J. Math. Anal. Appl. 473 (2019) 712–748

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Variational analysis and optimal control of dynamic unilateral contact models with friction ☆

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A R T I C L E   I N F O

Article history:
Received 27 April 2017
Available online 8 January 2019
Submitted by A. Daniilidis

Keywords:
Evolutionary variational–hemivariational inequality
Clarke subdifferential
Indicator function
Unilateral constraint
Weak solvability
Optimal control

A B S T R A C T

The target of the present work is to consider the optimal control for a class of evolutionary variational–hemivariational inequalities, which modeling the dynamic viscoelastic unilateral contact problems with normal damped response and friction. After studying the weak solvability of the unilateral contact model, we consider its optimal control by three aspects, the optimal control via external forces and initial conditions, the time optimal control problem and the maximum stay control problem. Finally, the conditions which guarantee the existence of optimal solutions to the corresponding control problems are delivered.

1. Introduction

The theory of hemivariational inequality, which is a natural generalization of traditional variational inequality, was firstly proposed by P.D. Panagiotopoulos in 1980s and has been of great interest recently. This is due to the intensive development of applications of hemivariational inequality to many engineering and economic fields, such as nonsmooth mechanics of solid, fluid mechanics, equilibrium problems, and so on. One may refer to the monographs [24] and [27] for more details.

Variational–hemivariational inequalities represent a special class of inequalities, in which both convex and nonconvex functionals are involved. But until now, there are still a few publications that treat these inequalities and study their applications in solid contact mechanics. In Han et al. [18] and Migorski et al. [25], the authors considered the existence and uniqueness of solution for a class of elliptic variational–

☆ The research is supported by Hundred Talent Program for Introducing the Overseas High-Level Talents of Guangxi Colleges and Universities, Guangxi Natural Science Foundation (No. 2017GXNSFBA198152), and Projects of Young Teachers Scientific Research Development Foundation of Guangxi University of Finance and Economics (No. 2017QNA03 and No. 2017QNB15).

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https://doi.org/10.1016/j.jmaa.2018.12.068
0022-247X/

hemivariational inequalities and applied these abstract results to study the static and quasistatic frictional contact problems with unilateral constraints, respectively. In Han et al. [17], the authors studied the adhesive unilateral contact between a viscoelastic body and a deformable foundation. They provided a result on unique solvability for a system consisting of a elliptic variational–hemivariational inequality and an ordinary differential equation. The existence of solutions to a class of evolutionary variational–hemivariational inequalities of the parabolic type have been considered by Carl et al. in [8] and Han et al. [19] who used the technique of lower and upper solutions and surjectivity theorem, respectively.

On the other hand, in optimal control problems we are looking for a control law in a given infinite dimensional system such that a certain optimality criterion is achieved. Such problems always include a cost functional which can be a function of state and control variables, cf. e.g. Lions [20] and Troltzsch [34]. In the last few decades, there have been published several monographs and papers focused on the topic of optimal control for hemivariational inequalities. For more details, cf. [3,10,11,15,22,23] and [29], etc.

But as far as we know, there is still no monograph or paper to study the optimal control problems for dynamic nonsmooth unilateral contact models, which modeling by a class of evolutionary variational–hemivariational inequalities. Motivated by the aforementioned contributions, in this work we are going to study the unique weak solvability and optimal control for a dynamic viscoelastic unilateral contact problem with normal damped response and friction. The present work represents a new contribution of optimal control theory for a class of parabolic variational–hemivariational inequalities and with applications to contact mechanics. The idea of this work mainly comes from the papers [11], [23] and our recently work [19].

2. The mechanical model and its weak formulation

The purpose of this section is to provide the motivation of studying abstract evolutionary variational–hemivariational inequalities in our paper, which modeling a class of dynamic unilateral contact problems between the viscoelastic body and foundation. We shall provide its classical and weak formulations. To this end, firstly, it is necessary to reaffirm some classical notation and function spaces in contact mechanics which will be useful in the sequel.

Assume the domain Ω is an open, bounded and connected set of $\mathbb{R}^d (d = 2, 3)$, and its boundary $\Gamma = \partial\Omega$ is Lipschitz continuous. Denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$, and we define the inner product and the corresponding norm of space $\mathbb{S}^d$ to

$$\sigma : \tau = \sigma_{ij}\tau_{ij} \quad \text{and} \quad \|\tau\|_{\mathbb{S}^d} = (\tau : \tau)^{1/2},$$

respectively, for all $\sigma = (\sigma_{ij}), \tau = (\tau_{ij}) \in \mathbb{S}^d$.

We introduce the spaces are as follows

$$\begin{aligned} &H = L^2(\Omega; \mathbb{R}^d), \quad \mathcal{H} = L^2(\Omega; \mathbb{S}^d) = \big\{\tau = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega)\big\} \\ &H_1 = \{v \in H \mid \varepsilon(u) \in \mathcal{H}\}, \qquad \mathcal{H}_1 = \big\{\tau \in \mathcal{H} \mid \operatorname{Div} \tau \in H\big\}, \end{aligned}$$

where notation ε and Div stand for the deformation and divergence operators, respectively, defined by

$$\varepsilon(u) = \big(\varepsilon_{ij}(u)\big), \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{1}$$

and

$$\operatorname{Div} \sigma = (\sigma_{ij,j}), \quad i, j = 1, \ldots, d. \tag{2}$$

So now, by endowing the inner products of

$$\langle u, v\rangle_H = \int_\Omega u \cdot v \, dx, \qquad \langle \sigma, \tau\rangle_{\mathcal{H}} = \int_\Omega \sigma : \tau \, dx,$$

$$\langle u, v\rangle_{H_1} = \langle u, v\rangle_H + \langle \varepsilon(u), \varepsilon(v)\rangle_{\mathcal{H}}, \quad \langle \sigma, \tau\rangle_{\mathcal{H}_1} = \langle \sigma, \tau\rangle_{\mathcal{H}} + \langle \operatorname{Div} \sigma, \operatorname{Div} \tau\rangle_H,$$

it is clear that $H, \mathcal{H}, H_1$ and $\mathcal{H}_1$ are Hilbert spaces.

2.1. Classical formulation

Consider a viscoelastic body occupies the open, bounded and connected domain Ω of $\mathbb{R}^d$, and its Lipschitz continuous boundary Γ is composed of three disjoint measurable parts Γ_D, Γ_N and Γ_C such that the measure of Γ_D is positive. Let $(0, T)$, with $T > 0$, denote the finite time interval of interest. The body is in contact on Γ_C with the foundation. The displacement field $u \colon \Omega \times (0, T) \to \mathbb{R}^d$ and the stress field $\sigma \colon \Omega \times (0, T) \to \mathbb{S}^d$ play the roles of unknowns in our frictional contact problem.

We model the viscoelastic body by the so-called Kelvin–Voigt constitutive equation, given by

$$\sigma(t) = \mathcal{R}\big(t, \varepsilon(u'(t))\big) + \mathcal{E}\varepsilon(u(t)) \quad \text{in } \Omega \times (0, T), \tag{3}$$

where the viscosity operator $\mathcal{R}$ may depend on both the time and the location of the point, the elasticity operator $\mathcal{E}$ is allowed to depend on the location of the point, the notation ε represents the linearized (or small) strain tensor which is defined by (1), and σ denotes the stress tensor.

The equation of motion, which is derived from the fundamental principle of momentum conservation (cf. [31] and the references therein), is used to describe the evolution of the mechanical state of the viscoelastic body, i.e.,

$$u''(t) = \operatorname{Div} \sigma(t) + f_0(t) \quad \text{in } \Omega \times (0, T), \tag{4}$$

f_0 stands for the density of applied volume forces, and Div represents the divergence operator given by (2). Without loss of generality, we assume in (4) that the mass density is equal to one.

Assume that the viscoelastic body is fixed on the part Γ_D of the surface, which means that the displacement field vanishes on Γ_D, hence

$$u(t) = 0 \quad \text{on } \Gamma_D \times (0, T). \tag{5}$$

We suppose that the surface tractions of density f_N act on Γ_N, which implies that

$$\sigma(t)\nu = f_N(t) \text{ on } \Gamma_N \times (0, T), \tag{6}$$

where ν denotes the unit outward normal vector on the boundary Γ.

Next, we turn to the description of the normal contact condition on Γ_C, i.e.,

$$\begin{cases} u'_\nu(t) \le g, \ \sigma_\nu(t) + p(t, u'_\nu(t)) \le 0, \\ \big(u'_\nu(t) - g\big)\big(\sigma_\nu(t) + p(t, u'_\nu(t))\big) = 0 \end{cases} \quad \text{on } \Gamma_C \times (0, T), \tag{7}$$

where σ_ν and u'_ν denote the normal stress and the normal velocity, respectively, $g \ge 0$ represents a non-negative constant, and the normal damping function $p \ge 0$ depend on both the time and the location of the point, which is reasonable since the values of the contact function p can change over time if the material temperature is changed during the contact process. We mention that this type of contact condition is called unilateral contact law with normal damped response and has been studied in [14], where the normal damping function p was assumed to be independent of time variable. Even more specifically, letting $p = 0$ and $g = 0$ in (7), we get the Signorini-type contact condition in velocities of the form

$$u'_\nu(t) \le 0, \quad \sigma_\nu(t) \le 0, \quad u'_\nu(t)\,\sigma_\nu(t) = 0 \quad \text{on} \quad \Gamma_C \times (0,T),$$

which has been proposed in some works, cf. e.g., [13] and [33], etc.

We employ the so-called slip rate-dependent friction law to describe the tangential friction condition on the part Γ_C, i.e.,

$$\begin{cases} u'_\tau(t) = 0 \implies \|\sigma_\tau(t)\|_{\mathbb{R}^d} \le F_b(t, u'(t)), \\ u'_\tau(t) \neq 0 \implies -\sigma_\tau(t) = F_b(t, u'(t)) \dfrac{u'_\tau(t)}{\|u'_\tau(t)\|_{\mathbb{R}^d}} \end{cases} \quad \text{on } \Gamma_C \times (0,T), \tag{8}$$

in which the friction bound $F_b \ge 0$ is supposed to depend on the time variable, the location of the point, and the velocity on the contact boundary Γ_C. For the details of physical background on friction law (8), cf. e.g., [6], [14], [24], [30] and [32].

Finally, we give the initial values of the displacement and velocity in our model, i.e.,

$$u(0) = a \;\text{ and }\; u'(0) = b \;\text{ in }\; \Omega. \tag{9}$$

So now, collecting the equations and conditions (3)–(9), and providing the classical formulation of the dynamic unilateral contact problem is as follows.

Problem 2.1. Find a displacement field $u\colon \Omega \times (0,T) \to \mathbb{R}^d$ and a stress field $\sigma\colon \Omega \times (0,T) \to \mathbb{S}^d$ such that for a.e. $t \in (0,T)$,

$$\sigma(t) = \mathcal{R}\big(t, \varepsilon(u'(t))\big) + \mathcal{E}\varepsilon(u(t)) \quad \text{in} \quad \Omega, \tag{10}$$

$$u''(t) = \mathrm{Div}\,\sigma(t) + f_0(t) \quad \text{in} \quad \Omega, \tag{11}$$

$$u(t) = 0 \quad \text{on} \quad \Gamma_D, \tag{12}$$

$$\sigma(t)\nu = f_N(t) \quad \text{on} \quad \Gamma_N, \tag{13}$$

$$\begin{cases} u'_\nu(t) \le g,\ \sigma_\nu(t) + p(t, u'_\nu(t)) \le 0, \\ \big(u'_\nu(t) - g\big)\big(\sigma_\nu(t) + p(t, u'_\nu(t))\big) = 0 \end{cases} \quad \text{on} \quad \Gamma_C, \tag{14}$$

$$\begin{cases} u'_\tau(t) = 0 \implies \|\sigma_\tau(t)\|_{\mathbb{R}^d} \le F_b(t, u'(t)), \\ u'_\tau(t) \neq 0 \implies -\sigma_\tau(t) = F_b(t, u'(t)) \dfrac{u'_\tau(t)}{\|u'_\tau(t)\|_{\mathbb{R}^d}} \end{cases} \quad \text{on} \quad \Gamma_C, \tag{15}$$

$$u(0) = a,\ u'(0) = b \quad \text{in} \quad \Omega. \tag{16}$$

2.2. Variational formulation

To the end of deriving the variational formulation of Problem 2.1 in this subsection, we first define a closed subspace of H_1 by

$$V = \big\{\, v \in H_1 \,|\, v = 0 \;\text{ on }\; \Gamma_D \,\big\}.$$

Meanwhile, V^* denotes its dual space. Since the condition $\mathrm{meas}(\Gamma_D) > 0$ and Korn's inequality, we observe that the space V is a real Hilbert space equipped with the inner product

$$\langle u, v\rangle_V = \langle \varepsilon(u), \varepsilon(v)\rangle_{\mathcal{H}} \quad \text{for all} \quad u, v \in V$$

and the associated norm $\|\cdot\|_V$. Furthermore, by the continuity of the trace operator, we have

$$\|v\|_{L^2(\Gamma_C;\mathbb{R}^d)} \le C_0\|v\|_V \ \text{ for all } \ v \in V \tag{17}$$

with the constant $C_0 > 0$ which depend on only the domain Ω, Γ_D and Γ_C.

Next, it is necessary to list some hypotheses on the data of Problem 2.1. Assume that

$\underline{H(\mathcal{R})}$: the nonlinear viscosity operator $\mathcal{R}\colon \Omega \times (0,T) \times \mathbb{S}^d \to \mathbb{S}^d$ is such that

(a) $\mathcal{R}(\cdot,\cdot,\varepsilon)$ is measurable on $\Omega \times (0,T)$, for all $\varepsilon \in \mathbb{S}^d$,
(b) $\mathcal{R}(x,t,\cdot)$ is continuous on $\mathbb{S}^d$, for a.e. $(x,t) \in \Omega \times (0,T)$,
(c) $\|\mathcal{R}(x,t,\varepsilon)\|_{\mathbb{S}^d} \le \bar{a}_0(x,t) + \bar{a}_1\|\varepsilon\|_{\mathbb{S}^d}$ for all $\varepsilon \in \mathbb{S}^d$ and a.e. $(x,t) \in \Omega \times (0,T)$ with $\bar{a}_0 \in L^2(\Omega \times (0,T))$, $\bar{a}_0 \ge 0$ and $\bar{a}_1 > 0$,
(d) $\big(\mathcal{R}(x,t,\varepsilon_1) - \mathcal{R}(x,t,\varepsilon_2)\big) : \big(\varepsilon_1 - \varepsilon_2\big) \ge 0$ for all ε_1, $\varepsilon_2 \in \mathbb{S}^d$, a.e. $(x,t) \in \Omega \times (0,T)$,
(e) $\mathcal{R}(x,t,\varepsilon) : \varepsilon \ge m_{\mathcal{R}}\|\varepsilon\|^2_{\mathbb{S}^d}$ for all $\varepsilon \in \mathbb{S}^d$, a.e. $(x,t) \in \Omega \times (0,T)$ with $m_{\mathcal{R}} > 0$,
(f) $\big(\mathcal{R}(x,t,\varepsilon_1) - \mathcal{R}(x,t,\varepsilon_2)\big) : \big(\varepsilon_1 - \varepsilon_2\big) \ge \alpha_{\mathcal{R}}\|\varepsilon_1 - \varepsilon_2\|^2_{\mathbb{S}^d}$, for all ε_1, $\varepsilon_2 \in \mathbb{S}^d$, a.e. $(x,t) \in \Omega \times (0,T)$ with $\alpha_{\mathcal{R}} > 0$.

$\underline{H(\mathcal{E})}$: the elasticity operator $\mathcal{E}\colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is such that

(a) $\mathcal{E}(x,\varepsilon) = E(x)\varepsilon$, for all $\varepsilon \in \mathbb{S}^d$, a.e. $x \in \Omega$,
(b) $E(x) = (E_{ijkl}(x))$ with $E_{ijkl} = E_{jikl} = E_{lkij} \in L^\infty(\Omega)$,
(c) $E_{ijkl}(x)\varepsilon_{ij}\varepsilon_{kl} \ge 0$ for all symmetric tensors $\varepsilon = (\varepsilon_{ij}) \in \mathbb{S}^d$ and a.e. $x \in \Omega$.

$\underline{H(p)}$: the normal damping function $p\colon \Gamma_C \times (0,T) \times \mathbb{R} \to \mathbb{R}_+$ is such that

(a) $p(\cdot,\cdot,r)$ is measurable on $\Gamma_C \times (0,T)$ for all $r \in \mathbb{R}$,
(b) for a.e. $(x,t) \in \Gamma_C \times (0,T)$, $p(x,t,\cdot)$ is continuous on $\mathbb{R}$, and

$$p(x,t,r) \le c_{0\nu}(x,t) + c_{1\nu}|r|$$

for all $r \in \mathbb{R}$ with $c_{0\nu} \in L^\infty(\Gamma_C \times (0,T))$, $c_{0\nu}$, $c_{1\nu} \ge 0$,
(c) $|p(x,t,r_1) - p(x,t,r_2)| \le L_p|r_1 - r_2|$ for all r_1, $r_2 \in \mathbb{R}$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ with $L_p \ge 0$.

$\underline{H(F_b)}$: the friction bound $F_b\colon \Gamma_C \times (0,T) \times \mathbb{R}^d \to \mathbb{R}_+$ is such that

(a) $F_b(\cdot,\cdot,s)$ is measurable on $\Gamma_C \times (0,T)$ for all $s \in \mathbb{R}^d$,
(b) for a.e. $(x,t) \in \Gamma_C \times (0,T)$, $F_b(x,t,\cdot)$ is continuous on $\mathbb{R}^d$, and

$$F_b(x,t,s) \le c_{0\tau}(x,t) + c_{1\tau}\|s\|_{\mathbb{R}^d}$$

for all $s \in \mathbb{R}^d$ with $c_{0\tau} \in L^\infty(\Gamma_C \times (0,T))$, $c_{0\tau}, c_{1\tau} \ge 0$,
(c) $|F_b(x,t,s_1) - F_b(x,t,s_2)| \le L_{F_b}\|s_1 - s_2\|_{\mathbb{R}^d}$ for all s_1, $s_2 \in \mathbb{R}^d$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ with $L_{F_b} \ge 0$.

