J. Math. Anal. Appl. 464 (2018) 546–556

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

r-Almost Newton–Ricci solitons immersed into a Riemannian manifold

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ARTICLE INFO

Article history:
Received 15 January 2018
Available online 11 April 2018
Submitted by J. Lenells

Keywords:
Space forms
r-Almost Newton–Ricci solitons
Totally geodesic hypersurfaces
Locally symmetric spaces
Einstein manifolds

ABSTRACT

We establish the new concept of r-almost Newton–Ricci soliton for hypersurfaces immersed in a Riemannian manifold, which involves the r-th Newton and the Ricci tensors and extends in a natural way the notion of immersed almost Ricci solitons introduced by Barros et al. [3]. In this setting, our purpose is to investigate the existence of these geometric objects. After exhibit some examples of r-almost Newton–Ricci solitons, we obtain sufficient conditions to guarantee that they must be totally geodesic under suitable constraints on the potential function and using appropriate maximum principles. Furthermore, a particular study of 1-almost Newton–Ricci solitons immersed in a locally symmetric Einstein manifold is also made.

1. Introduction

The concept of Ricci soliton, which was introduced by Hamilton in his seminal work [12], corresponds to a natural generalization of the definition of Einstein metric. Let us recall that an n-dimensional Riemannian manifold (M^n, g) is said to be a *Ricci soliton* if there exist a complete vector field X on M^n and a constant $\lambda \in \mathbb{R}$ satisfying the following equation

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda g, \tag{1.1}$$

where Ric and $\mathcal{L}$ stand for the Ricci tensor and the Lie derivative on M^n, respectively.

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https://doi.org/10.1016/j.jmaa.2018.04.026
0022-247X/

We note that Ricci solitons also corresponds to selfsimilar solutions of Hamilton's Ricci flow [12] and often arise as limits of dilations of singularities in the Ricci flow. They can be viewed as fixed points of the Ricci flow, as a dynamical system on the space of Riemannian metrics modulo diffeomorphisms and scalings. For more details on this subject, we recommend the survey due to Cao [7] and the references therein.

Recently Pigola et al. [16] extended the definition of Ricci solitons by adding the condition on the parameter λ in (1.1) to be a real smooth function on M^n, attracting a lot of attention in the mathematical community. It is worth to remark that they arise from the Ricci–Bourguignon flow as discovered recently by Catino et al. [9]. In this more general setting, we refer to equation (1.1) as being the fundamental equation of an *almost Ricci soliton* (M^n, g, X, λ). For $\lambda > 0$ the almost Ricci soliton is *shrinking*, for $\lambda = 0$ it is *steady*, and for $\lambda < 0$ it is *expanding*. Otherwise, it is called *indefinite*.

In the particular case where the vector field X is the gradient of a smooth function $f : M^n \to \mathbb{R}$, the manifold will be called a *gradient almost Ricci soliton*. The function f is called the *potential function* of the gradient almost Ricci soliton. In this case, it is easy to see that equation (1.1) becomes

$$\mathrm{Ric} + \mathrm{Hess} f = \lambda g, \tag{1.2}$$

where $\mathrm{Hess} f$ stands for the Hessian of the potential function f. The almost gradient Ricci soliton equation (1.2) links geometric information about the curvature of the manifold through the Ricci tensor and the geometry of the level sets of the potential function by means of their second fundamental form. Hence, classifying almost gradient Ricci solitons under some curvature conditions is a natural problem.

In [3], Barros et al. studied isometric immersions of an almost Ricci soliton (M^n, g, X, λ) into a Riemannian manifold $\overline{M}^{n+p}$. In this context, they presented some obstruction results in order to obtain a minimal immersion under conditions on the sectional curvature of $\overline{M}^{n+p}$. In particular, when $\overline{M}^{n+p}$ has nonpositive sectional curvature, they proved that if (M^n, g, X, λ) is a traditional Ricci soliton and X has integrable norm on M^n, then M^n cannot be minimal. Moreover, it was showed by Wylie [20] that if (M^n, g, X, λ) is a shrinking Ricci soliton, with X having bounded norm on M^n, then M^n must be compact. In particular, when $\overline{M}^{n+p}$ is a space form of nonpositive sectional curvature, then such an immersion cannot be minimal. We refer to [14] for further discussions about this subject.

