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Compact homomorphisms between algebras of $C(K)$-valued Lipschitz functions

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ABSTRACT

We give a complete description of homomorphisms between two Banach algebras of Lipschitz functions with values in continuous functions. We also characterize the compactness of those homomorphisms.

1. Introduction

Let X be a compact metric space with metric d_X and $\mathcal{A}$ a commutative Banach algebra with norm $\|\cdot\|_{\mathcal{A}}$. By $C(X,\mathcal{A})$, we denote the Banach algebra of all $\mathcal{A}$-valued continuous functions on X, with norm

$$\|f\|_{C(X,\mathcal{A})} = \sup\{\|f(x)\|_{\mathcal{A}} : x \in X\}.$$

If an $\mathcal{A}$-valued function f on X satisfies

$$\mathcal{L}_{X,\mathcal{A}}(f) = \sup_{\substack{x,x' \in X \\ x \neq x'}} \frac{\|f(x) - f(x')\|_{\mathcal{A}}}{d_X(x,x')} < \infty,$$

then we say that f is a *Lipschitz function*. By $\mathrm{Lip}(X,\mathcal{A})$, we denote the set of all $\mathcal{A}$-valued Lipschitz functions on X. Clearly, $\mathrm{Lip}(X,\mathcal{A}) \subset C(X,\mathcal{A})$ and $\mathrm{Lip}(X,\mathcal{A})$ is a Banach algebra with norm

$$\|f\|_{\mathrm{Lip}(X,\mathcal{A})} = \|f\|_{C(X,\mathcal{A})} + \mathcal{L}_{X,\mathcal{A}}(f).$$

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In case that $\mathcal{A} = \mathbb{C}$, we write $C(X) = C(X, \mathbb{C})$ and $\mathrm{Lip}(X) = \mathrm{Lip}(X, \mathbb{C})$. The Lipschitz algebra $\mathrm{Lip}(X)$ has been well studied. The researches on this subject may be found in the book [7]. Here a mapping between two Banach algebras is said to be a *homomorphism* if it preserving addition, scalar multiplication and multiplication. Moreover if it maps unit to unit, then we say that it is *unital*. In [6], Sherbert characterized unital homomorphisms between Lipschitz algebras.

Theorem A *(Sherbert, [6]). Suppose that X and Y are compact metric spaces with metrics d_X and d_Y respectively. Then T is a unital homomorphism from $\mathrm{Lip}(X)$ into $\mathrm{Lip}(Y)$, if and only if there exists a mapping $\varphi : Y \to X$ with*

$$\sup_{\substack{y,y' \in Y \\ y \neq y'}} \frac{d_X(\varphi(y), \varphi(y'))}{d_Y(y, y')} < \infty,$$

such that

$$(Tf)(y) = f(\varphi(y)) \qquad (y \in Y)$$

for all $f \in \mathrm{Lip}(X)$.

This theorem has been developed in several directions. In [1], F. Botelho and J. Jamison replaced $\mathrm{Lip}(X)$ by $\mathrm{Lip}(X, \mathcal{A})$, where $\mathcal{A}$ is the Banach algebra $\mathbf{c}$ of convergent sequences or the Banach algebra ℓ^∞ of bounded sequences. They determined the unital homomorphisms from $\mathrm{Lip}(X, \mathbf{c})$ into $\mathrm{Lip}(Y, \mathbf{c})$ and those from $\mathrm{Lip}(X, \ell^\infty)$ into $\mathrm{Lip}(Y, \ell^\infty)$, where X and Y are compact metric spaces.

In general, if K is a compact Hausdorff space, then $C(K)$ denotes the Banach algebra of all complex-valued continuous functions on K, with norm $\|f\|_{C(K)} = \sup_{\xi \in K} |f(\xi)|$. In [4], S. Oi took up the algebra $\mathrm{Lip}(X, C(K))$ and proved the following theorem:

Theorem B *(Oi, [4]). Suppose that X and Y are as in Theorem A, and that K and M are compact Hausdorff spaces. Assume that Y is connected. Then T is a unital homomorphism from $\mathrm{Lip}(X, C(K))$ into $\mathrm{Lip}(Y, C(M))$ if and only if there exist a class $\{\varphi_\eta\}_{\eta \in M}$ of mappings from Y to X with the properties* (a) *and* (b) *below and a continuous mapping $\psi : M \to K$ such that*

$$\big[(Tf)(y)\big](\eta) = \big[f(\varphi_\eta(y))\big](\psi(\eta)) \qquad (y \in Y,\ \eta \in M)$$

for all $f \in \mathrm{Lip}(X, C(K))$.

(a) *For each $y \in Y$, the mapping $\eta \mapsto \varphi_\eta(y)$ from M to X is continuous.*

(b) $\displaystyle \sup_{\eta \in M} \sup_{\substack{y,y' \in Y \\ y \neq y'}} \frac{d_X(\varphi_\eta(y), \varphi_\eta(y'))}{d_Y(y, y')} < \infty.$

This theorem leads to the result of Botelho and Jamison mentioned above. Here we turn our attention to two assumptions in Theorem B. One is that Y is connected and the other is that T is unital. These assumptions seem to be inessential but they simplify the statement of theorem. In order to remove these assumptions and to state a general result, we consider a function f in $\mathrm{Lip}(X, C(K))$ as a function of two variables $x \in X$ and $\xi \in K$. So we write $f(x, \xi)$ instead of $\big[f(x)\big](\xi)$. Let f be a function on $X \times K$. With $x \in X$ we associate a function f_x defined on K by $f_x(\xi) = f(x, \xi)$. Similarly, if $\xi \in K$, f^ξ is the function defined on X by $f^\xi(x) = f(x, \xi)$. In general, for any mapping of two variables, we use the same expression:

For example, if $\psi : Y \times M \to K$, then $\psi^\eta : Y \to K$ and $\psi_y : M \to K$ are defined by $\psi^\eta(y) = \psi(y, \eta)$ and $\psi_y(\eta) = \psi(y, \eta)$.

A subset A of a topological space is said to be *clopen*, if A is both open and closed. We do not exclude the possibility that a clopen set is empty. We understand that the statement about an empty set is true.

Theorem 1. *Suppose that X and Y are compact metric spaces with metrics d_X and d_Y respectively, and that K and M are compact Hausdorff spaces. If T is a homomorphism from $\mathrm{Lip}(X, C(K))$ into $\mathrm{Lip}(Y, C(M))$, then there exist a clopen subset $\mathcal{D}$ of $Y \times M$ and two continuous mappings $\varphi : \mathcal{D} \to X$ and $\psi : \mathcal{D} \to K$ with* (i) *and* (ii) *such that T has the form:*

$$(Tf)(y, \eta) = \begin{cases} f\big(\varphi(y, \eta), \psi(y, \eta)\big) & \big((y, \eta) \in \mathcal{D}\big) \\ 0 & \big((y, \eta) \in (Y \times M) \setminus \mathcal{D}\big) \end{cases} \tag{1}$$

for all $f \in \mathrm{Lip}(X, C(K))$.

(i) *There exists a bound $L \geq 0$ such that*

$$(y, \eta), (y', \eta) \in \mathcal{D} \text{ and } y \neq y' \text{ imply } \frac{d_X\big(\varphi(y, \eta), \varphi(y', \eta)\big)}{d_Y(y, y')} \leq L. \tag{2}$$

(ii) *For any $\eta \in M$, the set $\mathcal{D}^\eta = \{y \in Y : (y, \eta) \in \mathcal{D}\}$ is a union of finitely many disjoint clopen subsets $V_1^\eta, \ldots, V_{n_\eta}^\eta$ of Y such that*

$$\psi^\eta \text{ is constant on } V_i^\eta \text{ for } i = 1, \ldots, n_\eta,$$

and

$$d_Y(V_i^\eta, V_j^\eta) \geq r \qquad (i \neq j). \tag{3}$$

Here r is a positive constant independent of η.