And moreover, the densities of volume forces and surface tractions satisfy

$$f_0 \in L^2(0,T;H),\ f_N \in L^2(0,T;L^2(\Gamma_N;\mathbb{R}^d)). \tag{18}$$

Finally, the given initial displacement and velocity satisfy

$$a,\, b \in V, \quad b_\nu(x) \in (-\infty, g) \text{ for a.e. } x \in \Gamma_C. \tag{19}$$

In the following, we define two new functions $j_\nu \colon \Gamma_C \times (0,T) \times \mathbb{R} \to \mathbb{R}$ and $j_\tau \colon \Gamma_C \times (0,T) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ by

$$j_\nu(x,t,r) = \int_0^r p(x,t,\bar{r})\, d\bar{r}, \tag{20}$$

$$j_\tau(x,t,s,\bar{s}) = F_b(x,t,s)g(\bar{s}), \;\; g(\bar{s}) = \|\bar{s}\|_{\mathbb{R}^d}, \tag{21}$$

respectively, for all $r \in \mathbb{R}$, s, $\bar{s} \in \mathbb{R}^d$ and a.e. $(x,t) \in \Gamma_C \times (0,T)$.

In view of the hypotheses of $H(p)(a)$–(c) and $H(F_b)(a)$–(c), we may obtain the following two lemmas, respectively.

Lemma 2.2. *Let the hypotheses $H(p)(a)$, (b) hold. Then the function j_ν defined by (20) satisfies that*
$\underline{P(j_\nu)}$: *$j_\nu \colon \Gamma_C \times (0,T) \times \mathbb{R} \to \mathbb{R}$ is such that*

(a) $j_\nu(\cdot,\cdot,r)$ is measurable on $\Gamma_C \times (0,T)$ for all $r \in \mathbb{R}$ and there exists a function $e_\nu \in L^2(\Gamma_C)$ such that $j_\nu(\cdot,\cdot,e_\nu(\cdot)) \in L^1\big(\Gamma_C \times (0,T)\big)$,
(b) $j_\nu(x,t,\cdot)$ is locally Lipschitz on $\mathbb{R}$, for a.e. $(x,t) \in \Gamma_C \times (0,T)$,
(c) $|\zeta| \le c_{0\nu}(x,t) + c_{1\nu}|r|$ for all $r \in \mathbb{R}$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ with $\zeta \in \partial j_\nu(x,t,r)$, $c_{0\nu} \in L^\infty(\Gamma_C \times (0,T))$ and $c_{0\nu}, c_{1\nu} \ge 0$, ∂j_ν denotes the Clarke subdifferential of j_ν with respect to the last variable,
(d) either $j_\nu(x,t,\cdot)$ or $-j_\nu(x,t,\cdot)$ is regular on $\mathbb{R}$ for a.e. $(x,t) \in \Gamma_C \times (0,T)$.

Moreover, in addition, if the hypothesis $H(p)(c)$ holds, then

(e) $(\zeta_1 - \zeta_2)(r_1 - r_2) \ge -L_p|r_1 - r_2|^2$ for all $\zeta_i \in \partial j_\nu(x,t,r_i)$, $r_i \in \mathbb{R}$, $i = 1,2$ and a.e. $(x,t) \in \Gamma_C \times (0,T)$ with $L_p \ge 0$.

Lemma 2.3. *Assume that the hypotheses $H(F_b)(a)$, (b) hold. Then the function j_τ defined by (21) fulfills the condition*
$\underline{P(j_\tau)}$: *$j_\tau \colon \Gamma_C \times (0,T) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is such that*

(a) $j_\tau(\cdot,\cdot,s,\bar{s})$ is measurable on $\Gamma_C \times (0,T)$ for all s, $\bar{s} \in \mathbb{R}^d$ and there exists a function $e_\tau \in L^2(\Gamma_C;\mathbb{R}^d)$ such that $j_\tau\big(\cdot,\cdot,w(\cdot),e_\tau(\cdot)\big) \in L^1\big(\Gamma_C \times (0,T)\big)$ for all $w \in L^2(\Gamma_C;\mathbb{R}^d)$,
(b) for a.e. $(x,t) \in \Gamma_C \times (0,T)$, $j_\tau(x,t,\cdot,\bar{s})$ is continuous on $\mathbb{R}^d$ for all $\bar{s} \in \mathbb{R}^d$, and $j_\tau(x,t,s,\cdot)$ is locally Lipschitz on $\mathbb{R}^d$ for all $s \in \mathbb{R}^d$,
(c) $\|\xi\|_{\mathbb{R}^d} \le c_{0\tau}(x,t) + c_{1\tau}(\|s\|_{\mathbb{R}^d} + \|\bar{s}\|_{\mathbb{R}^d})$ for all s, $\bar{s} \in \mathbb{R}^d$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ with $\xi \in \partial j_\tau(x,t,s,\bar{s})$, $c_{0\tau} \in L^\infty(\Gamma_C \times (0,T))$ and $c_{0\tau}$, $c_{1\tau} \ge 0$, ∂j_τ denotes the Clarke subdifferential of j_τ with respect to the last variable,
(d) either $j_\tau(x,t,s,\cdot)$ or $-j_\tau(x,t,s,\cdot)$ is regular on $\mathbb{R}^d$ for all $s \in \mathbb{R}^d$, a.e. $(x,t) \in \Gamma_C \times (0,T)$,
(e) $j^0_\tau(x,t,\cdot,\cdot;\rho)$ is upper semicontinuous on $\mathbb{R}^d \times \mathbb{R}^d$ for all $\rho \in \mathbb{R}^d$, a.e. $(x,t) \in \Gamma_C \times (0,T)$.

Furthermore, in addition, if the hypothesis $H(F_b)(c)$ holds, then

(f) $(\xi_1 - \xi_2)\cdot(\bar{s}_1 - \bar{s}_2) \ge -L_{F_b}(\|s_1 - s_2\|_{\mathbb{R}^d} + \|\bar{s}_1 - \bar{s}_2\|_{\mathbb{R}^d})\|\bar{s}_1 - \bar{s}_2\|_{\mathbb{R}^d}$ for all $\xi_i \in \partial j_\tau(x,t,s_i,\bar{s}_i)$, s_i, $\bar{s}_i \in \mathbb{R}^d$, $i = 1,2$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ with $L_{F_b} \ge 0$.

Remark 2.4. Like the expression in Lemma 2.2 and Lemma 2.3, we should mention that the Clarke subdifferential which is appeared in the sequel that is always understood with respect to the last variable of the corresponding nonsmooth function.

Proof of Lemma 2.2. Note that the property $P(j_\nu)(a)$ can be directly obtained under the hypotheses $H(p)(a)$, (b). To prove $P(j_\nu)(b)$, let r_1, $r_2 \in B(r,\epsilon)$, $r \in \mathbb{R}$, $\epsilon > 0$. By the definition (20), we have

$$|j_\nu(x,t,r_1) - j_\nu(x,t,r_2)| \le \Big|\int_{r_1}^{r_2} p(x,t,\bar r)\,d\bar r\Big| \le L_\nu(r,\epsilon)|r_1 - r_2|$$

with $L_\nu(r,\epsilon) = \max_{\hat r\in B(r,\epsilon)} p(x,t,\hat r)$ for a.e. $(x,t) \in \Gamma_C \times (0,T)$. This shows that the function $j_\nu(x,t,\cdot)$ is locally Lipschitz on $\mathbb{R}$.

Next, since the function $p(x,t,\cdot)$ is continuous on $\mathbb{R}$, for a.e. $(x,t) \in \Gamma_C \times (0,T)$, which together with the definition of Clarke subdifferential, it is clear that

$$\partial j_\nu(x,t,r) = p(x,t,r) \quad \text{for all} \quad r \in \mathbb{R} \text{ and a.e. } (x,t) \in \Gamma_C \times (0,T). \tag{22}$$

Therefore, the property $P(j_\nu)(c)$ holds under the hypothesis $H(p)(b)$. Moreover, since the function $j_\nu(x,t,\cdot)$ is strictly differentiable, hence $P(j_\nu)(d)$ is satisfied.

To conclude the proof, it remains to show the property $P(j_\nu)(e)$. In fact, under the hypothesis $H(p)(c)$, this conclusion follows directly from (22). We finish the proof. □

Proof of Lemma 2.3. Similarly as the proof of Lemma 2.2, the properties $P(j_\tau)(a)$, (b), (d), (e) can be verified under the hypotheses $H(F_b)(a)$, (b) and the definition (21). To show $P(j_\tau)(c)$, we use the definitions of the Clarke subdifferential and (21) again, to see that

$$\partial j_\tau(x,t,s,\bar s) = \begin{cases} F_b(x,t,s)\bar B(0,1) & \text{if } \bar s = 0, \\ F_b(x,t,s)\dfrac{\bar s}{\|\bar s\|_{\mathbb{R}^d}} & \text{if } \bar s \ne 0 \end{cases} \tag{23}$$

for a.e. $(x,t) \in \Gamma_C \times (0,T)$ and all s, $\bar s \in \mathbb{R}^d$, in which $\bar B(0,1)$ denotes the closed unit ball in $\mathbb{R}^d$. Therefore, taking into account $H(F_b)(b)$, a simple computation shows that the condition $P(j_\tau)(c)$ holds.

Finally, we check the property $P(j_\tau)(f)$ by using the hypothesis $H(F_b)(c)$ and (23). Indeed, from the relation $\xi_i \in \partial j_\tau(x,t,s_i,\bar s_i)$, for all ξ_i, s_i, $\bar s_i \in \mathbb{R}^d$, $i = 1,2$ and a.e. $(x,t) \in \Gamma_C \times (0,T)$, we get

$$\begin{aligned}(\xi_1 - \xi_2)\cdot(\bar s_1 - \bar s_2) &\ge -L_{F_b}\|s_1 - s_2\|_{\mathbb{R}^d}\|\bar s_1 - \bar s_2\|_{\mathbb{R}^d} \\ &\ge -L_{F_b}(\|s_1 - s_2\|_{\mathbb{R}^d} + \|\bar s_1 - \bar s_2\|_{\mathbb{R}^d})\|\bar s_1 - \bar s_2\|_{\mathbb{R}^d}\end{aligned}$$

with $L_{F_b} \ge 0$. This completes the proof. □

Lemma 2.5. *The frictional contact conditions (14)–(15) lead to the following subdifferential forms*

$$\begin{cases} -\sigma_\nu(t) \in \partial j_\nu\big(t, u'_\nu(t)\big) + \partial_c I_{(-\infty,g]}\big(u'_\nu(t)\big), \\ -\sigma_\tau(t) \in \partial j_\tau\big(t, u'(t), u'_\tau(t)\big) \end{cases} \quad \text{on } \Gamma_C \tag{24}$$

for a.e. $t \in (0,T)$, *in which* $I_{(-\infty,\,g]}\colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ *represents the indicator function (normal cone) of the interval* $(-\infty,\, g]$ *defined by*

$$I_{(-\infty,\,g]}(r)=\begin{cases}0 & \textit{if}\;\; r\in(-\infty,\,g],\\ +\infty & \textit{otherwise},\end{cases} \tag{25}$$

and $\partial_c I_{(-\infty,g]}$ *denotes the convex subdifferential of* $I_{(-\infty,g]}$.

Proof. Firstly, it is easy to formulate the contact condition (14) by the following way

$$\begin{cases}-\sigma_\nu(t)=p\big(t,u'_\nu(t)\big) & \text{if}\;\; u'_\nu(t)<g,\\ -\sigma_\nu(t)\geq p\big(t,u'_\nu(t)\big) & \text{if}\;\; u'_\nu(t)=g\end{cases}\qquad \text{on}\;\Gamma_C,\;\;\text{for a.e.}\;\; t\in(0,T),$$

which, associated with (22), to see that

$$\begin{cases}-\sigma_\nu(t)=\partial j_\nu\big(t,u'_\nu(t)\big) & \text{if}\;\; u'_\nu(t)<g,\\ -\sigma_\nu(t)\geq \partial j_\nu\big(t,u'_\nu(t)\big) & \text{if}\;\; u'_\nu(t)=g\end{cases}\qquad \text{on}\;\Gamma_C,\;\;\text{for a.e.}\;\; t\in(0,T).$$

So now, we use the fact of

$$\partial_c I_{(-\infty,\,g]}(r)=\begin{cases}0 & \text{if}\;\; r\in(-\infty,\,g),\\ [0,+\infty) & \text{if}\;\; r=g,\\ \emptyset & \text{otherwise},\end{cases}$$

to obtain that the first subdifferential form of (24) holds.

In what follows, we check the second line of (24). Let $\xi\in\mathbb{R}^d$ and in accordance with the friction law (15), we have, for a.e. $t\in(0,T)$ and on Γ_C,

$$\begin{aligned}\sigma_\tau(t)\cdot\big(\xi-u'_\tau(t)\big)&=-F_b(t,u'(t))\frac{u'_\tau(t)}{\|u'_\tau(t)\|_{\mathbb{R}^d}}\cdot\big(\xi-u'_\tau(t)\big)\\ &\geq F_b(t,u'(t))\|u'_\tau(t)\|_{\mathbb{R}^d}-F_b(t,u'(t))\|\xi\|_{\mathbb{R}^d}\;\;\text{if}\;\; u'_\tau(t)\neq 0,\end{aligned} \tag{26}$$

which is due to $u'_\tau\cdot\xi\leq\|u'_\tau\|_{\mathbb{R}^d}\|\xi\|_{\mathbb{R}^d}$ and the non-negativity of friction coefficient F_b. And moreover, if $u'_\tau(t)=0$ for a.e. $t\in(0,T)$, it follows from (15) that

$$\|\sigma_\tau(t)\|_{\mathbb{R}^d}\leq F_b(t,u'(t))\qquad\text{on}\;\Gamma_C,\;\;\text{for a.e.}\;\; t\in(0,T),$$

which implies that, for a.e. $t\in(0,T)$ and on Γ_C,

$$\begin{aligned}\sigma_\tau(t)\cdot\big(\xi-u'_\tau(t)\big)&=\sigma_\tau(t)\cdot\xi\geq-\|\sigma_\tau(t)\|_{\mathbb{R}^d}\|\xi\|_{\mathbb{R}^d}\geq -F_b(t,u'(t))\|\xi\|_{\mathbb{R}^d}\\ &=F_b(t,u'(t))\|u'_\tau(t)\|_{\mathbb{R}^d}-F_b(t,u'(t))\|\xi\|_{\mathbb{R}^d}\;\;\text{if}\;\; u'_\tau(t)=0.\end{aligned} \tag{27}$$

Thus now, applying (21), (26), (27) and basing on the definition of Clarke subdifferential, we finally get

$$-\sigma_\tau(t)\in F_b(t,u'(t))\partial\,\|u'_\tau(t)\|_{\mathbb{R}^d}=\partial j_\tau\big(t,u'(t),u'_\tau(t)\big)\qquad\text{on}\;\Gamma_C,$$

for a.e. $t\in(0,T)$. This completes the proof. □

So now, let us assuming that the functions u and σ are sufficiently smooth and solve Problem 2.1. According to the Green formula and using the similar approach as in [19] or in [25], we may obtain the following variational formulation of Problem 2.1, in terms of the displacement field.

Problem 2.6. Find a displacement field $u\colon (0,T) \to V$ such that for all $v \in V$ and a.e. $t \in (0,T)$

$$\begin{aligned}
&\big\langle u''(t), v - u'(t)\big\rangle_{V^*\times V} + \big\langle \mathcal{R}\big(t, \varepsilon(u'(t))\big) + \mathcal{E}\varepsilon(u(t)), \varepsilon(v) - \varepsilon(u'(t))\big\rangle_{\mathcal{H}} \\
&\quad + \int_{\Gamma_C} \Big(j_\nu^0\big(t, u_\nu'(t); v_\nu - u_\nu'(t)\big) + j_\tau^0\big(t, u'(t), u_\tau'(t); v_\tau - u_\tau'(t)\big)\Big)\, d\,\Gamma \\
&\quad + \int_{\Gamma_C} I_{(-\infty,g]}\big(v_\nu\big)\, d\,\Gamma - \int_{\Gamma_C} I_{(-\infty,g]}\big(u_\nu'(t)\big)\, d\Gamma \geq \langle f(t), v - u'(t)\rangle_{V^*\times V}
\end{aligned} \tag{28}$$

with $u(0) = a$, $u'(0) = b$ and $f\colon (0,T) \to V^*$

$$\langle f(t), v\rangle_{V^*\times V} = \langle f_0(t), v\rangle_H + \langle f_N(t), v\rangle_{L^2(\Gamma_N;\mathbb{R}^d)}. \tag{29}$$

We conclude that the inequality (28) in Problem 2.6 is called an evolutionary variational–hemivariational inequality. In this inequality, the functions j_ν and j_τ are, in general, nonconvex and model the contact and friction phenomena on the contact surface Γ_C while the unbounded index function $I_{(-\infty,g]}$ describes the unilateral constraint. A couple of functions (σ, u) which satisfies (10) and (28) is called a weak solution to Problem 2.1. The weak solution admits the regularity of

$$\begin{aligned}
&u \in W^{1,2}(0,T;V), \quad u'' \in L^2(0,T;V^*), \\
&\sigma \in L^2(0,T;\mathcal{H}) \text{ and } \operatorname{Div}\sigma \in L^2(0,T;V^*).
\end{aligned}$$

3. Solvability of the frictional contact problem

In this section we will provide a result on the unique weak solvability of the frictional contact problem (10)–(16), or equivalently, on the existence and uniqueness of solutions to Problem 2.6.

We first introduce some further notation. Let $Z = H^{1-\delta}(\Omega;\mathbb{R}^d)$ with $\delta \in (0,1/2)$, Z^* denotes its dual space. Thus it is clear that the spaces (V, H, V^*) and (Z, H, Z^*) form two Gelfand triples of spaces with continuous embeddings $V \subset Z \subset H \subset Z^* \subset V^*$, the embedding $V \subset Z$ is compact with the embedding constant denoted by $c_e > 0$. As usual in the study of evolutionary problems, we need the notation of spaces

$$\begin{aligned}
&\mathcal{V} = L^2(0,T;V), \quad \mathcal{V}^* = L^2(0,T;V^*), \quad \mathcal{W} = \{v \in \mathcal{V} \,|\, v' \in \mathcal{V}^*\}, \\
&\mathcal{Z} = L^2(0,T;Z), \quad \mathcal{Z}^* = L^2(0,T;Z^*), \quad \widehat{\mathcal{H}} = L^2(0,T;H).
\end{aligned}$$

The duality pairing between $\mathcal{V}^*$ and $\mathcal{V}$ is represented by

$$\langle u^*, u\rangle_{\mathcal{V}^*\times\mathcal{V}} = \int_0^T \langle u^*(t), u(t)\rangle_{V^*\times V}\, dt \quad \text{for all} \quad (u^*, u) \in \mathcal{V}^* \times \mathcal{V}.$$

Main theorem on the existence of solutions to Problem 2.6 states that

Theorem 3.1. *Let the hypotheses $H(\mathcal{R})(a)$–(e), $H(\mathcal{E})(a)$–(c), $H(p)(a)$–(b), $H(F_b)(a)$–(b), (18) and (19) hold. If*

$$2C_0^2\big(\sqrt{3}c_{1\nu} + 2\sqrt{5}c_{1\tau}\big) < m_{\mathcal{R}}, \tag{30}$$

then Problem 2.6 has at least one solution $u \in \mathcal{V}$ such that $u' \in \mathcal{W}$.