Here, we introduce the new concept of r-almost Newton–Ricci soliton and study the properties of this new object. Our approach is based on the use of the so-called Newton transformations P_r and their associated second order differential operators L_r (see Section 2 for more details), which allows us to extend in a natural way the immersed almost Ricci solitons introduced in [3]. For this, let $\varphi : M^n \to \overline{M}^{n+1}$ be an oriented hypersurface immersed into an $(n+1)$-dimensional Riemannian manifold $\overline{M}^{n+1}$. We say that M^n is an *r-almost Newton–Ricci soliton*, for some $0 \le r \le n$, if there exist a smooth function $f : M^n \to \mathbb{R}$ such that the following equation holds:

$$\mathrm{Ric} + P_r \circ \mathrm{Hess} f = \lambda g, \tag{1.3}$$

where λ is a smooth function on M^n and $P_r \circ \mathrm{Hess} f$ stands for the tensor given by

$$P_r \circ \mathrm{Hess} f(X, Y) = \langle P_r \nabla_X \nabla f, Y \rangle,$$

for tangent vector fields $X, Y \in \mathfrak{X}(M)$. In particular, when $r = 0$ we recover the definition of a gradient almost Ricci soliton.

This manuscript is organized in the following way: In Section 2 we recall some basic facts and notations that will appear along the paper. Afterwards, in Section 3 we exhibit some examples of immersions satisfying the equation of r-almost Newton–Ricci solitons (1.3), for every $1 \le r \le n$, and we establish our first main results concerning the existence of these geometric objects. Finally, in Section 4 we consider the particular case of 1-almost Newton–Ricci solitons immersed in a locally symmetric Einstein manifold.

2. Preliminaries

Let $\varphi : M^n \to \overline{M}^{n+1}$ be an oriented and connected hypersurface immersed into an $(n+1)$-dimensional Riemannian manifold $\overline{M}^{n+1}$. It is well known that the Gauss equation of the immersion is given by

$$R(X,Y)Z = (\overline{R}(X,Y)Z)^\top + \langle AX, Z\rangle AY - \langle AY, Z\rangle AX$$

for every tangent vector fields $X, Y, Z \in \mathfrak{X}(M)$, where $(\ )^\top$ denotes the tangential component of a vector field in $\mathfrak{X}(\overline{M})$ along M^n. Here, $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ stands for the second fundamental form (or shape operator) of M^n in $\overline{M}^{n+1}$ with respect to a fixed orientation, $\overline{R}$ and R denote the curvature tensors of $\overline{M}^{n+1}$ and M^n, respectively. In particular, the scalar curvature S of the hypersurface M^n satisfies

$$S = \sum_{i,j}^{n} \langle \overline{R}(e_i, e_j)e_j, e_i\rangle + n^2H^2 - |A|^2, \tag{2.1}$$

where $\{e_1, \dots, e_n\}$ is an orthonormal frame on TM and $|\cdot|$ denotes the Hilbert–Schmidt norm. When $\overline{M}^{n+1}$ is a space form of constant sectional curvature c, we have the identity

$$S = n(n-1)c + n^2H^2 - |A|^2. \tag{2.2}$$

Associated to second fundamental form A of the hypersurface M^n there are n algebraic invariants, which are the elementary symmetric functions S_r of its principal curvatures $k_1, \dots, k_n$, given by

$$S_0 = 1 \quad \text{and} \quad S_r = \sum_{i_1 < \dots < i_r} k_{i_1} \cdots k_{i_r}.$$

The r-th mean curvature H_r of the immersion is defined by

$$\binom{n}{r} H_r = S_r.$$

In the case $r = 1$, we have $H_1 = \frac{1}{n} tr(A) = H$ the mean curvature of M^n.

For each $0 \le r \le n$, one defines the r-th *Newton transformation* $P_r : \mathfrak{X}(M) \to \mathfrak{X}(M)$ of the hypersurface M^n by setting $P_0 = I$ (the identity operator) and, for $1 \le r \le n$, via the recurrence relation

$$P_r = \sum_{j=0}^{r} (-1)^{r-j} \binom{n}{j} H_j A^{r-j}, \tag{2.3}$$

where $A^{(j)}$ denotes the composition of A with itself, j times ($A^{(0)} = I$). Let us remember that associated to each Newton transformation P_r one has the second order linear differential operator $L_r : C^\infty(M) \to C^\infty(M)$ defined by