Conversely, if $\mathcal{D}$, φ, ψ are given as above, then T defined by (1) is a homomorphism from $\mathrm{Lip}(X, C(K))$ into $\mathrm{Lip}(Y, C(M))$. Moreover, T is unital if and only if $\mathcal{D} = Y \times M$.

In (3), $d_Y(A, B)$ denotes the usual distance between two sets $A, B \subset Y$, that is, $d_Y(A, B) = \inf\{d_Y(y, y') : y \in A, y' \in B\}$. (If $A = \emptyset$ or if $B = \emptyset$, then we set $d_Y(A, B) = \infty$).

Next we consider the following problem:

When is a homomorphism between Lipschitz algebras compact?

In [3], H. Kamowitz and S. Scheinberg answered to this problem as follows:

Theorem C *(Kamowitz and Scheinberg, [3]). Let T be a unital homomorphism from $\mathrm{Lip}(X)$ into $\mathrm{Lip}(Y)$ described in Theorem A. Then T is compact if and only if*

$$\lim_{d_Y(y, y') \to 0} \frac{d_X(\varphi(y), \varphi(y'))}{d_Y(y, y')} = 0.$$

In this paper, we give a necessary and sufficient condition for T in Theorem 1 to be compact.

Theorem 2. *Let X, Y, K, M be as in Theorem 1. Suppose that T is a homomorphism from* $\mathrm{Lip}(X, C(K))$ *into* $\mathrm{Lip}(Y, C(M))$ *with the form* (1) *in Theorem 1. Then T is compact if and only if* (iii) *and* (iv) *hold.*

(iii) *For any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$(y, \eta), (y', \eta) \in \mathcal{D} \text{ and } 0 < d_Y(y, y') < \delta \text{ imply } \frac{d_X\big(\varphi(y, \eta), \varphi(y', \eta)\big)}{d_Y(y, y')} < \varepsilon. \tag{4}$$

(iv) *For any $y \in Y$, the set $\mathcal{D}_y = \{\eta \in M : (y, \eta) \in \mathcal{D}\}$ is a union of finitely many disjoint clopen sets $\Omega_y^1, \ldots, \Omega_y^{n_y}$ such that*

$$\psi_y \text{ is constant on } \Omega_y^i \text{ for } i = 1, \ldots, n_y.$$

2. Preliminaries

As mentioned in Introduction, we consider a function f in $\mathrm{Lip}(X, C(K))$ as a function on $X \times K$.

Proposition 2.1. *Let f be a complex-valued function on $X \times K$. Then $f \in \mathrm{Lip}(X, C(K))$ if and only if $f \in C(X \times K)$ and*

$$\mathcal{L}_{X,C(K)}(f) = \sup_{\substack{x,x' \in X \\ x \neq x'}} \frac{\|f_x - f_{x'}\|_{C(K)}}{d_X(x, x')} < \infty. \tag{5}$$

Moreover, $\|f\|_{C(X,C(K))} = \|f\|_{C(X \times K)}$.

Proof. Straightforward. $\square$

The next proposition implies that $\mathrm{Lip}(X, C(K))$ separates the points of $X \times K$.

Proposition 2.2. *For any $(x_0, \xi_0) \in X \times K$ and for any open neighborhood $\mathcal{U}$ of (x_0, ξ_0), there exist an $f \in \mathrm{Lip}(X, C(K))$ and $m > 0$ such that $0 \le f \le 1$, $f(x_0, \xi_0) = 1$ and $f(x, \xi) \le m < 1$ for all $(x, \xi) \in (X \times K) \setminus \mathcal{U}$.*

Proof. Let $(x_0, \xi_0) \in X \times K$ and let $\mathcal{U}$ be an open neighborhood of (x_0, ξ_0). Then there exist an open neighborhood U of x_0 in X and an open neighborhood Θ of ξ_0 in K such that $(x_0, \xi_0) \in U \times \Theta \subset \mathcal{U}$.

Let h be a function on X defined by

$$h(x) = 1 - \frac{d_X(x, x_0)}{\mathrm{diam}(X)} \qquad (x \in X),$$

where $\mathrm{diam}(X) = \sup\{d_X(x, x') : x, x' \in X\}$. We easily see that $h \in \mathrm{Lip}(X)$, $0 \le h \le 1$, $h(x_0) = 1$ and $h(x) < 1$ for all $x \in X \setminus \{x_0\}$. By Urysohn's lemma, there is a $u \in C(K)$ such that $0 \le u \le 1$, $u(\xi_0) = 1$ and $u(\xi) = 0$ for all $\xi \in K \setminus \Theta$. Now, put $f(x, \xi) = h(x)u(\xi)$ for $(x, \xi) \in X \times K$. Then we can verify that f has the desired properties. Here m may be taken as the maximum of f on the compact set $(X \times K) \setminus \mathcal{U}$. $\square$

Here we summarize a fundamental fact on the Banach algebra $\mathrm{Lip}(X, C(K))$.

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Proposition 2.3. $\mathrm{Lip}(X, C(K))$ *is a semisimple unital commutative Banach algebra and its maximal ideal space is identified with* $X \times K$. *In fact, for any multiplicative linear functional* Ψ *on* $\mathrm{Lip}(X, C(K))$, *there exists a unique point* $(x, \xi) \in X \times K$ *such that* $\Psi(f) = f(x, \xi)$ *for all* $f \in \mathrm{Lip}(X, C(K))$.

We can prove this proposition by the well-known argument in theory of Banach algebras. The details may be found in [4, Propositions 11 and 12].

3. Proof of Theorem 1

In this section we prove Theorem 1.

3.1. Proof of sufficiency

We first settle the converse statement. Suppose that $\mathcal{D}$ is a clopen subset of $Y \times M$, that $\varphi : \mathcal{D} \to X$ and $\psi : \mathcal{D} \to K$ are continuous mappings with (i) and (ii), and that T is defined by (1).

Lemma 3.1. *If*

$$\rho = \inf\bigl\{d_Y(y, y') \,:\, (y, \eta) \in \mathcal{D} \text{ and } (y', \eta) \in (Y \times M) \setminus \mathcal{D} \text{ for some } \eta \in M\bigr\},$$

then $\rho > 0$.

If there is no pair $(y, y') \in Y \times Y$ such that $(y, \eta) \in \mathcal{D}$ and $(y', \eta) \in (Y \times M) \setminus \mathcal{D}$ for some $\eta \in K$, then we understand that $\rho = \infty$.

Proof. Conversely, assume that $\rho = 0$. Then for each $n = 1, 2, \ldots$, there exist $(y_n, \eta_n) \in \mathcal{D}$ and $(y'_n, \eta_n) \in (Y \times M) \setminus \mathcal{D}$ such that $d_Y(y_n, y'_n) < 1/n$. Since $\mathcal{D}$ is compact, there exist a net $\{n_\alpha\}$ and a point $(y, \eta) \in \mathcal{D}$ such that $y_{n_\alpha} \to y$ and $\eta_{n_\alpha} \to \eta$. Then $d_Y(y_{n_\alpha}, y'_{n_\alpha}) < 1/n_\alpha \to 0$. Hence $y'_{n_\alpha} \to y$ and so $(y'_{n_\alpha}, \eta_{n_\alpha}) \to (y, \eta)$. Since $(Y \times M) \setminus \mathcal{D}$ is closed, we get $(y, \eta) \in (Y \times M) \setminus \mathcal{D}$. This contradicts the fact that $(y, \eta) \in \mathcal{D}$. Consequently, we have $\rho > 0$. $\square$

Lemma 3.2. *For any* $f \in \mathrm{Lip}(X, C(K))$, $Tf \in \mathrm{Lip}(Y, C(M))$.

Proof. Let $f \in \mathrm{Lip}(X, C(K))$. By Proposition 2.1, we have $f \in C(X \times K)$ and (5).

