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A maximum principle at infinity with applications to geometric vector fields

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A B S T R A C T

We derive a new form of maximum principle, applicable to a vector field with nonnegative divergence in a connected, oriented, complete noncompact Riemannian manifold. We then use it to obtain some applications to Killing vector fields. More precisely, we first show that, under a reasonable condition at infinity, an orientable, connected, complete noncompact hypersurface of a Riemannian manifold, transversal to a Killing vector field of constant norm and with nonnegative second fundamental form, is totally geodesic. We also deal with the case of a hypersurface of constant mean curvature – instead of nonnegative second fundamental form, and show that it has to be totally geodesic too.

1. Introduction

In Differential Geometry, several important geometric situations are analytically modeled and studied with the aid of linear or quasilinear elliptic partial differential operators. Therefore, it is not surprising that several versions of the maximum principle play a major role in the theory, as a quick glance through [3] corroborates.

In this paper, we derive, in Theorem 2.2, a form of maximum principle which is appropriate for controlling the behavior of a smooth vector field, satisfying a suitable set of hypotheses, on a connected, complete noncompact Riemannian manifold. It is the analogue of the simple fact that, on such a manifold, a nonnegative subharmonic function that vanishes at infinity actually vanishes identically.

Then, in Section 3 we present some relevant geometric applications of our maximum principle to Killing vector fields. Further applications, to Lorentzian and Kählerian Geometry, will appear in [1] and [2].

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2. A maximum principle for vector fields

Our version of maximum principle for vector fields is the object of the coming result. In order to properly state it, if M is a connected, complete noncompact Riemannian manifold, we let $d(\cdot, o) : M \to [0, +\infty)$ stand for the Riemannian distance of M, measured from a fixed point $o \in M$. Thus, if $f \in \mathcal{C}^0(M)$ satisfies

$$\lim_{d(x,o)\to+\infty} f(x) = 0,$$

we shall simply say that f *converges to zero at infinity.*

For later use, we recall Proposition 2.1 of [4].

Proposition 2.1 *([4]). Let M be oriented and $\mathcal{L}^1(M)$ denote the space of Lebesgue integrable functions on M. If $Y \in \mathfrak{X}(M)$ is such that $|Y| \in \mathcal{L}^1(M)$ and $\operatorname{div}_M(Y)$ does not change sign on M, then $\operatorname{div}_M(Y) \equiv 0$ on M.*

Theorem 2.2. *Let M be a connected, oriented, complete noncompact Riemannian manifold, and let $X \in \mathfrak{X}(M)$ be a vector field on M. Assume that there exists a nonnegative, non-identically vanishing function $f \in \mathcal{C}^\infty(M)$, converging to zero at infinity and such that $\langle \nabla f, X\rangle \geq 0$. If $\operatorname{div}_M X \geq 0$ on M, then:*

(a) $\langle \nabla f, X\rangle \equiv 0$ *on* M.
(b) $\operatorname{div}_M X \equiv 0$ *on* $M \setminus f^{-1}(0)$.
(c) $\operatorname{div}_M X \equiv 0$ *on* M *if* $f^{-1}(0)$ *has zero Lebesgue measure.*

Proof. Let m denote the Lebesgue measure on M, and $\mathcal{L}^1(M)$ the space of Lebesgue integrable functions on M. The hypotheses on f assure that it is bounded above on M. Assume, with no loss of generality, that $\sup_M f = a > 1$.

We shall first show that there exist C^1 functions $\phi, \psi : [0, a] \to \mathbb{R}_+$, with strictly positive derivatives and satisfying the following conditions:

(i) $\phi \circ f = \phi(f) \in \mathcal{L}^1(M)$;
(ii) $(\psi \circ f)|X| = \psi(f)|X|$ is bounded on M.