$$L_r u = \operatorname{tr}(P_r \circ \text{Hess } u).$$

When $r = 0$, we note that L_0 is just the Laplacian operator. Moreover, it is not difficult to see that

$$\begin{aligned} \operatorname{div}_M(P_r \nabla u) &= \sum_{i=1}^{n} \langle (\nabla_{e_i} P_r)(\nabla u), e_i\rangle + \sum_{i=1}^{n} \langle P_r(\nabla_{e_i} \nabla u), e_i\rangle \\ &= \langle \operatorname{div}_M P_r, \nabla u\rangle + L_r u, \end{aligned} \tag{2.4}$$

where the divergence of P_r on M^n is given by

$$\mathrm{div}_M P_r = \mathrm{tr}(\nabla P_r) = \sum_{i=1}^{n} (\nabla_{e_i} P_r)(e_i).$$

In particular, when the ambient space has constant sectional curvature equation (2.4) reduces to $L_r u = \mathrm{div}_M(P_r \nabla u)$, because $\mathrm{div}_M P_r = 0$ (see [17] for more details).

For our purpose, it also will be appropriate to deal with the so-called traceless second fundamental form of the hypersurface, which is given by $\Phi = A - HI$. Observe that $\mathrm{tr}\ \Phi = 0$ and $|\Phi|^2 = \mathrm{tr}(\Phi^2) = |A|^2 - nH^2 \geq 0$, with equality if and only if M^n is totally umbilical.

3. On the existence of r-almost Newton–Ricci solitons

Before present our first main results, we will exhibit some examples of immersions satisfying the equation of r-almost Newton–Ricci solitons (1.3).

Example 1. We consider the standard immersion of $\mathbb{S}^n$ into $\mathbb{S}^{n+1}$, which we know that is totally geodesic. In particular, $P_r \equiv 0$ for all $1 \leq r \leq n$, and choosing $\lambda = (n-1)/n$, we obtain that the immersion satisfies equation (1.3).

Example 2. Let us consider the standard immersion of the n-sphere $\mathbb{S}^n$ into Euclidean space $\mathbb{R}^{n+1}$ endowed with induced metric g. According to [4], by choosing the functions

$$\lambda_a(x) = -\langle x, a\rangle + n - 1 \text{ and } f_a(x) = -\lambda_a + c,$$

where $a \in \mathbb{R}^{n+1}$, $a \neq 0$, $c \in \mathbb{R}$ and $x = (x_1, \ldots, x_{n+1}) \in \mathbb{S}^n$ is the position vector, we have that $(\mathbb{S}^n, g, \nabla f_a, \lambda_a)$ satisfies

$$\mathrm{Ric} + \mathrm{Hess} f_a = \lambda_a g.$$

On the other hand, it is well known that $\mathbb{S}^{n+1}$ is totally umbilical with r-th mean curvature $H_r = 1$ and second fundamental form $A = I$. In particular, for every $0 \leq r \leq n-1$ the Newton tensors are given by

$$P_r = \alpha I,$$

where $\alpha = \sum_{j=0}^{r} (-1)^{r-j} \binom{n}{j}$. Hence, taking the smooth function $f = \alpha^{-1} f_a$ we get that the immersion satisfies equation (1.3).

Example 3. We recall that the *Gaussian soliton* is the Euclidean space $\mathbb{R}^n$ endowed with its standard metric $|\,|$ and the potential function $f(x) = \dfrac{\lambda}{4}|x|^2$ (see, for instance, Section 4.2 of [10]). On the other hand, it is well known that the horospheres of the hyperbolic space $\mathbb{H}^{n+1}$ are totally umbilical hypersurfaces isometric to $\mathbb{R}^n$, having r-th mean curvature $H_r = 1$ and second fundamental form $A = I$. Hence, we can reason as in Example 2 to verify that the horospheres $\mathbb{R}^n \hookrightarrow \mathbb{H}^{n+1}$ satisfy equation (1.3).

In order to state and prove our first result let us quote the following maximum principle due to Caminha et al. [6] (for more details, see Proposition 1 of [6]). In what follows, for each $p \geq 1$ we will use the notation

$$L^p(M) = \{u : M^n \to \mathbb{R}\ ;\ \int_M |u|^p dM < +\infty\}.$$

Lemma 1. *Let X be a smooth vector field on the n-dimensional, complete, noncompact, oriented Riemannian manifold M^n, such that $\operatorname{div}_M X$ does not change sign on M^n. If $|X| \in L^1(M)$, then $\operatorname{div}_M X = 0$.*

The following result extends Theorem 1.2 in [3].