We first show that $Tf \in C(Y \times M)$. Since $\varphi : \mathcal{D} \to X$ and $\psi : \mathcal{D} \to K$ are continuous and since $f \in C(X \times K)$, the first line in (1) implies that Tf is continuous on $\mathcal{D}$. Of course, the second one implies that it is so on $(Y \times M) \setminus \mathcal{D}$. Noting that $\mathcal{D}$ is clopen, we see that Tf is continuous on $Y \times M$.

To see that $Tf \in \mathrm{Lip}(Y, C(M))$, it suffices to show that

$$\mathcal{L}_{Y,C(M)}(Tf) = \sup_{\substack{y, y' \in Y \\ y \neq y'}} \frac{\|(Tf)_y - (Tf)_{y'}\|_{C(M)}}{d_Y(y, y')} < \infty. \tag{6}$$

For this end, choose $y, y' \in Y$ so that $y \neq y'$ and let $\eta \in M$. We consider three cases.

[Case 1] $(y, \eta), (y', \eta) \in \mathcal{D}$: By (ii), $y \in V_i^\eta$ and $y' \in V_{i'}^\eta$ for some $i, i' \in \{1, \ldots, n_\eta\}$. We first consider the case $i = i'$. Then $y, y' \in V_i^\eta$. Since ψ^η is constant on V_i^η, $\psi(y, \eta) = \psi^\eta(y) = \psi^\eta(y') = \psi(y', \eta)$. Put $x = \varphi(y, \eta)$, $x' = \varphi(y', \eta)$ and $\xi = \psi(y, \eta) = \psi(y', \eta)$. Using (1), we compute

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$$\begin{aligned} \big|(Tf)(y,\eta)-(Tf)(y',\eta)\big| &= \big|f(\varphi(y,\eta),\psi(y,\eta))-f(\varphi(y',\eta),\psi(y',\eta))\big| \\ &= |f(x,\xi)-f(x',\xi)| = |f_x(\xi)-f_{x'}(\xi)| \\ &\le \|f_x-f_{x'}\|_{C(K)} \\ &\le \mathcal{L}_{X,C(K)}(f)\, d_X(x,x') \\ &= \mathcal{L}_{X,C(K)}(f)\, d_X\big(\varphi(y,\eta),\varphi(y',\eta)\big) \\ &\le \mathcal{L}_{X,C(K)}(f)\, L\, d_Y(y,y'), \end{aligned} \tag{7}$$

where the fourth and last lines follow from (5) and (2), respectively.

On the other hand, if $i \ne i'$, then (3) yields $d_Y(y,y') \ge d_Y(V_i^\eta, V_{i'}^\eta) \ge r$. Hence

$$\frac{\big|(Tf)(y,\eta)-(Tf)(y',\eta)\big|}{d_Y(y,y')} \le \frac{\big|(Tf)(y,\eta)\big|+\big|(Tf)(y',\eta)\big|}{r} \le \frac{2\,\|f\|_{C(X\times K)}}{r}. \tag{8}$$

[Case 2] $(y,\eta) \in \mathcal{D}$ and $(y',\eta) \in (Y \times M) \setminus \mathcal{D}$: Then Lemma 3.1 says that $d_Y(y,y') \ge \rho > 0$. By (1), we get

$$\frac{\big|(Tf)(y,\eta)-(Tf)(y',\eta)\big|}{d_Y(y,y')} \le \frac{\big|f(\varphi(y,\eta),\psi(y,\eta))-0\big|}{\rho} \le \frac{\|f\|_{C(X\times K)}}{\rho}. \tag{9}$$

[Case 3] $(y,\eta), (y',\eta) \in (Y \times M) \setminus \mathcal{D}$: By (1),

$$(Tf)(y,\eta)-(Tf)(y',\eta) = 0. \tag{10}$$

Combining (7)–(10), we can arrive at (6). Indeed, if we put $C = \max\big\{L, 2/r, 1/\rho\big\}$, then we have

$$\mathcal{L}_{Y,C(M)}(Tf) = \sup_{\substack{y,y'\in Y\\ y\ne y'}} \sup_{\eta\in M} \frac{\big|(Tf)(y,\eta)-(Tf)(y',\eta)\big|}{d_Y(y,y')} \le C\,\|f\|_{\mathrm{Lip}(X,C(K))}, \tag{11}$$

because $\mathcal{L}_{X,C(K)}(f) \le \|f\|_{\mathrm{Lip}(X,C(K))}$ and $\|f\|_{C(X\times K)} \le \|f\|_{\mathrm{Lip}(X,C(K))}$. □

Lemma 3.2 says that T maps $\mathrm{Lip}(X, C(K))$ into $\mathrm{Lip}(Y, C(M))$. While the form (1) shows that T is a homomorphism. Thus we obtain the converse statement of Theorem 1.

Remark. From (1), we see that $\|Tf\|_{C(Y\times M)} \le \|f\|_{C(X\times K)}$. Using this and (11), we obtain the norm estimate

$$\|T\| = \sup_{\|f\|_{\mathrm{Lip}(X,C(K))}\le 1} \|Tf\|_{\mathrm{Lip}(Y,C(M))} \le C+1.$$

This estimate is not sharp, but it seems to be difficult to give an exact expression of $\|T\|$.

3.2. Proof of necessity

We turn to the proof of the main statement of Theorem 1. Suppose that T is an arbitrary homomorphism from $\mathrm{Lip}(X, C(K))$ into $\mathrm{Lip}(Y, C(M))$. Since $\mathrm{Lip}(Y, C(M))$ is semisimple, we know from [5, Theorem 11.10] that T is continuous. Thus the norm $\|T\|$ is determined as a bounded linear operator T.

If $T = O$, then we only take $\mathcal{D} = \emptyset$. So, we assume that $T \ne O$.

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Lemma 3.3. *There exist a clopen subset $\mathcal{D}$ of $Y \times M$ and two mapping $\varphi : \mathcal{D} \to X$ and $\psi : \mathcal{D} \to K$ such that (1) holds.*

Proof. Let $\mathbf{1}$ denote the unit of $\mathrm{Lip}(X, C(K))$, namely, the constant 1 function on $X \times K$. Since $(T\mathbf{1})^2 = T(\mathbf{1}^2) = T\mathbf{1}$, we have $(T\mathbf{1})(y, \eta) \in \{1, 0\}$ for all $(y, \eta) \in Y \times M$. Put

$$\mathcal{D} = \{(y, \eta) \in Y \times M : (T\mathbf{1})(y, \eta) = 1\}. \tag{12}$$

Then

$$(Y \times M) \setminus \mathcal{D} = \{(y, \eta) \in Y \times M : (T\mathbf{1})(y, \eta) = 0\}.$$

Since $T\mathbf{1}$ is continuous on $Y \times M$, both $\mathcal{D}$ and $(Y \times M) \setminus \mathcal{D}$ are closed. Hence $\mathcal{D}$ is clopen.

To determine the mappings $\varphi : \mathcal{D} \to X$ and $\psi : \mathcal{D} \to K$, fix any $(y, \eta) \in \mathcal{D}$. Define a functional $\Psi_{(y,\eta)}$ on $\mathrm{Lip}(X, C(K))$ by

$$\Psi_{(y,\eta)}(f) = (Tf)(y, \eta) \qquad \big(f \in \mathrm{Lip}(X, C(K))\big).$$

Then $\Psi_{(y,\eta)}$ is a homomorphism from $\mathrm{Lip}(X, C(K))$ into $\mathbb{C}$. Moreover, (12) yields $\Psi_{(y,\eta)}(\mathbf{1}) = (T\mathbf{1})(y, \eta) = 1$. Hence $\Psi_{(y,\eta)}$ is a multiplicative linear functional on $\mathrm{Lip}(X, C(K))$. Thus Proposition 2.3 gives a unique point $(x, \xi) \in X \times K$ such that

$$\Psi_{(y,\eta)}(f) = f(x, \xi) \qquad \big(f \in \mathrm{Lip}(X, C(K))\big).$$

By putting $\varphi(y, \eta) = x$ and $\psi(y, \eta) = \xi$, we determine the mappings $\varphi : \mathcal{D} \to X$ and $\psi : \mathcal{D} \to K$. Then, for any $f \in \mathrm{Lip}(X, C(K))$,

$$(Tf)(y, \eta) = \Psi_{(y,\eta)}(f) = f(x, \xi) = f\big(\varphi(y, \eta), \psi(y, \eta)\big). \tag{13}$$