To this end, for each integer $k \geq 1$, let

$$A_k = \Big\{x \in M; f(x) > \frac{1}{k}\Big\}.$$

Note that $A_k \subset A_{k+1}$ and $0 < m(A_k) < \infty$ for every $k \geq 1$. This second relation follows from the fact that $f \geq 0$ is smooth, non identically zero and converges to 0 at infinity. Moreover, the nonnegativity of f also gives

$$M = f^{-1}(0) \cup \bigcup_{k\geq 1} A_k.$$

In order to get (i), set $\phi(a) = \frac{1}{m(A_1)}$, $\phi\big(\frac{1}{k}\big) = \frac{1}{2^k m(A_{k+1})}$ and note that $\phi\Big(\frac{1}{k+1}\Big) < \phi\Big(\frac{1}{k}\Big) < \phi(a)$ for $k \geq 1$. Now, extend ϕ to a C^1 strictly increasing function on $(0, a]$ satisfying

$$0 < \phi'(t) < 2\left(\frac{\phi(\frac{1}{k}) - \phi(\frac{1}{k+1})}{\frac{1}{k} - \frac{1}{k+1}}\right) \tag{1}$$

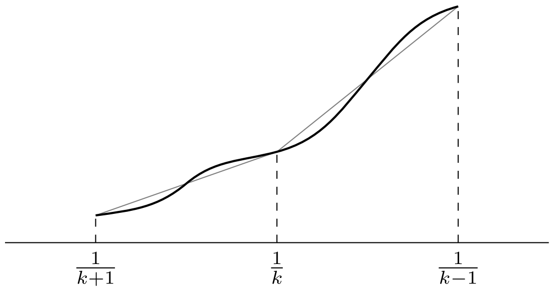

Fig. 1. Extending ϕ to $(0, a]$.

for every $t \in \left[\frac{1}{k+1}, \frac{1}{k}\right]$ (Fig. 1 shows that, in order to accomplish this, one has to ensure that ϕ does not oscillate too fast).

Letting $\phi(0) = 0$, we claim that ϕ is C^1 in $[0, a]$. Firstly, for $0 < t < \frac{1}{j}$, there exists $k \geq j$ such that $\frac{1}{k+1} \leq t < \frac{1}{k}$. Therefore,

$$0 < \frac{\phi(t)}{t} < \frac{\phi(\frac{1}{k})}{\frac{1}{k+1}} = \frac{k+1}{2^k m(A_{k+1})} \xrightarrow{k} 0,$$

whence $\phi'(0) = 0$. On the other hand,

$$\frac{\phi(\frac{1}{k}) - \phi(\frac{1}{k+1})}{\frac{1}{k} - \frac{1}{k+1}} = \frac{k(k+1)(2m(A_{k+2}) - m(A_{k+1}))}{2^{k+1}m(A_{k+2})m(A_{k+1})} < \frac{k(k+1)}{2^k m(A_{k+1})} \xrightarrow{k} 0,$$

so that (1) gives $\lim_{t\to 0} \phi'(t) = 0 = \phi'(0)$, as wished.

Finally, since $\phi(f) = 0$ in $f^{-1}(0)$, we have

$$\begin{aligned}\int_M \phi(f)dM &= \int_{A_1} \phi(f)dM + \sum_{k=1}^{\infty} \int_{A_{k+1}\setminus A_k} \phi(f)dM \\ &\leq \Big(\sup_{A_1} \phi(f)\Big) m(A_1) + \sum_{k=1}^{\infty} \Big(\sup_{A_{k+1}\setminus A_k} \phi(f)\Big) m(A_{k+1} \setminus A_k) \\ &< \phi(a)m(A_1) + \sum_{k=1}^{\infty} \phi\left(\frac{1}{k}\right) m(A_{k+1}) \\ &= 1 + \sum_{k=1}^{\infty} \frac{1}{2^k} = 2,\end{aligned}$$

and this establishes (i).