Theorem 1. *Let M^n be a complete r-almost Newton–Ricci soliton immersed into a Riemannian manifold M_c^{n+1} of constant sectional curvature c, with bounded second fundamental form and potential function $f : M^n \to \mathbb{R}$ such that $|\nabla f| \in L^1(M)$. We have:*

(i) *If $c \leq 0$ and $\lambda > 0$, then M^n can not be minimal;*
(ii) *If $c < 0$ and $\lambda \geq 0$, then M^n can not be minimal;*
(iii) *If $c = 0$, $\lambda \geq 0$ and M^n is minimal, then M^n is isometric to the $\mathbb{R}^n$.*

Proof. For its further reference, we observe that since the ambient space has constant sectional curvature, by equation (2.4) the operator L_r is a divergent type operator. On the other hand, because M^n has bounded second fundamental form it follows from (2.3) that the Newton transformation P_r has bounded norm. In particular,

$$|P_r \nabla f| \leq |P_r||\nabla f| \in L^1(M).$$

Regarding to (i) and (ii), let us assume by contradiction that M^n is minimal. Then, equation (2.2) jointly with the assumption $c \leq 0$ ($c < 0$) imply that the scalar curvature of M^n satisfies $S \leq 0$ ($S < 0$). Hence, contracting equation (1.3) we have $L_r f = n\lambda - S > 0$ in both the cases, which contradicts Lemma 1, since the remark aforementioned. This completes the proof of the two first assertions.

For the third claim, because the ambient space has constant sectional curvature $c = 0$ and M^n is minimal, we deduce from equation (2.2) that

$$S = -|A|^2 \leq 0.$$

So, since $\lambda \geq 0$ we have that $L_r(f) = n\lambda - S \geq 0$. By using that $L_r u = \operatorname{div}_M(P_r \nabla u)$ and $|P_r \nabla f| \in L^1(M)$ we have once more from Lemma 1 that $L_r f = 0$ on M^n. Hence, we conclude that $0 \geq S = \lambda n \geq 0$, that is, $S = \lambda = 0$. This implies that $|A|^2 = 0$. Therefore, the r-almost Newton–Ricci soliton M^n must be totally geodesic and flat. $\square$

In order to obtain our next result we will need of the following key lemma, which corresponds to Theorem 3 of [22].

Lemma 2. *Let u be a nonnegative smooth subharmonic function on a complete Riemannian manifold M^n. If $u \in L^p(M)$, for some $p > 1$, then u is constant.*

Next, we are in condition to prove the following result, which holds when the ambient space is an arbitrary Riemannian manifold.

Theorem 2. *Let M^n be a complete r-almost Newton–Ricci soliton immersed into a Riemannian manifold $\overline{M}^{n+1}$ of sectional curvature $\overline{K}$, such that P_r is bounded from above (in the sense of quadratic forms) and its potential function $f : M^n \to \mathbb{R}$ is nonnegative and $f \in L^p(M)$ for some $p > 1$. We have:*

(i) *If $\overline{K} \leq 0$ and $\lambda > 0$, then M^n can not be minimal;*
(ii) *If $\overline{K} < 0$ and $\lambda \geq 0$, then M^n can not be minimal;*
(iii) *If $\overline{K} \leq 0$, $\lambda \geq 0$ and M^n is minimal, then M^n is flat and totally geodesic.*

Proof. Let us prove (i). Supposing by contradiction that M^n is minimal, our assumption on the sectional curvature of the ambient space and equation (2.1) imply that $S \le 0$. Hence, contracting equation (1.3) we have $L_r f = n\lambda - R > 0$. Thus, since we are assuming that P_r is bounded from above, there exists a positive constant β such that

$$\beta \Delta f \ge L_r f > 0.$$

In particular, from Lemma 2 we get that f must be constant, which gives a contradiction. Finally, reasoning as in the proof of Theorem 1 we conclude (ii) and (iii). □

Remark 1. Let us recall that a Riemannian manifold M^n is said to be *parabolic* if the only subharmonic functions $f \in C^\infty(M)$ with $\sup_M f < +\infty$ are the constant ones. In Theorem 2, we can replace the hypothesis that f is nonnegative and $f \in L^p(M)$, for some $p > 1$, by assuming that f is bounded from above and M^n being parabolic. It was observed by Colding and Minicozzi [11] that a complete surface M^2 satisfying quadratic area growth $\mathrm{Vol}(B_s^M) \le Cs^2$ must be parabolic, where B_s^M denotes an intrinsic geodesic ball in M^2. Furthermore, it was showed in [8] by Cao and Zhou that in a shrinking gradient Ricci soliton there is a uniform constant $C > 0$ so that $\mathrm{Vol}(B_s^M) \le Cs^n$. Hence, all 2-dimensional shrinking gradient Ricci soliton M^2 must be parabolic.