Furthermore, we have the following uniqueness result.

Theorem 3.2. *Assume that the assumptions of Theorem 3.1 are satisfied. In addition, if the hypotheses $H(\mathcal{R})(f)$, $H(p)(c)$, $H(F_b)(c)$ and the smallness condition*

$$C_0^2\big(L_p + 2L_{F_b}\big) < \alpha_{\mathcal{R}} \tag{31}$$

hold, then Problem 2.6 admits a unique solution.

The proofs of the above two theorems will be provided as consequences of results on the existence and uniqueness of solutions to an abstract evolutionary inclusion, which will be formulated and studied in the upcoming Subsections 3.1 and 3.2.

3.1. Evolutionary inclusion

We start with the definitions of abstract operators $A\colon (0,T) \times V \to V^*$ and $B\colon V \to V^*$ which are introduced by

$$\begin{cases} \langle A(t,u), v\rangle_{V^*\times V} = \langle \mathcal{R}(t,\varepsilon(u)), \varepsilon(v)\rangle_{\mathcal{H}}, \\ \langle Bu, v\rangle_{V^*\times V} = \langle \mathcal{E}\varepsilon(u), \varepsilon(v)\rangle_{\mathcal{H}}, \end{cases} \tag{32}$$

for all u, $v \in V$ and a.e. $t \in (0,T)$.

Lemma 3.3. *Let the hypotheses $H(\mathcal{R})(a)$–(e) and $H(\mathcal{E})(a)$–(c) hold. Then the operators A and B defined by (32) satisfy*

(A1) $A(\cdot,u)$ is measurable on $(0,T)$ for all $u \in V$,
(A2) $A(t,\cdot)$ is continuous and monotone on V, for a.e. $t \in (0,T)$, and consequently, it is pseudomonotone on V, for a.e. $t \in (0,T)$,
(A3) for all $u \in V$ and a.e. $t \in (0,T)$, we have

$$\|A(t,u)\|_{V^*} \le \sqrt{2}(a_0(t) + a_1\|u\|_V)$$

with $a_0(t) \in L^2(0,T)$, $a_0(t) = \|\bar{a}_0(t)\|_{L^2(\Omega)}$ and $a_1 = \bar{a}_1$,
(A4) A is coercive, i.e., for all $u \in V$ and a.e. $t \in (0,T)$,

$$\langle A(t,u), u\rangle_{V^*\times V} \ge m_{\mathcal{R}}\|u\|_V^2 \quad \textit{with} \quad m_{\mathcal{R}} > 0,$$

(B1) *$B \in \mathcal{L}(V,V^*)$ and $\langle Bu,u\rangle_{V^*\times V} \ge 0$ for all $u \in V$,*
(B2) B is a symmetric operator, that is, $\langle Bu,v\rangle_{V^*\times V} = \langle Bv,u\rangle_{V^*\times V}$ for all u, $v \in V$.

Furthermore, in addition, if the hypothesis $H(\mathcal{R})(f)$ holds, then

(A5) for all u_1, $u_2 \in V$ and a.e. $t \in (0,T)$, we have

$$\langle A(t,u_1) - A(t,u_2), u_1 - u_2\rangle_{V^*\times V} \ge \alpha_{\mathcal{R}}\|u_1 - u_2\|_V^2 \quad \textit{with} \quad \alpha_{\mathcal{R}} > 0.$$

Proof. For the proof of the properties $(A1)$–$(A4)$, $(B1)$ and $B(2)$, we refer to Lemmas 3 and 4 in [21]. Next, we check the condition $(A5)$. Indeed, using the hypothesis $H(\mathcal{R})(f)$ and the definition (32), it is obvious that

$$\begin{aligned}\langle A(t,u_1)-A(t,u_2),u_1-u_2\rangle_{V^*\times V} &= \langle \mathcal{R}(t,\varepsilon(u_1))-\mathcal{R}(t,\varepsilon(u_2)),\varepsilon(u_1)-\varepsilon(u_2)\rangle_{\mathcal{H}} \\ &= \int_\Omega \big(\mathcal{R}(x,t,\varepsilon(u_1))-\mathcal{R}(x,t,\varepsilon(u_2))\big) : \big(\varepsilon(u_1)-\varepsilon(u_2)\big)\,dx \\ &\geq \alpha_{\mathcal{R}} \int_\Omega \|\varepsilon(u_1-u_2)\|^2_{\mathbb{S}^d}\,dx = \alpha_{\mathcal{R}}\|u_1-u_2\|^2_V\end{aligned}$$

for all u_1, $u_2 \in V$ and a.e. $t \in (0,T)$, which finishes the proof. $\square$

Subsequently, we introduce two functionals

$$\begin{cases} J_1\colon (0,T)\times L^2(\Gamma_C)\to\mathbb{R}, \\ J_2\colon (0,T)\times L^2(\Gamma_C;\mathbb{R}^d)\times L^2(\Gamma_C;\mathbb{R}^d)\to\mathbb{R}, \end{cases}$$

which are defined by

$$\begin{cases} J_1(t,w) = \displaystyle\int_{\Gamma_C} j_\nu(x,t,w(x))\,d\Gamma, \\ J_2(t,u,z) = \displaystyle\int_{\Gamma_C} j_\tau(x,t,u(x),z(x))\,d\Gamma, \end{cases} \tag{33}$$

for all $w \in L^2(\Gamma_C)$, u, $z \in L^2(\Gamma_C;\mathbb{R}^d)$ and a.e. $t \in (0,T)$.

From the proof of Theorem 3.47 in [24], we may deduce that

Lemma 3.4. *If the hypotheses $P(j_\nu)(a)$–(d) and $P(j_\tau)(a)$–(e) hold, then the functionals J_1 and J_2 defined by (33) fulfill the following conditions*

- *(J1) $J_1(\cdot,w)$ and $J_2(\cdot,u,z)$ are measurable on $(0,T)$ for all $w \in L^2(\Gamma_C)$ and u, $z \in L^2(\Gamma_C;\mathbb{R}^d)$,*
- *(J2) for a.e. $t \in (0,T)$, $J_1(t,\cdot)$ is locally Lipschitz on $L^2(\Gamma_C)$, and $J_2(t,u,\cdot)$ is locally Lipschitz on $L^2(\Gamma_C;\mathbb{R}^d)$ for all $u \in L^2(\Gamma_C;\mathbb{R}^d)$,*
- *(J3) $\|w^*\|_{L^2(\Gamma_C)} \leq \sqrt{2\,\mathrm{meas}(\Gamma_C)}\,c_{0\nu}(t) + c_{1\nu}\|w\|_{L^2(\Gamma_C)}$ for all w, $w^* \in L^2(\Gamma_C)$ with $w^* \in \partial J_1(t,w)$, a.e. $t \in (0,T)$, where ∂J_1 stands for the Clarke subdifferential of $J_1(t,\cdot)$,*
- *(J4) for all u, z, $z^* \in L^2(\Gamma_C;\mathbb{R}^d)$ and a.e. $t \in (0,T)$, we have*

$$\|z^*\|_{L^2(\Gamma_C;\mathbb{R}^d)} \leq \sqrt{3\,\mathrm{meas}(\Gamma_C)}\,c_{0\tau}(t) + c_{1\tau}\|u\|_{L^2(\Gamma_C;\mathbb{R}^d)} + c_{1\tau}\|z\|_{L^2(\Gamma_C;\mathbb{R}^d)}$$

  with $z^* \in \partial J_2(t,u,z)$, where ∂J_2 denotes the Clarke subdifferential of $J_2(t,u,\cdot)$,
- *(J5) either $J_1(t,\cdot)$ or $-J_1(t,\cdot)$ is regular on $L^2(\Gamma_C)$ for a.e. $t \in (0,T)$, and, either $J_2(t,u,\cdot)$ or $-J_2(t,u,\cdot)$ is regular on $L^2(\Gamma_C;\mathbb{R}^d)$ for all $u \in L^2(\Gamma_C;\mathbb{R}^d)$ and a.e. $t \in (0,T)$,*
- *(J6) the multifunction $\partial J_2\colon L^2(\Gamma_C;\mathbb{R}^d)\times L^2(\Gamma_C;\mathbb{R}^d)\to 2^{L^2(\Gamma_C;\mathbb{R}^d)}$ has a closed graph in the topology of $L^2(\Gamma_C;\mathbb{R}^d)\times L^2(\Gamma_C;\mathbb{R}^d)\times(w\text{-}L^2(\Gamma_C;\mathbb{R}^d))$, where w-means the weak convergence.*

Moreover, if we add the hypotheses $P(j_\nu)(e)$ and $P(j_\tau)(f)$, then it follows

- *(J7) for all w_1, w_2, ζ_1, $\zeta_2 \in L^2(\Gamma_C)$, u_1, u_2, z_1, z_2, ξ_1, $\xi_2 \in L^2(\Gamma_C;\mathbb{R}^d)$, and a.e. $t \in (0,T)$, we have*

$$\langle \zeta_1-\zeta_2, w_1-w_2\rangle_{L^2(\Gamma_C)} \geq -L_p\|w_1-w_2\|^2_{L^2(\Gamma_C)},$$

$$\langle \xi_1 - \xi_2, z_1 - z_2 \rangle_{L^2(\Gamma_C;\mathbb{R}^d)} \geq -L_{F_b} \|u_1 - u_2\|_{L^2(\Gamma_C;\mathbb{R}^d)} \|z_1 - z_2\|_{L^2(\Gamma_C;\mathbb{R}^d)}$$
$$-L_{F_b} \|z_1 - z_2\|^2_{L^2(\Gamma_C;\mathbb{R}^d)}$$

with $\zeta_i \in \partial J_1(t, w_i)$, $\xi_i \in \partial J_2(t, u_i, z_i)$, $i = 1, 2$, *and* L_p, $L_{F_b} \geq 0$.

Finally, we define the functional $\bar{\Psi}\colon L^2(\Gamma_C) \to [0, +\infty]$ by

$$\bar{\Psi}(w) = \int_{\Gamma_C} I_{(-\infty,g]}(w(x))\, d\Gamma \quad \text{for all} \quad w \in L^2(\Gamma_C), \tag{34}$$

where $I_{(-\infty,g]}$ is the indicator function defined by (25).

Lemma 3.5. *If the hypothesis (19) holds, then the functional* $\bar{\Psi}$ *defined by (34) satisfies the following conditions*

- *(P1)* $\bar{\Psi}$ *is proper, convex and lower semicontinuous, and* $b_\nu \in dom(\bar{\Psi})$,
- *(P2)* $\partial_c \bar{\Psi}(\cdot + b_\nu)$ *is strongly quasi-bounded,*
- *(P3) the graph of* $\partial_c \bar{\Psi}$ *is closed in the topology of* $L^2(\Gamma_C) \times (w\text{-}L^2(\Gamma_C))$, *where w-means the weak convergence.*

Proof. From the definitions (25) and (34), combined with the hypothesis (19), it is clear that the functional $\bar{\Psi}$ is proper, convex and $b_\nu \in dom(\bar{\Psi})$. To prove that $\bar{\Psi}$ is lower semicontinuous, let $w_n \to w$ strongly in $L^2(\Gamma_C)$, as $n \to \infty$. Subsequently, passing to a subsequence, if necessary, we have

$$w_n(x) \to w(x) \text{ in } \mathbb{R} \text{ for a.e. } x \in \Gamma_C.$$

Since the indicator function $I_{(-\infty,g]}$ is lower semicontinuous, we get

$$I_{(-\infty,g]}(w(x)) \leq \liminf I_{(-\infty,g]}(w_n(x)) \quad \text{for a.e.} \quad x \in \Gamma_C,$$

which entails that

$$\int_{\Gamma_C} I_{(-\infty,g]}(w(x))\, dx \leq \int_{\Gamma_C} \liminf I_{(-\infty,g]}(w_n(x))\, dx. \tag{35}$$

Next, employing the result of Lemma 2.5 in [26], we see that

$$\int_{\Gamma_C} \liminf I_{(-\infty,g]}(w_n(x))\, dx \geq -k_1 \int_{\Gamma_C} |w_n(x)|\, dx - k_2 \,\mathrm{meas}(\Gamma_C)$$
$$\geq -k_1 \sqrt{\mathrm{meas}(\Gamma_C)} \|w_n\|_{L^2(\Gamma_C)} - k_2 \,\mathrm{meas}(\Gamma_C)$$

with k_1, $k_2 \geq 0$. Hence, we are in a position to apply Fatou's lemma and obtain

$$\int_{\Gamma_C} \liminf I_{(-\infty,g]}(w_n(x))\, dx \leq \liminf \int_{\Gamma_C} I_{(-\infty,g]}(w_n(x))\, dx. \tag{36}$$

Therefore, combining (35), (36) and (34), we have $\bar{\Psi}(w) \leq \liminf \bar{\Psi}(w_n)$. The proof of property $(P1)$ is completed.

Next, we pass to the proof of $(P2)$. By the condition of

$$b_\nu(x) < g \quad \text{for a.e. } \ x \in \Gamma_C,$$

we deduce that there exists a closed ball $S_\epsilon(b_\nu(x))$ with the center at $b_\nu(x)$ and a radius $\epsilon(x) > 0$ small enough, for a.e. $x \in \Gamma_C$ such that $S_\epsilon(b_\nu(x)) \subset (-\infty, g)$. Thus, we may suppose that there exist some elements $w \in L^2(\Gamma_C)$ with $w(x) \in S_\epsilon(b_\nu(x))$ for a.e. $x \in \Gamma_C$ which satisfy

$$|w(x) - b_\nu(x)| \le \epsilon(x) \quad \text{for a.e. } \ x \in \Gamma_C.$$

This implies that

$$\|w - b_\nu\|_{L^2(\Gamma_C)} \le \sqrt{\text{meas}(\Gamma_C)}\, \|\epsilon\|_{L^\infty(\Gamma_C)} \quad \text{for some } w \in L^2(\Gamma_C), \tag{37}$$

where $\text{meas}(\Gamma_C)$ denotes the measure of Γ_C.

On the other hand, by the definition of the indicator function $I_{(-\infty,g]}$ and the fact of that $w(x) \in int(-\infty, g]$, it is easy to derive that for all $w^*(x) \in \partial_c I_{(-\infty,g]}(w(x))$, we have

$$w^*(x) = 0 \ \text{ for a.e. } x \in \Gamma_C.$$

Therefore, by Lemma 7.1 in [19] and the definition (34), we know

$$w^* = 0 \quad \text{ with } \ w^* \in \partial_c \bar{\Psi}(w) \ \text{ for some } w \in L^2(\Gamma_C). \tag{38}$$

Now, let u, $u^* \in L^2(\Gamma_C)$, $u^* \in \partial_c \bar{\Psi}(u + b_\nu)$, $\langle u^*, u\rangle_{L^2(\Gamma_C)} \le M$ and $\|u\|_{L^2(\Gamma_C)} \le M$ with $M > 0$. We will show that there exists a constant $K(M) > 0$ such that

$$\|u^*\|_{L^2(\Gamma_C)} \le K(M). \tag{39}$$

Using the monotonicity of the multivalued operator $\partial_c \bar{\Psi}(\cdot + b_\nu)$ (being a consequence of the monotonicity of $\partial_c \bar{\Psi}(\cdot)$), and from the conditions

$$(u, u^*), (w - b_\nu, w^*) \in Gr\big(\partial_c \bar{\Psi}(\cdot + b_\nu)\big),$$

we have

$$\langle u^* - w^*, u - w + b_\nu\rangle_{L^2(\Gamma_C)} \ge 0.$$

This inequality together with (38) implies that

$$\langle u^*, u\rangle_{L^2(\Gamma_C)} \ge \langle u^*, w - b_\nu\rangle_{L^2(\Gamma_C)} + \langle w^*, u - w + b_\nu\rangle_{L^2(\Gamma_C)} = \langle u^*, w - b_\nu\rangle_{L^2(\Gamma_C)}.$$

Since (37), we know the above inequality holds for all $(w - b_\nu) \in L^2(\Gamma_C)$ with

$$\|w - b_\nu\|_{L^2(\Gamma_C)} \le \sqrt{\text{meas}(\Gamma_C)}\, \|\epsilon\|_{L^\infty(\Gamma_C)} := r,$$

and combining with the fact (cf. Proposition 14 of [5]) of

$$\sup_{\|w - b_\nu\|_{L^2(\Gamma_C)} \le r} \big\langle u^*, w - b_\nu \big\rangle_{L^2(\Gamma_C)} = r\|u^*\|_{L^2(\Gamma_C)},$$

we may directly get the desired estimation

$$\|u^*\|_{L^2(\Gamma_C)} \le \frac{1}{r}\langle u^*, u\rangle_{L^2(\Gamma_C)} = \frac{1}{\sqrt{\operatorname{meas}(\Gamma_C)}\,\|\epsilon\|_{L^\infty(\Gamma_C)}}\langle u^*, u\rangle_{L^2(\Gamma_C)}, \tag{40}$$

i.e., the condition (39) holds. This completes the proof of property $(P2)$.

To conclude the proof of Lemma 3.5, it remains to show that the property $(P3)$ holds. Assume that $\{u_n\} \subset L^2(\Gamma_C)$ and $\eta_n \in \partial_c\bar{\Psi}(u_n)$ are such that

$$u_n \to u \ \text{ strongly in } \ L^2(\Gamma_C) \ \text{ and } \ \eta_n \to \eta \ \text{ weakly in } L^2(\Gamma_C), \ \text{as} \ n \to \infty.$$

According to the definition of the convex subdifferential and the property of the upper semicontinuity of $-\bar{\Psi}$, it is clear that $\eta \in \partial_c\bar{\Psi}(u)$. This concludes the proof of the lemma. □

Considering now the embedding and the trace operators by

$$\begin{cases} i\colon H^{1/2-\delta}(\Gamma_C;\mathbb{R}^d) \to L^2(\Gamma_C;\mathbb{R}^d), \\ \gamma_1\colon Z \to H^{1/2-\delta}(\Gamma_C;\mathbb{R}^d). \end{cases}$$

And let $\gamma\colon Z \to L^2(\Gamma_C;\mathbb{R}^d)$ be the trace operator introduced by

$$\gamma(v) = i(\gamma_1(v)) \qquad \text{for} \ \ v \in Z.$$

Using γ, we define the normal and the tangential trace operators $\gamma_\nu\colon Z \to L^2(\Gamma_C)$ and $\gamma_\tau\colon Z \to L^2(\Gamma_C;\mathbb{R}^d)$ by

$$\gamma_\nu(v) = (\gamma(v))_\nu \quad \text{and} \quad \gamma_\tau(v) = (\gamma(v))_\tau \quad \text{for} \ \ v \in Z,$$

respectively. By $\|\gamma\|$, $\|\gamma_\nu\|$ and $\|\gamma_\tau\|$, we denote the norms of

$$\|\gamma\|_{\mathcal{L}(Z;L^2(\Gamma_C;\mathbb{R}^d))}, \quad \|\gamma_\nu\|_{\mathcal{L}(Z;L^2(\Gamma_C))} \quad \text{and} \quad \|\gamma_\tau\|_{\mathcal{L}(Z;L^2(\Gamma_C;\mathbb{R}^d))},$$

respectively. The notation γ^*, γ_ν^* and γ_τ^* stand for their adjoint operators.