Finally, if $(y, \eta) \in (Y \times M) \setminus \mathcal{D}$, then $(T\mathbf{1})(y, \eta) = 0$ and so for any $f \in \mathrm{Lip}(X, C(K))$, $Tf = T(f\mathbf{1}) = (Tf)(T\mathbf{1})$ and so

$$(Tf)(y, \eta) = (Tf)(y, \eta)\,(T\mathbf{1})(y, \eta) = 0.$$

Together with (13), we establish (1). $\square$

Lemma 3.4. *The mappings $\varphi : \mathcal{D} \to X$ and $\psi : \mathcal{D} \to K$ are continuous.*

Proof. Define a mapping $\Phi : \mathcal{D} \to X \times K$ by

$$\Phi(y, \eta) = \big(\varphi(y, \eta), \psi(y, \eta)\big) \qquad \big((y, \eta) \in \mathcal{D}\big).$$

We prove the lemma by verifying that Φ is continuous at each point $(y_0, \eta_0) \in \mathcal{D}$. Let $\mathcal{U}$ be an arbitrary open neighborhood of $\Phi(y_0, \eta_0)$ in $X \times K$. By Proposition 2.2, there exists an $f \in \mathrm{Lip}(X, C(K))$ such that $0 \le f \le 1$, $f(\Phi(y_0, \eta_0)) = 1$ and

$$0 \le f(x, \xi) \le m < 1 \qquad \big((x, \xi) \in (X \times K) \setminus \mathcal{U}\big). \tag{14}$$

Put $\mathcal{V} = \big\{(y, \eta) \in \mathcal{D} : |(Tf)(y, \eta)| > m\big\}$. Since Tf is continuous on $Y \times M$, $\mathcal{V}$ is open. Also, $(y_0, \eta_0) \in \mathcal{V}$ because

$$(Tf)(y_0,\eta_0) = f\big(\varphi(y_0,\eta_0), \psi(y_0,\eta_0)\big) = f\big(\Phi(y_0,\eta_0)\big) = 1 > m.$$

Moreover, if $(y,\eta) \in \mathcal{V}$, then

$$\big|f(\Phi(y,\eta))\big| = \big|f\big(\varphi(y,\eta), \psi(y,\eta)\big)\big| = \big|(Tf)(y,\eta)\big| > m$$

and (14) forces that $\Phi(y,\eta) \in \mathcal{U}$. Hence $\Phi(\mathcal{V}) \subset \mathcal{U}$. Thus Φ is continuous at (y_0,η_0), as desired. □

Lemma 3.5. *φ satisfies* (i).

Proof. Let $(y,\eta), (y',\eta) \in \mathcal{D}$ with $y \neq y'$. Put $x_0 = \varphi(y',\eta)$ and

$$f(x) = d_X(x, x_0) \qquad (x \in X).$$

Then $f \in \mathrm{Lip}(X)$ and $\|f\|_{\mathrm{Lip}(X)} \leq \mathrm{diam}(X)+1$. Extend f to $X \times K$ by $\hat{f}(x,\xi) = f(x)$ for all $(x,\xi) \in X \times K$. Clearly $\hat{f} \in \mathrm{Lip}(X, C(K))$ and $\|\hat{f}\|_{\mathrm{Lip}(X,C(K))} = \|f\|_{\mathrm{Lip}(X)}$. Moreover, we have

$$\begin{aligned}
d_X\big(\varphi(y,\eta), \varphi(y',\eta)\big) &= \big|d_X\big(\varphi(y,\eta), x_0\big) - d_X\big(\varphi(y',\eta), x_0\big)\big| \\
&= \big|f(\varphi(y,\eta)) - f(\varphi(y',\eta))\big| \\
&= \big|\hat{f}\big(\varphi(y,\eta), \psi(y,\eta)\big) - \hat{f}\big(\varphi(y',\eta), \psi(y',\eta)\big)\big| \\
&= \big|(T\hat{f})(y,\eta) - (T\hat{f})(y',\eta)\big| = \big|(T\hat{f})_y(\eta) - (T\hat{f})_{y'}(\eta)\big| \\
&\leq \big\|(T\hat{f})_y - (T\hat{f})_{y'}\big\|_{C(M)} \\
&\leq \mathcal{L}_{Y,C(M)}(T\hat{f})\, d_Y(y,y').
\end{aligned}$$

Since $\mathcal{L}_{(Y,C(M))}(T\hat{f}) \leq \|T\hat{f}\|_{\mathrm{Lip}(Y,C(M))} \leq \|T\|\,\|\hat{f}\|_{\mathrm{Lip}(X,C(K))} \leq \|T\|(\mathrm{diam}(X)+1)$, we obtain

$$\frac{d_X\big(\varphi(y,\eta), \varphi(y',\eta)\big)}{d_Y(y,y')} \leq \|T\|\ (\mathrm{diam}(X)+1),$$

which is (i). □

Lemma 3.6. *There exists an $r > 0$ such that*

$$(y,\eta), (y',\eta) \in \mathcal{D} \text{ and } \ d_Y(y,y') < r \text{ imply } \ \psi^\eta(y) = \psi^\eta(y').$$

Proof. Take r so that $0 < r < 1/\|T\|$. Choose $(y,\eta), (y',\eta) \in \mathcal{D}$ with $d_Y(y,y') < r$ and assume that $\psi^\eta(y) \neq \psi^\eta(y')$. By Urysohn's lemma, we find a $u \in C(K)$ such that $0 \leq u \leq 1$, $u(\psi^\eta(y)) = 1$ and $u(\psi^\eta(y')) = 0$. Define a function on $X \times K$ as $\tilde{u}(x,\xi) = u(\xi)$ for all $(x,\xi) \in X \times K$. Then $\tilde{u} \in \mathrm{Lip}(X, C(K))$ and $\|\tilde{u}\|_{\mathrm{Lip}(X,C(K))} = \|u\|_{C(K)} = 1$. Moreover we have

$$\begin{aligned}
1 &= \big|u(\psi^\eta(y)) - u(\psi^\eta(y'))\big| = \big|u(\psi(y,\eta)) - u(\psi(y',\eta))\big| \\
&= \big|\tilde{u}\big(\varphi(y,\eta), \psi(y,\eta)\big) - \tilde{u}\big(\varphi(y',\eta), \psi(y',\eta)\big)\big| \\
&= \big|(T\tilde{u})(y,\eta) - (T\tilde{u})(y',\eta)\big| \\
&\leq \big\|(T\tilde{u})_y - (T\tilde{u})_{y'}\big\|_{C(M)} \\
&\leq \mathcal{L}_{Y,C(M)}(T\tilde{u})\ d_Y(y,y') \\
&< \|T\tilde{u}\|_{\mathrm{Lip}(Y,C(M))}\, r \leq \|T\|\,\|\tilde{u}\|_{\mathrm{Lip}(X,C(K))}\, r = \|T\|\, r < 1,
\end{aligned}$$

a contradiction. Hence $\psi^\eta(y) = \psi^\eta(y')$. □

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Lemma 3.7. *ψ satisfies* (ii)*.*

Proof. Fix any $\eta \in M$ and put $\mathcal{D}^\eta = \{y \in Y : (y, \eta) \in \mathcal{D}\}$. Since $\mathcal{D}$ is clopen, $\mathcal{D}^\eta$ is a clopen subset of Y. For any $y \in \mathcal{D}^\eta$, put

$$V_y = \{z \in \mathcal{D}^\eta : \psi^\eta(z) = \psi^\eta(y)\}. \tag{15}$$

Clearly, ψ^η is constant on V_y. Also, we have

$$V_y \cap V_{y'} \neq \emptyset \implies V_y = V_{y'}. \tag{16}$$

Since ψ^η is continuous by Lemma 3.4, V_y is a closed subset of $\mathcal{D}^\eta$. To see that V_y is an open subset of $\mathcal{D}^\eta$, let $z \in V_y$ and consider an r-ball $B(z; r) = \{w \in \mathcal{D}^\eta : d_Y(w, z) < r\}$, where r is given in Lemma 3.6. If $w \in B(z; r)$, then $(w, \eta), (z, \eta) \in \mathcal{D}$ and $d_Y(w, z) < r$. Hence Lemma 3.6 implies that $\psi^\eta(w) = \psi^\eta(z) = \psi^\eta(y)$, and so $w \in V_y$. Therefore $B(z; r) \subset V_y$. Thus V_y is an open subset of $\mathcal{D}^\eta$. Consequently, V_y is a clopen subset of Y.