In our next result we extended Theorem 1.5 of [3] for the case when $X = \nabla f$, giving conditions for an r-almost Newton–Ricci soliton immersed be totally umbilical since it has bounded second fundamental form. More precisely, we prove the following

Theorem 3. *Let M^n be a complete r-almost Newton–Ricci soliton immersed into a Riemannian manifold $\overline{M}_c^{n+1}$ of constant sectional curvature c, with bounded second fundamental form and potential function $f : M^n \to \mathbb{R}$ such that $|\nabla f| \in L^1(M)$. Then:*

(i) *If $\lambda \ge (n-1)c + nH^2$, then M^n is totally geodesic, with $\lambda = (n-1)c$ and scalar curvature $S = n(n-1)c$;*
(ii) *If M^n is compact and $\lambda \ge (n-1)c + nH^2$, then M^n is isometric to a Euclidean sphere;*
(iii) *If $\lambda \ge (n-1)(c + H^2)$, then M^n is totally umbilical. In particular, the scalar curvature $S = n(n-1)K_M$ is constant, where $K_M = \frac{\lambda}{n-1}$ is the sectional curvature of M^n.*

Proof. To prove (i), we observe that by definition of r-almost Newton–Ricci soliton jointly with equation (2.2) we find

$$L_r f = n(\lambda - (n-1)c - nH^2) + |A|^2. \tag{3.1}$$

Then, from our assumption on λ we get that $L_r f$ is a nonnegative function on M^n. In particular, by Lemma 1 we obtain that $L_r f$ vanishes identically. Hence, from equation (3.1) we conclude that M^n is totally geodesic and $\lambda = (n-1)c$. Moreover, it is clear from equation (2.2) that $S = n(n-1)c$, which proves the claim (i).

If M^n is compact, as it is totally geodesic, then the ambient space must be necessarily a sphere $\mathbb{S}^{n+1}$ and M^n is isometric to the Euclidean sphere $\mathbb{S}^n$, proving (ii).

For the third assertion, let us begin observing that equation (3.1) can be written in terms of the traceless second fundamental form Φ as

$$L_r f = n(\lambda - (n-1)(c + H^2)) + |\Phi|^2. \tag{3.2}$$

So, our hypothesis on λ gives $L_r f \ge 0$. Then, by applying Lemma 1 once more we have $L_r(f) = 0$. This implies that $|\Phi|^2 = 0$, that is, M^n is a totally umbilical hypersurface. In particular, the principal curvature

κ of M^n is constant and M^n has constant sectional curvature given by $K_M = c + \kappa^2$. This jointly with equation (3.2) assures that

$$\lambda = (n-1)(c + H^2) = (n-1)(c + \kappa^2) = (n-1)K_M,$$

which implies that $S = n(n-1)K_M$, as desired. $\square$

From Theorem 3 we get the following consequence

Corollary 1. *Let M^n be a compact r-almost Newton–Ricci soliton immersed into $\mathbb{R}^{n+1}$. If $\lambda \geq (n-1)H^2$, then M^n is isometric to $\mathbb{S}^n$.*

Theorem 1.6 of [3] asserts that a nontrivial almost Ricci soliton M^n, minimally immersed in $\mathbb{S}^{n+1}$ with $S \geq n(n-2)$ and such that the norm of the second fundamental form attains its maximum, must be isometric to $\mathbb{S}^n$. Applying Theorem 3 we obtain an extension of this result.