In order to get (ii), let $s_k = \sup_{A_k} |X| + 1$, $\psi(a) = \frac{1}{s_1}$ and $\psi\left(\frac{1}{k}\right) = \frac{1}{2^k s_{k+1}}$ for $k \geq 1$. Here again, note that $\psi\left(\frac{1}{k+1}\right) < \psi\left(\frac{1}{k}\right) < \psi(a)$ for $k \geq 1$. Setting $\psi(0) = 0$ and arguing as for ϕ, we can extend ψ to a strictly increasing C^1 function on $[0, a]$, with positive derivative. It now suffices to note that, since $\psi(f)|X| = 0$ in $f^{-1}(0)$, $\psi(f)|X| < 1$ in A_1 and $\psi(f)|X| < \frac{1}{2^k}$ in $A_{k+1} \setminus A_k$ and for every $k \geq 1$, we have $\psi(f)|X|$ bounded on M.

Having constructed ϕ and ψ as above, let $Y = \phi(f)\psi(f)X$, so that $Y \in C^1(M)$ and $|Y| \in \mathcal{L}^1(M)$. Moreover, the properties of ϕ and ψ, together with our hypotheses on f and X, give

$$\mathrm{div}_M(Y) = \phi(f)\psi(f)\mathrm{div}_M(X) + (\phi'(f)\psi(f) + \phi(f)\psi'(f))\langle \nabla f, X\rangle \geq 0 \tag{2}$$

on M. Hence, Proposition 2.1 guarantees that $\mathrm{div}_M Y \equiv 0$ on M.

Now, since $\phi'(f)\psi(f) + \phi(f)\psi'(f) > 0$ outside $f^{-1}(0)$, (2) gives $\langle \nabla f, X\rangle = 0$ outside $f^{-1}(0)$. On the other hand, for $x \in f^{-1}(0)$ we must have $\nabla f(x) = 0$, since x is a point of minimum of f. Thus, $\langle \nabla f, X\rangle = 0$ on M. Relation (2) now reduces to $\phi(f)\psi(f)\mathrm{div}_M(X) = 0$, so that $\mathrm{div}_M(X) = 0$ outside $f^{-1}(0)$ (for $\phi(f)$ and $\psi(f)$ are positive there). Then, the open set $\{p \in M; \mathrm{div}_M(X)(p) > 0\}$ is contained in $f^{-1}(0)$, so that it is empty if $m\big(f^{-1}(0)\big) = 0$. $\square$

3. Applications to Killing vector fields

In this section, we consider a Riemannian manifold $(\overline{M}^{n+1}, \langle\cdot,\cdot\rangle)$ endowed with a non-vanishing Killing vector field Z of constant norm. Replacing Z by $\frac{Z}{|Z|}$, if necessary, we can assume that $|Z| = 1$, and we do so hereafter.

If $M^n \subset \overline{M}^{n+1}$ is an orientable hypersurface of $\overline{M}$ transversal to Z, then M can be oriented by the choice of a unit normal vector field N such that $\langle N, Z\rangle > 0$.

We let $\theta : M \to [0, \frac{\pi}{2})$ denote the acute angle between N and Z at each point, given by the equality

$$\langle N, Z\rangle = \cos\theta.$$

If θ vanishes identically, then N is identical to Z. In such a case, for $p \in M$, $u, v \in T_pM$ and letting $\overline{\nabla}$ denote the Levi-Civita connection of $\overline{M}$, the Killing condition of X allows us to compute

$$\langle A_p u, v\rangle = -\langle \overline{\nabla}_u N, v\rangle = -\langle \overline{\nabla}_u Z, v\rangle = \langle \overline{\nabla}_v Z, u\rangle = -\langle A_p v, u\rangle,$$

where A stands for the Weingarten operator (or second fundamental form) of M with respect to N. Since A is symmetric, this implies $A_p = 0$, and the immateriality of the choice of p shows that M is totally geodesic.

If θ does not vanish identically, we want suitable conditions on A that still force M to be totally geodesic. To this end, if M is complete and noncompact, we say that N *converges to Z at infinity* provided θ converges to 0 at infinity.