Note that

$$\mathcal{D}^\eta = \bigcup_{y \in \mathcal{D}^\eta} V_y.$$

Since $\mathcal{D}^\eta$ is compact, we can select finitely many $y_1, \ldots, y_n \in \mathcal{D}^\eta$ such that

$$\mathcal{D}^\eta = \bigcup_{i=1}^{n} V_{y_i}.$$

By (16), we may assume that $V_{y_1}, \ldots, V_{y_n}$ are disjoint.

Finally we show that $d_Y(V_{y_i}, V_{y_j}) \geq r$ $(i \neq j)$. Assume that $d_Y(V_{y_i}, V_{y_j}) < r$. Then there exist $z_i \in V_{y_i}$ and $z_j \in V_{y_j}$ such that $d_Y(z_i, z_j) < r$. By Lemma 3.6, $\psi^\eta(z_i) = \psi^\eta(z_j)$, and hence (15) and (16) yield $V_{y_i} = V_{y_j}$. Since $V_{y_1}, \ldots, V_{y_n}$ are disjoint, we must have $d_Y(V_{y_i}, V_{y_j}) \geq r$ $(i \neq j)$.

Putting $n_\eta = n$ and writing V_i^η for V_{y_i} $(i = 1, \ldots, n_\eta)$, we obtain (ii). $\square$

Thus the proof of Theorem 1 is completed.

4. Proof of Theorem 2

In this section, we prove Theorem 2. Throughout this section, T is a homomorphism from $\mathrm{Lip}(X, C(K))$ into $\mathrm{Lip}(Y, C(M))$ with the form (1) in Theorem 1. Of course, the set $\mathcal{D}$ and the mappings φ and ψ are as in Theorem 1. Since T is bounded, we use its norm $\|T\|$ again. Let $\mathbb{B}_{\mathrm{Lip}(X,C(K))}$ be the unit ball of $\mathrm{Lip}(X, C(K))$, that is,

$$\mathbb{B}_{\mathrm{Lip}(X,C(K))} = \big\{f \in \mathrm{Lip}(X, C(K)) : \|f\|_{\mathrm{Lip}(X,C(K))} \leq 1\big\}.$$

4.1. Proof of sufficiency

We first show the "if" part in Theorem 2.

Suppose that φ and ψ satisfy (iii) and (iv) respectively. We prove that T is compact. Here we may assume that $T \neq O$, otherwise there is nothing to prove.

Lemma 4.1. *Let $(y_0, \eta_0) \in \mathcal{D}$. For any $\varepsilon > 0$, there exists an open neighborhood Θ of η_0 in M such that*

$$\eta \in \Theta \quad \textit{implies} \quad \sup_{f \in \mathbb{B}_{\mathrm{Lip}(X,C(K))}} \big|(Tf)(y_0, \eta) - (Tf)(y_0, \eta_0)\big| < \varepsilon. \tag{17}$$

Proof. Put $\mathcal{D}_{y_0} = \{\eta \in M : (y_0, \eta) \in \mathcal{D}\}$. Since $\eta_0 \in \mathcal{D}_{y_0}$, by (iv), there exists $j \in \{1, \ldots, n_{y_0}\}$ such that $\eta_0 \in \Omega^j_{y_0}$. Then $\Omega^j_{y_0}$ is a clopen subset on which ψ_{y_0} is constant. Hence if $\eta \in \Omega^j_{y_0}$, then $\psi(y_0, \eta) = \psi_{y_0}(\eta) = \psi_{y_0}(\eta_0) = \psi(y_0, \eta_0)$. Let $\varepsilon > 0$ and put

$$\Theta = \big\{\eta \in \Omega^j_{y_0} : d_X\big(\varphi_{y_0}(\eta), \varphi_{y_0}(\eta_0)\big) < \varepsilon\big\}.$$

Since $\varphi_{y_0} : \mathcal{D}_{y_0} \to X$ is continuous, Θ is an open neighborhood of η_0 in $\mathcal{D}_{y_0}$. For any $\eta \in \Theta$, put $x = \varphi(y_0, \eta)$, $x_0 = \varphi(y_0, \eta_0)$ and $\xi = \psi(y_0, \eta) = \psi(y_0, \eta_0)$. Then, for any $f \in \mathbb{B}_{\mathrm{Lip}(X,C(K))}$, we have

$$\begin{aligned}
\big|(Tf)(y_0, \eta) - (Tf)(y_0, \eta_0)\big| &= \big|f\big(\varphi(y_0, \eta), \psi(y_0, \eta)\big) - f\big(\varphi(y_0, \eta_0), \psi(y_0, \eta_0)\big)\big| \\
&= \big|f(x, \xi) - f(x_0, \xi)\big| = \big|f_x(\xi) - f_{x_0}(\xi)\big| \\
&\le \|f_x - f_{x_0}\|_{C(K)} \\
&\le \mathcal{L}_{X,C(K)}(f)\, d_X(x, x_0) \\
&= \mathcal{L}_{X,C(K)}(f)\, d_X\big(\varphi(y_0, \eta), \varphi(y_0, \eta_0)\big) \\
&= \mathcal{L}_{X,C(K)}(f)\, d_X\big(\varphi_{y_0}(\eta), \varphi_{y_0}(\eta_0)\big) \\
&\le \|f\|_{\mathrm{Lip}(X,C(K))}\, \varepsilon \le \varepsilon.
\end{aligned}$$

Hence we obtain (17). □

Lemma 4.2. *In $C(Y \times M)$, the closure of $T\big(\mathbb{B}_{\mathrm{Lip}(X,C(K))}\big)$ is compact.*

Proof. According to Arzelá–Ascoli theorem ([2, Theorem IV.6.7]), we show that $T\big(\mathbb{B}_{\mathrm{Lip}(X,C(K))}\big)$ is bounded and equicontinuous on $Y \times M$.

The boundedness follows from an easy computation:

$$\big|(Tf)(y, \eta)\big| \le \|Tf\|_{C(Y \times M)} \le \|Tf\|_{\mathrm{Lip}(Y,C(M))} \le \|T\|\, \|f\|_{\mathrm{Lip}(X,C(K))} \le \|T\|$$

for all $(y, \eta) \in Y \times M$ and all $f \in \mathbb{B}_{\mathrm{Lip}(X,C(K))}$.