Corollary 2. *Let M^n be a complete r-almost Newton–Ricci soliton minimally immersed in $\mathbb{S}^{n+1}$. Suppose that $S \geq n(n-2)$, the norm of the second fundamental form attains its maximum and $\lambda \geq n-1$. Then, M^n is isometric to $\mathbb{S}^n$.*

Proof. Since the immersion is minimal with $S \geq n(n-2)$, from (2.2) we get that

$$|A|^2 = n(n-1) - S \leq n.$$

From Simons' formula [18], we obtain

$$\Delta|A|^2 = |\nabla A|^2 + (n - |A|^2)|A|^2 \geq 0.$$

Thus, we can apply Hopf's strong maximum principle to get that $\nabla A = 0$ on M^n. Therefore, Proposition 1 of [13] assures that M^n must be compact and, hence, the result follows from Theorem 3. $\square$

As another application of Lemma 2 we also get

Theorem 4. *Let M^n be a complete r-almost Newton–Ricci soliton immersed into a Riemannian manifold $\overline{M}_c^{n+1}$ of constant sectional curvature c, such that P_r is bounded from above (in the sense of quadratic forms) and its potential function $f : M^n \to \mathbb{R}$ is nonnegative and $f \in L^p(M)$ for some $p > 1$. Then:*

(i) *If $\lambda \geq (n-1)c + nH^2$, then M^n is totally geodesic, with $\lambda = (n-1)c$ and scalar curvature $S = n(n-1)c$.*
(ii) *If $\lambda \geq (n-1)(c + H^2)$, then M^n is totally umbilical. In particular, the scalar curvature $S = n(n-1)K_M$ is constant, where $K_M = \frac{\lambda}{n-1}$ is the sectional curvature of M^n.*

Proof. Let us begin observing that by equation (3.1) and assumption on λ we get

$$L_r f = n(\lambda - (n-1)c - nH^2) + |A|^2 \geq 0. \tag{3.3}$$

Since we are assuming that P_r is bounded from above, there is a positive constant β such that

$$\beta \Delta f \geq L_r f \geq 0.$$

By Lemma 2, we have that f must be constant. Therefore $L_r f = 0$, and by equation (3.3) we conclude that M^n is totally geodesic, $\lambda = (n-1)c$ and $S = n(n-1)c$, proving item (i). Finally, reasoning as in Theorem 3, it is not difficult to prove item (ii). □

4. 1-almost Newton–Ricci solitons in locally symmetric spaces

In this section we will extend our previous results to the case of 1-almost Newton–Ricci solitons immersed into a locally symmetric Riemannian manifold. To this end, we will work with locally symmetric spaces obeying a standard curvature constraint.

Before stating the main results, let us recall that a Riemannian manifold is said to be *locally symmetric* when all the covariant derivative components of its curvature tensor vanish identically. In this setting, such spaces consist in an interesting generalization of constant curvature spaces. Hence, it is a natural question to revisit in this ambient spaces the known results of constant curvature spaces.

Proceeding, let M^n be a hypersurface immersed into a locally symmetric Riemannian manifold $\overline{M}^{n+1}$. In what follows we introduce our curvature constraint, which will be assumed in the main results of this section. More specifically, we will assume that there is a constant c_1 such that the sectional curvature $\overline{K}$ of the ambient space $\overline{M}^{n+1}$ satisfies the following equality:

$$\overline{K}(\eta, v) = \frac{c_1}{n}, \tag{4.1}$$

for vectors $\eta \in T^\perp M$ and $v \in TM$.

Remark 2. As mentioned above, a Riemannian manifold $\overline{M}_c^{n+1}$ of constant sectional curvature c is a locally symmetric space and it is easy to see that the curvature condition (4.1) is satisfied for every hypersurface Σ^n immersed into $\overline{M}_c^{n+1}$, with $c_1/n = c$. Therefore, in some sense our assumption is a natural generalization of the case where the ambient space has constant sectional curvature. Moreover, when the ambient manifold is a Riemannian product of two Riemannian manifolds of constant sectional curvature, say $\overline{M} = M_1(k_1) \times M_2(k_2)$, then $\overline{M}$ is also locally symmetric and, if $k_1 = 0$ and $k_2 \geq 0$, then every hypersurface of the type $\Sigma = \Sigma_1 \times M_2(k_2)$, where Σ_1 is an orientable and connected hypersurface immersed in $M_1(k_1)$, satisfies the curvature constraint (4.1) with $c_1 = 0$ (for more details, see Remark 3.1 of [1]).