Combining the notation (32)–(34), the definitions of the Clarke and convex subdifferentials, and the regularity of j_ν and j_τ, we are able to prove that Problem 2.6 can be equivalently formulated in the form of following evolutionary inclusion.

Problem 3.6. Find $u \in \mathcal{V}$ such that $u' \in \mathcal{W}$, and for a.e. $t \in (0,T)$,

$$\begin{cases} u''(t) + A(t,u'(t)) + Bu(t) + \gamma_\nu^*\zeta(t) + \gamma_\tau^*\xi(t) + \gamma_\nu^*\eta(t) = f(t), \\ \zeta(t) \in \partial J_1\big(t,\gamma_\nu u'(t)\big),\ \xi(t) \in \partial J_2\big(t,\gamma\, u'(t),\gamma_\tau u'(t)\big), \\ \eta(t) \in \partial_c\bar{\Psi}\big(\gamma_\nu u'(t)\big), \end{cases} \tag{41}$$

with $u(0) = a$, $u'(0) = b$.

3.2. Existence and uniqueness of solutions

In a subsection, we shall consider the solvability of Problem 3.6. To this end, we firstly recall an important surjectivity result which is very useful in the study of existence of solutions to a class of parabolic partial differential equations and evolutionary hemivariational inequalities. For a detailed exposition and its proof, we refer to Theorem 2.2 in [19] and Theorem 3.1 in [16].

Theorem 3.7. *Let X be a reflexive Banach space which is strictly convex, and the operator $L\colon X \supset D(L) \to X^*$ be linear and maximal monotone. If an operator $A\colon X \to 2^{X^*}$ is coercive, bounded and L-pseudomonotone, and $B\colon X \to 2^{X^*}$ is a maximal monotone operator which is strongly quasi-bounded with $0 \in D(B)$, then $L + A + B$ is surjective, so $R(L + A + B) = X^*$.*

Theorem 3.8. *Assume that the hypotheses $(A1)$–$(A4)$, $(B1)$–$(B2)$, $(J1)$–$(J6)$, $(P1)$–$(P2)$, (18) and (19) hold. If*

$$2\big(\sqrt{3}c_{1\nu}\|\gamma_\nu\|^2 + \sqrt{5}c_{1\tau}\|\gamma\|\|\gamma_\tau\| + \sqrt{5}c_{1\tau}\|\gamma_\tau\|^2\big)\, c_e^2 < m_{\mathcal{R}}, \tag{42}$$

then Problem 3.6 admits at least one solution.

Proof. The proof is mainly based on the surjectivity result of Theorem 3.7. We first introduce the operator $K\colon \mathcal{V} \to C(0,T;V)$ given by

$$Kv(t) = \int_0^t v(s)\,ds + a \quad \text{for all} \quad v \in \mathcal{V}. \tag{43}$$

Then, define the Nemitsky (superposition) operators corresponding to the translations of $A(t,\cdot)$, $B(K\cdot)$, $\partial J_1(t,\gamma_\nu\cdot)$, $\partial J_2(t,\gamma\cdot,\gamma_\tau\cdot)$ and $\partial_c\bar{\Psi}(\gamma_\nu\cdot)$ by

$$(\mathcal{A}_0 v)(t) = A(t, v(t) + b), \quad (\mathcal{B}_0 v)(t) = B\big(K(v+b)(t)\big), \tag{44}$$

$$(\mathcal{N}_0 v)(t) = \big\{\zeta \in L^2(0,T;L^2(\Gamma_C)) \mid \zeta(t) \in \partial J_1(t, \gamma_\nu(v(t)+b))\big\}, \tag{45}$$

$$\begin{aligned}(\bar{\mathcal{N}}_0 v)(t) = \big\{\xi \in L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d)) \mid \\ \xi(t) \in \partial J_2(t, \gamma(v(t)+b), \gamma_\tau(v(t)+b))\big\},\end{aligned} \tag{46}$$

$$(\mathcal{M}_0 v)(t) = \big\{\eta \in L^2(0,T;L^2(\Gamma_C)) \mid \eta(t) \in \partial_c\bar{\Psi}(\gamma_\nu(v(t)+b))\big\} \tag{47}$$

for all $v \in \mathcal{V}$ and a.e. $t \in (0,T)$. Hence, Problem 3.6 is equivalent to the following operator inclusion

$$\begin{cases} \text{find } v \in \mathcal{W} \text{ such that} \\ v' + \mathcal{A}_0 v + \mathcal{B}_0 v + \widehat{\gamma}_\nu^* \mathcal{N}_0 v + \widehat{\gamma}_\tau^* \bar{\mathcal{N}}_0 v + \widehat{\gamma}_\nu^* \mathcal{M}_0 v \ni f, \\ v(0) = 0, \end{cases} \tag{48}$$

where $\widehat{\gamma}_\nu^*\colon L^2(0,T;L^2(\Gamma_C)) \to \mathcal{Z}^*$ and $\widehat{\gamma}_\tau^*\colon L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d)) \to \mathcal{Z}^*$ introduced by

$$(\widehat{\gamma}_\nu^*\zeta)(t) = \gamma_\nu^*\zeta(t), \qquad (\widehat{\gamma}_\tau^*\xi)(t) = \gamma_\tau^*\xi(t)$$

for all $\zeta \in L^2(0,T;L^2(\Gamma_C))$, $\xi \in L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d))$ and a.e. $t \in (0,T)$, represent the Nemitsky operators corresponding to γ_ν^* and γ_τ^*, respectively.

So now, to the purpose of solve Problem 3.6, we only need to show the existence of solutions to operator inclusion (48). Having in mind the operator defined by (43), we note that

$$\begin{cases} v \in \mathcal{W} \text{ is a solution to (48) if and only if} \\ u = K(v+b) \text{ with } u \in \mathcal{V}, u' \in \mathcal{W} \ \text{ solves Problem 3.6.} \end{cases} \tag{49}$$

Define the operator $L\colon \mathcal{V} \supset D(L) \to \mathcal{V}^*$ by $Lv = v'$ which is considered with the domain

$$D(L) = \{\, u \in \mathcal{W} \mid v(0) = 0 \,\}.$$

Thus it is clear that L is a linear and maximal monotone operator. Next, referring to Lemmas 3.9 and 3.10 which are showed later in this subsection, we know that the multivalued operator $\mathcal{T}_0\colon \mathcal{V} \to 2^{\mathcal{V}^*}$ given by $\mathcal{T}_0 = \mathcal{A}_0 + \mathcal{B}_0 + \widehat{\gamma}_\nu^* \mathcal{N}_0 + \widehat{\gamma}_\tau^* \bar{\mathcal{N}}_0$ is coercive, bounded and L-pseudomonotone, and moreover, $\widehat{\gamma}_\nu^* \mathcal{M}_0$ is a maximal monotone operator which is strongly quasi-bounded with $0 \in D(\widehat{\gamma}_\nu^* \mathcal{M}_0)$. Hence, in view of the surjectivity result in Theorem 3.7, we infer that the inclusion (48) has at least one solution $v \in \mathcal{W}$. This, together with (49) implies that Problem 3.6 has at least one solution $u \in \mathcal{V}$ such that $u' \in \mathcal{W}$. This concludes the proof. □

In the following we provide two lemmas we used in the proof of Theorem 3.8.

Lemma 3.9. *Let $\mathcal{T}_0\colon \mathcal{V} \to 2^{\mathcal{V}^*}$ be the operator defined by $\mathcal{T}_0 = \mathcal{A}_0 + \mathcal{B}_0 + \widehat{\gamma}_\nu^* \mathcal{N}_0 + \widehat{\gamma}_\tau^* \bar{\mathcal{N}}_0$, where the operators $\mathcal{A}_0$, $\mathcal{B}_0$, $\mathcal{N}_0$ and $\bar{\mathcal{N}}_0$ are given by (44)–(46). If the hypotheses $(A1)$–$(A4)$, $(B1)$, $(B2)$, $(J1)$–$(J6)$ hold, and the smallness condition*

$$2\big(\sqrt{3}c_{1\nu}\|\gamma_\nu\|^2 + \sqrt{5}c_{1\tau}\|\gamma\|\|\gamma_\tau\| + \sqrt{5}c_{1\tau}\|\gamma_\tau\|^2\big)\, c_e^2 < m_{\mathcal{R}} \tag{50}$$

is satisfied, then the operator $\mathcal{T}_0$ is coercive, bounded and L-pseudomonotone.

Proof. The proof will carried out in several steps.

Step 1. We prove that the multivalued operator $\mathcal{T}_0$ is coercive in the sense of Definition A.3. To this end, let $v \in \mathcal{V}$ and $v^* = \mathcal{T}_0 v$, i.e., $v^* = \mathcal{A}_0 v + \mathcal{B}_0 v + \widehat{\gamma}_\nu^* \zeta + \widehat{\gamma}_\tau^* \xi$ with $\zeta \in \mathcal{N}_0 v$ and $\xi \in \bar{\mathcal{N}}_0 v$. According to the hypotheses $(A3)$, $(A4)$, $(B1)$ and $(B2)$, we have

$$\begin{aligned}
\langle \mathcal{A}_0 v + \mathcal{B}_0 v, v\rangle_{\mathcal{V}^*\times\mathcal{V}} &= \int_0^T \langle A(t, v(t) + b) + B(K(v+b)(t)), v(t)\rangle_{V^*\times V}\, dt \\
&\geq m_{\mathcal{R}} \int_0^T \|v(t) + b\|_V^2\, dt - \int_0^T \langle A(t, v(t) + b), b\rangle_{V^*\times V}\, dt \\
&\quad + \int_0^T \langle B(K(v+b)(t)), (K(v+b))'(t) - b\rangle_{V^*\times V}\, dt \\
&\geq \frac{m_{\mathcal{R}}}{2}\|v\|_{\mathcal{V}}^2 - (\hat{a}_1 + \hat{b}_1)\|v\|_{\mathcal{V}} - (\hat{a}_2 + \hat{b}_2)
\end{aligned} \tag{51}$$

for all $v \in \mathcal{V}$ with

$$\begin{aligned}
&\hat{a}_1 = \sqrt{2T}a_1\|b\|_V, \quad \hat{b}_1 = T\sqrt{T}\|B\|\|b\|_V, \\
&\hat{a}_2 = \sqrt{2T}\|a_0\|_{L^2(0,T)}\|b\|_V + (m_{\mathcal{R}} + \sqrt{2}a_1 T)\|b\|_V^2, \\
&\hat{b}_2 = 1/2\|B\|\|a\|_V^2 + T\|B\|\|a\|_V\|b\|_V + T^2\|B\|\|b\|_V^2.
\end{aligned}$$

Since $\zeta \in \mathcal{N}_0 v$ and $\xi \in \bar{\mathcal{N}}_0 v$, it is clear that

$$\zeta(t) \in \partial J_1\big(t, \gamma_\nu(v(t) + b)\big), \tag{52}$$

$$\xi(t) \in \partial J_2\big(t, \gamma(v(t) + b), \gamma_\tau(v(t) + b)\big) \tag{53}$$

for all $v \in \mathcal{V}$ and a.e. $t \in (0, T)$. Hence, taking into account the hypotheses $(J3)$ and $(J4)$, we obtain

$$
\begin{aligned}
&\|\widehat{\gamma}_\nu^* \zeta\|_{\mathcal{V}^*}^2 = \int_0^T \|\gamma_\nu^* \zeta(t)\|_{V^*}^2 \, dt \le \|\gamma_\nu^*\|^2 \, c_e^2 \int_0^T \|\zeta(t)\|_{L^2(\Gamma_C)}^2 \, dt \\
&\le \|\gamma_\nu\|^2 \, c_e^2 \int_0^T \Big(\sqrt{2 \operatorname{meas}(\Gamma_C)} \, c_{0\nu}(t) + c_{1\nu} \|\gamma_\nu(v(t) + b)\|_{L^2(\Gamma_C)}\Big)^2 \, dt \\
&\le \|\gamma_\nu\|^2 \, c_e^2 \int_0^T 3\big(2 \operatorname{meas}(\Gamma_C) \, c_{0\nu}^2(t) + c_{1\nu}^2 \|\gamma_\nu\|^2 c_e^2 \|v(t)\|_V^2 + c_{1\nu}^2 c_e^2 \|\gamma_\nu\|^2 \|b\|_V^2\big) \, dt \\
&\le 6T \operatorname{meas}(\Gamma_C) \|c_{0\nu}\|_{L^\infty(0,T)}^2 \|\gamma_\nu\|^2 \, c_e^2 \\
&\qquad + 3 c_{1\nu}^2 \, c_e^4 \|\gamma_\nu\|^4 \|v\|_{\mathcal{V}}^2 + 3T c_{1\nu}^2 \, c_e^4 \|\gamma_\nu\|^4 \|b\|_V^2
\end{aligned}
\tag{54}
$$

and

$$
\begin{aligned}
&\|\widehat{\gamma}_\tau^* \xi\|_{\mathcal{V}^*}^2 \le \|\gamma_\tau\|^2 \, c_e^2 \int_0^T \|\xi(t)\|_{L^2(\Gamma_C;\mathbb{R}^d)}^2 \, dt \\
&\le 15T \operatorname{meas}(\Gamma_C) \|c_{0\tau}\|_{L^\infty(0,T)}^2 \|\gamma_\tau\|^2 \, c_e^2 \\
&\quad + 5 \, c_{1\tau}^2 \, c_e^4 (\|\gamma\|^2 + \|\gamma_\tau\|^2) \|\gamma_\tau\|^2 \|v\|_{\mathcal{V}}^2 \\
&\quad + 5T \, c_{1\tau}^2 \, c_e^4 (\|\gamma\|^2 + \|\gamma_\tau\|^2) \|\gamma_\tau\|^2 \|b\|_V^2,
\end{aligned}
\tag{55}
$$

for all $v \in \mathcal{V}$. Thus, we combine (54) and (55) to see that

$$
\begin{aligned}
\langle \widehat{\gamma}_\nu^* \zeta + \widehat{\gamma}_\tau^* \xi, v \rangle_{\mathcal{V}^* \times \mathcal{V}} &\ge -(\|\widehat{\gamma}_\nu^* \zeta\|_{\mathcal{V}^*} + \|\widehat{\gamma}_\tau^* \xi\|_{\mathcal{V}^*}) \|v\|_{\mathcal{V}} \\
&\ge -\big(\sqrt{3} c_{1\nu} \|\gamma_\nu\|^2 + \sqrt{5} c_{1\tau} \|\gamma\| \|\gamma_\tau\| + \sqrt{5} c_{1\tau} \|\gamma_\tau\|^2\big) \, c_e^2 \|v\|_{\mathcal{V}}^2 - \hat{c} \|v\|_{\mathcal{V}}
\end{aligned}
$$

for all $v \in \mathcal{V}$ with $\hat{c} \ge 0$. Hence, in accordance with (51) and the smallness condition (50), we deduce that the multivalued operator $\mathcal{T}_0$ is coercive in the sense of Definition A.3.

Step 2. We prove that $\mathcal{T}_0$ is a bounded operator. According to $(A3)$ and $(B1)$, we have

$$
\begin{aligned}
&\|\mathcal{A}_0 v + \mathcal{B}_0 v\|_{\mathcal{V}^*}^2 \le \int_0^T \|A(t, v(t) + b)\|_{V^*}^2 \, dt + \int_0^T \|B(K(v + b)(t))\|_{V^*}^2 \, dt \\
&\le 2 \int_0^T (a_0(t) + a_1 \|v(t)\|_V + a_1 \|b\|_V)^2 \, dt + \|B\|^2 \int_0^T \|K(v + b)(t)\|_V^2 \, dt \\
&\le 6 \|a_0\|_{L^2(0,T)}^2 + (6a_1^2 T + 3T^3 \|B\|^2) \|b\|_V^2 \\
&\qquad + 3T \|B\|^2 \|a\|_V^2 + (6a_1^2 + 3T^2 \|B\|^2) \|v\|_{\mathcal{V}}^2
\end{aligned}
$$

for all $v \in \mathcal{V}$. On the other hand, exploiting (54) and (55), we have

$$
\|\widehat{\gamma}_\nu^* \mathcal{N}_0 v + \widehat{\gamma}_\tau^* \bar{\mathcal{N}}_0 v\|_{\mathcal{V}^*} \le c(1 + \|v\|_{\mathcal{V}})
$$

for all $v \in \mathcal{V}$ with $c > 0$. Hence, we easily obtain that the operator $\mathcal{T}_0$ is bounded.

Step 3. We verify that the multivalued operator $\mathcal{T}_0$ satisfies the condition (a) in Definition A.4. In fact, by Proposition 3.23(iv) in [24], we know that the values of ∂J_1 and ∂J_2 are nonempty and convex subsets of $L^2(\Gamma_C)$ and $L^2(\Gamma_C;\mathbb{R}^d)$, respectively. These imply that the sets $\mathcal{T}_0 v$ are nonempty and convex for all $v \in \mathcal{V}$. To prove that $\mathcal{T}_0 v$ is closed in $\mathcal{V}^*$ for all $v \in \mathcal{V}$, we need to check only that both $\mathcal{N}_0 v$ and $\bar{\mathcal{N}}_0 v$ are closed in $\mathcal{V}^*$ for all $v \in \mathcal{V}$. For this purpose, let $\{\zeta_n\} \subset \mathcal{N}_0 v$, $\{\xi_n\} \subset \bar{\mathcal{N}}_0 v$, $n \geq 1$ be such that

$$\begin{aligned} &\zeta_n \to \zeta \quad \text{strongly in } \ L^2(0,T;L^2(\Gamma_C)), \quad \text{ as } \ n \to \infty, \\ &\xi_n \to \xi \quad \text{strongly in } \ L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d)), \ \text{ as } \ n \to \infty. \end{aligned}$$

Thus, subsequently, by passing to a subsequence, if necessary, we may assume that

$$\begin{cases} \zeta_n(t) \to \zeta(t) \quad \text{strongly in } \ L^2(\Gamma_C) \quad \text{for a.e. } t \in (0,T), \\ \xi_n(t) \to \xi(t) \quad \text{strongly in } \ L^2(\Gamma_C;\mathbb{R}^d) \quad \text{for a.e. } t \in (0,T), \end{cases}$$

as $n \to \infty$. From the hypothesis $(J6)$ and Proposition 3.23(v) in [24], we deduce that $\zeta \in \mathcal{N}_0 v$ and $\zeta \in \bar{\mathcal{N}}_0 v$ for all $v \in \mathcal{V}$. Consequently, the sets $\mathcal{T}_0 v$ are closed in $\mathcal{V}^*$ for all $v \in \mathcal{V}$. Furthermore, by Step 2, it is clear that the sets $\mathcal{T}_0 v$ are also bounded in $\mathcal{V}^*$ for all $v \in \mathcal{V}$.

