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# Sharp Landen transformation inequalities for hypergeometric functions, with applications

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ABSTRACT

The authors present sharp Landen transformation inequalities for the hypergeometric functions ${}_2F_1(a,b;a+b;x)$ and ${}_2F_1(a,b;(a+b+1)/2;x)$, by showing the monotonicity properties of certain combinations defined in terms of one of these two hypergeometric functions and linear (or rational) functions, thus giving complete solutions of the problem on extending the well-known Landen transformation identities for the complete elliptic integrals of the first kind to these two hypergeometric functions, and substantially improving the related known results. As applications of these results, sharp Landen transformation inequalities are obtained for the generalized Grötzsch ring functions and the modular functions, which appear in Ramanujan's modular equations. Some other properties of hypergeometric functions and several properties of the Ramanujan constant are obtained, too.


## 1. Introduction

Let $\mathbb{N}=\{n|\ n$ is a positive integer$\}$ as usual, and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. For real numbers $a,b$ and $c$ with $c\neq 0,-1,-2,\cdots$, the Gaussian hypergeometric function is defined by

$$F(a,b;c;x)={}_2F_1(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)n!}x^n,\ \ |x|<1, \tag{1.1}$$

where $(a,0)=1$ for $a\neq 0$, and $(a,n)=a(a+1)(a+2)\cdots(a+n-1)$ for $n\in\mathbb{N}$ is the shifted factorial function. The function $F(a,b;c;x)$ is said to be zero-balanced if $c=a+b$. It is well known that $F(a,b;c;x)$, as a very important special function, has wide applications in mathematics, physics, as well as in some fields of engineering, and many other special functions in mathematical physics and even some elementary functions are particular or limiting cases of $F(a,b;c;x)$ (cf. [6,2,1,8,7,5,12–16,19,20,22,28,39,40]).

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Throughout this paper, by the symmetry of the parameters $a$ and $b$ in the function $F(a,b;a+b;x)$, we assume that $a \le b$, without loss of generality. For each $r \in [0,1]$, we let $r' = \sqrt{1-r^2}$, and for $a,b \in (0,\infty)$ with $c = a+b$ and $C = (c+1)/2$, let

$$\begin{cases} F(r) = F(a,b;c;r), F_+(r) = F(a+1,b+1;c+2;r), F_0(r) = F\left(\frac{1}{2},\frac{1}{2};1;r\right), \\ F_{3/2}(r) = F\left(\frac{3}{2},\frac{3}{2};3;r\right), G_0(r) = F\left(\frac{1}{2},\frac{1}{2};2;r\right), \overline{G}_0(r) = F\left(\frac{3}{2},\frac{3}{2};2;r\right), \\ \overline{F}(r) = F(a,b;C;r), G(r) = F(a,b;c+1;r), \overline{G}(r) = F(a+1,b+1;C+1;r). \end{cases} \tag{1.2}$$

Observe that the conditions $a \le b$ and $c = a+b$ imply that

$$a \le c/2 \le b \quad \text{and} \quad ab = a(c-a) \le c^2/4. \tag{1.3}$$

By (1.3), for $a,b \in (0,\infty)$ with $c = a+b$, we have the following simple relations

$$\begin{cases} c \le 1 \Rightarrow ab \le c^2/4 \le c/4 \le 1/4, \; 1/a + 1/b \le 4 \Leftrightarrow ab \ge c/4, \\ ab \ge (11-7c)/16 \Rightarrow \left(c^2/4\right) - (11-7c)/16 = (c-1)(4c+11)/16 \ge 0 \Leftrightarrow c \ge 1, \\ ab \ge (c+1)/8 \Rightarrow \left(c^2/4\right) - (c+1)/8 = (c-1)(2c+1)/8 \ge 0 \Leftrightarrow c \ge 1, \\ ab \ge c/4 \Rightarrow \left(c^2/4\right) - c/4 \ge 0 \Leftrightarrow c \ge 1, \; (5/4) - c < ab < (c+1)/8 \Rightarrow c > 1, \end{cases} \tag{1.4}$$

which will be frequently used in our results and their proofs.

One of the most important special cases of $F(a,b;c;x)$ is the complete elliptic integral $\mathscr{K}(r)$ of the first kind associated with the modulus $r \in (0,1)$, which is defined by

$$\mathscr{K}(r) = \frac{\pi}{2} F\left(\frac{1}{2},\frac{1}{2};1;r^2\right) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-r^2\sin^2 t}}. \tag{1.5}$$

The complete elliptic integral of the first kind associated with the complementary modulus $r'$ is denoted by $\mathscr{K}'(r) = \mathscr{K}(r')$. Clearly, $\mathscr{K}(0) = \pi/2$ and $\mathscr{K}(1^-) = \infty$. It is well known that $\mathscr{K}(r)$ and $\mathscr{K}'(r)$ satisfies the following Landen transformation identities

$$\mathscr{K}\left(\frac{2\sqrt{r}}{1+r}\right) = (1+r)\mathscr{K}(r), \; \mathscr{K}\left(\frac{1-r}{1+r}\right) = \frac{1+r}{2}\mathscr{K}'(r) \tag{1.6}$$

(cf. [6,1,7,5,19]), or equivalently,

$$F_0\left(\frac{4r}{(1+r)^2}\right) = (1+r)F_0\left(r^2\right), \; F_0\left(\left(\frac{1-r}{1+r}\right)^2\right) = \frac{1+r}{2}F_0\left(r'^2\right). \tag{1.7}$$

More general forms of (1.7) were given in [13, Entries 3 & 5, p.50]. Some other transformations and identities were given in [1, pp.560-561] and in [13–17]. The following question is natural:

**Question 1.1.** To what extent, can the identities in (1.7) be extended to the zero-balanced hypergeometric function $F(a,b;a+b;x)$ or even some other hypergeometric functions for $a,b \in (0,\infty)$ and $r \in (0,1)$?

In [26], the authors tried to extend (1.7) to $F(a,b;a+b;r)$ by studying the monotonicity properties of the function $\varphi(r) \equiv (1+\sqrt{r})F(r) - F(x)$ for $a,b \in (0,\infty)$ and $(0,1)$, where $x = 4\sqrt{r}/(1+\sqrt{r})^2$. It was stated in [26, Theorem 1.2(1)] that for $a,b \in (0,1)$ with $c = a+b \le 1$, $\varphi$ is increasing from $(0,1)$ onto $(0,\delta)$ so that $F(4r/(1+r)^2) \le (1+r)F(r^2) \le \delta + F(4r/(1+r)^2)$, where $\delta = (R - \log 16)/B$.

Here and hereafter, $B = B(a,b)$ and $R = R(a,b)$ are defined by (2.1) and (2.4) in Section 2, respectively. Unfortunately, in [26, Proof of Theorem 1.2(1)], the derivative $dx/dr = x(1-\sqrt{r})/[2r(1+\sqrt{r})]$ was misprinted as $dx/dr = x(1-\sqrt{r})/[2\sqrt{r}(1+\sqrt{r})]$, thus resulting in a gap as pointed out by A. Baricz in his email to the second author of [26] in June 2005. A correct proof of [26, Theorem 1.2(1)] was given in [29]. Moreover, it was proved in [29, Lemma 3.2] that if $a + b \le 1$ $(1/a + 1/b \le 4)$, then $\varphi$ is increasing (decreasing) from $(0, 1)$ onto $(0, \delta)$ $((\delta, 0)$, respectively). The following results were also proved in [29].

**Theorem 1.2.** *(1)* [29, Theorem 2.2 and Corollary 2.3] *For $a, b \in (0, \infty)$ with $ab \le 1/4$, and for $r \in (0, 1)$,*

$$1 \le (1+r)F\left(r^2\right)/F\left(4r/(1+r)^2\right) \le B/\pi, \tag{1.8}$$

$$\pi F\left(r^2\right)/B < F\left(4r/(1+r)^2\right) < 2F\left(r^2\right). \tag{1.9}$$

*If $1/a + 1/b \le 4$, then for $r \in (0, 1)$, each inequality in (1.8) is reversed, and*

$$F\left(r^2\right) < F\left(4r/(1+r)^2\right) < 2\pi F\left(r^2\right)/B. \tag{1.10}$$

*(2)* ([29, Theorem 2.4]) *For $r \in (0, 1)$ and $a, b \in (0, \infty)$ with $a + b \le 1$,*

$$0 \le (1+r)F\left(r^2\right) - F\left(4r/(1+r)^2\right) \le (R - \log 16)/B. \tag{1.11}$$

*If $1/a + 1/b \le 4$, then each inequality in (1.11) is reversed.*

*(3)* ([29, Theorem 2.5]) *For $r \in (0, 1)$ and $a, b \in (0, \infty)$ with $ab \le 1/4$,*

$$1/2 \le F\left(((1-r)/(1+r))^2\right) \Big/ \left[(1+r)F\left(r'^2\right)\right] \le B/(2\pi). \tag{1.12}$$

*If $a, b \in (0, \infty)$ with $a + b \le 1$, then*

$$(1+r)F\left(r'^2\right) \le 2F\left(((1-r)/(1+r))^2\right) \le (1+r)\left[F\left(r'^2\right) + (R - \log 16)/B\right]. \tag{1.13}$$

*If $a, b \in (0, \infty)$ with $1/a + 1/b \le 4$, then each inequality in (1.12)–(1.13) is reversed.*

In [38], Z.H. Yang, Y.M. Chu and M.K. Wang improved Theorem 1.2 and [29, Lemmas 3.1 & 3.2] (see also [30]). Recently, M.K. Wang and Y.M. Chu studied another kind of generalizations of (1.7). They revealed some monotonicity properties of the function $r \mapsto (1+\sqrt{r})\overline{F}(r) - \overline{F}(4\sqrt{r}/(1+\sqrt{r})^2)$ for $r \in (0, 1)$, and proved that the following Landen inequality

$$\overline{F}\left(4r/(1+r)^2\right) \le (1+r)\overline{F}\left(r^2\right) \tag{1.14}$$

holds if and only if $a+b \le 1$, while (1.14) is reversed if and only if $ab \ge (a+b+1)/8$ (see [31, Theorem 2.1]). They also obtained in [31, Theorem 2.2] some other Landen transformation inequalities similar to (1.11). In [9–11,30,32,33,35], some other transformation inequalities and Landen-type inequalities were obtained.

The main purpose of this paper is to give complete answers to Question 1.1 for $F(a, b; a + b; r)$ and $F(a, b; (a+b+1)/2; r)$, and present sharp double Landen transformation inequalities for them, by showing the monotonicity properties of certain combinations defined in terms of $F(a, b; a+b; r)$ or $F(a, b; (a+b+1)/2; r)$ and linear or rational functions with innovative ways, thus deepening the studies of the extensions of (1.7) above-mentioned, and substantially improving and perfecting all the related known results. (See the theorems and their corollaries proved in Sections 4–5.) In Section 6, we shall apply these results to obtain several sharp Landen transformation inequalities for the generalized Grötzsch ring functions and the modular functions, which appear in Ramanujan's modular equations. To meet the needs of the proofs of our main results, we shall show several important properties of the Ramanujan constant $R(a,b)$ and some other properties of hypergeometric functions in Section 2 and Section 3, respectively.

## 2. Preliminaries

In this section, we introduce some more notations for later use, show several properties of the Ramanujan constant $R(a,b)$ defined by (2.4) below, and establish a technical lemma.