Let $\overline{M}^{n+1}$ be a locally symmetric Riemannian manifold satisfying condition (4.1) and let $\{e_1, \dots, e_{n+1}\}$ be an orthonormal frame on $T\overline{M}$. Then, its scalar curvature $\overline{S}$ is given by

$$\begin{aligned}
\overline{S} &= \sum_{i=1}^{n+1} \overline{\mathrm{Ric}}(e_i, e_i) \\
&= \sum_{i,j=1}^{n} \langle \overline{R}(e_i, e_j)e_i, e_j\rangle + 2\sum_{i=1}^{n} \langle \overline{R}(e_{n+1}, e_i)e_{n+1}, e_i\rangle \\
&= \sum_{i,j=1}^{n} \langle \overline{R}(e_i, e_j)e_i, e_j\rangle + 2c_1.
\end{aligned}$$

Moreover, it is well known that the scalar curvature of a locally symmetric Riemannian manifold is constant. Thus, $\sum_{i,j=1}^{n} \langle \overline{R}(e_i, e_j)e_i, e_j\rangle$ is a constant naturally attached to a locally symmetric Riemannian manifold satisfying condition (4.1). So, for the sake of simplicity, we will adopt the following notation $\overline{\mathcal{S}} := \frac{1}{n(n-1)} \sum_{i,j=1}^{n} \langle \overline{R}(e_i, e_j)e_i, e_j\rangle$. It is worth pointing out that when $\overline{M}^{n+1}$ is a space of constant sectional curvature, then the constant $\overline{\mathcal{S}}$ agrees with its sectional curvature.

Before to present our next results, we exhibit an example of r-almost Newton–Ricci soliton immersed in a locally symmetric Riemannian manifold satisfying the curvature constraint (4.1).

Example 4. Since the canonical immersion $\mathbb{S}^n \hookrightarrow \mathbb{S}^n \times \mathbb{R}$ is totally geodesic, proceeding as in Example 1 we see that this immersion satisfies equation (1.3) for all $1 \le r \le n$ and $\lambda = (n-1)/n$.

The following result extends Theorem 1 for the context of 1-almost Newton–Ricci soliton immersed in a locally symmetric Einstein manifold.

Theorem 5. *Let $\overline{M}^{n+1}$ be a locally symmetric Einstein manifold satisfying the curvature condition (4.1). Let M^n be a complete 1-almost Newton–Ricci soliton immersed into $\overline{M}^{n+1}$ with bounded second fundamental form and potential function $f : M^n \to \mathbb{R}$ such that $|\nabla f| \in L^1(M)$. We have:*

(i) *If $\overline{\mathcal{S}} \le 0$ and $\lambda > 0$, then M^n can not be minimal;*
(ii) *If $\overline{\mathcal{S}} < 0$ and $\lambda \ge 0$, then M^n can not be minimal;*
(iii) *If $\overline{\mathcal{S}} = 0$, $\lambda \ge 0$ and M^n is minimal, then M^n is totally geodesic.*

Proof. To prove (i), let us reason as in the proof of Theorem 1 assuming by contradiction that M^n is minimal. Then, by our hypothesis on the constant $\overline{\mathcal{S}}$ we get from equation (2.1) that the scalar curvature of M^n satisfies $S \le 0$, which implies $L_r f = n\lambda - S > 0$.

On the other hand, we recall from the discussion in the Preliminaries that the differential operator L_1 satisfies

$$L_1 f = \mathrm{div}_M(P_1 \nabla f) - \langle \mathrm{div}_M P_1, \nabla f\rangle. \tag{4.2}$$

In particular, taking an orthonormal frame $\{e_1, \ldots, e_n\}$ on TM and denoting by N the orientation of M^n, it follows from Lemma 25 of [2] that

$$\langle \mathrm{div}_M P_1, \nabla f\rangle = \sum_{i=1}^{n} \langle \overline{R}(N, e_i)\nabla f, e_i\rangle = \overline{\mathrm{Ric}}(N, \nabla f).$$

Hence, since $\overline{M}^{n+1}$ is assumed to be Einstein we conclude by equation (4.2) jointly with the previous identity that

$$L_1 f = \mathrm{div}_M(P_1 \nabla f).$$

Moreover, as was observed in Theorem 1 we get from our assumption on second fundamental form that $|P_r \nabla f| \in L^1(M)$. Therefore, we are in position to apply Lemma 1 to conclude that $L_r f = 0$, which gives a contradiction.