Step 4. We prove that the multivalued operator $\mathcal{T}_0 \colon \mathcal{V} \to 2^{\mathcal{V}^*}$ is upper semicontinuous in the topology of $\mathcal{V} \times (w\text{-}\mathcal{V}^*)$. By Proposition 3.8 in [24], it is enough to check that the set

$$\mathcal{T}_0^-(\mathcal{K}) = \{\, v \in \mathcal{V} \mid \mathcal{T}_0 v \cap \mathcal{K} \neq \emptyset \,\}$$

is closed in $\mathcal{V}$ for every weakly closed set $\mathcal{K} \subset \mathcal{V}^*$. For this purpose, let

$$v_n \to v \ \text{ strongly in } \mathcal{V}, \quad \text{as } \ n \to \infty,$$

where $v_n \in \mathcal{T}_0^-(\mathcal{K})$, so there exists $v_n^* \in \mathcal{V}^*$ such that $v_n^* \in \mathcal{T}_0(v_n) \cap \mathcal{K}$. By the boundedness of the operator $\mathcal{T}_0$ (cf. Step 2), we may assume that there exists a subsequence, for simplicity, the subscript is still denoted by n such that

$$v_n^* \to v^* \ \text{ weakly in } \mathcal{V}^*, \quad \text{as } \ n \to \infty.$$

It follows from the weak closedness of the set $\mathcal{K}$ that

$$v^* \in \mathcal{K}. \tag{56}$$

In what follows, it is enough to verify that $v^* \in \mathcal{T}_0 v$. To this end, we prove that the single-valued operators $\mathcal{A}_0$ and $\mathcal{B}_0$ are demicontinuous, and the multivalued operators $\mathcal{N}_0$ and $\bar{\mathcal{N}}_0$ have closed graphs in the topologies $\mathcal{V} \times \big(w\text{-}L^2(0,T;L^2(\Gamma_C))\big)$ and $\mathcal{V} \times \mathcal{V} \times \big(w\text{-}L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d))\big)$, respectively.

Since both operators A and B are pseudomonotone and bounded, so in view of Proposition 27.7 in [35], we know that the operators A and B are demicontinuous. According to (44) and the definition of demicontinuity, we still have the demicontinuity of the Nemitsky operators $\mathcal{A}_0$ and $\mathcal{B}_0$.

On the other hand, using the hypothesis $(J6)$ and Proposition 3.23(v) in [24] again, similarly as the above proofs of closedness of operators $\mathcal{N}_0$ and $\bar{\mathcal{N}}_0$, we may derive that the multivalued operators $\mathcal{N}_0$ and $\bar{\mathcal{N}}_0$ have closed graphs in the topologies of $\mathcal{V} \times \big(w\text{-}L^2(0,T;L^2(\Gamma_C))\big)$ and $\mathcal{V} \times \mathcal{V} \times \big(w\text{-}L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d))\big)$, respectively. Hence,

$$v^* \in \mathcal{T}_0 v. \tag{57}$$

By (56), we obtain $v^* \in \mathcal{T}_0 v \cap \mathcal{K}$, i.e., $\mathcal{T}_0^-(\mathcal{K})$ is a closed set in $\mathcal{V}$ for every weakly closed set $\mathcal{K} \subset \mathcal{V}^*$. Therefore, the multivalued operator $\mathcal{T}_0 \colon \mathcal{V} \to 2^{\mathcal{V}^*}$ is upper semicontinuous from $\mathcal{V}$ to $\mathcal{V}^*$ endowed with the weak topology.

Step 5. It remains to check the condition (d) in Definition A.4. Let $\{v_n\} \subset D(L)$, $v_n^* \in \mathcal{T}_0 v_n$, $v_n \to v$ weakly in $\mathcal{W}$, $v_n^* \to v^*$ weakly in $\mathcal{V}^*$ and

$$\limsup \langle v_n^*, v_n - v\rangle_{\mathcal{V}^* \times \mathcal{V}} \le 0. \tag{58}$$

We prove that $v^* \in \mathcal{T}_0 v$ and $\langle v_n^*, v_n\rangle_{\mathcal{V}^* \times \mathcal{V}} \to \langle v^*, v\rangle_{\mathcal{V}^* \times \mathcal{V}}$.

Since the embedding $\mathcal{W} \subset \mathcal{Z}$ is compact, so we get

$$v_n \to v \quad \text{strongly in } \mathcal{Z}. \tag{59}$$

Next, using the boundedness of the operators $\mathcal{N}_0$ and $\bar{\mathcal{N}}_0$, we may assume that

$$\begin{cases} \zeta_n \to \zeta \ \ \text{weakly in } \ L^2(0,T;L^2(\Gamma_C)) \ \ \text{and} \\ \xi_n \to \xi \ \ \text{weakly in } \ L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d)), \end{cases} \tag{60}$$

as $n \to \infty$, where $\zeta_n \in \mathcal{N}_0 v_n$ and $\xi_n \in \bar{\mathcal{N}}_0 v_n$. Thus, combining (59), (60) and the fact that both $\mathcal{N}_0$ and $\bar{\mathcal{N}}_0$ have closed graphs in the topologies of $\mathcal{Z} \times \big(w\text{-}L^2(0,T;L^2(\Gamma_C))\big)$ and $\mathcal{Z} \times \mathcal{Z} \times \big(w\text{-}L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d))\big)$, respectively, we deduce

$$\zeta \in \mathcal{N}_0 v \quad \text{ and } \quad \xi \in \bar{\mathcal{N}}_0 v. \tag{61}$$

On the other hand, by (58) and (60), we have

$$\begin{aligned} \limsup \langle \mathcal{A}_0 v_n + \mathcal{B}_0 v_n, v_n - v\rangle_{\mathcal{V}^* \times \mathcal{V}} &\le \limsup \langle v_n^*, v_n - v\rangle_{\mathcal{V}^* \times \mathcal{V}} \\ &+ \lim \langle \widehat{\gamma}_\nu^* \zeta_n + \widehat{\gamma}_\tau^* \xi_n, v - v_n\rangle_{\mathcal{Z}^* \times \mathcal{Z}} \le 0. \end{aligned}$$

Therefore, from the L-pseudomonotonicity of the operators $\mathcal{A}_0$ and $\mathcal{B}_0$ (obtained in Theorem 2(b) in [4]), we have

$$\mathcal{A}_0 v_n + \mathcal{B}_0 v_n \to \mathcal{A}_0 v + \mathcal{B}_0 v \quad \text{weakly in } \mathcal{V}^* \tag{62}$$

and

$$\langle \mathcal{A}_0 v_n + \mathcal{B}_0 v_n, v_n - v\rangle_{\mathcal{V}^* \times \mathcal{V}} \to 0. \tag{63}$$

Finally, collecting (59)–(63) and passing to the limit in the equation

$$v_n^* = \mathcal{A}_0 v_n + \mathcal{B}_0 v_n + \widehat{\gamma}_\nu^* \zeta_n + \widehat{\gamma}_\tau^* \xi_n,$$

we infer that $v^* = \mathcal{A}_0 v + \mathcal{B}_0 v + \widehat{\gamma}_\nu^* \zeta + \widehat{\gamma}_\tau^* \xi \in \mathcal{A}_0 v + \mathcal{B}_0 v + \widehat{\gamma}_\nu^* \mathcal{N}_0 v + \widehat{\gamma}_\tau^* \bar{\mathcal{N}}_0 v = \mathcal{T}_0 v$ and

$$\begin{aligned} \lim \langle v_n^*, v_n\rangle_{\mathcal{V}^* \times \mathcal{V}} &= \lim \langle \mathcal{A}_0 v_n + \mathcal{B}_0 v_n, v_n - v\rangle_{\mathcal{V}^* \times \mathcal{V}} \\ &+ \lim \langle \mathcal{A}_0 v_n + \mathcal{B}_0 v_n, v\rangle_{\mathcal{V}^* \times \mathcal{V}} + \lim \langle \widehat{\gamma}_\nu^* \zeta_n + \widehat{\gamma}_\tau^* \xi_n, v_n\rangle_{\mathcal{Z}^* \times \mathcal{Z}} \\ &= \lim \langle \mathcal{A}_0 v + \mathcal{B}_0 v, v\rangle_{\mathcal{V}^* \times \mathcal{V}} + \lim \langle \widehat{\gamma}_\nu^* \zeta + \widehat{\gamma}_\tau^* \xi, v\rangle_{\mathcal{Z}^* \times \mathcal{Z}} = \langle v^*, v\rangle_{\mathcal{V}^* \times \mathcal{V}}. \end{aligned}$$

Now, we are in a position to combine *Steps 3–5* to see that the operator $\mathcal{T}_0$ is pseudomonotone with respect to $D(L)$. This finishes the proof of the lemma. □

Lemma 3.10. *Let the hypotheses* $(P1)$ *and* $(P2)$ *hold. Then the multivalued operator* $\widehat{\gamma}_\nu^*\mathcal{M}_0\colon \mathcal{V} \to 2^{\mathcal{V}^*}$ *with* $\mathcal{M}_0$ *defined by (47), is maximal monotone and strongly quasi-bounded with* $0 \in D(\widehat{\gamma}_\nu^*\mathcal{M}_0)$.

Proof. From Proposition A.7 and the definition (47), we can observe that for all $v \in \mathcal{V}$, $\eta \in \mathcal{M}_0 v$ if and only if

$$\eta \in \partial_c \widehat{\Psi}\big(\widehat{\gamma}_\nu(v+b)\big) \qquad \text{for all } \ v \in \mathcal{V} \tag{64}$$

where

$$\widehat{\Psi}(z) = \int_0^T \bar{\Psi}(z(t))\,dt \quad \text{for all } \ z \in L^2(0,T;L^2(\Gamma_C)). \tag{65}$$

Moreover, using a technique similar to that used to obtain the condition $(P1)$ in Lemma 3.5, we know the functional $\widehat{\Psi}$ is proper, convex and lower semicontinuous, thus its convex subdifferential $\partial_c\widehat{\Psi}$ is maximal monotone.

We show the operator $\widehat{\gamma}_\nu^*\mathcal{M}_0$ is monotone. To this end, let $(u, \widehat{\gamma}_\nu^* u^*)$, $(v, \widehat{\gamma}_\nu^* v^*) \in Gr(\widehat{\gamma}_\nu^*\mathcal{M}_0)$. It is clear that

$$\langle \widehat{\gamma}_\nu^*(u^*-v^*), u-v\rangle_{\mathcal{V}^*\times\mathcal{V}} = \langle u^*-v^*, \widehat{\gamma}_\nu(u+b-v-b)\rangle_{L^2(0,T;L^2(\Gamma_C))}$$

with $u^* \in \partial_c\widehat{\Psi}\big(\widehat{\gamma}_\nu(u+b)\big)$ and $v^* \in \partial_c\widehat{\Psi}\big(\widehat{\gamma}_\nu(v+b)\big)$. Using the monotonicity of $\partial_c\widehat{\Psi}$, it is obvious that the operator $\widehat{\gamma}_\nu^*\mathcal{M}_0$ is monotone.

Next, assume that for all $(v, \widehat{\gamma}_\nu^* v^*) \in Gr\big(\widehat{\gamma}_\nu^*\mathcal{M}_0\big)$, it holds

$$\langle \widehat{\gamma}_\nu^* v^* - \widehat{\gamma}_\nu^*\eta, v - z\rangle_{\mathcal{V}^*\times\mathcal{V}} \ge 0 \quad \text{for all } \ z \in \mathcal{V}.$$

Hence, we have

$$\langle \widehat{\gamma}_\nu^* v^* - \widehat{\gamma}_\nu^*\eta, v - z\rangle_{\mathcal{V}^*\times\mathcal{V}} = \langle v^*-\eta, \widehat{\gamma}_\nu(v+b-z-b)\rangle_{L^2(0,T;L^2(\Gamma_C))} \ge 0.$$

Due to the maximal monotonicity of the convex subdifferential $\partial_c\widehat{\Psi}$, we obtain $\eta \in \partial_c\widehat{\Psi}(\widehat{\gamma}_\nu(z+b))$, which implies the maximal monotonicity of the operator $\widehat{\gamma}_\nu^*\mathcal{M}_0$.

In what follows, we check that the operator $\widehat{\gamma}_\nu^*\mathcal{M}_0$ is strongly quasi-bounded. In fact, by (64), this is equivalent to verify the strong quasi-boundedness of the operator $\widehat{\gamma}_\nu^*\partial_c\widehat{\Psi}\big(\widehat{\gamma}_\nu(\cdot+b)\big)$. To this end, we suppose that for all $(v, \widehat{\gamma}_\nu^*\eta) \in Gr\big(\widehat{\gamma}_\nu^*\partial_c\widehat{\Psi}(\widehat{\gamma}_\nu(\cdot+b))\big)$, there exists a constant $\hat{k} > 0$ such that

$$\|v\|_{\mathcal{V}} \le \hat{k} \quad \text{ and } \quad \langle \widehat{\gamma}_\nu^*\eta, v\rangle_{\mathcal{V}^*\times\mathcal{V}} \le \hat{k}.$$

Then, by the definition (65) and Proposition A.7, using a technique analogous to that we used in the proof of (40) (cf. the proof of Lemma 3.5), we have

$$\begin{aligned} \|\widehat{\gamma}_\nu^*\eta\|_{\mathcal{V}^*} &\le \|\gamma_\nu^*\|c_e\|\eta\|_{L^2(0,T;L^2(\Gamma_C))} \\ &\le \frac{\|\gamma_\nu^*\|c_e}{\sqrt{T\,\mathrm{meas}(\Gamma_C)}\|\epsilon\|_{L^\infty(\Gamma_C)}}\,\langle \widehat{\gamma}_\nu^*\eta, v\rangle_{\mathcal{V}^*\times\mathcal{V}} \end{aligned} \tag{66}$$

with $\eta \in \partial_c\widehat{\Psi}(\widehat{\gamma}_\nu(v+b))$. Therefore, we deduce that $\widehat{\gamma}_\nu^*\partial_c\widehat{\Psi}\big(\widehat{\gamma}_\nu(\cdot+b)\big)$ is strongly quasi-bounded, and so is the operator $\widehat{\gamma}_\nu^*\mathcal{M}_0$. Furthermore, the property $0 \in D(\widehat{\gamma}_\nu^*\mathcal{M}_0)$ follows directly from the facts $dom(\bar{\Psi}) \subset D(\partial_c\bar{\Psi})$ and $(P1)$, and the definition (47).

Finally, to conclude the proof of Lemma 3.10, it remains to show the operator $\mathcal{M}_0$ is well defined. Multiplying (48) in duality by $v \in \mathcal{V}$, and using the proof of coercivity of the operator $\mathcal{T}_0$ in Lemma 3.9, we obtain

$$k_1 \|v\|_{\mathcal{V}}^2 \le k_2 \|v\|_{\mathcal{V}} + k_3 \tag{67}$$

for all $v \in \mathcal{V}$ with k_1, k_2, $k_3 > 0$, where the coefficients k_2 and k_3 depend on the data $\|a\|_V$, $\|b\|_V$ and $\|f\|_{\mathcal{V}^*}$. Hence, we immediately get

$$\|v\|_{\mathcal{V}} \le c_1 \quad \text{with} \;\; c_1 > 0. \tag{68}$$

Moreover, taking into account the boundedness of the operator $\mathcal{T}_0$, one has

$$\langle \widehat{\gamma}_\nu^* \eta, v\rangle_{\mathcal{V}^* \times \mathcal{V}} = \langle f - v' - \mathcal{T}_0 v, v\rangle_{\mathcal{V}^* \times \mathcal{V}} \le \big(\|f\|_{\mathcal{V}^*} + \|\mathcal{T}_0 v\|_{\mathcal{V}^*}\big)\|v\|_{\mathcal{V}} \le c_2 \tag{69}$$

for $\eta \in \mathcal{M}_0 v$, for all $v \in \mathcal{V}$ with $c_2 > 0$. So, combining (68) and (69) with the fact of that the operator $\widehat{\gamma}_\nu^* \mathcal{M}_0$ is strongly quasi-bounded, we know that

$$\|\widehat{\gamma}_\nu^* \eta\|_{\mathcal{V}^*} \le c_3 \;\; \text{for } \eta \in \mathcal{M}_0 v, \;\; v \in \mathcal{V} \tag{70}$$

with $c_3 > 0$, which means that the operator $\mathcal{M}_0$ is well defined. The proof of the lemma is completed. □

Theorem 3.11. *Assume the hypotheses of Theorem 3.8. If the hypotheses* $(A5)$, $(J7)$ *hold, and the smallness condition*

$$\big(L_p \|\gamma_\nu\|^2 + L_{F_b}\|\gamma\|\|\gamma_\tau\| + L_{F_b}\|\gamma_\tau\|^2\big) c_e^2 < \alpha_{\mathcal{R}} \tag{71}$$

is satisfied, then the solution of Problem 3.6 is unique.