We can now state and prove a first Bernstein-type result, which reads as follows.

Theorem 3.1. *Let $\overline{M}^{n+1}$ be a Riemannian manifold and $M^n \subset \overline{M}^{n+1}$ be a connected, orientable, complete noncompact hypersurface of $\overline{M}$, transversal to a unit Killing vector field Z of $\overline{M}$ and oriented by the choice of a unit normal vector field N such that $\langle N, Z\rangle > 0$. If the second fundamental form A of M with respect to N is nonnegative and N converges to Z at infinity, then M is totally geodesic.*

Proof. Setting $f = 1 - \langle N, Z\rangle$ on M, we get $f \geq 0$. If f vanishes identically, then N is identical to Z and we have already seen that M is totally geodesic. Therefore, we may assume that f does not vanish identically.

Let $Z^\top$ stand for the orthogonal projection of Z on M. The Killing condition on Z gives the classical formulas

$$\langle \nabla f, Z^\top\rangle = \langle AZ^\top, Z^\top\rangle \ \text{ and } \ \mathrm{div}_M(Z^\top) = \langle N, Z\rangle \mathrm{tr}(A),$$

where $\mathrm{tr}(A)$ stands for the trace of the second fundamental form.

In turn, the expressions above, together with the nonnegativity of A and the choice of N, give

$$\langle \nabla f, Z^\top\rangle \geq 0 \ \text{ and } \ \mathrm{div}_M(Z^\top) \geq 0.$$

Moreover, since N converges to Z at infinity, we get that f converges to 0 at infinity. Hence, Theorem 2.2 allows us to conclude that $\langle \nabla f, Z^\top \rangle \equiv 0$ on M and $\operatorname{div}_M(Z^\top) \equiv 0$ on $M \setminus F$, where $F = f^{-1}(0)$. However, since $\langle N, Z \rangle > 0$ on M, we conclude that $\operatorname{tr}(A) \equiv 0$ on $M \setminus F$, and hence $A \equiv 0$ on $M \setminus F$.

Now, note that $F = \{x \in M;\, N(x) = Z(x)\}$. If x is in the interior of F, then $N = Z$ in a neighborhood of x, whence $A \equiv 0$ in such a neighborhood. In particular, $A \equiv 0$ in the interior of F and, since it already vanishes in $M \setminus F$, we conclude that $A \equiv 0$ on all of M. Therefore, M is totally geodesic. $\square$

Essentially the same argument as the one presented in the proof of the previous result allows us to approach the case of a connected, complete noncompact spacelike hypersurface of a Lorentzian ambient space (in such a case, orientability comes for free). This gives a second Bernstein-type result, quoted below.

Theorem 3.2. *Let $\overline{M}^{n+1}$ be a Lorentzian manifold and $M^n \subset \overline{M}^{n+1}$ be a connected, complete noncompact spacelike hypersurface of $\overline{M}$, transversal to a unit timelike Killing vector field Z of $\overline{M}$ and oriented by the choice of a unit timelike normal vector field N such that $\langle N, Z \rangle < 0$. If the second fundamental form A of M with respect to N is nonnegative and N converges to Z at infinity, then M is totally geodesic.*

Interesting particular cases of Theorems 3.1 and 3.2 are those of a Riemannian or Lorentzian group, namely, a Lie group equipped with a biinvariant Riemannian or Lorentzian metric, respectively. Indeed, in either such case, every element of the Lie algebra of the group is a Killing vector field of constant norm. Hence, we get the following corollaries, which encompass $\mathbb{R}^{n+1}$ and $\mathbb{L}^{n+1}$ with their standard metrics.