For $x, y \in (0,\infty)$, the classical gamma, psi (digamma) and beta functions are defined as

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt,\ \psi(x)=\frac{d}{dx}\log\Gamma(x),\ B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{2.1}$$

respectively (cf. [6,1,5,28]). Let $\gamma = \lim_{n\to\infty}[\sum_{k=1}^n(1/k)-\log n] = 0.577215664\cdots$ be the Euler–Mascheroni constant, and $\zeta(x)=\sum_{n=1}^\infty n^{-x}$ the Riemann zeta function. The following formulas are well-known (cf. [1, 6.1.15, 6.3.5, 6.3.16 & 6.4.10])

$$x\Gamma(x)=\Gamma(x+1),\ \psi^{(n)}(x+1)=\psi^{(n)}(x)+(-1)^n n!x^{-n-1},\ n\in\mathbb{N}_0, \tag{2.2}$$

$$\psi(x)=-\gamma-\frac{1}{x}+\sum_{k=1}^\infty\frac{x}{k(k+x)},\ \psi^{(n)}(x)=\sum_{k=1}^\infty\frac{(-1)^{n+1}n!}{(k+x)^{n+1}},\ n\in\mathbb{N}. \tag{2.3}$$

Define the function $R$ on $(0,\infty)\times(0,\infty)$ by

$$R(a,b)=-2\gamma-\psi(a)-\psi(b), \tag{2.4}$$

which is called the Ramanujan constant in literature although it is in fact a function of $a$ and $b$ (cf. [23]). Let

$$R_c(a)=R(a,c-a)\equiv-2\gamma-\psi(a)-\psi(c-a),\ a+b=c, \tag{2.5}$$

$$R(a)=R(a,1-a)=-2\gamma-\psi(a)-\psi(1-a),\ a+b=1, \tag{2.6}$$

$$B(a)=B(a,1-a)=\Gamma(a)\Gamma(1-a)=\pi/\sin(\pi a),\ a+b=1. \tag{2.7}$$

$R(a)$ is also said to be the Ramanujan constant in literature (cf. [23]). By (2.6)–(2.7) and [1, 6.3.3],

$$B(1/2)=\pi,\ R(1/2)=-2[\gamma+\psi(1/2)]=\log 16. \tag{2.8}$$

By the symmetry, we always assume that $a\in(0,c/2]$ in (2.5), and $a\in(0,1/2]$ in (2.6)–(2.7), without loss of generality. In the sequel, we let $\alpha=ab/(a+b)$, $B=B(a,b)$, $R=R(a,b)$, and by (2.2) and (2.4),

$$B_+=B(a+1,b+1)=\frac{\alpha B}{a+b+1},\ R_+=R(a+1,b+1)=R-\frac{1}{\alpha}. \tag{2.9}$$

Next we recall the following well-known formulas

$$F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\ c>a+b, \tag{2.10}$$

$$\frac{d}{dx}F(a,b;c;x)=\frac{ab}{c}F(a+1,b+1;c+1;x), \tag{2.11}$$

$$F(a,b;c;x)=(1-x)^{c-a-b}F(c-a,c-b;c;x), \tag{2.12}$$

$$BF(a,b;a+b;r)=R-\log(1-r)+O((1-r)\log(1-r)) \tag{2.13}$$

as $r \to 1$ (see [1, 15.1.20, 15.2.1,15.3.3, & 15.3.10] and [6,8]). It follows from (2.10)–(2.12) that

$$F'(r) = \frac{\alpha G(r)}{1-r},\ F_0'(r) = \frac{G_0(r)}{4(1-r)},\ G(1) = \frac{1}{\alpha B},\ G_0(1) = \frac{4}{\pi}. \tag{2.14}$$

Now we present some properties of $R(a,b)$ in Lemmas 2.1 and 2.2.

**Lemma 2.1.** *(1) For each $c \in (0,\infty)$, as functions of $a$, $g_1(a) \equiv R_c(a) = -2\gamma - \psi(a) - \psi(c-a)$ and $g_2(a) \equiv B(a, c-a)$ are both strictly decreasing and convex on $(0, c/2]$.*

*(2) For $a, b \in (0,\infty)$, $a+b = c$, set $x = ab$. Then $R$ can be expressed by the following function of $x$ and $c$*

$$R(a,b) = g_3(x,c) \equiv \frac{c}{x} - \sum_{k=1}^{\infty} \frac{ck+2x}{k(k^2+ck+x)}, \tag{2.15}$$

*and $g_3$ is strictly decreasing and convex in $x \in (0, c^2/4]$, and in $c \in (0,\infty)$, with $g_3(0^+, c) = \infty$, while $g_4(c) \equiv g_3(c^2/4, c)$ is strictly decreasing and convex from $(0,\infty)$ onto $(-\infty,\infty)$ with $g_4(1) = R(1/2) = \log 16$ and $g_4(2) = R(1,1) = 0$.*

*(3) For each $c \in (0,\infty)$, the function $g_5(x) \equiv x g_3(x,c)$ is strictly decreasing from $(0,\infty)$ onto $(-\infty, c)$.*

**Proof.** (1) By the monotonicity property of $\psi'$, $g_1'(a) = \psi'(c-a) - \psi'(a)$ is strictly increasing from $(0, c/2]$ onto $(-\infty, 0]$. This yields the result for $g_1$.

By the monotonicity property of $\psi$, $g_2'(a)/g_2(a) = \psi(a) - \psi(c-a)$ is strictly increasing in $a$ from $(0, c/2]$ onto $(-\infty, 0]$, so that $g_2$ is strictly decreasing and convex on $(0, c/2]$ with $g_2(c/2) = \Gamma(c/2)^2/\Gamma(c)$, since $-g_2'(a) = g_2(a)[\psi(c-a) - \psi(a)]$ is a product of two positive decreasing functions.

(2) The monotonicity and convexity properties of $g_3$ in $c \in (0,\infty)$ follow from the monotonicity and concavity of $\psi$ and (2.5). It follows from (2.3) and (2.4) that

$$\begin{aligned} R(a,b) &= \frac{1}{a} + \frac{1}{b} - \sum_{k=1}^{\infty} \frac{a}{k(k+a)} - \sum_{k=1}^{\infty} \frac{b}{k(k+b)} \\ &= \frac{c}{ab} - \sum_{k=1}^{\infty} \frac{ck+2ab}{k(k^2+ck+ab)}, \end{aligned} \tag{2.16}$$

and hence $R(a,b)$ has the expression (2.15). Differentiation gives

$$\frac{\partial g_3}{\partial x} = -\left[\frac{c}{x^2} + \sum_{k=1}^{\infty} \frac{2k+c}{(k^2+ck+x)^2}\right],$$

yielding the monotonicity and convexity of $g_3$ in $x \in (0, c^2/4]$. Clearly, $g_3(0^+, c) = \infty$. By (2.15) and (2.3),

$$g_4(c) = g_3\left(c^2/4, c\right) = \frac{4}{c} - \sum_{k=1}^{\infty} \frac{c}{k(k+c/2)} = -2\left[\gamma + \psi\left(\frac{c}{2}\right)\right]. \tag{2.17}$$

Hence $g_4$ is strictly decreasing and convex from $(0,\infty)$ onto $(-\infty,\infty)$ by the monotonicity and concavity of $\psi$, with $g_4(1) = g_3(1/4, 1) = -2[\gamma + \psi(1/2)] = \log 16 = R(1/2)$ and $g_4(2) = 0$ by [1, 6.3.2 & 6.3.3].

(3) Part (3) follows from (2.15). □

**Lemma 2.2.** *Let $c = a + b$ for $a, b \in (0,\infty)$.*

*(1) If $ab \le 1/4$, then*

$$R(a,b) \ge \log 16, \tag{2.18}$$

*with the equality if and only if $a = b = 1/2$.*

*(2) If $ab \ge c/4$, then the inequality (2.18) is reversed.*

*(3) If $1/4 < ab < c/4$, then $R(a,b)$ and $\log 16$ are not directly comparable, that is, neither (2.18) nor its inverse inequality holds.*

*(4) If $ab = 1/4$ and $c \in [1,\infty)$, then*

$$abR(a,b) - c = \frac{1}{4}R(a,b) - c \le \log 2 - 1, \tag{2.19}$$

*and the second equality holds if and only if $c = 1$.*

**Proof.** (1) Let $g_4$ be as in Lemma 2.1. First, if $c = a + b \le 1$, then $ab \le c^2/4 \le 1/4$ by (1.3), and by Lemma 2.1(2),

$$R(a,b) \ge g_4(c) \ge g_4(1) = \log 16. \tag{2.20}$$

The first and second equalities in (2.20) simultaneously hold if and only if $a = b = 1/2$.

Second, let $ab \le 1/4$ and $c = a + b > 1$. Then by Lemma 2.1(2),

$$R(a,b) \ge g_6(c) \equiv g_3(1/4, c) = 4c - \sum_{k=1}^{\infty} \frac{ck + 1/2}{k(k^2 + ck + 1/4)}, \tag{2.21}$$

$$g_6'(c) = 4 - \sum_{k=1}^{\infty} \frac{k^2 - 1/4}{(k^2 + ck + 1/4)^2}. \tag{2.22}$$

The first equality in (2.21) holds if and only if $ab = 1/4$. Clearly, $g_6'$ is strictly increasing on $(0,\infty)$. It follows from (2.3) and [1, 6.4.4 & Table 23.3] that for $c > 1$,

$$\begin{aligned} g_6'(c) > g_6'(1) &= 4 - \sum_{k=1}^{\infty} \frac{k - 1/2}{(k+1/2)^3} = 4 - \sum_{k=1}^{\infty} \frac{1}{(k+1/2)^2} + \sum_{k=1}^{\infty} \frac{1}{(k+1/2)^3} \\ &= 4 - \psi'\left(\frac{1}{2}\right) - \frac{1}{2}\psi''\left(\frac{1}{2}\right) = 4 - \frac{\pi^2}{2} + 7\zeta(3) = 7.479596\cdots, \end{aligned}$$

so that $g_6$ is strictly increasing and convex on $(1,\infty)$ and

$$R(a,b) \ge g_6(c) > g_6(1) = g_4(1) = \log 16 \tag{2.23}$$

by (2.21). Hence (2.18) and its equality case follow from (2.20) and (2.23).

(2) If $ab \ge c/4$, then $c \ge 1$ by (1.3), and it follows from Lemma 2.1(2) that

$$R(a,b) \le g_7(c) \equiv g_3(c/4, c) = 4 - \sum_{k=1}^{\infty} \frac{c(k+1/2)}{k(k^2 + ck + c/4)}. \tag{2.24}$$

It is easy to verify that $g_7$ is strictly decreasing and convex on $(0,\infty)$, and hence by (2.24),

$$R(a,b) \le g_7(c) \le g_7(1) = g_3(1/4, 1) = g_4(1) = \log 16$$

for $c \ge 1$, showing that (2.18) is reversed.

(3) If $1/4 < ab < c/4$, then by Lemma 2.1(2), (2.21) and (2.24), we obtain

$$g_7(c) = g_3(c/4, c) = \lim_{ab \to c/4} R(a,b) \le R(a,b) \le \lim_{ab \to 1/4} R(a,b) = g_3(1/4, c) = g_6(c), \tag{2.25}$$

$$\lim_{c\to\infty} \lim_{ab \to (1/4)^+} R(a,b) = g_6(\infty) = \infty, \lim_{c\to\infty} \lim_{ab \to (c/4)^-} R(a,b) = g_7(\infty) = -\infty. \tag{2.26}$$

This shows that $R(a,b) < \log 16$ when $ab$ is close to $c/4$ and $c$ is sufficiently large, and $R(a,b) > \log 16$ when $ab$ is close to $1/4$ and $c$ is sufficiently large.