Proof. Let u_1, $u_2 \in \mathcal{V}$ be two solutions to Problem 3.6 such that u_1', $u_2' \in \mathcal{W}$, i.e., they meet the following system

$$\begin{cases} u_i''(t) + A(t, u_i'(t)) + Bu_i(t) + \gamma_\nu^* \zeta_i(t) + \gamma_\tau^* \xi_i(t) + \gamma_\nu^* \eta_i(t) = f(t), \\ \zeta_i(t) \in \partial J_1\big(t, \gamma_\nu u_i'(t)\big), \;\; \xi_i(t) \in \partial J_2\big(t, \gamma\, u_i'(t), \gamma_\tau u_i'(t)\big), \\ \eta_i(t) \in \partial_c \bar{\Psi}\big(\gamma_\nu u_i'(t)\big), \; u_i(0) = a, \; u_i'(0) = b \end{cases} \tag{72}$$

for a.e. $t \in (0, T)$ and $i = 1$, 2. According to the hypotheses $(A5)$, $(B1)$, $(B2)$, by a simple calculation, we have

$$\int_0^T \big\langle u_1''(t) - u_2''(t), u_1'(t) - u_2'(t)\big\rangle_{V^* \times V}\, dt$$
$$= \frac{1}{2}\int_0^T \frac{d}{dt}\|u_1'(t) - u_2'(t)\|_H^2\, dt = \frac{1}{2}\|u_1'(T) - u_2'(T)\|_H^2, \tag{73}$$

$$\int_0^T \big\langle A(t, u_1'(t)) - A(t, u_2'(t)), u_1'(t) - u_2'(t)\big\rangle_{V^* \times V}\, dt \ge \alpha_{\mathcal{R}}\|u_1' - u_2'\|_{\mathcal{V}}^2, \tag{74}$$

$$\begin{aligned}
&\int_0^T \big\langle Bu_1(t) - Bu_2(t), u_1'(t) - u_2'(t)\big\rangle_{V^*\times V}\, dt \\
&\quad = \frac{1}{2}\int_0^T \frac{d}{dt}\big\langle Bu_1(t) - Bu_2(t), u_1(t) - u_2(t)\big\rangle_{V^*\times V}\, dt \\
&\quad = \frac{1}{2}\big\langle Bu_1(T) - Bu_2(T), u_1(T) - u_2(T)\big\rangle_{V^*\times V} \geq 0.
\end{aligned} \tag{75}$$

Next, because of the hypotheses $(J7)$ and $(P1)$, we know that

$$\begin{aligned}
&\int_0^T \big\langle \zeta_1(t) - \zeta_2(t), \gamma_\nu u_1'(t) - \gamma_\nu u_2'(t)\big\rangle_{V^*\times V}\, dt \\
&\quad + \int_0^T \big\langle \xi_1(t) - \xi_2(t), \gamma_\tau u_1'(t) - \gamma_\tau u_2'(t)\big\rangle_{V^*\times V}\, dt \\
&\quad\quad + \int_0^T \big\langle \eta_1(t) - \eta_2(t), \gamma_\nu u_1'(t) - \gamma_\nu u_2'(t)\big\rangle_{V^*\times V}\, dt \\
&\geq -\big(L_p\|\gamma_\nu\|^2 + L_{F_b}\|\gamma_\tau\|^2\big)c_e^2\|u_1' - u_2'\|_{\mathcal{V}}^2 - L_{F_b}\|\gamma\|\|\gamma_\tau\|c_e^2\|u_1' - u_2'\|_{\mathcal{V}}^2,
\end{aligned} \tag{76}$$

with $\zeta_i(t) \in \partial J_1\big(t, \gamma_\nu u_i'(t)\big)$, $\xi_i(t) \in \partial J_2\big(t, \gamma\, u_i'(t), \gamma_\tau u_i'(t)\big)$ and $\eta_i(t) \in \partial_c \bar{\Psi}\big(\gamma_\nu u_i'(t)\big)$ for a.e. $t \in (0, T)$, $i = 1, 2$. Hence, we combine (72)–(76) to see that

$$\alpha_{\mathcal{R}}\|u_1' - u_2'\|_{\mathcal{V}}^2 - \big(L_p\|\gamma_\nu\|^2 + L_{F_b}\|\gamma\|\|\gamma_\tau\| + L_{F_b}\|\gamma_\tau\|^2\big)c_e^2\|u_1' - u_2'\|_{\mathcal{V}}^2 \leq 0,$$

which, in view of the smallness condition (71) and the equality

$$u(t) = \int_0^t u'(s)\, ds + a \quad \text{for} \quad t \in (0, T)$$

implies that Problem 3.6 has a unique solution $u \in \mathcal{V}$ such that $u' \in \mathcal{W}$. We finish the proof. $\square$

Lemma 3.12. *Let the hypotheses of Theorem 3.8 hold and u be a solution to Problem 3.6. Then there exists a positive constant $c > 0$ such that*

$$\|u\|_{\mathcal{V}} + \|u'\|_{\mathcal{V}} + \|u''\|_{\mathcal{V}^*} \leq c, \tag{77}$$

where the constant $c > 0$ depends only on $\|a\|_V$, $\|b\|_V$ and $\|f\|_{\mathcal{V}^*}$.

Proof. According to (49) and (68), we see that

$$\|u' - b\|_{\mathcal{V}} = \|v\|_{\mathcal{V}} \leq \bar{c}_1 \quad \text{with} \quad \bar{c}_1 > 0, \tag{78}$$

where $v \in \mathcal{W}$ represents solutions to inclusion (48). This implies that

$$\|u'\|_{\mathcal{V}} \leq \bar{c}_2 \quad \text{with} \quad \bar{c}_2 > 0. \tag{79}$$

Note that for all $u \in W^{1,2}(0,T;V)$, the following relation holds

$$u(t) = a + \int_0^t u'(s)\,ds \quad \text{for all} \quad t \in [0,T].$$

Hence, by (79) and the Hölder inequality, we have

$$\|u\|_{\mathcal{V}} \le \bar{c}_3 \quad \text{with} \quad \bar{c}_3 > 0. \tag{80}$$

Next, we show the bound for $\|u''\|_{\mathcal{V}^*}$. From the boundedness of the operator $\mathcal{T}_0$, (78), (70), (48) and (49), we obtain

$$\begin{aligned} \|u''\|_{\mathcal{V}^*} = \|v'\|_{\mathcal{V}^*} &\le \|\mathcal{A}_0 v\|_{\mathcal{V}^*} + \|\mathcal{B}_0 v\|_{\mathcal{V}^*} + \|\widehat{\gamma}_\nu^* \zeta\|^2_{\mathcal{V}^*} \\ &+ \|\widehat{\gamma}_\tau^* \xi\|_{\mathcal{V}^*} + \|\widehat{\gamma}_\nu^* \eta\|_{\mathcal{V}^*} + \|f\|_{\mathcal{V}^*} \le \bar{c}_4, \end{aligned} \tag{81}$$

where $\zeta \in \mathcal{N}_0 v$, $\xi \in \bar{\mathcal{N}}_0 v$ and $\eta \in \mathcal{M}_0 v$ for all $v \in \mathcal{V}$ with $\bar{c}_4 > 0$. The estimate (77) is now a consequence of the inequalities (79), (80) and (81). The proof is completed. $\square$

We conclude Section 3 with the proofs of Theorems 3.1 and 3.2 which are based on our abstract results.

Proof of Theorem 3.1. From Lemmas 2.2–2.3 and Lemmas 3.3–3.5, we know that under the hypotheses $H(\mathcal{R})(a)$–(e), $H(\mathcal{E})(a)$–(c), $H(p)(a)$–(b), $H(F_b)(a)$–(b), (18) and (19), the conditions $(A1)$–$(A4)$, $(B1)$–$(B2)$, $(J1)$–$(J6)$, $(P1)$–$(P2)$, (18) and (19) of Theorem 3.8 are satisfied by the definitions (20), (21), (32), (33) and (34). The smallness condition (42) directly follows from (30), (17) and the fact that $\|\gamma\| = C_0$, $\|\gamma_\nu\| \le C_0$ and $\|\gamma_\tau\| \le C_0$. Therefore, by Theorem 3.8, we immediately obtain that Problem 2.6 has at least one solution $u \in \mathcal{V}$ such that $u' \in \mathcal{W}$. The proof is completed. $\square$

Proof of Theorem 3.2. It follows from Lemmas 2.2–2.3 and Lemmas 3.3–3.5 that the conditions $(A5)$ and $(J7)$ hold under the hypotheses $H(\mathcal{R})(f)$, $H(p)(c)$ and $H(F_b)(c)$. Analogously as in the proof of existence, the condition (71) is derived from (31), (17) and the fact that $\|\gamma\| = C_0$, $\|\gamma_\nu\| \le C_0$ and $\|\gamma_\tau\| \le C_0$. Then, by Theorem 3.11, we obtain that the solution of Problem 2.6 is unique, which completes the proof. $\square$

4. Optimal control problems

The main goal of this section is to study the optimal control problem of the unilateral contact problem with friction. We are concerned with the optimal control via the external forces and initial conditions, the time optimal control problem, and the maximum stay control problem. Consequently, we are lead to study optimal control problems for the evolutionary inclusion in Problem 3.6. We deliver conditions which guarantee the existence of optimal solutions to the corresponding control problems.

4.1. Optimal control via external forces and initial conditions

We start with a problem in which the control variable is given by $q = (f, a, b) \in \mathcal{Q}$, f represents the density of external forces (such as boundary traction and gravity, etc.), a and b denote the initial displacement and velocity, respectively, and $\mathcal{Q} = \mathcal{V}^* \times V \times V$ stands for the space of controls. For every $q \in \mathcal{Q}$, we denote the solution set of Problem 3.6 by

$$\mathcal{S}(q) = \left\{\, y \in \mathcal{S} \mid y = y(q) = (u(q), u'(q)) \text{ is a solution of Problem 3.6} \,\right\},$$

where $\mathcal{S} = \mathcal{V} \times \mathcal{W}$. Note that if the hypotheses of Theorem 3.8 hold, then the solution set $\mathcal{S}(q) \subset \mathcal{S}$ can contain more than one element for every $q \in \mathcal{Q}$. In this case, we can consider the multivalued mapping $\mathcal{S} \colon \mathcal{Q} \to 2^{\mathcal{V}\times\mathcal{W}}$ which to a control $q \in \mathcal{Q}$ assigns the solution set $\mathcal{S}(q)$.

Next, given a nonempty admissible set of controls $\mathcal{Q}_{ad} \subset \mathcal{Q}$, and an objective functional $F \colon \mathcal{Q} \times \mathcal{S} \to \mathbb{R}$, we formulate the optimal control problem as follows

$$\begin{aligned} &\text{find a control } q^* \in \mathcal{Q}_{ad} \text{ and a state } y^* = y(q^*) \in \mathcal{S} \text{ such that} \\ &F(q^*, y^*) = \inf\big\{F(q,y) \mid (q,y) \in \mathcal{Q}_{ad} \times \mathcal{S}(q)\big\}. \end{aligned} \tag{82}$$

In what follows, to study the above optimal control problem, we need to introduce assumptions on the admissible set of controls and the objective functional. Assume that

$(\mathcal{Q}_{ad})$ the admissible set of controls $\mathcal{Q}_{ad}$ is a weakly compact subset of $\mathcal{Q}$,
(F) the objective functional F is lower semicontinuous with respect to the topology of w-$(\mathcal{Q} \times \mathcal{S})$.

We begin with the following crucial result on the dependence of the solution set of Problem 3.6 on the control.

Lemma 4.1. *Assume that the hypotheses of Theorem 3.8, $(\mathcal{Q}_{ad})$ and (F) hold. Then the solution map $\mathcal{S} \colon \mathcal{Q} \ni q \mapsto \mathcal{S}(q) \subset \mathcal{S}$ has a closed graph in $(w\text{-}\mathcal{Q}) \times (w\text{-}\mathcal{S})$ topology.*

Proof. We observe that according to Theorem 3.8, for every $q \in \mathcal{Q}$, the set $\mathcal{S}(q)$ is nonempty. Let $\{q_n\} \subset \mathcal{Q}$, $q_n \to q$ weakly in $\mathcal{Q}$, $y_n \in \mathcal{S}(q_n)$, $y_n \to y$ weakly in $\mathcal{V} \times \mathcal{W}$. Hence, there is a sequence $\{(f_n, a_n, b_n)\} \subset \mathcal{V}^* \times V \times V$ such that

$$f_n \to f \quad \text{weakly in} \quad \mathcal{V}^*, \tag{83}$$

$$a_n \to a \quad \text{weakly in} \quad V, \tag{84}$$

$$b_n \to b \quad \text{weakly in} \quad V, \tag{85}$$

as $n \to \infty$, and for each $n \in \mathbb{N}$, $(u_n, u_n') \in \mathcal{V} \times \mathcal{W}$ solves the following inclusion

$$\begin{cases} u_n''(t) + A(t, u_n'(t)) + Bu_n(t) + \gamma_\nu^*\zeta_n(t) + \gamma_\tau^*\xi_n(t) + \gamma_\nu^*\eta_n(t) = f_n(t), \\ \zeta_n(t) \in \partial J_1\big(t, \gamma_\nu u_n'(t)\big),\ \xi_n(t) \in \partial J_2\big(t, \gamma\, u_n'(t), \gamma_\tau u_n'(t)\big), \\ \eta_n(t) \in \partial_c \bar{\Psi}\big(\gamma_\nu u_n'(t)\big),\ u_n(0) = a_n,\ u_n'(0) = b_n \end{cases} \tag{86}$$

for a.e. $t \in (0, T)$. By the estimates obtained in Lemma 3.12, we directly get

$$u_n \to u \quad \text{weakly in} \quad \mathcal{V}, \tag{87}$$

$$u_n' \to u' \quad \text{weakly in} \quad \mathcal{V}, \tag{88}$$

$$u_n'' \to u'' \quad \text{weakly in} \quad \mathcal{V}^*, \tag{89}$$

as $n \to \infty$. So now, it is enough to check that $(u, u') \in \mathcal{S}$ is a solution to the limit problem, i.e., (u, u') satisfies (41).

Firstly, since the embeddings $W^{1,2}(0,T;V) \subset C(0,T,V)$, $\mathcal{W} \subset C(0,T,H)$ and $V \subset H$ are continuous, hence by (84), (85), (87)–(89), and Lemma 2.55 in [24], we have

$$u(0) = a \quad \text{and} \quad u'(0) = b.$$

Secondly, because of the compact embedding $\mathcal{W} \subset \mathcal{Z}$, we immediately get

$$u_n'(t) \to u'(t) \quad \text{strongly in} \quad Z, \quad \text{for a.e. } t \in (0,T), \tag{90}$$

as $n \to \infty$. By $(J3)$, $(J4)$ and the estimate (77), we know that

$$\|\zeta_n\|_{L^2(0,T;L^2(\Gamma_C))} \leq \tilde{\varrho}_1 \quad \text{and} \quad \|\xi_n\|_{L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d))} \leq \tilde{\varrho}_2, \tag{91}$$

where $\zeta_n(t) \in \partial J_1(t, \gamma_\nu u_n'(t))$ and $\xi_n(t) \in \partial J_2(t, \gamma\, u_n'(t), \gamma_\tau u_n'(t))$ for a.e. $t \in (0,T)$ with $\tilde{\varrho}_i > 0$, $i = 1, 2$. This, together with (77) and the first equality in (86) imply that

$$\|\eta_n\|_{L^2(0,T;L^2(\Gamma_C))} \leq \tilde{\varrho}_3 \tag{92}$$

with $\eta_n(t) \in \partial_c \bar{\Psi}(\gamma_\nu u_n'(t))$ for a.e. $t \in (0,T)$ with $\tilde{\varrho}_3 > 0$. Hence, we may suppose that there exist subsequences, all subscripts are still denoted by n, $\{\zeta_n\}$, $\{\xi_n\}$ and $\{\eta_n\}$ such that

$$\zeta_n \to \zeta \quad \text{weakly in } L^2(0,T;L^2(\Gamma_C)), \tag{93}$$
$$\xi_n \to \xi \quad \text{weakly in } L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d)), \tag{94}$$
$$\eta_n \to \eta \quad \text{weakly in } L^2(0,T;L^2(\Gamma_C)), \tag{95}$$

as $n \to \infty$ with

$$\zeta_n(t) \in \partial J_1(t, \gamma_\nu u_n'(t)), \tag{96}$$
$$\xi_n(t) \in \partial J_2(t, \gamma\, u_n'(t), \gamma_\tau u_n'(t)), \tag{97}$$
$$\eta_n(t) \in \partial_c \bar{\Psi}(\gamma_\nu u_n'(t)), \tag{98}$$

for a.e. $t \in (0,T)$. So, combining $(J6)$, $(P3)$, Proposition 3.23 in [24], (90) and (93)–(98), we see that

$$\zeta(t) \in \partial J_1(t, \gamma_\nu u'(t)), \quad \xi(t) \in \partial J_2(t, \gamma\, u'(t), \gamma_\tau u'(t)) \quad \text{and} \quad \eta(t) \in \partial_c \bar{\Psi}(\gamma_\nu u'(t))$$

for a.e. $t \in (0,T)$.

Subsequently, by a reasoning similar to the *Step 5* in the proof of Lemma 3.9, we will prove that

$$\mathcal{A}u_n' + \mathcal{B}u_n \to \mathcal{A}u' + \mathcal{B}u_n \quad \text{weakly in} \quad \mathcal{V}^*, \tag{99}$$

where $\mathcal{A}$ and $\mathcal{B}$ are the Nemyckii operators corresponding to A and B, respectively. Indeed, based on the weak continuity of $\mathcal{B}$, it is clear that

$$\mathcal{B}u_n \to \mathcal{B}u_n \quad \text{weakly in} \quad \mathcal{V}^*. \tag{100}$$

Next, since

$$\begin{aligned}
&\limsup \langle \mathcal{B}u_n, u' - u_n' \rangle_{\mathcal{V}^* \times \mathcal{V}} \\
&= \limsup \big(- \langle \mathcal{B}u - \mathcal{B}u_n, u' - u_n' \rangle_{\mathcal{V}^* \times \mathcal{V}} + \langle \mathcal{B}u, u' - u_n' \rangle_{\mathcal{V}^* \times \mathcal{V}} \big) \\
&\leq \limsup \langle \mathcal{B}u, u' - u_n' \rangle_{\mathcal{V}^* \times \mathcal{V}} \leq 0,
\end{aligned}$$

which, together with the facts of $\langle f, u_n' - u' \rangle_{\mathcal{V}^* \times \mathcal{V}} \to 0$, $\langle \widehat{\gamma}_\nu^* \zeta_n + \widehat{\gamma}_\tau^* \xi_n + \widehat{\gamma}_\nu^* \eta_n, u_n' - u' \rangle_{\mathcal{Z}^* \times \mathcal{Z}} \to 0$ and the first equality in (86), we may obtain

$$\limsup\langle \mathcal{A}u_n', u_n' - u'\rangle_{\mathcal{V}^*\times\mathcal{V}} \le \limsup\langle u_n'' + \mathcal{B}u_n, u' - u_n'\rangle_{\mathcal{V}^*\times\mathcal{V}}$$
$$\le -\frac{1}{2}\|u_n'(T) - u'(T)\|_H^2 + \lim\langle u'', u' - u_n'\rangle_{\mathcal{V}^*\times\mathcal{V}} \le 0.$$

Hence, from Lemma 5.5 (e) in [24] and (100), we deduce (99).

