\partial_t v, Av \in L^{s, \infty}(0, \infty; L_{\sigma}^{q, \infty}). \quad (3.3)$$

By (1.11) and the weak Hölder inequality, it holds that

$$\|u \cdot \nabla u\|_{L^{s, \infty}(0, \infty; L^{q, \infty})} \leq C\|u\|_{L^{s_0, \infty}(0, \infty; L^{q_0, \infty})}\|\nabla u\|_{L^{s_1, \infty}(0, \infty; L^{q_1, \infty})},$$

and hence it follows from Lemma 2.4 that w is a solution of (2.20) for $a = 0$ and $f = -u \cdot \nabla u$ with the property that

$$\partial_t w, Aw \in L^{s, \infty}(0, \infty; L_{\sigma}^{q, \infty}). \quad (3.4)$$

Since $u = v + w$, we see from (3.3) and (3.4) that u satisfies (1.12). Obviously, u satisfies (1.2). This completes the proof of Theorem 1.1. \square

Proof of Corollary 1.1. Let us define spaces X_0, Y_0 and \mathcal{Y}_0 by

$$\begin{aligned} X_0 &\equiv \{a \in X; a(\lambda x) = \lambda^{-1}a(x), \forall x \in \mathbb{R}^n, \forall \lambda > 0\}, \\ Y_0 &\equiv \{f \in Y; f(\lambda x, \lambda^2 t) = \lambda^{-3}f(x, t), \forall x \in \mathbb{R}^n, \forall t > 0, \forall \lambda > 0\}, \\ \mathcal{Y}_0 &\equiv \{u \in \mathcal{Y}; u(\lambda x, \lambda^2 t) = \lambda^{-1}u(x, t), \forall x \in \mathbb{R}^n, \forall t > 0, \forall \lambda > 0\}. \end{aligned}$$

Obviously, X_0 , Y_0 and \mathcal{Y}_0 are closed subspaces of X , Y and \mathcal{Y} , respectively. By [Lemmata 2.1, 2.2, 2.3](#) and a direct calculation of change of scaling, we have

$$e^{-tA}a \in \mathcal{Y}_0, \quad \int_0^t e^{-(t-s)A} Pf(\tau) d\tau \in \mathcal{Y}_0, \quad \int_0^t e^{-(t-s)A} P(u \cdot \nabla v)(\tau) d\tau \in \mathcal{Y}_0$$

for all $a \in X_0$, $f \in Y_0$ and all $u, v \in \mathcal{Y}_0$, and hence [Lemma 3.1](#) holds with X , Y and \mathcal{Y} replaced by X_0 , Y_0 and \mathcal{Y}_0 , respectively. Then, applying the implicit function theorem to the map $S : X_0 \times Y_0 \times \mathcal{Y}_0 \rightarrow \mathcal{Y}_0$ defined by [\(3.1\)](#), as in the proof of [Theorem 1.1](#), we obtain the desired self-similar solution under the condition [\(1.7\)](#). This proves [Corollary 1.1](#). \square

3.2. Local existence of solutions; Proof of [Theorem 1.2](#)

For construction of the mild solution locally on $(0, T_*)$ for some $0 < T_* \leq T$, we make use of the successive approximation $\{u_j\}_{j=0}^\infty$ as

$$\begin{aligned} u_0(t) &= e^{-tA}a + \int_0^t e^{-(t-\tau)A} Pf(\tau) d\tau, \\ u_{j+1}(t) &= u_0(t) - N(u_j, u_j)(t), \quad j = 0, 1, \dots \end{aligned}$$

Let us define $M_j = M_j(T_*)$ by

$$M_j(T_*) = \|u_j\|_{L^{s_0, \infty}(0, T_*; L^{q_0, \infty})} + \|\nabla u_j\|_{L^{s_1, \infty}(0, T_*; L^{q_1, \infty})}, \quad j = 0, 1, \dots$$

By [Lemma 2.1](#) with $p = n$ and [Lemmata 2.2 and 2.3](#), we have

$$M_{j+1} \leq M_0 + C_* M_j^2 \quad j = 0, 1, \dots, \quad (3.5)$$

where $C_* = C_*(n, q, q_0, q_1)$ is independent of j and T_* . Hence, if we may choose T_* so that

$$M_0 = M_0(T_*) < \frac{1}{4C_*}, \quad (3.6)$$

then it follows from [\(3.5\)](#) that

$$M_j \leq \frac{1 - \sqrt{1 - 4C_* M_0}}{2C_*} \quad \text{for all } j = 0, 1, \dots$$

By the standard method as in Kato [\[11\]](#), we see that such a bound of $\{M_j\}_{j=0}^\infty$ yields the desired mild solution u of [\(N-S\)](#) on $(0, T_*)$ in the class [\(1.18\)](#).