(4) It is easy to verify that for each $k \in \mathbb{N}$, the function $c \mapsto (ck+1/2)/(k^2+ck+1/4)$ is strictly increasing on $[1,\infty)$. If $ab = 1/4$ and $c \in [1,\infty)$, then it follows from (2.3) and (2.15) that

$$\begin{aligned} abR - c =& \frac{1}{4}R - c = -\frac{1}{4}\sum_{k=1}^{\infty} \frac{ck+1/2}{k\,(k^2+ck+1/4)} \\ &\le -\frac{1}{4}\sum_{k=1}^{\infty} \frac{k+1/2}{k\,(k^2+k+1/4)} = -\frac{1}{4}\sum_{k=1}^{\infty} \frac{1}{k(k+1/2)} \\ &= -\frac{1}{2}[\psi(1/2)+\gamma+2] = \log 2 - 1, \end{aligned}$$

and the third equality holds if and only if $c = 1$. □

**Remark 2.3.** (1) A more specifical and direct proof of Lemma 2.2(3) is as follows:

Taking $ab = (c+1)/8$, then $1/4 < ab < c/4$, and by Lemma 2.1(2),

$$R(a,b) = g_8(c) \equiv g_3\left(\frac{c+1}{8}, c\right) = \frac{8c}{c+1} - \sum_{k=1}^{\infty} \frac{ck+(c+1)/4}{k\,[k^2+ck+(c+1)/8]}, \tag{2.27}$$

with $g_8(1) = g_4(1) = \log 16$ and $g_8(\infty) = -\infty$. It is easy to show that the function

$$g_9(c) \equiv (c+1)^2 g_8'(c) = 8 - (c+1)^2 \sum_{k=1}^{\infty} \frac{k^2+(k/4)-1/8}{[k^2+ck+(c+1)/8]^2} \tag{2.28}$$

is strictly decreasing on $(1,\infty)$, with $g_9(\infty) = -\infty$ and

$$\begin{aligned} g_9(1) &= 4\left[2 - \sum_{k=1}^{\infty} \frac{k^2+(k/4)-1/8}{(k+1/2)^4}\right] = 4\left[2 - \sum_{k=1}^{\infty} \frac{1}{(k+1/2)^2} + \frac{3}{4}\sum_{k=1}^{\infty} \frac{1}{(k+1/2)^3}\right] \\ &= 4\left[2 - \psi'\left(\frac{1}{2}\right) - \frac{3}{8}\psi''\left(\frac{1}{2}\right)\right] = 4\left[2 - \frac{\pi^2}{2} + \frac{21}{4}\zeta(3)\right] = 3.375996\cdots. \end{aligned}$$

Hence by (2.28), $g_9$ has a unique zero $c_1 \in (1,\infty)$ such that $g_8$ is strictly increasing on $(1, c_1]$, and decreasing on $[c_1,\infty)$. This shows that there exists a number $c_2 \in (c_1,\infty)$ such that $g_8(c) > \log 16$ for $c \in (1, c_2)$, $g_8(c_2) = g_8(1) = \log 16$, and $g_8(c) < \log 16$ for $c \in (c_2,\infty)$. Hence $R(a,b) - \log 16$ changes sign, and neither (2.18) nor its inverse inequality holds for $a, b \in (0,\infty)$ with $1/4 < ab < c/4$.

(2) The conditions for several main results in [30,38] contain the comparisons between $R(a,b)$ and $\log 16$, and between $R(a,b)$ and $R(a_1,b_1)$ for distinct $(a,b)$ and $(a_1,b_1)$, which are not easy for checking computations. Lemmas 2.1 and 2.2 provide us a method to verify whether these conditions are satisfied.

In [38, Theorem 2.1] (see also [30, Lemma 1.1]), Z.H. Yang, Y.M. Chu and M.K. Wang proved a good criterion for the monotonicity of the quotient $\varphi(x) \equiv A(x)/B(x)$, where $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) =$

$\sum_{n=0}^{\infty} b_n x^n$ are of a common radius $r$ of convergence. They use the sign of the limiting values $H_{A,B}(r^-)$ of the function $H_{A,B} = (A'B/B') - A$ to determine the monotonicity of $\varphi$ in certain cases. Since $H_{A,B}(x) = B(x)^2\varphi'(x)/B'(x)$, it is easy to see that [38, Theorem 2.1] can be changed to the following more natural one.

**Lemma 2.4.** *Suppose that $r \in (0,\infty)$ is the common radius of convergence of the real power series $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ with $b_n > 0$, and $\{a_n/b_n\}$ is a non-constant sequence. Let $\varphi(x) = A(x)/B(x)$.*

*(1) If there is an $n_0 \in \mathbb{N}$ such that the sequence $\{a_n/b_n\}$ is increasing (decreasing) for $0 \le n \le n_0$, and decreasing (increasing) for $n \ge n_0$, then $\varphi$ is increasing (decreasing) on $(0,r)$ if and only if $\varphi'(r^-) \ge 0$ ($\varphi'(r^-) \le 0$, respectively).*

*(2) If there is an $n_0 \in \mathbb{N}$ such that the sequence $\{a_n/b_n\}$ is increasing (decreasing) for $0 \le n \le n_0$, and decreasing (increasing) for $n \ge n_0$, and if $\varphi'(r^-) < 0$ ($\varphi'(r^-) > 0$), then there exists a number $x_0 \in (0,r)$ such that $\varphi$ is strictly increasing (decreasing) on $(0,x_0]$ and decreasing (increasing, respectively) on $[x_0,r)$.*

## 3. Some properties of hypergeometric functions

In this section, we shall show some relations between the functions $F(r)$ and $F_0(r)$, $G(r)$ and $G_0(r)$, $\overline{F}(r)$ and $F_0(r)$, $\overline{G}(r)$ and $\overline{G}_0(r)$.

**Lemma 3.1.** *For $a,b \in (0,\infty)$ with $a+b=c$, and for $r \in (0,1)$, let $f_1(r) \equiv F(r)/F_0(r)$.*

*(1) If $ab \le 1/4$, or if $1/4 < ab < c/4$ with $R(a,b) \ge \log 16$, then $f_1$ is decreasing from $(0,1)$ onto $(\pi/B,1)$.*

*(2) If $ab \ge c/4$, then $f_1$ is increasing from $(0,1)$ onto $(1,\pi/B)$.*

*(3) In other cases not stated in parts (1)–(2), that is, $1/4 < ab < c/4$ with $R(a,b) < \log 16$, there exists a number $r_1 = r_1(a,b) \in (0,1)$ such that $f_1$ is decreasing on $(0,r_1]$ and increasing on $[r_1,1)$, with $f_1(0)=1$ and $f_1(1^-) = \pi/B$.*

*(4) If the conditions in parts (1)–(2) are satisfied, then for $r \in (0,1)$,*

$$\min\left\{1, \frac{\pi}{B}\right\} \le \frac{F(a,b;c;r)}{F(1/2,1/2;1;r)} \le \max\left\{1, \frac{\pi}{B}\right\}, \tag{3.1}$$

*with equality in each instance if and only if $a=b=1/2$. If the conditions in part (3) are satisfied, then the second inequality in (3.1) is still valid.*

**Proof.** Clearly, $f_1(0)=1$, and $f_1(1^-) = \pi/B$ by (2.13). For $n \in \mathbb{N}_0$, put

$$a_{1,n} = \frac{(a,n)(b,n)}{(c,n)n!}, b_{1,n} = \left[\frac{(1/2,n)}{n!}\right]^2, c_{1,n} = \frac{a_{1,n}}{b_{1,n}}, \Delta_1(n,ab,c) = \left(ab - \frac{1}{4}\right)n + ab - \frac{c}{4}.$$

Then by (1.1), and by differentiation,

$$f_1(r) = \frac{\sum_{n=0}^{\infty} a_{1,n} r^n}{\sum_{n=0}^{\infty} b_{1,n} r^n}, \quad \frac{c_{1,n+1}}{c_{1,n}} = 1 + \frac{\Delta_1(n,ab,c)}{(n+c)(n+1/2)^2}, \tag{3.2}$$

$$f_1'(r) = \frac{4\alpha F_0(r)G(r) - F(r)G_0(r)}{4(1-r)F_0(r)^2}. \tag{3.3}$$

Clearly, $f_1'(0) = \alpha - 1/4$. By (2.8) and (2.13)–(2.14), it is easy to obtain

$$\lim_{r\to 1}\frac{4\alpha BG(r)-\pi G_0(r)}{(1-r)F_0(r)}=\pi(1-4ab),\quad \lim_{r\to 1}\frac{\log(1/(1-r))}{F_0(r)}=\pi, \tag{3.4}$$

$$\lim_{r\to 1}[4\alpha BG(r)\log 16-\pi RG_0(r)]=4(\log 16-R), \tag{3.5}$$

and by l'Hôpital's rule,

$$\lim_{r\to 1}\frac{4\alpha BG(r)\log 16-\pi RG_0(r)}{(1-r)F_0(r)^2}=0 \text{ if } R=\log 16. \tag{3.6}$$

It follows from (2.13) and (3.3)–(3.6) that

$$\begin{aligned} f_1'(1^-) &= \lim_{r\to 1}\frac{1}{4(1-r)F_0(r)^2}\left[\frac{4\alpha}{\pi}G(r)\log\frac{16}{1-r}-\frac{G_0(r)}{B}\log\frac{e^R}{1-r}+O((1-r)\log(1-r))\right]\\ &= \frac{1}{4\pi B}\lim_{r\to 1}\left[\frac{4\alpha BG(r)\log 16-\pi RG_0(r)}{(1-r)F_0(r)^2}+\frac{4\alpha BG(r)-\pi G_0(r)}{(1-r)F_0(r)}\cdot\frac{\log(1/(1-r))}{F_0(r)}\right]\\ &= \frac{\pi(1-4ab)}{4B}+\frac{1}{4\pi B}\lim_{r\to 1}\frac{4\alpha BG(r)\log 16-\pi RG_0(r)}{(1-r)F_0(r)^2}\\ &= \begin{cases}\infty, & \text{if } R<\log 16,\\ \pi(1-4ab)/(4B), & \text{if } R=\log 16,\\ -\infty, & \text{if } R>\log 16.\end{cases} \end{aligned} \tag{3.7}$$

(1) If $4ab\le\min\{1,c\}$, then $\Delta_1(n,ab,c)\le 0$, and $c_{1,n}$ is decreasing in $n\in\mathbb{N}_0$ by (3.2). Hence by [29, Lemma 2.1], $f_1$ is decreasing on $(0,1)$.

If $1<4ab<c$ and $R\ge\log 16$, then it follows from (3.2) and (3.7) that $c_{1,n}$ is decreasing and then increasing in $n\in\mathbb{N}_0$, and $f_1'(1^-)<0$. Hence by Lemma 2.4(1), $f_1$ is decreasing on $(0,1)$.

Clearly, if $4ab\le\min\{1,c\}$, then $4ab\le 1$. Conversely, if $4ab\le 1$ and $c>1$ ($4ab\le 1$ and $c\le 1$), then $4ab\le\min\{1,c\}=1$ ($4ab\le c^2\le c=\min\{1,c\}$ by (1.3), respectively). Hence the condition $4ab\le\min\{1,c\}$ can be simplified as $ab\le 1/4$.

(2) If $4ab\ge\max\{1,c\}$, then $\Delta_1(n,ab,c)\ge 0$, and $c_{1,n}$ is increasing in $n\in\mathbb{N}_0$ by (3.2). Hence $f_1$ is increasing on $(0,1)$ by [29, Lemma 2.1].

Since $4ab\le c^2$ by (1.3), the condition $4ab\ge\max\{1,c\}$ implies that $c^2\ge\max\{1,c\}$, so that $c\ge 1$. Hence the condition $4ab\ge\max\{1,c\}$ can be simplified as $ab\ge c/4$.

(3) If $1/4<ab<c/4$ and $R<\log 16$, then $c_{1,n}$ is decreasing and then increasing in $n\in\mathbb{N}_0$, and $f_1'(1^-)=\infty$. Hence part (3) follows from Lemma 2.4(2).