Finally, passing to the limit in the first equality of (86), we directly get that $(u, u') \in \mathcal{S}$ satisfies (41). This completes the proof of the lemma. □

Theorem 4.2. *Under the hypotheses of Theorem 3.8 and the assumptions ($\mathcal{Q}_{ad}$) and (F), the problem (82) has an optimal solution.*

Proof. Let $\{(q_n, y_n)\} \subset \mathcal{Q}_{ad} \times \mathcal{S}(q_n)$ be a minimizing sequence for the functional F in the problem (82), i.e.,

$$\lim_{n\to\infty} F(q_n, y_n) = \inf\{F(q, y) \mid (q, y) \in \mathcal{Q}_{ad} \times \mathcal{S}(q)\}. \tag{101}$$

By the assumption ($\mathcal{Q}_{ad}$), we may choose a subsequence $\{q_n\} \subset \mathcal{Q}_{ad}$ such that

$$q_n \to q^* \text{ weakly in } \mathcal{Q} \text{ with } q^* \in \mathcal{Q}_{ad}. \tag{102}$$

Combining with Lemma 4.1, we have

$$y_n \to y^* \text{ weakly in } \mathcal{S} \text{ with } y_n \in \mathcal{S}(q_n) \text{ and } y^* \in \mathcal{S}(q^*). \tag{103}$$

Hence, (102), (103) and (F) imply that

$$F(q^*, y^*) \le \liminf_{n\to\infty} F(q_n, y_n).$$

So, in view of (101), we get $F(q^*, y^*) = \inf\{F(q, y) \mid (q, y) \in \mathcal{Q}_{ad} \times \mathcal{S}(q)\}$. The proof is completed. □

We conclude this subsection with a simple example of the objective functional $F\colon \mathcal{Q} \times \mathcal{S} \to \mathbb{R}$ which satisfies the assumption (F).

Example 4.3. Let the element $y_d = (u_d, u_d', u_d'') \in \mathcal{V} \times \mathcal{V} \times \mathcal{V}^*$ denote a fixed target. Consider the objective functional $F\colon \mathcal{Q} \times \mathcal{S} \to \mathbb{R}$ of the following form

$$F(q, y) = \|q\|_{\mathcal{Q}} + \|y - y_d\|_{\mathcal{S}} = \|f\|_{\mathcal{V}^*} + \|a\|_V + \|b\|_V$$
$$+ \|u - u_d\|_{\mathcal{V}} + \|u' - u_d'\|_{\mathcal{V}} + \|u' - u_d'\|_{\mathcal{V}^*},$$

where $q = (f, a, b) \in \mathcal{Q} = \mathcal{V}^* \times V \times V$ and $y = (u, u') \in \mathcal{S} = \mathcal{V} \times \mathcal{W}$. It is clear that this functional is convex and strongly continuous, and thus it is lower semicontinuous with respect to the topology of w-($\mathcal{Q} \times \mathcal{S}$).

For other examples of cost functionals, refer to Lions [20] and Troltzsch [34].

4.2. Time optimal control problem

In this subsection, we turn to the study of a time optimal control problem. We recall the definition of convergence of sets in the sense of Kuratowski.

Definition 4.4. Let (X, τ) be a Hausdorff topological space and let $\{A_n\}$, $n \geq 1$, be a sequence of nonempty subsets to X. We define

$$\tau\text{-}\liminf A_n = \{\, x \in X \mid x = \tau\text{-}\lim x_n,\, x_n \in A_n \,\} \text{ and}$$
$$\tau\text{-}\limsup A_n = \{\, x \in X \mid x = \tau\text{-}\lim\, x_{n_k}, x_{n_k} \in A_{n_k},\, n_1 < n_2 < \ldots < n_k < \ldots \}.$$

The sets $\tau\text{-}\liminf A_n$ and $\tau\text{-}\limsup A_n$ are called τ-Kuratowski lower and upper limits, respectively, of the sets A_n. Furthermore, if

$$A = \tau\text{-}\liminf A_n = \tau\text{-}\limsup A_n,$$

then we say that A is τ-Kuratowski limit of the sets A_n.

The proof of next auxiliary statement can be found in Proposition 4.7.44 of [12].

Lemma 4.5. *Let $(\mathcal{O}, \Sigma, \mu)$ be a σ-finite measure space and E be a Banach space. Assume that for f_n, $f \in L^p(\mathcal{O}; E)$ with $p \in [1, +\infty)$,*

$$f_n \to f \quad \textit{weakly in} \quad L^p(\mathcal{O}; E),$$

and $f_n(u) \in G(u) \in P_{wk}(E)$ for μ-a.e. $u \in \mathcal{O}$ and $n \in \mathbb{N}$, where the notation $P_{wk}(E)$ represents the sets of all nonempty, weakly compact subsets of E. Then

$$f(u) \in \overline{conv}\big(w\text{-}\limsup\{f_n(u)\}_{n\in\mathbb{N}}\big) \quad \textit{for } \mu\textit{-a.e. on } \mathcal{O},$$

where the symbol $\overline{conv}(B)$ denotes the closed convex hull of set B.

We consider the control system associated to Problem 3.6 and described by the following inclusion

$$\begin{cases} u''(t) + A(t, u'(t)) + Bu(t) + \gamma_\nu^*\zeta(t) \\ \qquad + \gamma_\tau^*\xi(t) + \gamma_\nu^*\eta(t) = f(t) + G(t)\phi(t), \\ \zeta(t) \in \partial J_1\big(t, \gamma_\nu u'(t)\big),\ \xi(t) \in \partial J_2\big(t, \gamma\, u'(t), \gamma_\tau u'(t)\big), \\ \eta(t) \in \partial_c \bar{\Psi}\big(\gamma_\nu u'(t)\big),\, u(0) = a,\, u'(0) = b \end{cases} \tag{104}$$

for a.e. $t \in (0, T)$, in which the data A, B, J_1, J_2, $\bar{\Psi}$, f, a and b have been given in Section 3, ϕ represents the control parameter which provide “source-like” densities of external forces by the controller operator G.

We suppose that the control parameter ϕ belongs to the space $L^2(0, T; Y)$, Y is a separable and reflexive Banach space which stands for the space of control variables, and we assume that the controller G satisfies

$$G \in L^\infty(0, T; \mathcal{L}(Y, V^*)). \tag{105}$$

Referring to Theorem 3.8, we know that under the hypotheses of this theorem and the assumption (105), the problem (104) has at least one solution $y = (u(\phi), u'(\phi)) \in \mathcal{V} \times \mathcal{W}$ for each $\phi \in L^2(0, T; Y)$. Hence, we may define the solution map by

$$\bar{\mathcal{S}} \colon L^2(0, T; Y) \ni \phi \to \bar{\mathcal{S}}(\phi) \subset \mathcal{V} \times \mathcal{W},$$

where $\bar{\mathcal{S}}(\phi)$ represents the solution set of the problem (104) corresponding to ϕ.

Our main interest in this section is to look for the minimal time to reach a moving target set, i.e., changing in time, a set of desirable states for (u, u'). So in what follows, we shall give two assumptions on the target set $\mathcal{T}$ and the control constraint set $\mathcal{C}$. Assume that

$H(\mathcal{T})$ the multivalued function $\mathcal{T}\colon [0,T] \to 2^{V\times H}$ is measurable and its graph is closed in the topology of $[0,T]\times V\times H$,

$H(\mathcal{C})$ $\mathcal{C}\colon [0,T] \to 2^{Y}\backslash\{\emptyset\}$ is a multivalued function such that $\mathcal{C}(t)$ is closed and convex for all $t\in[0,T]$, and $\|\mathcal{C}(t)\|_Y = \sup\{\|\phi\|_Y \mid \phi\in\mathcal{C}(t)\} \in L^\infty(0,T)$.

Before we formulate the time optimal control problem under consideration, we give two more assumptions of the controllability-type and on the elasticity operator $\mathcal{E}$.

$A(\mathcal{C})$ there exists $\phi \in L^2(0,T;Y)$ with $\phi(t)\in\mathcal{C}(t)$ such that $y(\tau)\in\mathcal{T}(\tau)$ for some $\tau\in(0,T)$, where $y\in\bar{\mathcal{S}}(\phi)$.

The elasticity operator $\mathcal{E}(x,\varepsilon) = E(x)\varepsilon$ satisfies the coercivity condition

$$E_{ijkl}(x)\varepsilon_{ij}\varepsilon_{kl} \geq \beta\,\varepsilon_{ij}\varepsilon_{kl} \ \text{ for all symmetric tensors } \ \varepsilon = (\varepsilon_{ij}) \in \mathbb{S}^d \tag{106}$$

and a.e. $x\in\Omega$ with constant $\beta>0$.

Remark 4.6. If the assumption (106) holds, then the operator $B\colon V\to V^*$ defined by (32) is such that

$$\langle Bu, u\rangle_{V^*\times V} \geq \beta\|u\|_V^2 \quad \text{for all} \ \ u\in V \ \ \text{with} \ \ \beta>0. \tag{107}$$

The time optimal control problem under consideration is described as follows

$$\begin{cases} \text{find the control and state } (\phi, y)\in L^2(0,T;Y)\times\bar{\mathcal{S}}(\phi) \text{ such that} \\ y(t_0)\in\mathcal{T}(t_0),\ t_0 = \inf\big\{\tau\in(0,T)\,|\,\tau \text{ satisfies } A(\mathcal{C})\big\}. \end{cases} \tag{108}$$

Remark 4.7. As far as we know, the time optimal control problem is very important in solid contact mechanics. The above problem can be simply understood by looking for the shortest time so that the deformation of the material reaches the desired value. Furthermore, $H(\mathcal{T})$, $H(\mathcal{C})$ and $A(\mathcal{C})$, respectively, represent the assumptions of target set (solution set (u,u')), control constraint set (requirements of ϕ) and controllability condition of our time optimal control problem (108).

Lemma 4.8. *Assume the hypotheses of Theorem 3.8, (105) and (107). If*

$$\phi_n \to \phi \quad \textit{weakly in} \ \ L^2(0,T;Y), \quad \textit{as} \quad n\to\infty, \tag{109}$$

and $y_n\in\bar{\mathcal{S}}(\phi_n)$, then there exists a subsequence of $\{y_n\}$, still denoted in the same way, such that

$$y_n \to y \quad \textit{strongly in} \ \ C(0,T;V\times H) \ \ \textit{with} \ \ y\in\bar{\mathcal{S}}(\phi).$$

Proof. Let $\{\phi_n\}\subset L^2(0,T;Y)$ be a sequence such that (109) holds. According to Theorem 3.8, Lemma 3.12 and the hypothesis (105), it is clear that for every ϕ_n, there exists at least one solution $y_n = (u_n, u_n')\in\bar{\mathcal{S}}(\phi_n)$ of (104) which satisfies the estimate

$$\|u_n\|_{\mathcal{V}} + \|u_n'\|_{\mathcal{W}} \leq c\big(1 + \|a\|_V + \|b\|_V + \|f\|_{\mathcal{V}^*} + \|G\|_{L^\infty(0,T;\mathcal{L}(Y,V^*))}\|\phi_n\|_{L^2(0,T;Y)}\big)$$

with $c > 0$. Hence, by passing to a subsequence, if necessary, we get

$$y_n \to y \quad \text{weakly in} \quad \mathcal{V} \times \mathcal{W}, \quad \text{as} \quad n \to \infty.$$

Next, similarly as in the proof of Lemma 4.1, we can verify that $y \in \bar{\mathcal{S}}(\phi)$. To conclude the proof, we are going to check that $y_n \to y$ strongly in $C(0,T;V \times H)$. From the facts $y_n = (u_n, u_n') \in \bar{\mathcal{S}}(\phi_n)$ and $y = (u, u') \in \bar{\mathcal{S}}(\phi)$, we have

$$\begin{cases} u_n''(t) + A(t, u_n'(t)) + Bu_n(t) + \gamma_\nu^* \zeta_n(t) \\ \qquad + \gamma_\tau^* \xi_n(t) + \gamma_\nu^* \eta_n(t) = f(t) + G(t)\phi_n(t), \\ \zeta_n(t) \in \partial J_1\big(t, \gamma_\nu u_n'(t)\big),\ \xi_n(t) \in \partial J_2\big(t, \gamma\, u_n'(t), \gamma_\tau u_n'(t)\big), \\ \eta_n(t) \in \partial_c \bar{\Psi}\big(\gamma_\nu u_n'(t)\big),\ u_n(0) = a,\ u_n'(0) = b \end{cases} \tag{110}$$

for a.e. $t \in (0,T)$ and

$$\begin{cases} u''(t) + A(t, u'(t)) + Bu(t) + \gamma_\nu^* \zeta(t) \\ \qquad + \gamma_\tau^* \xi(t) + \gamma_\nu^* \eta(t) = f(t) + G(t)\phi(t), \\ \zeta(t) \in \partial J_1\big(t, \gamma_\nu u'(t)\big),\ \xi(t) \in \partial J_2\big(t, \gamma\, u'(t), \gamma_\tau u'(t)\big), \\ \eta(t) \in \partial_c \bar{\Psi}\big(\gamma_\nu u'(t)\big),\ u(0) = a,\ u'(0) = b \end{cases} \tag{111}$$

for a.e. $t \in (0,T)$. By the hypotheses $(A2)$, $(B1)$–$(B2)$ and (107), we obtain

$$\begin{aligned} &\int_0^t \big\langle A(s, u_n'(s)) - A(s, u'(s)) + Bu_n(s) - Bu(s), u_n'(s) - u'(s)\big\rangle_{V^* \times V}\, ds \\ &\quad \geq \frac{1}{2}\int_0^t \frac{d}{ds}\big\langle Bu_n(s) - Bu(s), u_n(s) - u(s)\big\rangle_{V^* \times V}\, ds \\ &\qquad \geq \beta \|u_n(t) - u(t)\|_V^2 \end{aligned} \tag{112}$$

for all $t \in [0,T]$. Exploiting $(J3)$ and $(J4)$, we get

$$\begin{aligned} &\int_0^t \big\langle \gamma_\nu^* \zeta_n(s) - \gamma_\nu^* \zeta(s) + \gamma_\tau^* \xi_n(s) - \gamma_\tau^* \xi(s), u_n'(s) - u'(s)\big\rangle_{V^* \times V}\, ds \\ &= \int_0^t \big\langle \zeta_n(s) - \zeta(s), \gamma_\nu u_n'(s) - \gamma_\nu u'(s)\big\rangle_{L^2(\Gamma_C)}\, ds \\ &+ \int_0^t \big\langle \xi_n(s) - \xi(s), \gamma_\tau u_n'(s) - \gamma_\tau u'(s)\big\rangle_{L^2(\Gamma_C;\mathbb{R}^d)}\, ds \\ &\leq \hat{C}_1(1 + \|u_n'\|_{\mathcal{Z}} + \|u'\|_{\mathcal{Z}})\|u_n' - u'\|_{\mathcal{Z}} \end{aligned} \tag{113}$$

for all $t \in [0,T]$ with $\hat{C}_1 > 0$, where $\zeta_n(t) \in \partial J_1(t, \gamma_\nu u_n'(t))$, $\zeta(t) \in \partial J_1(t, \gamma_\nu u'(t))$, $\xi_n(t) \in \partial J_2(t, \gamma\, u_n'(t), \gamma_\tau u_n'(t))$ and $\xi(t) \in \partial J_2(t, \gamma\, u'(t), \gamma_\tau u'(t))$ for a.e. $t \in (0,T)$. Now, we use (110)–(113), the monotonicity of the operator $\partial_c \bar{\Psi}$ and the equality

$$\int_0^t \langle u_n''(s) - u''(s), u_n'(s) - u'(s)\rangle_{V^*\times V}\, ds = \frac{1}{2}\|u_n'(t) - u'(t)\|_H^2$$

for all $t \in [0,T]$, to derive the estimate

$$\begin{aligned}\|u_n'(t) - u'(t)\|_H^2 + \beta\|u_n(t) - u(t)\|_V^2 &\le \hat{C}_1(1 + \|u_n'\|_{\mathcal{Z}} + \|u'\|_{\mathcal{Z}})\|u_n' - u'\|_{\mathcal{Z}} \\ &\quad + c_e\|G\|_{L^\infty(0,T;\mathcal{L}(Y;V^*))}\|\phi_n - \phi\|_{L^2(0,T;Y)}\|u_n' - u'\|_{\mathcal{Z}}\end{aligned} \tag{114}$$

for all $t \in [0,T]$ with $\hat{C}_1 > 0$. Since the embedding $\mathcal{W} \subset \mathcal{Z}$ is compact, we deduce that $u_n' \to u$ strongly in $\mathcal{Z}$. This convergence, together with the inequality (114) implies

$$y_n = (u_n, u_n') \to (u, u') = y \quad \text{strongly in} \quad C(0,T;V \times H).$$

This concludes the proof of the lemma. □

The following result provides sufficient conditions for existence of time optimal solution to optimal control problem (108).

Theorem 4.9. *Assume the hypotheses of Theorem 3.8 and the assumptions (105) and (107). Then the time optimal control problem (108) admits an optimal solution* $(\phi, y) \in L^2(0,T;Y) \times \bar{\mathcal{S}}(\phi)$.

Proof. First, from the definition of t_0, we may assume that there exists a sequence $\{t_n\} \subset (0,T)$ which satisfies $A(\mathcal{C})$ and such that

$$t_n \to t_0, \quad \text{as} \quad n \to \infty.$$

By the controllability-type assumption $A(\mathcal{C})$ and Theorem 3.8, we know that for each t_n, there is a control $\phi_n \in L^2(0,T;Y)$ and a state $y_n \in \bar{\mathcal{S}}(\phi_n)$ such that $\phi_n(t) \in \mathcal{C}(t)$ for a.e. $t \in (0,T)$ and $y_n(t_n) \in \mathcal{T}(t_n)$. Next, since the sequence $\{\phi_n\}$ is bounded in $L^2(0,T;Y)$ (this follows from the assumption $H(\mathcal{C})$), thus we may suppose that

$$\phi_n \to \phi \quad \text{weakly in} \quad L^2(0,T;Y), \quad \text{as} \quad n \to \infty. \tag{115}$$

Because $\mathcal{C}(t)$ is a weak compact set of Y for a.e. $t \in (0,T)$, by using (115) and Lemma 4.5, we obtain

$$\phi(t) \in \overline{conv}\big(w\text{-}\limsup\{\phi_n(t)\}_{n\in\mathbb{N}}\big) \subset \overline{conv}(\mathcal{C}(t)) \quad \text{for a.e.} \quad t \in (0,T).$$

Hence, according to $H(\mathcal{C})$, we get $\phi(t) \in \mathcal{C}(t)$ for a.e. $t \in (0,T)$.