The equicontinuity will be shown as follows: Clearly, $T\big(\mathbb{B}_{\mathrm{Lip}(X,C(K))}\big)$ is equicontinuous on the clopen set $(Y \times M) \setminus \mathcal{D}$, because $Tf = 0$ on $(Y \times M) \setminus \mathcal{D}$ for all $f \in \mathrm{Lip}(X, C(K))$, by (1). To show that $T\big(\mathbb{B}_{\mathrm{Lip}(X,C(M))}\big)$ is equicontinuous at each $(y_0, \eta_0) \in \mathcal{D}$, let $\varepsilon > 0$. Take an open neighborhood Θ of η_0 in M as in Lemma 4.1, and put $V = \big\{y \in Y : d_Y(y, y_0) < \varepsilon/\|T\|\big\}$. Define an open neighborhood $\mathcal{W}$ of (y_0, η_0) in $Y \times M$ as

$$\mathcal{W} = (V \times \Theta) \;\cap\; \mathcal{D}.$$

Then, for any $(y, \eta) \in \mathcal{W}$ and $f \in \mathbb{B}_{\mathrm{Lip}(X,C(K))}$, we have

$$\begin{aligned}
\big|(Tf)(y, \eta) - (Tf)(y_0, \eta)\big| &\le \big\|(Tf)_y - (Tf)_{y_0}\big\|_{C(M)} \le \mathcal{L}_{Y,C(M)}(Tf)\, d_Y(y, y_0) \\
&\le \|Tf\|_{\mathrm{Lip}(Y,C(M))}\, \big(\varepsilon/\|T\|\big) \le \|T\|\, \|f\|_{\mathrm{Lip}(X,C(K))}\, \big(\varepsilon/\|T\|\big) \le \varepsilon,
\end{aligned}$$

because $y \in V$, while Lemma 4.1 implies

$$\big|(Tf)(y_0,\eta)-(Tf)(y_0,\eta_0)\big|<\varepsilon,$$

because $\eta\in\Theta$. Hence the triangle inequality shows that

$$(y,\eta)\in\mathcal{W}\ \text{ implies } \sup_{f\in\mathbb{B}_{\mathrm{Lip}(X,C(K))}}\big|(Tf)(y,\eta)-(Tf)(y_0,\eta_0)\big|<2\varepsilon.$$

Thus we conclude that $T\big(\mathbb{B}_{\mathrm{Lip}(X,C(K))}\big)$ is equicontinuous on $Y\times M$. □

Lemma 4.3. *For any $\varepsilon>0$, there exists a constant $c_\varepsilon>0$ such that*

$$\|Tf\|_{\mathrm{Lip}(Y,C(M))}\le\varepsilon+c_\varepsilon\|Tf\|_{C(Y\times M)}\tag{18}$$

for all $f\in\mathbb{B}_{\mathrm{Lip}(X,C(K))}$.

Proof. Fix $\varepsilon>0$. By (iii), there exists a $\delta_\varepsilon>0$ such that

$$(y,\eta),(y',\eta)\in\mathcal{D}\text{ and }0<d_Y(y,y')<\delta_\varepsilon\text{ imply }\frac{d_X\big(\varphi(y,\eta),\varphi(y',\eta)\big)}{d_Y(y,y')}<\varepsilon.\tag{19}$$

Let $f\in\mathbb{B}_{\mathrm{Lip}(X,C(K))}$, and choose $(y,\eta),(y',\eta)\in Y\times M$ with $y\ne y'$. We consider three cases.

[Case 1] $(y,\eta),(y',\eta)\in\mathcal{D}$: By (ii) in Theorem 1, $y\in V_i^\eta$ and $y'\in V_{i'}^\eta$ for some $i,i'\in\{1,\dots,n_\eta\}$. We first consider the case $i=i'$. If $d_Y(y,y')<\delta_\varepsilon$, then the computation (7) using (19) instead of (2) gives

$$\begin{aligned}\big|(Tf)(y,\eta)-(Tf)(y',\eta)\big|&\le\mathcal{L}_{X,C(K)}(f)\,d_X\big(\varphi(y,\eta),\varphi(y',\eta)\big)\\&\le\mathcal{L}_{X,C(K)}(f)\,\varepsilon\,d_Y(y,y')\\&\le\|f\|_{\mathrm{Lip}(X,C(K))}\,\varepsilon\,d_Y(y,y')\le\varepsilon\,d_Y(y,y').\end{aligned}\tag{20}$$

On the other hand, if $d_Y(y,y')\ge\delta_\varepsilon$, then

$$\frac{\big|(Tf)(y,\eta)-(Tf)(y',\eta)\big|}{d_Y(y,y')}\le\frac{|(Tf)(y,\eta)|+|(Tf)(y',\eta)|}{\delta_\varepsilon}\le\frac{2\|Tf\|_{C(Y\times M)}}{\delta_\varepsilon}.\tag{21}$$

In case that $i\ne i'$, we have $d_Y(y,y')\ge r$ by (3), and so

$$\frac{\big|(Tf)(y,\eta)-(Tf)(y',\eta)\big|}{d_Y(y,y')}\le\frac{2\|Tf\|_{C(Y\times M)}}{r}.\tag{22}$$

[Case 2] $(y,\eta)\in\mathcal{D}$ and $(y',\eta)\in(Y\times M)\setminus\mathcal{D}$: Then Lemma 3.1 says that $d_Y(y,y')\ge\rho$ and so

$$\frac{\big|(Tf)(y,\eta)-(Tf)(y',\eta)\big|}{d_Y(y,y')}\le\frac{|(Tf)(y,\eta)|}{\rho}\le\frac{\|Tf\|_{C(Y\times M)}}{\rho}.\tag{23}$$

[Case 3] $(y,\eta),(y',\eta)\in(Y\times M)\setminus\mathcal{D}$: By (1),

$$(Tf)(y,\eta)-(Tf)(y',\eta)=0.\tag{24}$$

Now, put $\tilde{c}_\varepsilon=\max\{2/\delta_\varepsilon,2/r,1/\rho\}$. We combine (20)–(24) to get

$$\mathcal{L}_{Y,C(M)}(Tf)=\sup_{\substack{y,y'\in Y\\y\ne y'}}\sup_{\eta\in M}\frac{\big|(Tf)(y,\eta)-(Tf)(y',\eta)\big|}{d_Y(y,y')}\le\max\{\varepsilon,\tilde{c}_\varepsilon\,\|Tf\|_{C(Y\times M)}\}.$$

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Hence

$$\|Tf\|_{\mathrm{Lip}(Y,C(M))} \le \varepsilon + (\tilde{c}_\varepsilon + 1)\|Tf\|_{C(Y\times M)},$$

which is (18). □

Lemma 4.4. *In* $\mathrm{Lip}(Y, C(M))$*, the closure of* $T\big(\mathbb{B}_{\mathrm{Lip}(X,C(K))}\big)$ *is compact.*

Proof. Let $\{f_n\}$ be an arbitrary sequence in $\mathbb{B}_{\mathrm{Lip}(X,C(K))}$. By Lemma 4.2, there exist a subsequence $\{f_{n_i}\}$ and a function $g \in C(Y \times M)$ such that $\|Tf_{n_i} - g\|_{C(Y\times M)} \to 0$. To see that $\{Tf_{n_i}\}$ is a Cauchy sequence in $\mathrm{Lip}(Y, C(M))$, let $\varepsilon > 0$. Since $\{Tf_{n_i}\}$ is a Cauchy sequence in $C(Y \times M)$, there exists an N such that $i, j \ge N$ implies $\|Tf_{n_i} - Tf_{n_j}\|_{C(Y\times M)} < \varepsilon/c_\varepsilon$. Substituting $f = \big(f_{n_i} - f_{n_j}\big)/2$ in (18), we see

$$i, j \ge N \text{ implies } \|Tf_{n_i} - Tf_{n_j}\|_{\mathrm{Lip}(Y,C(M))} \le 2\varepsilon + c_\varepsilon \|Tf_{n_i} - Tf_{n_j}\|_{C(Y\times M)} < 3\varepsilon.$$

Hence $\{Tf_{n_i}\}$ is a Cauchy sequence in $\mathrm{Lip}(Y, C(M))$, and so it converges to some function in $\mathrm{Lip}(Y, C(M))$. Thus we conclude that the closure of $T\big(\mathbb{B}_{\mathrm{Lip}(X,C(K))}\big)$ is compact in $\mathrm{Lip}(Y, C(M))$. □

Lemma 4.4 says that T is a compact operator from $\mathrm{Lip}(X, C(K))$ into $\mathrm{Lip}(Y, C(M))$, and the "if" part was proved.

4.2. Proof of necessity

In the sequels, we suppose that T is compact.