(4) Part (4) follows from parts (1)–(3). □

**Lemma 3.2.** *For $a,b\in(0,\infty)$ with $c=a+b$, and for $r\in(0,1)$, let $f_2(r)\equiv G(r)/G_0(r)$.*

*(1) If $ab\le 1/4$, then $f_2$ is decreasing from $(0,1)$ onto $(\pi/(4\alpha B),1)$.*

*(2) If $ab\ge(c+1)/8$, then $f_2$ is increasing from $(0,1)$ onto $(1,\pi/(4\alpha B))$.*

*(3) In other cases not stated in parts (1)–(2), that is, $1/4<ab<(c+1)/8$, there exists a number $r_2=r_2(a,b)\in(0,1)$ such that $f_2$ is decreasing on $(0,r_2]$ and increasing on $[r_2,1)$, with $f_2(0)=1$ and $f_2(1^-)=\pi/(4\alpha B)$.*

*(4) If $ab\le 1/4$ or $ab\ge(c+1)/8$, then for $r\in(0,1)$,*

$$\min\left\{1,\frac{\pi}{4\alpha B}\right\}\le\frac{F(a,b;c+1;r)}{F(1/2,1/2;2;r)}\le\max\left\{1,\frac{\pi}{4\alpha B}\right\}, \tag{3.8}$$

*with equality in each instance if and only if $a=b=1/2$. In other cases, that is, $1/4<ab<(c+1)/8$, the second inequality in (3.8) also holds.*

**Proof.** The limiting value $f_2(0) = 1$ is clear. By (2.14), we obtain $f_2(1) = \pi/(4\alpha B)$. For $n \in \mathbb{N}_0$, set

$$a_{2,\,n} = \frac{(a,n)(b,n)}{(c+1,n)n!}, b_{2,\,n} = \frac{(1/2,n)^2}{n!(n+1)!}, c_{2,\,n} = \frac{a_{2,\,n}}{b_{2,\,n}},$$

$$\Delta_2(n,ab,c) = \left(ab + c - \frac{5}{4}\right)n + 2\left(ab - \frac{c+1}{8}\right),$$

$$\zeta_1 = \frac{1}{c+1}\left(ab - \frac{c+1}{8}\right), \zeta_2 = \frac{\pi}{4\alpha B}(abR - c + 1 - \log 2).$$

Then by (1.1) and (2.11),

$$f_2(r) = \frac{\sum_{n=0}^{\infty} a_{2,\,n}r^n}{\sum_{n=0}^{\infty} b_{2,\,n}r^n}, \frac{c_{2,\,n+1}}{c_{2,\,n}} = 1 + \frac{\Delta_2(n,ab,c)}{(n+c+1)(n+1/2)^2}, \tag{3.9}$$

$$f_2'(r) = \frac{1}{G_0(r)^2}\left[\frac{ab}{c+1}F_+(r)G_0(r) - \frac{1}{8}F_{3/2}(r)G(r)\right]. \tag{3.10}$$

Clearly, $f_2'(0) = \zeta_1$. By (2.14) and l'Hôpital's rule,

$$\lim_{r\to 1}\left[\frac{c\pi}{B}G_0(r) - G(r)\right] = \frac{4c}{abB}\left(ab - \frac{1}{4}\right),$$

$$\lim_{r\to 1}\frac{[c\pi G_0(r)/B] - G(r)}{r'} = 0 \text{ if } ab = \frac{1}{4}.$$

Hence we can apply (2.13) and (3.10) to obtain

$$\begin{aligned} f_2'(1^-) &= \frac{1}{G_0(1)^2}\lim_{r\to 1}\left[\frac{abG_0(r)}{(c+1)B_+}\log\frac{e^{R_+}}{1-r} - \frac{G(r)}{8B(3/2,3/2)}\log\frac{e^{R(3/2,3/2)}}{1-r}\right] \\ &= \frac{\pi^2}{16}\lim_{r\to 1}\left[\frac{cG_0(r)}{B}\log\frac{e^{R-1/\alpha}}{1-r} - \frac{G(r)}{\pi}\log\frac{e^{R(1/2)-4}}{1-r}\right] \\ &= \zeta_2 + \frac{\pi}{16}\lim_{r\to 1}\left[\frac{c\pi}{B}G_0(r) - G(r)\right]\log\frac{1}{1-r} \\ &= \begin{cases} -\infty, & \text{if } ab < 1/4, \\ \zeta_2, & \text{if } ab = 1/4, \\ \infty, & \text{if } ab > 1/4. \end{cases} \end{aligned} \tag{3.11}$$

(1) If $ab \le \min\{(5/4) - c, (c+1)/8\}$, then $\Delta_2(n,ab,c) \le 0$, and $c_{2,\,n}$ is decreasing in $n \in \mathbb{N}_0$ by (3.9). Hence $f_2$ is decreasing on $(0,1)$ by [29, Lemma 2.1].

If $ab \le 1/4$ and $c \le 1$, then $\min\{(5/4) - c, (c+1)/8\} = (c+1)/8$, so that $ab \le c^2/4 \le (c+1)/8 = \min\{(5/4) - c, (c+1)/8\}$ by (1.3). Hence $f_2$ is decreasing on $(0,1)$.

If $ab \le 1/4$ and $c > 1$, and if $ab \le (5/4) - c$, then $ab - (c+1)/8 < ab - 1/4 \le 0$, so that $ab \le \min\{(5/4) - c, (c+1)/8\}$. Hence $f_2$ is decreasing on $(0,1)$.

If $ab \le 1/4$ and $c > 1$, and if $ab > (5/4) - c$, then $ab - (c+1)/8 < ab - 1/4 \le 0$, and $c_{2,\,n}$ is decreasing and then increasing in $n \in \mathbb{N}_0$ by (3.9), and $f_2'(1^-) < 0$ by (3.11) and Lemma 2.2(4). Hence by Lemma 2.4(1), $f_2$ is decreasing on $(0,1)$.

From the above discussion, we obtain part (1).

(2) If $ab \geq \max\{(5/4)-c,(c+1)/8\}$, then $c_{2,n}$ is increasing in $n \in \mathbb{N}_0$ by (3.9), and hence $f_2$ is increasing on $(0,1)$ by [29, Lemma 2.1]. By (1.3)

$$ab \geq \max\{(5/4)-c,(c+1)/8\} \Rightarrow (c+1)/8 \leq c^2/4 \Leftrightarrow c \geq 1,$$

so that $ab \geq \max\{(5/4)-c,(c+1)/8\} = (c+1)/8$, and hence part (2) follows.

(3) If $1/4 < ab < (c+1)/8$, then $c > 1$, and $(5/4)-c < 1/4 < ab < (c+1)/8$. Hence by (3.9) and (3.11), $c_{2,n}$ is decreasing and then increasing in $n \in \mathbb{N}_0$, and $f_2'(1^-) = \infty$, so that part (3) follows from Lemma 2.4(2).

(4) Part (4) follows from parts (1)–(3). □

The following corollary follows immediately from Lemmas 3.1 and 3.2.

**Corollary 3.3.** *For $a, b \in (0,\infty)$,*

$$\alpha B(a,b) \begin{cases} \geq \pi\alpha, & \textit{if} \ \ ab \leq 1/4, \ \textit{or} \ \ 1/4 < ab < c/4 \ \textit{with} \ \ R \geq \log 16, \\ \leq \pi\alpha, & \textit{if} \ \ ab \geq c/4, \\ \geq \pi/4, & \textit{if} \ \ ab \leq 1/4, \\ \leq \pi/4, & \textit{if} \ \ ab \geq (c+1)/8 \geq 1/4. \end{cases} \tag{3.12}$$

**Remark 3.4.** (1) The monotonicity properties of $f_1$ and $f_2$ defined in Lemmas 3.1–3.2 have been given in [38, Propositions 3.3 & 3.5] (cf. also [30, Lemma 2.2]). However, the formulations of the conditions in our Lemmas 3.1–3.2 are simpler and clearer than those in [38, Propositions 3.3 & 3.5] and in [30, Lemma 2.2], and our proofs of the monotonicity properties of $f_1$ and $f_2$ are more natural and simpler.

(2) By Lemma 2.2, $\{(a,b)|\ 1/4 < ab < c/4, R(a,b) < \log 16\} \neq \emptyset$ and $\{(a,b)|\ 1/4 < ab < c/4, R(a,b) \geq \log 16\} \neq \emptyset$.

(3) For $a, b, a_1, b_1 \in (0,\infty)$, $c = a+b$ and $c_1 = a_1 + b_1$, [30, Lemma 2.2] introduced the notations

$$\begin{aligned}
D_1 &= \{(a,b)|\ ab \leq a_1b_1, abc_1 - a_1b_1c \leq 0\}, \\
D_3 &= \{(a,b)|\ ab \geq a_1b_1, abc_1 - a_1b_1c \geq 0\}, \\
D_5 &= \{(a,b)|\ ab + c - (a_1b_1 + c_1) \leq 0, ab(c_1+1) - a_1b_1(c+1) \leq 0\}, \\
D_7 &= \{(a,b)|\ ab + c - (a_1b_1 + c_1) \geq 0, ab(c_1+1) - a_1b_1(c+1) \geq 0\}, \\
D_{21} &= \{(a,b)|\ ab < a_1b_1, abc_1 - a_1b_1c > 0, R(a,b) \leq R(a_1,b_1)\}, \\
D_{22} &= \{(a,b)|\ ab < a_1b_1, abc_1 - a_1b_1c > 0, R(a,b) > R(a_1,b_1)\}, \\
D_{41} &= \{(a,b)|\ ab > a_1b_1, abc_1 - a_1b_1c < 0, R(a,b) \geq R(a_1,b_1)\}, \\
D_{42} &= \{(a,b)|\ ab > a_1b_1, abc_1 - a_1b_1c < 0, R(a,b) < R(a_1,b_1)\}, \\
D_{61} &= \{(a,b)|\ ab \geq a_1b_1, ab + c - (a_1b_1 + c_1) < 0, ab(c_1+1) - a_1b_1(c+1) > 0\}, \\
D_{81} &= \{(a,b)|\ ab \leq a_1b_1, ab + c - (a_1b_1 + c_1) > 0, ab(c_1+1) - a_1b_1(c+1) < 0\},
\end{aligned}$$

and proved the following results: (i) $f_1(r) \equiv F(r)/F_0(r)$ is strictly decreasing (increasing) on $(0,1)$ if $(a,b) \in D_1 \cup D_{41}$ ($(a,b) \in D_3 \cup D_{21}$, respectively). Moreover, if $(a,b) \in D_{22}$ ($D_{42}$), then there exists $r_0 \in (0,1)$ ($r_0^* \in (0,1)$) such that $f_1$ is strictly increasing (decreasing) on $(0,r_0)$ ($(0,r_0^*)$), and decreasing (increasing) on $(r_0,1)$ ($(r_0^*,1)$, respectively). (ii) $f_2(r) \equiv G(r)/G_0(r)$ is strictly decreasing (increasing) on $(0,1)$ if $(a,b) \in D_5 \cup D_{81}$ ($(a,b) \in D_7 \cup D_{61}$, respectively).

(4) Observe that it is difficult to verify whether the conditions in $D_{21}$, $D_{22}$, $D_{41}$ and in $D_{42}$ are satisfied, without the help of Lemmas 2.1 and 2.2. For instance, by virtue of Lemma 2.1(2), it is clear that $D_{21} = D_{41} = \emptyset$ if $a + b = a_1 + b_1$.

(5) For $a_1 = b_1 = 1/2$ and $j = 1, 3, 5, 7, 22, 42, 61, 81$, still denote $D_j|_{a_1=b_1=1/2}$ by $D_j$. Then by Lemmas 2.1 and 2.2, and by (1.4), it is easy to verify that

$$\begin{aligned}
D_1 &= \{(a,b)|\ ab \le 1/4, ab \le c/4\} = \{(a,b)|\ ab \le 1/4\},\\
D_3 &= \{(a,b)|\ ab \ge 1/4, ab \ge c/4\} = \{(a,b)|\ ab \ge c/4\},\\
D_5 &= \{(a,b)|\ ab + c \le 5/4, ab \le (c+1)/8\} = \{(a,b)|\ c \le 1\} \cup \{(a,b)|\ c \ge 1, ab \le (5/4) - c\},\\
D_7 &= \{(a,b)|\ ab + c \ge 5/4, ab \ge (c+1)/8\} = \{(a,b)|\ ab \ge (c+1)/8\},\\
D_{22} &= \{(a,b)|\ c/4 < ab < 1/4, R(a,b) > \log 16\} = \emptyset,\\
D_{61} &= \{(a,b)|\ ab \ge 1/4, ab + c < 5/4, ab > (c+1)/8\} = \emptyset,\\
D_{42} &= \{(a,b)|\ 1/4 < ab < c/4, R(a,b) < \log 16\},\\
D_{81} &= \{(a,b)|\ 5/4 - c < ab \le 1/4\}.
\end{aligned}$$

Hence we see that Lemmas 3.1 and 3.2 improve [30, Lemma 2.2] in the case when $a_1 = b_1 = 1/2$.

The following two lemmas perfect [31, Lemma 3.2] and its proof.