Next, in view of Lemma 4.8 and (115), we have

$$y_n \to y \quad \text{strongly in} \quad C(0,T;V \times H) \quad \text{with} \quad y \in \bar{\mathcal{S}}(\phi),$$

which implies that

$$\begin{aligned}&\|y_n(t_n) - y(t_0)\|_{V\times H} \le \|y_n(t_n) - y(t_n)\|_{V\times H} + \|y(t_n) - y(t_0)\|_{V\times H} \\ &\le \sup_{t\in[0,T]} \|y_n(t) - y(t)\|_{V\times H} + \|y(t_n) - y(t_0)\|_{V\times H} \to 0, \quad \text{as} \quad n \to \infty.\end{aligned}$$

Therefore, $y_n(t_n) \to y(t_0)$ in $V \times H$, as $n \to \infty$. So now, we use the hypothesis $H(\mathcal{T})$ and the fact $y_n(t_n) \in \mathcal{T}(t_n)$, to obtain that $y(t_0) \in \mathcal{T}(t_0)$. The proof is completed. □

4.3. Maximum stay control problem

The goal of this subsection is to regulate the control system (104) in such a way that its state $y = (u, u') \in \mathcal{V} \times \mathcal{W}$ stays in a preferred region $\mathcal{T}(\cdot)$ in maximum time. The maximum stay control problem under consideration reads as follows

$$\begin{cases} \text{find the control and state } (\phi, y) \text{ which solves} \\ \sup \big\{\lambda(y) \mid y \in \bar{\mathcal{S}}(\phi),\, \phi \in L^2(0,T;Y),\, \phi(t) \in \mathcal{C}(t)\big\}, \end{cases} \tag{116}$$

where the function $\lambda\colon C(0,T;V \times H) \to \mathbb{R}_+$ is defined by

$$\lambda(y) = \text{meas}\big(\{t \in [0,T] \mid y(t) \in \mathcal{T}(t)\}\big).$$

Remark 4.10. In contrast to (108), we call (116) by the maximum stay control problem which is due to that we would like to control the material deformation as slow as possible by controller, which is also a major control issue in solid contact mechanics.

The following result concerns existence of solutions to problem (116).

Theorem 4.11. *Assume that the hypotheses of Theorem 4.9 hold. Then the problem (116) admits a solution.*

Proof. To start the proof, firstly, it is necessary to introduce a new function

$$\hat{\lambda} : L^2(0,T;Y) \ni \phi \to \sup\big\{\lambda(y) \mid y \in \bar{\mathcal{S}}(\phi)\big\},$$

thus the maximum stay control problem (116) can be written as

$$\sup\big\{\hat{\lambda}(\phi) \mid \phi \in L^2(0,T;Y) \text{ with } \phi(t) \in \mathcal{C}(t)\big\}.$$

The proof is based on the Weierstrass-type theorem which states that an upper semicontinuous function on a compact set attains a maximum value, and the nonempty set of maximizers is compact (cf. e.g. Theorem 2.43 of [1]). To this end, we shall prove that the function $\hat{\lambda}$ is upper semicontinuous on $L^2(0,T;Y)$ endowed with the weak topology, and the set of selections

$$\big\{\phi \in L^2(0,T;Y) \mid \phi(t) \in \mathcal{C}(t) \text{ for a.e. } t \in (0,T)\big\}$$

is a weakly compact subset of $L^2(0,T;Y)$.

By observation, we know that $\hat{\lambda}$ is a marginal function, thus, to prove the weakly upper semicontinuity of $\hat{\lambda}$, we only need to show

(i) the solution set

$$\bar{\mathcal{S}} : L^2(0,T;Y) \to 2^{C(0,T;V\times H)} \quad \text{is upper semicontinuous}$$

in topology of $(w - L^2(0,T;Y) \times C(0,T;V \times H))$,

(ii) the function $\lambda\colon C(0,T;V \times H) \to \mathbb{R}_+$ is upper semicontinuous.

In fact, the proof of condition (i) can be directly from Lemma 4.8.

Next, we are going to show the condition (ii). Let $\{y_n\} \subset C(0,T;V \times H)$ be a sequence such that $y_n \to y$ in $C(0,T;V \times H)$. Define

$$D_n = \{t \in [0,T] \mid y_n(t) \in \mathcal{T}(t)\} = \{t \in [0,T] \mid d(y_n(t), \mathcal{T}(t)) = 0\}.$$

Due to the assumption $H(\mathcal{T})$, we know that the multifunction $\mathcal{T}(\cdot)$ is measurable, so is $t \to d(y_n(t), \mathcal{T}(t))$. This implies that D_n is a measurable subset of $[0,T]$. Then, let $t \in \limsup_{n\to\infty} D_n$. By the definition of the Kuratowski upper limit of sets, there exists $t_{n_k} \in D_{n_k}$ such that $t_{n_k} \to t$, as $k \to \infty$. Therefore $y_{n_k}(t_{n_k}) \in \mathcal{T}(t_{n_k})$. Using the assumption $H(\mathcal{T})$ again and the convergence $y_n \to y$ in $C(0,T;V \times H)$, we immediately get $y(t) \in \mathcal{T}(t)$. So $t \in D$, where $D = \{t \in [0,T] \mid y(t) \in \mathcal{T}(t)\}$. Hence, we have showed that

$$\limsup_{n\to\infty} D_n \subset D.$$

Furthermore, we have

$$\begin{aligned}\limsup_{n\to\infty} \operatorname{meas}(D_n) &\le \limsup_{n\to\infty} \operatorname{meas}\Big(\bigcup_{k\ge n} D_k\Big) = \lim_{n\to\infty} \operatorname{meas}\Big(\bigcup_{k\ge n} D_k\Big)\\ &= \operatorname{meas}\Big(\bigcap_{n\ge 1}\bigcup_{k\ge n} D_k\Big) \le \operatorname{meas}\Big(\limsup_{n\to\infty} D_n\Big) \le \operatorname{meas}(D).\end{aligned}$$

This implies that the function λ is upper semicontinuous.

Finally, since the hypothesis $H(\mathcal{C})$, we know the set $\mathcal{C}(t)$ is weakly compact in Y for all $t \in [0,T]$, which together with Theorem 4.5.25 in [12], we see the set of selections

$$\big\{\phi \in L^2(0,T;Y) \mid \phi(t) \in \mathcal{C}(t) \text{ for a.e. } t \in (0,T)\big\}$$

is a weakly compact subset of $L^2(0,T;Y)$. So now, applying the aforementioned Weierstrass-type theorem, we obtain that the problem (116) has a solution. This concludes the proof of the theorem. □

Appendix A

In the finally part we recall some basic mathematical background material that is employed in our main paper. For the detailed exposition, we refer, e.g. to the monographs [7], [9], [12], [24], [27], [28] and [35].

We start with the preparatory material on the theory of monotone type operators. Let X be a Banach space, X^* denotes its dual space. We employ the classical notation

$$D(A) = \big\{u \in X \,|\, Au \neq \emptyset\big\} \quad \text{and} \quad R(A) = \bigcup_{u\in X} Au$$

to denote the domain and range of multivalued operator $A\colon X \to 2^{X^*}$, respectively. The graph of A is defined by

$$Gr(A) = \big\{(u, u^*) \in X \times X^* \,|\, u^* \in Au\big\}.$$

Recall that a multivalued operator $A\colon X \to 2^{X^*}$ is said to be

(i) **strongly quasi-bounded**, if from the conditions

$$\langle u^*, u\rangle_{X^*\times X} \le M,\ \|u\|_X \le M,\ (u,u^*) \in Gr(A),\ M > 0,$$

we can derive that $\|u^*\|_{X^*} \le K(M)$ with $K(M) > 0$,

(ii) **strongly monotone**, if for all (u, u^*), $(v, v^*) \in Gr(A)$, there exist two constants $\alpha > 0$ and $q > 1$ such that

$$\langle u^* - v^*, u - v\rangle_{X^*\times X} \geq \alpha\|u - v\|_X^q,$$

(iii) **maximal monotone**, if A is monotone, and the condition

$$\langle u^* - v^*, u - v\rangle_{X^*\times X} \geq 0 \ \text{ for all } \ (v, v^*) \in Gr(A) \text{ and } (u, u^*) \in X \times X^*$$

implies that $(u, u^*) \in Gr(A)$,

(iv) **hemicontinuous**, if the mapping $r \to \langle A(u + rv), w\rangle_{X^*\times X}$ is continuous on $[0, 1]$ for all u, v, $w \in X$,

(v) **demicontinuous**, if for all $u_n, u \in X$, $u_n \to u$ strongly in X implies that $Au_n \to Au$ weakly in X^*.

The space of linear and bounded operators between Banach spaces E and F is denoted by $\mathcal{L}(E; F)$. Next, we recall the definitions of the pseudomonotonicity and the generalized pseudomonotonicity for multivalued operator $A\colon X \to 2^{X^*}$.

Definition A.1. Let X be a reflexive Banach space. A multivalued operator $A\colon X \to 2^{X^*}$ is called

(i) **pseudomonotone**, if it satisfies the following three conditions

  (a) the sets Au are nonempty, bounded, convex and closed for all $u \in X$,

  (b) A is upper semicontinuous from each finite dimensional subspace of X to X^* endowed with the weak topology,

  (c) from the convergence $u_n \to u$ weakly in X, where $\{u_n\} \subset X$, $\{u_n^*\} \subset X^*$, $u_n^* \in Au_n$ for all $n \geq 1$, and

$$\limsup\langle u_n^*, u_n - u\rangle_{X^*\times X} \leq 0,$$

  we have that for each $v \in X$, there exists $u^*(v) \in Au$ such that

$$\langle u^*(v), u - v\rangle_{X^*\times X} \leq \liminf\,\langle u_n^*, u_n - v\rangle_{X^*\times X}.$$

(ii) **generalized pseudomonotone**, if for any sequences $\{u_n\} \subset X$, $\{u_n^*\} \subset X^*$ with $u_n^* \in Au_n$ such that $u_n \to u$ weakly in X, $u_n^* \to u^*$ weakly in X^* and

$$\limsup\langle u_n^*, u_n - u\rangle_{X^*\times X} \leq 0,$$

we have $u^* \in Au$ and $\langle u_n^*, u_n\rangle_{X^*\times X} \to \langle u^*, u\rangle_{X^*\times X}$.

By the definitions of the pseudomonotonicity and the generalized pseudomonotonicity for multivalued operator $A\colon X \to 2^{X^*}$, it clear that if A is pseudomonotone, then it is generalized pseudomonotone. The converse holds under the additional condition that the sets Au are nonempty, bounded, closed, and convex for all $u \in X$.

The proof of the following result can be found in Proposition 2.4 in [27].

Proposition A.2. *Let X be a reflexive Banach space. If the multivalued operators A_1, $A_2\colon X \to 2^{X*}$ are pseudomonotone, then so is $A_1 + A_2$.*

In what follows, we introduce the definitions of coercivity and L-pseudomonotonicity for multivalued operators.

Definition A.3. Let X be a Banach space. If there exists a function $c\colon \mathbb{R} \to \mathbb{R}$ with $\lim_{t\to+\infty} c(t) = +\infty$ such that

$$\langle \zeta, u\rangle_{X^*\times X} \geq c(\|u\|_X)\|u\|_X \quad \text{with} \quad \zeta \in Au \text{ for all } u \in X,$$

then we say that the multivalued operator $A\colon X \to 2^{X^*}$ is coercive.

Definition A.4. Assume that X is a reflexive Banach space, and an operator $L\colon X \supset D(L) \to X^*$ is linear and maximal monotone. We say that the multivalued operator $A\colon X \to 2^{X^*}$ is L-pseudomonotone or pseudomonotone with respect to $D(L)$, if the conditions (a) and (b) of Definition A.1 are satisfied and the following condition holds

(d) for any sequences $\{u_n\} \subset D(L)$ and $u_n^* \in Au_n$ such that $Lu_n \to Lu$ weakly in X^*, $u_n^* \to u^*$ weakly in X^* for all $n \geq 1$, and

$$\limsup \langle u_n^*, u_n - u\rangle_{X^*\times X} \leq 0,$$

we deduce that $u^* \in Au$ and $\langle u_n^*, u_n\rangle_{X^*\times X} \to \langle u^*, u\rangle_{X^*\times X}$.

Remark A.5. From Definitions A.1(i)–(ii) and A.4, it is clear that if a multivalued operator is pseudomonotone, then it is generalized pseudomonotone, and hence it is also L-pseudomonotone.

Below we list some notation, definitions and necessary properties on the convex subdifferential and its generalization.

Definition A.6 *(Convex subdifferential).* Assume that a function $\varphi\colon X \to \mathbb{R} \cup \{\infty\}$ is convex and defined on a Banach space X, and $u \in X$. Then the set of all $u^* \in X^*$ such that

$$\varphi(u + v) - \varphi(u) \geq \langle u^*, v\rangle_{X^*\times X} \quad \text{for all} \quad v \in X$$

is called the convex subdifferential of φ at point u, and it is denoted by $\partial_c\varphi(u)$.

Given any convex function $\varphi\colon X \to \mathbb{R} \cup \{\infty\}$, we denote its effective domain by $dom(\varphi) = \{x \in X \mid \varphi(x) < \infty\}$.

Let $\mathcal{O}$ be a bounded, open, and connected subset of $\mathbb{R}^n$, $n \geq 1$. Define the functional $\Psi\colon L^2(\mathcal{O}; X) \to \mathbb{R} \cup \{\infty\}$ by

$$\Psi(u) = \int_{\mathcal{O}} \varphi(u(x))\, dx \quad \text{for all} \quad u \in L^2(\mathcal{O}; X). \tag{117}$$

We may derive the following interesting property of the subdifferential of the convex integral functional. Its proof is presented here for completeness.

Proposition A.7. *Assume that the functional Ψ is given by (117) and the function $\varphi\colon X \to \mathbb{R}\cup\{\infty\}$ is convex. Then for all $(u, u^*) \in L^2(\mathcal{O}; X) \times L^2(\mathcal{O}; X^*)$, we have that $u^* \in \partial_c\Psi(u)$ if and only if $u^*(x) \in \partial_c\varphi(u(x))$ for a.e. $x \in \mathcal{O}$.*

Proof. Firstly, by the definition (117), since the function φ is convex, it follows that the functional $\Psi\colon L^2(\mathcal{O}; X) \to \mathbb{R} \cup \{\infty\}$ is convex. Hence according to Definition A.6, the subdifferential $\partial_c\Psi(\cdot)$ is well defined.

Next, assume that $(u, u^*) \in L^2(\mathcal{O}; X) \times L^2(\mathcal{O}; X^*)$ and $u^*(x) \in \partial_c \varphi(u(x))$ for a.e. $x \in \mathcal{O}$. In view of Definition A.6 again, we have

$$\varphi(v(x)) - \varphi(u(x)) \geq \langle u^*(x), v(x) - u(x) \rangle_{X^* \times X}$$

for all $v \in L^2(\mathcal{O}, X)$ and a.e. $x \in \mathcal{O}$. From (117), we obtain

$$\begin{aligned}\Psi(v) - \Psi(u) &= \int_{\mathcal{O}} \big(\varphi(v(x)) - \varphi(u(x))\big)\, dx \geq \int_{\mathcal{O}} \langle u^*(x), v(x) - u(x) \rangle_{X^* \times X}\, dx \\ &= \langle u^*, v - u \rangle_{L^2(\mathcal{O}; X^*) \times L^2(\mathcal{O}; X)} \text{ for all } v \in L^2(\mathcal{O}, X).\end{aligned}$$

This implies that $u^* \in \partial_c \Psi(u)$.

Conversely, let $(u, u^*) \in L^2(\mathcal{O}; X) \times L^2(\mathcal{O}; X^*)$ and $u^* \in \partial_c \Psi(u)$. By Definition A.6 once more, we deduce that for all $w \in L^2(\mathcal{O}, X)$, we have

$$\int_{\mathcal{O}} \big(\varphi(w(x)) - \varphi(u(x))\big)\, dx \geq \int_{\mathcal{O}} \langle u^*(x), w(x) - u(x) \rangle_{X^* \times X}\, dx. \tag{118}$$

Let S be any measurable subset of $\mathcal{O}$ and

$$w(x) = \begin{cases} \xi & \text{if } x \in S, \\ u(x) & \text{if } x \notin S, \end{cases}$$

where $\xi \in X$ is arbitrary. From (118), we directly get

$$\int_{S} \big(\varphi(\xi) - \varphi(u(x))\big)\, dx \geq \int_{S} \langle u^*(x), \xi - u(x) \rangle_{X^* \times X}\, dx.$$

Since S is arbitrary, we may conclude that $\varphi(\xi) - \varphi(u(x)) \geq \langle u^*(x), \xi - u(x) \rangle_{X^* \times X}$ for a.e. $x \in \mathcal{O}$. Hence $u^*(x) \in \partial_c \varphi(u(x))$ for a.e. $x \in \mathcal{O}$. This completes the proof. □

In what follows, let X be a Banach space. Recall that a function $\varphi \colon X \to \mathbb{R}$ is said to be locally Lipschitz, if there exists a neighborhood $\mathcal{N}(u)$ of $u \in X$ such that

$$|\varphi(y) - \varphi(z)| \leq K_u \|y - z\|_X \text{ for all } y, z \in \mathcal{N}(u) \text{ with } K_u > 0.$$

The generalized directional derivative in the sense of Clarke for φ at point $u \in X$ in the direction $v \in X$ is defined by

$$\varphi^0(u; v) = \limsup_{y \to u, \lambda \downarrow 0} \frac{\varphi(y + \lambda v) - \varphi(y)}{\lambda}.$$

Definition A.8 *(Clarke subdifferential).* Let $\varphi \colon X \to \mathbb{R}$ be a locally Lipschitz function defined on a Banach space X. The Clarke subdifferential (or the generalized gradient) of φ at point $u \in X$ is the subset of the dual space X^* given by

$$\partial \varphi(u) = \{ \zeta \in X^* \mid \varphi^0(u; v) \geq \langle \zeta, v \rangle_{X^* \times X} \text{ for all } v \in X \}.$$

To state the definition of a regular function in the sense of Clarke, we need to recall the definition of the classical directional derivative for function $\varphi\colon X \to \mathbb{R}$ at point $u \in X$ in the direction $v \in X$, which is given by

$$\varphi'(u; v) = \lim_{\lambda \downarrow 0} \frac{\varphi(u + \lambda v) - \varphi(u)}{\lambda}.$$

Definition A.9 *(Clarke regular function).* If the function $\varphi\colon X \to \mathbb{R}$ is locally Lipschitz on a Banach space X, then we say that φ is regular in the sense of Clarke at point $u \in X$, if

(a) the classical directional derivative $\varphi'(u; v)$ exists for all $v \in X$,
(b) $\varphi^0(u; v) = \varphi'(u; v)$ for all $v \in X$.

We conclude the paper with the definition of marginal function, for the details, refer e.g. to page 51 of the monograph [2].

Definition A.10 *(Marginal function).* Let X and Y be two sets, G be a set-valued map from Y to X and W be a real-valued function defined on $X \times Y$. Consider the family of maximization problems

$$p(y) = \sup_{x \in G(y)} W(x, y),$$

which defined upon the parameter y. Thus, the function p is called the marginal function.

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