Lemma 4.5. φ *satisfies* (iii)*.*

Proof. Assume, to reach a contradiction, that φ does not satisfy (iii). Then there exist an $\varepsilon_0 > 0$ and two sequences $\{(y_n, \eta_n)\}$ and $\{(y'_n, \eta_n)\}$ in $\mathcal{D}$ such that

$$0 < d_Y(y_n, y'_n) < \frac{1}{n^2} \qquad \text{and} \qquad \frac{d_X\big(\varphi(y_n, \eta_n), \varphi(y'_n, \eta_n)\big)}{d_Y(y_n, y'_n)} \ge \varepsilon_0.$$

Put $z_n = \varphi(y_n, \eta_n)$ and $z'_n = \varphi(y'_n, \eta_n)$ for $n = 1, 2, \ldots$. In order to arrange the distance d_X, we here introduce a function χ_n:

$$\chi_n(t) = \frac{1}{2n}(1 - e^{-nt}) \qquad (t \in [0, \infty)).$$

Clearly, $0 \le \chi_n \le 1/2n$ and χ_n is differentiable and $\chi'_n(t) = e^{-nt}/2$. Define

$$f_n(x) = \chi_n\big(d_X(x, z'_n)\big) \qquad (x \in X).$$

For any $x, x' \in X$ with $x \ne x'$, the mean value theorem gives a point s_n between $d_X(x, z'_n)$ and $d_X(x', z'_n)$ such that

$$\chi_n\big(d_X(x, z'_n)\big) - \chi_n\big(d_X(x', z'_n)\big) = \chi'_n(s_n)\,\big(d_X(x, z'_n) - d_X(x', z'_n)\big),$$

and so

$$|f_n(x) - f_n(x')| = |\chi_n'(s_n)| \left| d_X(x, z_n') - d_X(x', z_n') \right| \leq \frac{e^{-ns_n}}{2} d_X(x, x') \leq \frac{1}{2} d_X(x, x').$$

Hence $f_n \in \mathrm{Lip}(X)$ and $\|f_n\|_{\mathrm{Lip}(X)} = \|f_n\|_{C(X)} + \mathcal{L}_{X,\mathbb{C}}(f_n) \leq \frac{1}{2n} + \frac{1}{2} \leq 1$.

Now put $\hat{f}_n(x, \xi) = f_n(x)$ for all $(x, \xi) \in X \times K$. Then $\hat{f}_n \in \mathrm{Lip}(X, C(K))$ and $\|\hat{f}_n\|_{\mathrm{Lip}(X,C(K))} \leq 1$, that is, $\hat{f}_n \in \mathbb{B}_{\mathrm{Lip}(X,C(K))}$.

Next we estimate the norm $\|T\hat{f}_n\|_{\mathrm{Lip}(Y,C(M))}$. We use the mean value theorem again, we compute as follows:

$$\begin{aligned}
\left|(T\hat{f}_n)(y_n, \eta_n) - (T\hat{f}_n)(y_n', \eta_n)\right| &= \left|\hat{f}_n\big(\varphi(y_n, \eta_n), \psi(y_n, \eta_n)\big) - \hat{f}_n\big(\varphi(y_n', \eta_n), \psi(y_n', \eta_n)\big)\right| \\
&= \left|f_n(z_n) - f_n(z_n')\right| = \left|\chi_n\big(d_X(z_n, z_n')\big) - \chi_n(0)\right| \\
&= \left|\chi_n'(\sigma_n)\right| \left|d_X(z_n, z_n') - 0\right| \\
&= \frac{e^{-n\sigma_n}}{2}\, d_X\big(\varphi(y_n, \eta_n), \varphi(y_n', \eta_n)\big) \\
&\geq \frac{e^{-n\sigma_n}}{2}\, \varepsilon_0\, d_Y(y_n, y_n'),
\end{aligned}$$

where $0 \leq \sigma_n \leq d_X(z_n, z_n')$. Hence

$$\|T\hat{f}_n\|_{\mathrm{Lip}(Y,C(M))} \geq \mathcal{L}_{Y,C(M)}(T\hat{f}_n) \geq \frac{\left\|(T\hat{f}_n)_{y_n} - (T\hat{f}_n)_{y_n'}\right\|_{C(M)}}{d_Y(y_n, y_n')} \geq \frac{e^{-n\sigma_n}}{2}\varepsilon_0. \tag{25}$$

While (2) in Theorem 1 implies

$$0 \leq \sigma_n \leq d_X(z_n, z_n') = d_X\big(\varphi(y_n, \eta_n), \varphi(y_n', \eta_n)\big) \leq L\, d_Y(y_n, y_n') \leq L\frac{1}{n^2},$$

and so $n\sigma_n \to 0$. Thus (25) implies

$$\liminf_{n\to\infty} \|T\hat{f}_n\|_{\mathrm{Lip}(Y,C(M))} \geq \frac{\varepsilon_0}{2} \tag{26}$$

Recall that T is compact. Since $\{\hat{f}_n\} \subset \mathbb{B}_{\mathrm{Lip}(X,C(K))}$, there exist a subsequence $\{\hat{f}_{n_i}\}$ and a function $g \in \mathrm{Lip}(Y, C(M))$ such that $\|T\hat{f}_{n_i} - g\|_{\mathrm{Lip}(Y,C(M))} \to 0$. Since $\|T\hat{f}_{n_i} - g\|_{C(Y\times M)} \leq \|T\hat{f}_{n_i} - g\|_{\mathrm{Lip}(Y,C(M))}$, we have $(T\hat{f}_{n_i})(y, \eta) \to g(y, \eta)$ for each $(y, \eta) \in Y \times M$. If $(y, \eta) \in \mathcal{D}$, then

$$\left|(T\hat{f}_{n_i})(y, \eta)\right| = \left|\hat{f}_{n_i}\big(\varphi(y, \eta), \psi(y, \eta)\big)\right| = \left|f_{n_i}\big(\varphi(y, \eta)\big)\right| \leq \frac{1}{2n_i} \;\to\; 0,$$

while if $(y, \eta) \in (Y \times M) \setminus \mathcal{D}$, then $(T\hat{f}_{n_i})(y, \eta) = 0$. As a result, we have $g(y, \eta) = 0$ for all $(y, \eta) \in Y \times M$, and so

$$\|T\hat{f}_{n_i}\|_{\mathrm{Lip}(Y,C(M))} \;\to\; 0.$$

This contradicts (26). $\square$

Fix $y \in Y$ and put $\mathcal{D}_y = \{\eta \in M : (y, \eta) \in \mathcal{D}\}$.

Lemma 4.6. *For any $\eta_0 \in \mathcal{D}_y$, there exists an open neighborhood of η_0 in $\mathcal{D}_y$ on which ψ_y is constant.*

Proof. Since $\mathcal{D}_y$ is a compact subset of M, we can treat the Banach algebra $C(\mathcal{D}_y)$ and a projection P from $\mathrm{Lip}(Y, C(M))$ into $C(\mathcal{D}_y)$:

$$(Pg)(\eta) = g(y, \eta) \qquad \big(\eta \in \mathcal{D}_y,\ g \in \mathrm{Lip}(Y, C(M))\big).$$

Clearly P is a bounded linear operator from $\mathrm{Lip}(Y, C(M))$ into $C(\mathcal{D}_y)$.