**Lemma 3.5.** *For $a, b \in (0, \infty)$ with $c = a + b$, and for $r \in (0, 1)$, let $C = (c + 1)/2$ and $f_3(r) = \overline{F}(r)/F_0(r)$.*

*(1) If $c \le 1$, then $f_3$ is decreasing on $(0, 1)$, with $f_3(0) = 1$, $f_3(1^-) = 0$ ($f_3(1^-) = \sin(\pi a)$) if $c < 1$ ($c = 1$, respectively).*

*(2) If $ab \ge (c + 1)/8$, then $f_3$ is increasing from $(0, 1)$ onto $(1, \infty)$.*

*(3) In other cases not stated in parts (1)–(2), namely, $ab < (c + 1)/8$ and $c > 1$, there exists a number $r_3 = r_3(a, b) \in (0, 1)$ such that $f_3$ is decreasing on $(0, r_3]$ and increasing on $[r_3, 1)$, with $f_3(0) = 1$ and $f_3(1^-) = \infty$.*

**Proof.** Clearly, $f_3(0) = 1$. If $c = 1$, then by (2.13),

$$f_3(1^-) = \lim_{r\to 1} \frac{F(a, 1-a; 1; r)}{F_0(r)} = \frac{\pi}{B(a)} \lim_{r\to 1} \frac{\log\left[e^{R(a)}/(1-r)\right]}{\log[16/(1-r)]} = \frac{\pi}{B(a)} = \sin(\pi a). \tag{3.13}$$

If $c < 1$, then $C - c = (1 - c)/2 > 0$, so that $\overline{F}(1) = \Gamma(C)\Gamma(C - c)/[\Gamma(C - a)\Gamma(C - b)]$ by (2.10). Hence

$$f_3(1^-) = \lim_{r\to 1} \overline{F}(r)/F_0(r) = 0,\ \ c < 1. \tag{3.14}$$

If $c > 1$, then $c - C = C - [(C - a) + (C - b)] = (c - 1)/2 > 0$. Hence by (2.10) and (2.12),

$$f_3(1^-) = \lim_{r\to 1} \frac{\overline{F}(r)}{F_0(r)} = \lim_{r\to 1} \frac{F(C-a, C-b; C; r)}{(1-r)^{c-C} F_0(r)} = \infty,\ \ c > 1. \tag{3.15}$$

The monotonicity properties of $f_3$ were proved in [31, Lemma 3.2(1)]. One can also use the same method as in the proof of Lemma 3.1 to show the monotonicity properties of $f_3$. We would like to give a simpler and more natural proof of the piecewise monotonicity of $f_3$ stated in part (3), as follows.

If $1/4 < ab < (c+1)/8$, then $c > 1$, $C - [(C-a) + (C-b)] = C - 1 > 0$, and by (2.10)–(2.12) and (2.14),

$$\lim_{r\to 1}\left[4abF(C-a,C-b;C+1;r) - C\frac{G_0(r)}{F_0(r)}F(C-a,C-b;C;r)\right] = \frac{4C\Gamma(C)^2}{\Gamma(a)\Gamma(b)},$$

$$\begin{aligned} f_3'(r) &= \frac{1}{F_0(r)^2}\left[\frac{ab}{C}F_0(r)F(a+1,b+1;C+1;r) - \frac{G_0(r)}{4(1-r)}\overline{F}(r)\right] \\ &= \frac{1}{4C(1-r)^C F_0(r)}\left[4abF(C-a,C-b;C+1;r) - C\frac{G_0(r)}{F_0(r)}F(C-a,C-b;C;r)\right], \end{aligned}$$

and hence

$$f_3'(0) = \frac{1}{C}\left(ab - \frac{c+1}{8}\right) < 0,\ \ f_3'(1^-) = \infty. \tag{3.16}$$

In the case when $1/4 < ab < (c+1)/8$, it is easy to show that the ratio of the $n$th coefficient in the power series of $\overline{F}(r)$ to that of $F_0(r)$ is decreasing and then increasing in $n \in \mathbb{N}_0$ (see [31, Proof of Lemma 3.2(1)]). Hence part (3) follows from Lemma 2.4(2), (3.15) and (3.16). □

**Lemma 3.6.** *For $a, b \in (0,\infty)$ with $c = a+b$ and $r \in (0,1)$, let $C = (c+1)/2$, $\tau = [\sin(\pi a)]/[4a(1-a)]$ and $f_4(r) = \overline{G}(r)/\overline{G}_0(r)$. Then we have the following conclusions:*

*(1) If $c \le 1$, then $f_4$ is decreasing on $(0,1)$, with $f_4(0) = 1$ and $f_4(1^-) = 0$ ($f_4(1^-) = \tau$) if $c < 1$ ($c = 1$, respectively).*

*(2) If $ab \ge (11-7c)/16$, then $f_4$ is increasing from $(0,1)$ onto $(1,\infty)$.*

*(3) In other cases not stated in parts (1)–(2), that is, $ab < (11-7c)/16 < 1/4$, there exists $r_4 = r_4(a,b) \in (0,1)$ such that $f_4$ is decreasing on $(0, r_4]$ and increasing on $[r_4, 1)$, with $f_4(0) = 1$ and $f_4(1^-) = \infty$.*

**Proof.** Clearly, $f_4(0) = 1$. If $c = 1$, then by (2.10), (2.12), (2.2) and (2.7),

$$f_4(1^-) = \lim_{r\to 1}\frac{F(a+1,2-a;2;r)}{F(3/2,3/2;2;r)} = \lim_{r\to 1}\frac{F(a,1-a;2;r)}{F(1/2,1/2;2;r)} = \tau. \tag{3.17}$$

If $c < 1$, then $C < 1$, $C + 1 - [(a+1) + (b+1)] = -C$, $C + 1 - [(C-a) + C - b)] = C$, and by (2.10) and (2.12),

$$f_4(1^-) = \lim_{r\to 1}(1-r)^{1-C}\frac{F(C-a,C-b;C+1;r)}{F(1/2,1/2;2;r)} = 0,\ \ c < 1. \tag{3.18}$$

If $c > 1$, then $C + 1 - [(C-a) + (C-b)] = C > 1$, and it follows from (2.10) and (2.12) that

$$f_4(1^-) = \lim_{r\to 1}\frac{F(C-a,C-b;C+1;r)}{(1-r)^{C-1}F(1/2,1/2;2;r)} = \infty,\ \ c > 1. \tag{3.19}$$

The monotonicity properties of $f_4$ have been obtained in [31, Lemma 3.2(2)]. Here we give a simpler and more natural proof of the piecewise monotonicity of $f_4$ stated in part (3).

Suppose that $ab < (11-7c)/16 < 1/4$. Then $c > 1$ and $C+2-[(C-a)+(C-b)] > C+1-[(C-a)+(C-b)] = C > 1$. In this case, it is easy to show that the ratio of the $n$th coefficient in the power series of $\overline{G}(r)$ to that of $\overline{G}_0(r)$ is decreasing and then increasing in $n \in \mathbb{N}_0$ (cf. [31, Proof of Lemma 3.2]). Let

$$g_{10}(r) = \frac{(a+1)(b+1)}{C+1}G_0(r)F(C-a,C-b;C+2;r) - \frac{9}{8}F(1/2,1/2;3;r)F(C-a,C-b;C+1;r).$$

Then by (2.10)–(2.12), $g_{10}(1) = C(C-1)\Gamma(C)^2/[2\pi ab\Gamma(a)\Gamma(b)] > 0$ and $f_4'(r) = g_{10}(r)(1-r)^{-(C+2)}\overline{G}_0(r)^{-2}$. Hence $f_4'(1^-) = \infty$, so that part (3) follows from (3.19) and Lemma 2.4(2). □

Next, we prove two lemmas playing a key role in the proofs of our main results in Sections 4–5.

**Lemma 3.7.** *For $a, b \in (0, \infty)$ with $c = a + b$ and $r \in (0, 1)$, let $r_2$ be as in Lemma 3.2, $\beta = 1 - 4\alpha$, $\delta = (R - \log 16)/B$, and let $f_5(r) \equiv F(r) - F_0(r)F'(r)/F_0'(r)$.*

*(1) $f_5$ is increasing from $(0, 1)$ onto $(\beta, \delta)$ if and only if $ab \le 1/4$.*

*(2) $f_5$ is decreasing from $(0, 1)$ onto $(\delta, \beta)$ if and only if $ab \ge (c + 1)/8$.*

*(3) In other cases not stated in parts (1)–(2), that is, $1/4 < ab < (c + 1)/8$, then $f_5$ is increasing from $(0, r_2]$ onto $(\beta, f_5(r_2)]$, and decreasing from $[r_2, 1)$ onto $(\delta, f_5(r_2)]$.*

**Proof.** Let $f_2$ be as in Lemma 3.2. Then by (2.14),

$$F'(r)/F_0'(r) = 4\alpha G(r)/G_0(r) = 4\alpha f_2(r), \tag{3.20}$$

$$f_5(r) = F(r) - 4\alpha F_0(r)f_2(r), \tag{3.21}$$

$$f_5'(r) = -F_0(r)\frac{d}{dr}\left[\frac{F'(r)}{F_0'(r)}\right] = -4\alpha F_0(r)f_2'(r).$$

Hence the monotonicity properties of $f_5$ stated in parts (1)–(3) follow from Lemma 3.2.

By (3.21), $f_5(0) = 1 - 4\alpha = \beta$. By (2.11) and (2.13), and by l'Hôpital's rule, we obtain

$$\lim_{r\to 1}\frac{\pi G_0(r) - 4\alpha BG(r)}{r'} = \lim_{r\to 1} r'\left[\frac{4ab\alpha B}{c+1}F_+(r) - \frac{\pi}{8}F_{3/2}(r)\right] = 0. \tag{3.22}$$

From (2.8), (2.13)–(2.14) and (3.21)–(3.22), we obtain the limiting value

$$\begin{aligned} f_5(1^-) &= \lim_{r\to 1}\left[\frac{1}{B}\log\frac{e^R}{1-r} - \frac{4\alpha G(r)}{\pi G_0(r)}\log\frac{16}{1-r}\right] \\ &= \delta + \frac{1}{\pi B}\lim_{r\to 1}\frac{1}{G_0(r)}\cdot\frac{\pi G_0(r) - 4\alpha BG(r)}{r'}\cdot\left(r'\log\frac{1}{1-r}\right) = \delta. \quad \square \end{aligned} \tag{3.23}$$

**Lemma 3.8.** *For $a, b \in (0, \infty)$ with $c = a + b$ and $C = (c + 1)/2$, let $r_4$ be as in Lemma 3.6, and put*

$$\rho = \frac{ab}{C},\ \sigma = 1 - 4\rho,\ \overline{\delta} = \frac{R(a) - \log 16}{B(a)},\ \theta = \frac{\sin((C-a)\pi)}{\sin(C\pi)},$$

*and for $r \in (0, 1)$, let $f_6(r) \equiv \overline{F}(r) - F_0(r)\overline{F}'(r)/F_0'(r)$. Then we have the following conclusions:*

*(1) $f_6$ is increasing on $(0, 1)$ if and only if $c \le 1$.*

*(2) $f_6$ is decreasing on $(0, 1)$ if and only if $ab \ge (11 - 7c)/16$.*

*(3) In other cases not stated in parts (1)–(2), that is, $ab < (11 - 7c)/16 < 1/4$, $f_6$ is increasing on $(0, r_4]$, and decreasing on $[r_4, 1)$.*

*(4) $f_6(0) = \sigma$, $f_6(1^-) = \overline{\delta}$ if $c = 1$, $f_6(1^-) = \theta$ if $c < 1$, and $f_6(1^-) = -\infty$ if $c > 1$.*

**Proof.** Let $f_4$ as in Lemma 3.6. Then by (2.11), we obtain

$$\overline{F}'(r)/F_0'(r) = 4\rho\overline{G}(r)/\overline{G}_0(r) = 4\rho f_4(r), \tag{3.24}$$

$$f_6(r) = \overline{F}(r) - 4\rho F_0(r) f_4(r), \tag{3.25}$$
$$f_6'(r) = -F_0(r)\frac{d}{dr}\left[\frac{\overline{F}'(r)}{F_0'(r)}\right] = -4\rho F_0(r) f_4'(r).$$

Hence the monotonicity properties of $f_6$, which are stated in parts (1)–(3), follow from Lemma 3.6.