Corollary 3.3. *Let G^{n+1} be a Riemannian group with Lie algebra $\mathfrak{g}$, and $M^n \subset G^{n+1}$ be a connected, orientable, complete noncompact hypersurface of G, transversal to $Z \in \mathfrak{g}$ and oriented by the choice of a unit normal vector field N such that $\langle N, Z \rangle > 0$. If the second fundamental form A of M with respect to N is nonnegative and N converges to Z at infinity, then M is totally geodesic.*

Corollary 3.4. *Let G^{n+1} be a Lorentzian group with Lie algebra $\mathfrak{g}$, and $M^n \subset G^{n+1}$ be a connected, complete noncompact spacelike hypersurface of G, transversal to a timelike $Z \in \mathfrak{g}$ and oriented by the choice of a unit normal vector field N such that $\langle N, Z \rangle < 0$. If the second fundamental form A of M with respect to N is nonnegative and N converges to Z at infinity, then M is totally geodesic.*

We end this section by stating a relevant particular case of Corollary 3.3.

Corollary 3.5. *Let $M^n \subset \mathbb{R}^{n+1}$ be an entire graph over $\mathbb{R}^n$. If the second fundamental form of M with respect to the upward-pointing unit normal vector field N is nonnegative and N converges to a fixed vector V at infinity, then M is a hyperplane orthogonal to V.*

If we drop the requirement that the second fundamental form is nonnegative definite and add the hypothesis that the hypersurface is of constant mean curvature, then we get a second Bernstein-type result.

Theorem 3.6. *Let $\overline{M}^{n+1}$ be an Einstein Riemannian manifold and $M^n \subset \overline{M}^{n+1}$ be a connected, orientable, complete noncompact hypersurface of $\overline{M}$ with constant mean curvature, transversal to a unit Killing vector field Z of $\overline{M}$ and oriented by the choice of a unit normal vector field N such that $\langle N, Z \rangle > 0$. If N converges to Z at infinity, then M is totally geodesic.*

Proof. We denote by $\overline{\nabla}$ and ∇ the Levi-Civita connections of $\overline{M}$ and M, respectively, and by $\overline{R}$ the curvature tensor of $\overline{M}$.

Let A be the second fundamental form of M with respect to N, let H be the constant value of the mean curvature and $Z^\top$ be the orthogonal projection of Z on M. We shall compute $\operatorname{div}_M(AZ^\top)$ and in order to do this, we fix $p \in M$ and let $(e_1, \dots, e_n)$ be a local orthonormal frame field on M, geodesic at p and diagonalizing A at p. Then, at p, we get

$$\begin{aligned}
\mathrm{div}_M(AZ^\top) &= \sum_{i=1}^n \langle \nabla_{e_i} AZ^\top, e_i\rangle = \sum_{i=1}^n \langle -\overline{\nabla}_{e_i}\overline{\nabla}_{Z^\top} N, e_i\rangle \\
&= \sum_{i=1}^n \langle \overline{R}(Z^\top, e_i)N - \overline{\nabla}_{Z^\top}\overline{\nabla}_{e_i} N - \overline{\nabla}_{[e_i, Z^\top]} N, e_i\rangle \\
&= -\mathrm{Ric}_{\overline{M}}(Z^\top, N) + Z^\top(nH) + \sum_{i=1}^n \langle Ae_i, \overline{\nabla}_{e_i} Z^\top\rangle \\
&= \langle N, Z\rangle |A|^2.
\end{aligned}$$

As in the proof of Theorem 3.1, we take $f = 1 - \langle N, Z\rangle$ on M and assume that f does not vanish identically. Since f converges to 0 at infinity and the vector field $AZ^\top$ satisfies

$$\langle \nabla f, AZ^\top\rangle = |AZ^\top|^2 \geq 0 \quad \text{and} \quad \mathrm{div}_M(AZ^\top) \geq 0,$$

Theorem 2.2 gives $\langle \nabla f, AZ^\top\rangle \equiv 0$ on M and $\mathrm{div}_M(AZ^\top) \equiv 0$ on $M \setminus F$, where $F = f^{-1}(0)$. Arguing again as in the proof of Theorem 3.1, we conclude that M is totally geodesic. $\square$

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