Now put $S = PT$. Since T is compact, S is a compact operator from $\mathrm{Lip}(X, C(K))$ into $C(\mathcal{D}_y)$. Hence Arzelá–Ascoli theorem says that $S\big(\mathbb{B}_{\mathrm{Lip}(X,C(K))}\big)$ is equicontinuous on $\mathcal{D}_y$. Hence there exists an open neighborhood Θ of η_0 such that

$$\eta \in \Theta \text{ implies } \sup_{f \in \mathbb{B}_{\mathrm{Lip}(X,C(K))}} \big|(Sf)(\eta) - (Sf)(\eta_0)\big| < \frac{1}{2}. \tag{27}$$

Conversely, assume that there exists $\eta_1 \in \Theta$ such that $\psi_y(\eta_1) \neq \psi_y(\eta_0)$. By Urysohn's lemma, there exists a $u \in C(M)$ such that $0 \le u \le 1$, $u\big(\psi_y(\eta_1)\big) = 1$ and $u\big(\psi_y(\eta_0)\big) = 0$. Put $\tilde{u}(x, \xi) = u(\xi)$ for all $(x, \xi) \in X \times K$. Then $\tilde{u} \in \mathbb{B}_{\mathrm{Lip}(X,C(K))}$. Hence (27) implies

$$\big|(S\tilde{u})(\eta_1) - (S\tilde{u})(\eta_0)\big| < \frac{1}{2}.$$

But

$$\begin{aligned}
\big|(S\tilde{u})(\eta_1) - (S\tilde{u})(\eta_0)\big| &= \big|(PT\tilde{u})(\eta_1) - (PT\tilde{u})(\eta_0)\big| \\
&= \big|(T\tilde{u})(y, \eta_1) - (T\tilde{u})(y, \eta_0)\big| \\
&= \big|\tilde{u}\big(\varphi(y, \eta_1), \psi(y, \eta_1)\big) - \tilde{u}\big(\varphi(y, \eta_0), \psi(y, \eta_0)\big)\big| \\
&= \big|u\big(\psi(y, \eta_1)\big) - u\big(\psi(y, \eta_0)\big)\big| = \big|u\big(\psi_y(\eta_1)\big) - u\big(\psi_y(\eta_0)\big)\big| = 1
\end{aligned}$$

a contradiction. Thus we conclude that ψ_y is constant on Θ. □

Lemma 4.7. *ψ satisfies* (iv).

Proof. For any $\eta \in \mathcal{D}_y$, put

$$\Omega^\eta = \big\{\zeta \in \mathcal{D}_y : \psi_y(\zeta) = \psi_y(\eta)\big\}.$$

Clearly, ψ_y is constant on Ω^η. Also, we have

$$\Omega^\eta \cap \Omega^{\eta'} \neq \emptyset \implies \Omega^\eta = \Omega^{\eta'}. \tag{28}$$

Since ψ_y is continuous, Ω^η is a closed subset of $\mathcal{D}_y$. Also we can easily see that Lemma 4.6 implies that Ω^η is an open subset of $\mathcal{D}_y$. Thus Ω^η is a clopen subset of M.

Note that

$$\mathcal{D}_y = \bigcup_{\eta \in \mathcal{D}_y} \Omega^\eta.$$

Since $\mathcal{D}_y$ is compact, we can select finitely many $\eta_1, \dots, \eta_n \in \mathcal{D}_y$ such that

$$\mathcal{D}_y = \bigcup_{i=1}^n \Omega^{\eta_i}.$$

By (28), we may assume that $\Omega^{\eta_1}, \dots, \Omega^{\eta_n}$ are disjoint. Putting $n_y = n$ and writing $\Omega_y^i = \Omega^{\eta_i}$ $(i = 1, \dots, n_y)$, we obtain (iv). □

5. Applications

Consider the case that K is a one-point set. Then $\mathrm{Lip}(X, C(K))$ is isometrically isomorphic to $\mathrm{Lip}(X)$. On the other hand, if X is a one-point set, $\mathrm{Lip}(X, C(K))$ is isometrically isomorphic to $C(K)$.

Corollary 1. *Suppose that X and Y are compact metric spaces with metrics d_X and d_Y respectively.*

(I) *If T is a homomorphism from $\mathrm{Lip}(X)$ into $\mathrm{Lip}(Y)$, then there exist a clopen subset Y_0 of Y and a continuous mapping $\varphi : Y_0 \to X$ with*

$$\sup_{\substack{y,y \in Y_0 \\ y \neq y'}} \frac{d_X(\varphi(y), \varphi(y'))}{d_Y(y, y')} < \infty$$

such that T has the form:

$$(Tf)(y) = \begin{cases} f\big(\varphi(y)\big) & \big(y \in Y_0\big) \\ 0 & \big(y \in Y \smallsetminus Y_0\big) \end{cases} \tag{29}$$

for all $f \in \mathrm{Lip}(X)$. Conversely, if Y_0, φ are given as above, then T defined by (29) is a homomorphism from $\mathrm{Lip}(X)$ into $\mathrm{Lip}(Y)$. Moreover, T is unital if and only if $Y_0 = Y$.

(II) *Suppose that T is a homomorphism from $\mathrm{Lip}(X)$ into $\mathrm{Lip}(Y)$ with the form (29). Then T is compact if and only if*

$$\lim_{\substack{y,y' \in Y_0 \\ d_Y(y,y') \to 0}} \frac{d_X(\varphi(y), \varphi(y'))}{d_Y(y, y')} = 0.$$

Now we turn to another setting.

Corollary 2. *Suppose that K and M are compact Hausdorff spaces.*

(I) *If T is a homomorphism from $C(K)$ into $C(M)$, then there exist a clopen subset M_0 of M and a continuous mapping $\psi : M_0 \to K$ such that T has the form:*

$$(Tf)(\eta) = \begin{cases} f\big(\psi(\eta)\big) & \big(\eta \in M_0\big) \\ 0 & \big(\eta \in M \smallsetminus M_0\big) \end{cases} \tag{30}$$

for all $f \in C(K)$. Conversely, if M_0, ψ are given as above, then T defined by (30) is a homomorphism from $C(K)$ into $C(M)$. Moreover, T is unital if and only if $M_0 = M$.

(II) *Suppose that T is a homomorphism from $C(K)$ into $C(M)$ with the form (30). Then T is compact if and only if M_0 is a union of finitely many clopen subsets $M_1, \dots, M_n$ such that ψ is constant on each M_i for $i = 1, \dots, n$. Moreover, T is compact if and only if T has a finite rank.*

Corollary 3. *Suppose that X is a compact metric space with metric d_X, and that M is a compact Hausdorff space.*

(I) *If T is a homomorphism from $\mathrm{Lip}(X)$ into $C(M)$, then there exist a clopen subset M_0 of M and a continuous mapping $\varphi : M_0 \to X$ such that T has the form:*

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$$(Tf)(\eta) = \begin{cases} f\big(\varphi(\eta)\big) & \big(\eta \in M_0\big) \\ 0 & \big(\eta \in M \setminus M_0\big) \end{cases} \tag{31}$$

for all $f \in \mathrm{Lip}(X)$*. Conversely, if* M_0*,* φ *are given as above, then* T *defined by* (31) *is a homomorphism from* $\mathrm{Lip}(X)$ *into* $C(M)$*. Moreover,* T *is unital if and only if* $M_0 = M$*.*

(II) *Every homomorphism from* $\mathrm{Lip}(X)$ *into* $C(M)$ *is compact.*

Corollary 4. *Suppose that* Y *is a compact metric space with metric* d_Y*, and that* K *is a compact Hausdorff space.*

(I) *If* T *is a homomorphism from* $C(K)$ *into* $\mathrm{Lip}(Y)$*, then* Y *is a union of finitely many disjoint clopen subsets* $Y_0, Y_1, \dots, Y_n$ *and there exist constant mappings* $\psi_i : Y_i \to K$ $(i = 1, \dots, n)$ *such that* T *has the form:*

$$(Tf)(y) = \begin{cases} f\big(\psi_i(y)\big) & \big(y \in Y_i, i = 1, \dots, n\big) \\ 0 & \big(y \in Y_0\big) \end{cases} \tag{32}$$

for all $f \in C(K)$*. Conversely, if* $Y_0, Y_1, \dots, Y_n$*,* $\psi_1, \dots, \psi_n$ *are given as above, then* T *defined by* (32) *is a homomorphism from* $C(K)$ *into* $\mathrm{Lip}(Y)$*. Moreover,* T *is unital if and only if* $Y_0 = \emptyset$*.*

(II) *Every homomorphism from* $C(K)$ *into* $\mathrm{Lip}(Y)$ *has a finite rank.*

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