Next, we prove part (4). By (3.24) and Lemma 3.6, $f_6(0) = 1 - 4\rho = \sigma$. If $c = 1$, then by (2.10), (2.7) and by l'Hôpital's rule,

$$\begin{aligned}
&\lim_{r\to 1}[\pi G_0(r) - 4a(1-a)B(a)F(a,1-a;2;1)] = 0,\\
&\lim_{r\to 1}\frac{\pi G_0(r) - 4a(1-a)B(a)F(a,1-a;2;r)}{r'}\\
&= \lim_{r\to 1} r'\left[2a^2(1-a)^2 B(a)F(a+1,2-a;3;r) - \frac{\pi}{8}F\left(\frac{3}{2},\frac{3}{2};3;r\right)\right] = 0,
\end{aligned}$$

and hence by (2.11)–(2.13) and (3.25), we obtain

$$\begin{aligned}
f_6(1^-) &= \lim_{r\to 1}\left[F(a,1-a;1;r) - 4a(1-a)\frac{F(a+1,2-a;2;r)}{F(3/2,3/2;2;r)}F_0(r)\right]\\
&= \lim_{r\to 1}\left[F(a,1-a;1;r) - 4a(1-a)\frac{F(a,1-a;2;r)}{G_0(r)}F_0(r)\right]\\
&= \lim_{r\to 1}\left[\frac{1}{B(a)}\log\frac{e^{R(a)}}{1-r} - \frac{4a(1-a)F(a,1-a;2;r)}{\pi G_0(r)}\log\frac{16}{1-r}\right]\\
&= \overline{\delta} + \lim_{r\to 1}\frac{\pi G_0(r) - 4a(1-a)B(a)F(a,1-a;2;r)}{4B(a)r'}\cdot\left(r'\log\frac{1}{1-r}\right) = \overline{\delta}.
\end{aligned} \tag{3.26}$$

If $c < 1$, then $C - c = (1-c)/2 > 0$, $0 < C - a = (1+c-2a)/2 < 1 - a < 1$, $\overline{G}_0(r) = (1-r)^{-1}G_0(r)$ by (2.12) and $\overline{G}(r) = (1-r)^{-C}F(C-a, C-b; C+1; r)$. It follows from (2.7), (2.10) and (3.25) that

$$\begin{aligned}
f_6(1^-) &= \frac{\Gamma(C)\Gamma(C-c)}{\Gamma(C-a)\Gamma(C-b)} - 4\rho\lim_{r\to 1}(1-r)^{1-C}F_0(r)\frac{F(C-a,C-b;C+1;r)}{G_0(r)}\\
&= \frac{\Gamma(C)\Gamma(1-C)}{\Gamma(C-a)\Gamma(1-(C-a))} = \frac{\sin((C-a)\pi)}{\sin(C\pi)} = \theta.
\end{aligned} \tag{3.27}$$

If $c > 1$, then $C > 1$, $C - c = 1 - C = (1-c)/2 < 0$, $C - [(C-a)+(C-b)] = C - 1 > 0$ and $C + 1 - [(C-a)+(C-b)] = C > 1$. By (2.12) and (3.25),

$$f_6(r) = \frac{F_0(r)}{(1-r)^{(c-1)/2}}\left[\frac{F(C-a,C-b;C;r)}{F_0(r)} - 4\rho\frac{F(C-a,C-b;C+1;r)}{G_0(r)}\right], \tag{3.28}$$

so that $f_6(1^-) = -\infty$, since the function inside the brackets tends to $-\pi\Gamma(C)^2/[\Gamma(a)\Gamma(b)]$ as $r \to 1$. □

**Corollary 3.9.** *For $a, b \in (0,\infty)$ with $c = a + b$, let $C = (c+1)/2$ and $\rho = ab/C$.*

*(1) If $ab \le 1/4$, then the function $f_1$ defined in Lemma 3.1 is concave on $(0,1)$, and the function $f_7(r) \equiv [F(r) - 1]/[F_0(r) - 1]$ is decreasing from $(0,1)$ onto $(\pi/B, 4\alpha)$. If $ab \ge (c+1)/8$, then $f_7$ is increasing from $(0,1)$ onto $(4\alpha, \pi/B)$. In particular, if $ab \le 1/4$, then for $r \in (0,1)$,*

$$\frac{\pi}{B}F_0(r) \le \min\left\{\left[1 - \left(1 - \frac{\pi}{B}\right)r\right]F_0(r), 1 - \frac{\pi}{B} + \frac{\pi}{B}F_0(r)\right\} \le F(r) \le 1 - 4\alpha + 4\alpha F_0(r), \tag{3.29}$$

*and if $ab \ge (c+1)/8$, then for $r \in (0,1)$,*

$$1 - 4\alpha + 4\alpha F_0(r) \le F(r) \le 1 - \frac{\pi}{B} + \frac{\pi}{B} F_0(r), \tag{3.30}$$

*with equality in each instance if and only if $a = b = 1/2$.*

*(2) If $c \le 1$, then the function $f_3$ defined in Lemma 3.5 is concave on $(0,1)$, and $f_8(r) \equiv [\overline{F}(r) - 1]/[F_0(r) - 1]$ is decreasing on $(0,1)$ with $f_8(0^+) = 4\rho$, $f_8(1^-) = 0$ ($f_8(1^-) = \sin(\pi a)$) if $c < 1$ ($c = 1$, respectively). If $ab \ge (11 - 7c)/16$ and $c > 1$, then $f_8$ is increasing from $(0,1)$ onto $(4\rho, \infty)$. In particular, for $r \in (0,1)$,*

$$(1-r)F_0(r) < \overline{F}(r) < 1 - 4\rho + 4\rho F_0(r), \ (c < 1), \tag{3.31}$$

$$1 - \sin(\pi a) + F_0(r)\sin(\pi a) \le \overline{F}(r) \le 1 - 4a(1-a) + 4a(1-a)F_0(r), \ (c = 1), \tag{3.32}$$

$$1 - 4\rho + 4\rho F_0(r) < \overline{F}(r), \ \left(ab \ge \frac{11 - 7c}{16} \ \text{ and } \ c > 1\right), \tag{3.33}$$

*with equality in each inequality in (3.32) if and only if $a = b = 1/2$.*

**Proof.** Let $g_{11}(r) = F_0'(r)/F_0(r)^2$ for $r \in (0,1)$, and $f_5$ and $f_6$ be as in Lemma 3.7 and Lemma 3.8, respectively. Since $g_{11}(r) = G_0(r)/\left[4(1-r)F_0(r)^2\right]$ by (2.14), and since the function $r \mapsto (1-r)F_0(r)^2$ is strictly decreasing on $(0,1)$ by [3, Lemma 5.4(1)], $g_{11}$ is strictly increasing on $(0,1)$.

(1) Differentiation gives

$$f_1'(r) = \frac{F'(r)F_0(r) - F_0'(r)F(r)}{F_0(r)^2} = \frac{F_0'(r)}{F_0(r)^2}\left[\frac{F'(r)}{F_0'(r)} F_0(r) - F(r)\right] = -f_5(r)g_{11}(r), \tag{3.34}$$

which is decreasing on $(0,1)$ by Lemma 3.7(1) if $ab \le 1/4$. Hence the concavity of $f_1$ follows.

Since $\left\{\frac{d}{dr}[F(r) - 1]\right\}\left\{\frac{d}{dr}[F_0(r) - 1]\right\}^{-1} = F'(r)/F_0'(r) = 4\alpha f_2(r)$ by (3.20), the monotonicity properties of $f_7$ follow from Lemma 3.2 and the Monotone l'Hôpital's rule [5, Theorem 1.25]. By l'Hôpital's rule, $f_7(0^+) = 4\alpha$, and by (2.13), we obtain $f_7(1^-) = \pi/B$.

The first inequality in (3.29) holds by Corollary 3.3. The first lower bound in the second inequality in (3.29) follows from the concavity of $f_1$. The remaining inequalities in (3.29) and the double inequality (3.30) follow from the monotonicity of $f_7$.

(2) Similarly, the concavity of $f_3$ follows from Lemma 3.8(1), since $f_3'(r) = -f_6(r)g_{11}(r)$. The proofs of the remaining conclusions in part (2) are similar to the proofs of the monotonicity of $f_7$ and (3.29)–(3.30). □

## 4. Landen transformation inequalities for hypergeometric functions

In this section, we shall give a complete answer to Question 1.1 by proving Theorem 4.1 and its corollaries, which extend the identities in (1.7) to the zero-balanced hypergeometric function $F(a, b; a+b; r)$ and substantially improve all results obtained in [29] and all the related results presented in [30,38].

**Theorem 4.1.** *For $a, b \in (0, \infty)$ with $c = a + b$, let $\alpha = ab/c$, $\beta = 1 - 4\alpha$ and $\delta = (R - \log 16)/B$, and define the function $f$ on $(0,1)$ by*

$$f(r) = (1+r)F\left(a, b; c; r^2\right) - F\left(a, b; c; \frac{4r}{(1+r)^2}\right) - \beta r.$$

*(1) If $ab \le 1/4$, then $f$ is increasing from $[0,1)$ onto $[0,\delta-\beta)$. In particular, if $ab \le 1/4$, then for $r \in (0,1)$,*

$$\beta r \le (1+r)F\left(a,b;c;r^2\right) - F\left(a,b;c;\frac{4r}{(1+r)^2}\right) \le \delta - \beta(1-r), \tag{4.1}$$

*with equality in each instance if and only if $a=b=1/2$.*

*(2) If $ab \ge (c+1)/8$, then $f$ is decreasing from $[0,1)$ onto $(\delta-\beta,0]$. In particular, if $ab \ge (c+1)/8$, then for $r \in (0,1)$,*

$$\delta - \beta(1-r) \le (1+r)F\left(a,b;c;r^2\right) - F\left(a,b;c;\frac{4r}{(1+r)^2}\right) \le \beta r, \tag{4.2}$$

*with equality in each instance if and only if $a=b=1/2$.*

*(3) In other cases not stated in parts (1)–(2), that is, $1/4 < ab < (c+1)/8$, $f$ is not monotone on $(0,1)$, and neither (4.1) nor (4.2) holds for all $r \in (0,1)$ and for all $a,b \in (0,\infty)$ with $1/4 < ab < (c+1)/8$.*

**Proof.** Put $x = 4r/(1+r)^2$ for $r \in (0,1)$. Then $x > r > r^2$ for $r \in (0,1)$, and

$$\frac{dx}{dr} = \frac{4(1-r)}{(1+r)^3}, 1-x = \left(\frac{1-r}{1+r}\right)^2, \frac{1}{1-x}\frac{dx}{dr} = \frac{4}{1-r^2}. \tag{4.3}$$

Clearly, $f(0)=0$. By (2.13) and (4.3), we obtain

$$f(1^-) = \lim_{r\to 1}\left(\frac{1+r}{B}\log\frac{e^R}{1-r^2} - \frac{1}{B}\log\frac{e^R}{1-x}\right) - \beta = \delta - \beta. \tag{4.4}$$

By (2.14) and (4.3), and by differentiation,

$$f'(r) = F\left(r^2\right) + \frac{2\alpha r}{1-r}G\left(r^2\right) - \frac{4\alpha}{1-r^2}G(x) - \beta. \tag{4.5}$$

By (1.7), $F_0(x) = (1+r)F_0\left(r^2\right)$. Differentiating both sides of this identity with respect to $r$, and using (2.14) and (4.3), we obtain the following relation

$$G_0(x) = \left(1-r^2\right)F_0\left(r^2\right) + \frac{r(1+r)}{2}G_0\left(r^2\right). \tag{4.6}$$

(1) If $c \le 1$ ($ab \le 1/4$ and $c > 1$), then $\alpha \le c/4 \le 1/4$ by (1.3) ($\alpha \le 1/(4c) < 1/4$, respectively). Hence $\beta \ge 0$ for $ab \le 1/4$.