Finally, reasoning as above it is not difficult to prove (ii) and (iii). $\square$

Remark 3. Taking into account Theorem 5, it is natural to ask oneself about the existence of Einstein manifolds which are locally symmetric. In this direction, Tod [19] showed that four-dimensional Einstein manifolds which are also D'Atri spaces are necessarily locally symmetric. More recently, Brendle [5] proved that a compact Einstein manifold of dimension $n \ge 4$ having nonnegative isotropic curvature must be locally symmetric, extending a previous result of Micallef and Wang for $n = 4$ (see Theorem 4.4 of [15]). See also [21] for another sufficient conditions for an Einstein manifold to be locally symmetric.

We continue obtaining an analogous result to Theorem 3 in the case where $r = 1$ and the ambient space is a locally symmetric space. More precisely, we get the following

Theorem 6. *Let $\overline{M}^{n+1}$ be a locally symmetric Einstein manifold satisfying the curvature condition (4.1). Let M^n be a complete 1-almost Newton–Ricci soliton immersed into $\overline{M}^{n+1}$ with bounded second fundamental form and potential function $f : M^n \to \mathbb{R}$ such that $|\nabla f| \in L^1(M)$. Then:*

(i) *If $\lambda \geq (n-1)\overline{\mathcal{S}}+nH^2$, then M^n is totally geodesic, with $\lambda = (n-1)\overline{\mathcal{S}}$ and scalar curvature $S = n(n-1)\overline{\mathcal{S}}$;*
(ii) *If $\lambda \geq (n-1)(\overline{\mathcal{S}} + H^2)$, then M^n is totally umbilical. In particular, the scalar curvature $S = n(n-1)(\overline{\mathcal{S}} + \kappa^2)$ is constant, where κ is the principal curvature of M^n.*

Proof. The result follows as in the proof of Theorem 3. For the sake of completeness, we give the following argument that proves (i). Taking trace in (1.3) and using the definition of the constant $\overline{\mathcal{S}}$, we obtain from equation (2.1) that

$$L_r f = n(\lambda - (n-1)\overline{\mathcal{S}} - nH^2) + |A|^2, \tag{4.3}$$

which implies that $L_1 f \geq 0$ because our hypothesis on λ. Then, by Lemma 1 we get that $L_r f = 0$. Therefore, we conclude from equation (4.3) that M^n is totally geodesic with $\lambda = (n-1)\overline{\mathcal{S}}$ and $S = n(n-1)\overline{\mathcal{S}}$, proving the result. □

We close our paper quoting the following result, which can be obtained from the similar arguments used in the proofs of Theorems 4 and 6.

Theorem 7. *Let $\overline{M}^{n+1}$ be a locally symmetric Einstein manifold satisfying the curvature condition (4.1). Let M^n be a complete 1-almost Newton–Ricci soliton immersed into $\overline{M}^{n+1}$ such that P_r is bounded from above (in the sense of quadratic forms), its potential function $f : M^n \to \mathbb{R}$ is nonnegative and $f \in L^p(M)$ for some $p > 1$. Then:*

(i) *If $\lambda \geq (n-1)\overline{\mathcal{S}}+nH^2$, then M^n is totally geodesic, with $\lambda = (n-1)\overline{\mathcal{S}}$ and scalar curvature $S = n(n-1)\overline{\mathcal{S}}$.*
(ii) *If $\lambda \geq (n-1)(\overline{\mathcal{S}} + H^2)$, then M^n is totally umbilical. In particular, the scalar curvature $S = n(n-1)(\overline{\mathcal{S}} + \kappa^2)$ is constant, where κ is the principal curvature of M^n.*

Remark 4. We observe that in Theorems 5, 6 and 7 we can replace the hypothesis that the ambient space $\overline{M}^{n+1}$ is Einstein by the weaker assumption that the tensor $\overline{\mathrm{Ric}}_N : \mathfrak{X}(M) \to \mathfrak{X}(M)$, given by $\overline{\mathrm{Ric}}_N(X) = \overline{\mathrm{Ric}}(N, X)$, is identically zero. With this improvement, the configuration of Example 4 is contemplated by these theorems.

Acknowledgments

The first author would like to thank to Barnabé Lima and Rondinelle Batista for some comments and valuable suggestions and he also wishes to express his gratitude for the excellent support during his visit at Department of Mathematics – Universidade Federal de Campina Grande-PB, where part of this paper was carried out. The third author is partially supported by CNPq, Brazil, grant 303977/2015-9.

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