Hence, we may choose ϵ_0 in [\(1.17\)](#) and T_* in such a way that [\(3.6\)](#) holds. Indeed, it follows from [\(1.17\)](#) that there exist functions $a_0 \in L^{n, \infty}$ with $\|a\|_{L^{n, \infty}} < 2\epsilon_0$, $a_1 \in L^{n, \infty} \cap L^\infty$, $g_0 \in L^{s, \infty}(0, T)$ with $\|g_0\|_{L^{s, \infty}(0, T)} < 2\epsilon_0$ and $g_1 \in L^{s, \infty}(0, T) \cap L^\infty(0, T)$ such that

$$a(x) = a_0(x) + a_1(x), \quad \|f(t)\|_{L^{q, \infty}} = g_0(t) + g_1(t) \quad (3.7)$$

for all $x \in \mathbb{R}^n$ and all $t \in (0, T)$. By [Lemma 2.1](#), we have that

$$\|e^{-tA}Pa_0\|_{L^{s_0, \infty}(0, \infty; L^{q_0, \infty})} + \|\nabla e^{-tA}Pa_0\|_{L^{s_1, \infty}(0, \infty; L^{q_1, \infty})} \leq C\|a_0\|_{L^{n, \infty}} \leq C\epsilon_0, \quad (3.8)$$

where $C = C(n, q, q_0, q_1)$. Since $q_0 > n$, it holds $a_1 \in L^{q_0}$, and hence we have by (2.1) that

$$\|e^{-tA}Pa_1\|_{L^{s_0,\infty}(0,T_*;L^{q_0,\infty})} \leq \|e^{-tA}Pa_1\|_{L^{s_0}(0,T_*;L^{q_0})} \leq CT_*^{\frac{1}{s_0}}\|a_1\|_{L^{q_0}}, \quad (3.9)$$

where $C = C(n, q_0)$ is independent of T_* . Since $n < q_1 < \infty$ and since $2/s_1 + n/q_1 = 2$, we have $1 < s_1 < 2$, and hence it follows from (2.2) that

$$\begin{aligned} \|\nabla e^{-tA}Pa_1\|_{L^{s_1,\infty}(0,T_*;L^{q_1,\infty})} &\leq \|\nabla e^{-tA}Pa_1\|_{L^{s_1}(0,T_*;L^{q_1})} \\ &\leq C\|a_1\|_{L^{q_1}} \left(\int_0^{T_*} t^{-\frac{s_1}{2}} dt \right)^{\frac{1}{s_1}} \\ &\leq CT_*^{\frac{1}{s_1}-\frac{1}{2}}\|a_1\|_{L^{q_1}}, \end{aligned} \quad (3.10)$$

where $C = C(n, q_1)$ is independent of T_* . It is easy to see that

$$\|g_1\|_{L^{s,\infty}(0,T_*)} \leq \|g_1\|_{L^s(0,T_*)} \leq T_*^{\frac{1}{s}-\frac{1}{\alpha}}\|g_1\|_{L^\alpha(0,T)} \quad (3.11)$$

for some $\alpha > s$. Hence, it follows from (3.8)–(3.11) that

$$M_0(T_*) \leq C \left(\epsilon_0 + T_*^{\frac{1}{s_0}}\|a_1\|_{L^{q_0}} + T_*^{\frac{1}{s_1}-\frac{1}{2}}\|a_1\|_{L^{q_1}} + T_*^{\frac{1}{s}-\frac{1}{\alpha}}\|g_1\|_{L^\alpha(0,T)} \right), \quad (3.12)$$

where $C = C(n, q, q_0, q_1)$ is independent of T_* . Notice that all powers of T_* on the right hand side of (3.12) are positive. Now, taking first $\epsilon_0 < 1/(8C_*C)$ and then next choosing T_* so small that

$$T_*^{\frac{1}{s_0}}\|a_1\|_{L^{q_0}} + T_*^{\frac{1}{s_1}-\frac{1}{2}}\|a_1\|_{L^{q_1}} + T_*^{\frac{1}{s}-\frac{1}{\alpha}}\|g_1\|_{L^\alpha(0,T)} < 1/(8C_*C),$$

we see from (3.12) that (3.6) is achieved. This proves Theorem 1.2.