It follows from Lemma 3.2(1) and (4.6) that

$$G(x) \le \frac{G\left(r^2\right)}{G_0\left(r^2\right)}G_0(x) = \left(1-r^2\right)F_0\left(r^2\right)\frac{G\left(r^2\right)}{G_0\left(r^2\right)} + \frac{r(1+r)}{2}G\left(r^2\right). \tag{4.7}$$

Let $f_5$ be as in Lemma 3.7. Then it follows from (3.21), (4.7) and Lemma 3.7(1) that for $r \in (0,1)$,

$$\begin{aligned} f'(r) &\ge F\left(r^2\right) + \frac{2\alpha r}{1-r}G\left(r^2\right) - \frac{4\alpha G\left(r^2\right)}{\left(1-r^2\right)G_0\left(r^2\right)}G_0(x) - \beta \\ &= F\left(r^2\right) - 4\alpha F_0\left(r^2\right)\frac{G\left(r^2\right)}{G_0\left(r^2\right)} - \beta = f_5\left(r^2\right) - \beta \ge 0, \end{aligned} \tag{4.8}$$

and hence the monotonicity of $f$ follows. The double inequality (4.1) and its equality case are clear.

(2) If $ab \geq (c+1)/8$, then the inequalities in (4.7)–(4.8) are all reversed by Lemmas 3.2(2) and 3.7(2), and hence the monotonicity of $f$ follows. The remaining conclusions in part (2) are clear.

(3) Let $1/4 < ab < (c+1)/8$. By (2.11), (2.14), (4.3) and (4.5), and by l'Hôpital's rule,

$$\begin{aligned}
\lim_{r\to 0^+}\frac{f'(r)}{r} &= \lim_{r\to 0}\frac{\left(1-r^2\right)F\left(r^2\right)+2\alpha r(1+r)G\left(r^2\right)-4\alpha G(x)-\beta\left(1-r^2\right)}{r\left(1-r^2\right)} \\
&= \lim_{r\to 0}\left[\frac{F\left(r^2\right)-4\alpha G(x)-\beta}{r}-rF\left(r^2\right)+2\alpha(1+r)G\left(r^2\right)+\beta r\right] \\
&= 2\alpha+\lim_{r\to 0}\frac{F\left(r^2\right)-4\alpha G(x)-\beta}{r} \\
&= 2\alpha+\lim_{r\to 0}\left[\frac{2\alpha r}{1-r^2}G\left(r^2\right)-\frac{16ab\alpha(1-r)}{(c+1)(1+r)^3}F_+(x)\right]=\frac{16\alpha}{c+1}\left(\frac{c+1}{8}-ab\right).
\end{aligned} \tag{4.9}$$

On the other hand, by (4.5), (2.9) and (2.13)–(2.14), and by l'Hôpital's rule, we obtain

$$\begin{aligned}
f'(1^-) &= \lim_{r\to 1}\frac{\left(1-r^2\right)F\left(r^2\right)+2\alpha r(1+r)G\left(r^2\right)-4\alpha G(x)}{1-r^2}-\beta \\
&= \frac{1}{2}\lim_{r\to 1}\left[2rF\left(r^2\right)-2\alpha(1+3r)G\left(r^2\right)-\frac{4ab\alpha}{c+1}r^2(1+r)F_+\left(r^2\right)\right. \\
&\quad \left.+\frac{16ab\alpha(1-r)}{(c+1)(1+r)^3}F_+(x)\right]-\beta \\
&= \lim_{r\to 1}\left[rF\left(r^2\right)-\frac{2ab\alpha}{c+1}r^2(1+r)F_+\left(r^2\right)\right]-\frac{4}{B}-\beta \\
&= \frac{1}{B}\lim_{r\to 1}\left[r\log\frac{e^R}{1-r^2}-2abr^2(1+r)\log\frac{e^{R-1/\alpha}}{1-r^2}\right]-\frac{4}{B}-\beta \\
&= \frac{(1-4ab)R+4(c-1)}{B}-\beta+\frac{1}{B}\lim_{r\to 1}\left[r-2abr^2(1+r)\right]\log\frac{1}{1-r^2} \\
&= \begin{cases}\infty, & \text{if } ab<1/4,\\ -\infty, & \text{if } ab>1/4.\end{cases}
\end{aligned} \tag{4.10}$$

By (4.9) and (4.10), if $1/4 < ab < (c+1)/8$, then there exist $r_5, r_6 \in (0,1)$ with $r_5 < r_6$ such that $f'(r) > 0$ for $r \in (0, r_5)$, and $f'(r) < 0$ for $r \in (r_6, 1)$. Hence $f$ is not monotone on $(0,1)$, and neither (4.1) nor (4.2) holds. $\square$

**Corollary 4.2.** *For $a, b \in (0,\infty)$ with $c = a+b$, let $\alpha$, $\beta$ and $\delta$ be as in Theorem 4.1, and let*

$$\delta_1 = \min\{B/\pi, 1+\delta\}, \ \delta_2 = \max\{B/\pi, 1+\delta\}, \ \delta_3 = \max\{0, 1+\delta-\beta\}.$$

*(1) If $ab \leq 1/4$, then for $r \in (0,1)$,*

$$1 \leq \frac{(1+r)F\left(a,b;c;r^2\right)}{F\left(a,b;c;4r/(1+r)^2\right)} \leq \delta_1, \tag{4.11}$$

*or equivalently,*

$$\frac{1+r}{\delta_1}F\left(a,b;c;r^2\right) \leq F\left(a,b;c;\frac{4r}{(1+r)^2}\right) \leq (1+r)F\left(a,b;c;r^2\right), \tag{4.12}$$

*with equality in each instance if and only if* $a=b=1/2$.

*(2) If* $ab \geq c/4$, *then for* $r \in (0,1)$,

$$\delta_2 \leq \frac{(1+r)F\left(a,b;c;r^2\right)}{F\left(a,b;c;4r/(1+r)^2\right)} \leq 1, \tag{4.13}$$

*or equivalently,*

$$(1+r)F\left(a,b;c;r^2\right) \leq F\left(a,b;c;\frac{4r}{(1+r)^2}\right) \leq \frac{1+r}{\delta_2}F\left(a,b;c;r^2\right). \tag{4.14}$$

*If* $(c+1)/8 \leq ab \leq c/4$, *then for* $r \in (0,1)$,

$$\delta_3 \leq \frac{(1+r)F\left(a,b;c;r^2\right)}{F\left(a,b;c;4r/(1+r)^2\right)} \leq 1+\beta r \leq 1+\beta. \tag{4.15}$$

*Each equality in (4.13)–(4.15) holds if and only if* $a=b=1/2$.

*(3) If* $ab \leq 1/4 \leq c/4$, *then for* $r \in (0,1)$,

$$\frac{\pi}{B} \leq \frac{(1+r)F\left(a,b;c;r^2\right)}{F\left(a,b;c;4r/(1+r)^2\right)} \leq \frac{B}{\pi}, \tag{4.16}$$

*with equality in each instance if and only if* $a=b=1/2$. *If* $ab \geq c/4$, *then each inequality in (4.16) is reversed.*

**Proof.** (1) If $ab \leq 1/4$, then $\delta \geq \beta \geq 0$ by Theorem 4.1(1) and its proof. Hence it follows from (4.1) that

$$1+\frac{\beta r}{F\left(4r/(1+r)^2\right)} \leq \frac{(1+r)F\left(r^2\right)}{F\left(4r/(1+r)^2\right)} \leq 1+\frac{\delta-\beta(1-r)}{F\left(4r/(1+r)^2\right)} \leq 1+\frac{\delta}{F\left(4r/(1+r)^2\right)},$$

yielding the following double inequality

$$1 \leq \frac{(1+r)F\left(r^2\right)}{F\left(4r/(1+r)^2\right)} \leq 1+\delta. \tag{4.17}$$

On the other hand, it follows from (1.7), (2.13) and Lemma 3.1(1) that

$$\begin{aligned}\frac{(1+r)F\left(r^2\right)}{F\left(4r/(1+r)^2\right)} &= \frac{F\left(r^2\right)}{F_0\left(r^2\right)}\cdot\frac{F_0\left(4r/(1+r)^2\right)}{F\left(4r/(1+r)^2\right)} \leq \lim_{r\to 1}\frac{F_0\left(4r/(1+r)^2\right)}{F\left(4r/(1+r)^2\right)}\\ &= \lim_{r\to 1}\frac{F_0(r)}{F(r)} = \frac{B}{\pi}\lim_{r\to 1}\frac{\log(16/(1-r))}{\log\left(e^R/(1-r)\right)} = \frac{B}{\pi}.\end{aligned} \tag{4.18}$$

Hence (4.11) follows from (4.17) and (4.18).

(2) If $ab \geq c/4$, then $c \geq 1$ by (1.3) or (1.4), $ab \geq c/4 \geq (c+1)/8 \geq 1/4$, $\beta \leq 0$, and $\delta \leq 0$ by Lemma 2.2(1). It follows from (4.2) that

$$1+\frac{\delta}{F\left(4r/(1+r)^2\right)} \leq 1+\frac{\delta+\beta(r-1)}{F\left(4r/(1+r)^2\right)} \leq \frac{(1+r)F\left(r^2\right)}{F\left(4r/(1+r)^2\right)} \leq 1+\frac{\beta r}{F\left(4r/(1+r)^2\right)}, \tag{4.19}$$

which yields

$$1+\delta \le \frac{(1+r)F\left(r^2\right)}{F\left(4r/(1+r)^2\right)} \le 1, \tag{4.20}$$

with equality in each instance if and only if $a=b=1/2$. By Lemma 3.1(2) and the first equality in (4.18),

$$\frac{(1+r)F\left(r^2\right)}{F\left(4r/(1+r)^2\right)} \ge \lim_{r\to 1}\frac{F_0\left(4r/(1+r)^2\right)}{F\left(4r/(1+r)^2\right)} = \frac{B}{\pi}. \tag{4.21}$$

Hence (4.13) follows from (4.20) and (4.21).

If $(c+1)/8 \le ab \le c/4$, then $\beta \ge 0$, and by Theorem 4.1(2),

$$\begin{aligned} 1+\delta-\beta \le 1+\frac{\delta-\beta+\beta r}{F\left(4r/(1+r)^2\right)} &\le \frac{(1+r)F\left(r^2\right)}{F\left(4r/(1+r)^2\right)} \\ &\le 1+\frac{\beta r}{F\left(4r/(1+r)^2\right)} \le 1+\beta r \le 1+\beta \end{aligned} \tag{4.22}$$

for $r\in(0,1)$, and hence the second lower bound $1+\delta-\beta$ and the second and third inequalities in (4.15) follow. The first lower bound 0 in (4.15) and the equality cases of (4.13)–(4.15) are clear.

(3) If $ab \le 1/4$ and $c \ge 1$, then (4.18) is still valid by Lemma 3.1(1). Furthermore, by (2.13) and Lemma 3.1(1), we obtain

$$\frac{(1+r)F\left(r^2\right)}{F\left(4r/(1+r)^2\right)} = \frac{F\left(r^2\right)}{F_0\left(r^2\right)}\cdot\frac{F_0\left(4r/(1+r)^2\right)}{F\left(4r/(1+r)^2\right)} \ge \lim_{r\to 1}\frac{F(r^2)}{F_0\left(r^2\right)} = \frac{\pi}{B}. \tag{4.23}$$

Hence (4.16) follows from (4.18) and (4.23). The equality case of (4.16) is clear.