References

- [1] O.A. Barraza, Self-similar solutions in weak L^p -spaces of the Navier–Stokes equations, *Rev. Mat. Iberoam.* 12 (1996) 411–439.
- [2] J. Bourgain, N. Pavlović, Ill-posedness of the Navier–Stokes equations in a critical space in 3D, *J. Funct. Anal.* 255 (2008) 2233–2247.
- [3] M. Cannone, G. Karch, Smooth or singular solutions to the Navier–Stokes system, *J. Differential Equations* 197 (2004) 247–274.
- [4] M. Cannone, F. Planchon, Self-similar solutions for Navier–Stokes equations in \mathbb{R}^3 , *Comm. Partial Differential Equations* 21 (1996) 179–194.
- [5] M. Cannone, F. Planchon, On the nonstationary Navier–Stokes equations with an external force, *Adv. Differential Equations* 4 (1999) 697–730.
- [6] R. Farwig, H. Sohr, W. Varnhorn, Optimal initial value conditions for local strong solutions of the Navier–Stokes equations, *Ann. Univ. Ferrara* 55 (2009) 89–110.
- [7] H. Fujita, T. Kato, On the Navier–Stokes initial value problem I, *Arch. Ration. Mech. Anal.* 16 (1964) 269–315.
- [8] Y. Giga, T. Miyakawa, Solutions in L^r of the Navier–Stokes initial value problem, *Arch. Ration. Mech. Anal.* 89 (1985) 267–281.
- [9] Y. Giga, H. Sohr, Abstract L^p estimates for the Cauchy problem with applications to the Navier–Stokes equations in exterior domains, *J. Funct. Anal.* 102 (1991) 72–94.
- [10] T. Iwabuchi, Navier–Stokes equations and nonlinear heat equations in modulation spaces with negative derivative indices, *J. Differential Equations* 248 (2010) 1972–2002.
- [11] T. Kato, Strong L^p -solution of the Navier–Stokes equation in \mathbb{R}^m , with applications to weak solutions, *Math. Z.* 187 (1984) 471–480.
- [12] H. Koch, D. Tataru, Well-posedness for the Navier–Stokes equations, *Adv. Math.* 157 (2001) 22–35.

- [13] H. Kozono, M. Yamazaki, Semilinear heat equations and the Navier–Stokes equation with distributions in new function spaces as initial data, *Comm. Partial Differential Equations* 19 (1994) 959–1014.
- [14] H. Kozono, M. Yamazaki, Local and global unique solvability of the Navier–Stokes exterior problem with Cauchy data in the space $L^{n,\infty}$, *Houston J. Math.* 21 (1995) 755–799.
- [15] H. Kozono, M. Yamazaki, Uniqueness criterion of weak solutions to the stationary Navier–Stokes equations in exterior domains, *Nonlinear Anal. Ser. A, Theory Methods* 38 (1999) 959–970.
- [16] P.G. Lemarie-Rieusset, *The Navier–Stokes Problem in the 21st Century*, CRC Press, Boca Raton–London–New York, 2016.
- [17] P.L. Lions, N. Masmoudi, Uniqueness of mild solutions of the Navier–Stokes equations in L^N , *Comm. Partial Differential Equations* 26 (1996) 179–194.
- [18] T. Nakatsuka, On uniqueness of stationary solutions to the Navier–Stokes equations in exterior domains, *Nonlinear Anal. Ser. A, Theory Methods* 75 (2012) 3457–3464.
- [19] J. Serrin, The initial value problem for the Navier–Stokes equations, in: R.E. Ranger (Ed.), *Nonlinear Problems*, Univ. Wisconsin Press, 1963, pp. 69–98.
- [20] E.M. Stein, G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton, N.J., 1971.
- [21] B. Wang, Ill-posedness for the Navier–Stokes equations in critical Besov spaces $\dot{B}_{\infty,q}^{-1}$, *Adv. Math.* 268 (2015) 350–372.
- [22] M. Yamazaki, The Navier–Stokes equations in the weak- L^n space with time-dependent external force, *Math. Ann.* 317 (2000) 635–675.
- [23] T. Yoneda, Ill-posedness of the 3D-Navier–Stokes equations in Besov spaces near BMO^{-1} , *J. Funct. Anal.* 258 (2010) 3376–3387.