If $ab \ge c/4$, then each inequality in (4.18) and in (4.23) is reversed by Lemma 3.1(2), so that each inequality in (4.16) is reversed. □

**Remark 4.3.** (1) In Corollary 4.2, $B/\pi$ and $1+\delta$ are not directly comparable. We explain this as follows.

To simplify the explaining, we take $c=1$, as an example. Then $B=B(a)$ and $R=R(a)$ for $a\in(0,1/2]$, and by [34, Theorem 2.2]) and [24, Lemma 2.2], $R(a)$ and $B(a)$ have the following series expansions

$$\begin{aligned} R(a) &= \frac{1}{a}+\sum_{n=1}^{\infty}\left[1+(-1)^n\right]\zeta(n+1)a^n \\ &= \log 16+4\sum_{n=1}\lambda(2n+1)(1-2a)^{2n}, \end{aligned} \tag{4.24}$$

$$\begin{aligned} B(a) &= \frac{1}{a}+\sum_{n=1}^{\infty}\left[1+(-1)^{n+1}\right]\eta(n+1)a^n \\ &= 4\sum_{n=0}\beta(2n+1)(1-2a)^{2n}, \end{aligned} \tag{4.25}$$

where $\lambda(\cdot)$, $\eta(\cdot)$ and $\beta(\cdot)$ are defined as follows

$$\lambda(n+1)=\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{n+1}},\ \eta(n)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^n},\ \beta(n)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^n},\ n\in\mathbb{N}$$

(see [1, 23.2.19–23.2.21]). By (2.7) and (4.24)–(4.25), we obtain

$$\lim_{a\to 0}\frac{\pi^2}{B(a)}\left[(1+\delta)-\frac{B(a)}{\pi}\right]=\lim_{a\to 0}[B(a)+R(a)-\log 16]\sin^2(\pi a)-\pi$$
$$=\pi^2\lim_{a\to 0}\left\{a^2[B(a)+R(a)]\right\}\cdot\left(\frac{\sin(\pi a)}{\pi a}\right)^2-\pi=-\pi,$$

which shows that

$$1+\delta<B(a)/\pi \quad \text{for sufficiently small} \quad a. \tag{4.26}$$

Next, $\beta(1)=\pi/4$ by [1, 23.2.30], and it follows from (2.7), (4.24)–(4.25) and [1, Table 23.3] that

$$\lim_{a\to 1/2}\frac{B(a)}{(1-2a)^2}\left[(1+\delta)-\frac{B(a)}{\pi}\right]=\lim_{a\to 1/2}\frac{1}{(1-2a)^2}\left[B(a)+R(a)-\log 16-\frac{1}{\pi}B(a)^2\right]$$
$$=\lim_{a\to 1/2}\frac{1}{(1-2a)^2}\left[4\sum_{n=1}^{\infty}[\lambda(2n+1)+\beta(2n+1)](1-2a)^{2n}-\pi\cot^2(\pi a)\right]$$
$$=4[\lambda(3)+\beta(3)]-\pi\left[\lim_{a\to 1/2}\frac{\cos(\pi a)}{1-2a}\right]^2=4[\lambda(3)+\beta(3)]-\frac{\pi^3}{4}=0.33141\cdots,$$

and hence

$$1+\delta>B(a)/\pi \quad \text{when} \quad a \quad \text{is close to} \quad 1/2. \tag{4.27}$$

From (4.26) and (4.27), we see that $B/\pi$ and $1+\delta$ are not directly comparable.

(2) It is clear that Corollary 4.2 improves [29, Theorem 2.2 and Corollary 2.3].

(3) (4.12) and (4.14) in Corollary 4.2 give an answer to the open problem in [4, p.79].

The following two corollaries generalize the second identity in (1.7) and substantially improves [29, Theorem 2.5], that is, Theorem 1.2(3).

**Corollary 4.4.** *For $a,b\in(0,\infty)$ with $c=a+b$, let $\alpha$, $\beta$ and $\delta$ be as in Theorem 4.1, and for $r\in(0,1)$, let*

$$g(r)=\frac{2}{1+r}F\left(a,b;c;\left(\frac{1-r}{1+r}\right)^2\right)-F\left(a,b;c;1-r^2\right)-\beta\frac{1-r}{1+r}.$$

*(1) If $ab\le 1/4$, then $g$ is decreasing from $(0,1]$ onto $[0,\delta-\beta)$. In particular, if $ab\le 1/4$, then for $r\in(0,1)$,*

$$\beta(1-r)\le 2F\left(a,b;c;\left(\frac{1-r}{1+r}\right)^2\right)-(1+r)F\left(a,b;c;1-r^2\right)\le\beta(1-r)+(\delta-\beta)(1+r), \tag{4.28}$$

*with equality in each instance if and only if $a=b=1/2$.*

*(2) If $ab\ge(c+1)/8$, then $g$ is increasing from $(0,1]$ onto $(\delta-\beta,0]$. In particular, if $ab\ge(c+1)/8$, then for $r\in(0,1)$,*

$$\beta(1-r)+(\delta-\beta)(1+r)\le 2F\left(a,b;c;\left(\frac{1-r}{1+r}\right)^2\right)-(1+r)F\left(a,b;c;1-r^2\right)\le\beta(1-r), \tag{4.29}$$

*with equality in each instance if and only if $a=b=1/2$.*

*(3) In other cases not stated in parts (1)–(2), that is, $1/4<ab<(c+1)/8$, $g$ is not monotone on $(0,1)$, and neither (4.28) nor (4.29) holds for all $r\in(0,1)$ and for all $a,b\in(0,\infty)$ with $1/4<ab<(c+1)/8$.*

**Proof.** Let $f$ be as in Theorem 4.1, and put $t = (1-r)/(1+r)$. Then $2/(1+r) = 1+t$, $1-r^2 = 4t/(1+t)^2$, and $g(r) = f(t)$. Hence the results follow from Theorem 4.1. $\square$

**Corollary 4.5.** *For $a, b \in (0,\infty)$ with $c = a+b$, let $\delta_1$, $\delta_2$ and $\delta_3$ be as in Corollary 4.2.*

*(1) If $ab \le 1/4$, then for $r \in (0,1)$,*

$$\frac{1}{2} \le \frac{F\left(a,b;c;((1-r)/(1+r))^2\right)}{(1+r)F\left(a,b;c;1-r^2\right)} \le \frac{1}{2}\delta_1, \tag{4.30}$$

*with equality in each instance if and only if $a = b = 1/2$.*

*(2) For $r \in (0,1)$, if $ab \ge c/4$, then*

$$\frac{1}{2}\delta_2 \le \frac{F\left(a,b;c;((1-r)/(1+r))^2\right)}{(1+r)F\left(a,b;c;1-r^2\right)} \le \frac{1}{2}, \tag{4.31}$$

*and if $(c+1)/8 \le ab \le c/4$, then*

$$\frac{1}{2}\delta_3 \le \frac{F\left(a,b;c;((1-r)/(1+r))^2\right)}{(1+r)F\left(a,b;c;1-r^2\right)} \le \frac{1+\beta}{2}, \tag{4.32}$$

*with equality in each instance if and only if $a = b = 1/2$.*

*(3) If $ab \le 1/4 \le c/4$, then for $r \in (0,1)$,*

$$\frac{\pi}{2B} \le \frac{F\left(a,b;c;((1-r)/(1+r))^2\right)}{(1+r)F\left(a,b;c;1-r^2\right)} \le \frac{B}{2\pi}, \tag{4.33}$$

*with equality in each instance if and only if $a = b = 1/2$. If $ab \ge c/4$, then each inequality in (4.33) is reversed.*

**Proof.** For $r \in (0,1)$, let $t = (1-r)/(1+r)$. Then

$$\frac{F\left(a,b;c;((1-r)/(1+r))^2\right)}{(1+r)F\left(a,b;c;1-r^2\right)} = \frac{(1+t)F\left(a,b;c;t^2\right)}{2F\left(a,b;c;4t/(1+t)^2\right)},$$

and hence the results follow from Corollary 4.2. $\square$

## 5. Another kind of Landen transformation inequalities

In this section, we shall refine the related main results in [31] by proving the following theorem and its corollaries.

**Theorem 5.1.** *For $a, b \in (0,\infty)$ with $c = a+b$ and $C = (c+1)/2$, let $\rho = ab/C$, $\sigma = 1-4\rho$, $\delta = (R-\log 16)/B$,*

$$\theta = \frac{\sin((C-a)\pi)}{\sin(C\pi)} \quad \textit{for} \quad c<1, \quad \omega = \begin{cases} \delta, & \textit{if} \quad c=1, \\ \theta, & \textit{if} \quad c<1, \\ -\infty, & \textit{if} \quad c>1, \end{cases}$$

*and define the function $h$ on $(0,1)$ by*

$$h(r) = (1+r)F\left(a,b;C;r^2\right) - F\left(a,b;C;\frac{4r}{(1+r)^2}\right) - \sigma r.$$

*(1) If $c \le 1$, then $h$ is increasing from $[0,1)$ onto $[0,\omega-\sigma)$. In particular, if $c \le 1$, then for $r \in (0,1)$,*

$$\sigma r \le (1+r)F\left(a,b;C;r^2\right) - F\left(a,b;C;\frac{4r}{(1+r)^2}\right) \le \omega - \sigma + \sigma r, \tag{5.1}$$

*with equality in each instance if and only if $a = b = 1/2$.*

*(2) If $ab \ge (11-7c)/16$ and $c > 1$, then $h$ is decreasing from $[0,1)$ onto $(-\infty,0]$. In particular, if $ab \ge (11-7c)/16$ and $c > 1$, then for $r \in (0,1)$,*

$$(1+r)F\left(a,b;C;r^2\right) - F\left(a,b;C;\frac{4r}{(1+r)^2}\right) < \sigma r. \tag{5.2}$$

*(3) In other cases note stated in parts (1)–(2), that is, $ab < (11-7c)/16 < 1/4$, $h$ is not monotone on $(0,1)$, and neither (5.1) nor (5.2) holds for all $r \in (0,1)$ and for all $a,b \in (0,\infty)$ with $ab < (11-7c)/16 < 1/4$.*

**Proof.** Put $x = 4r/(1+r)^2$ for $r \in (0,1)$. Then $x > r > r^2$ for $r \in (0,1)$, and equalities in (4.3) hold. By (1.7), $F_0(x) = (1+r)F_0\left(r^2\right)$. Differentiating both sides of this identity with respect to $r$, we obtain

$$\overline{G}_0(x) = \frac{(1+r)^3}{1-r}\left[F_0\left(r^2\right) + \frac{r(1+r)}{2}\overline{G}_0\left(r^2\right)\right]. \tag{5.3}$$

Differentiation gives

$$h'(r) = \overline{F}\left(r^2\right) + 2\rho r(1+r)\overline{G}\left(r^2\right) - \frac{4\rho(1-r)}{(1+r)^3}\overline{G}(x) - \sigma. \tag{5.4}$$

Clearly, $h(0) = 0$. If $c = 1$, then $h(r) = (1+r)F\left(a,1-a;1;r^2\right) - F(a,1-a;1;x) - \sigma r$, and by (2.13),

$$h(1^-) = \frac{1}{B(a)}\lim_{r\to 1}\left[(1+r)\log\frac{e^{R(a)}}{1-r^2} - \log\frac{e^{R(a)}}{1-x}\right] - \sigma = \delta - \sigma = \omega - \sigma. \tag{5.5}$$

If $c < 1$, then $C - c = 1 - C = (1-c)/2 > 0$, $0 < C - a = (1+b-a)/2 < (1+c)/2 < 1$, $0 < C - b = 1 - (C-a) < 1$, and hence by (2.10) and (2.7),

$$h(1^-) = F(a,b;C;1) - \sigma = \frac{\Gamma(C)\Gamma(1-C)}{\Gamma(C-a)\Gamma(C-b)} - \sigma = \theta - \sigma = \omega - \sigma. \tag{5.6}$$

If $c > 1$, then $C > 1$, $C - c = (1-c)/2 = 1 - C < 0$, $C - [(C-a)+(C-b)] = C - 1 > 0$. It follows from (2.12) and (4.3) that

$$\begin{aligned} h(r) &= (1+r)\left(1-r^2\right)^{C-c}F\left(C-a,C-b;C;r^2\right) - (1-x)^{C-c}F(C-a,C-b;C;x) - \sigma r \\ &= \frac{(1+r)^{1+3(C-c)}(1-r)^{c-C}F\left(C-a,C-b;C;r^2\right) - F(C-a,C-b;C;x)}{(1-r)^{c-1}(1+r)^{2(C-c)}} - \sigma r. \end{aligned}$$

Hence we obtain

$$h(1^-) = \omega - \sigma = -\infty,\ c > 1, \tag{5.7}$$

since by (2.10),

$$\lim_{r\to 1}\left[(1+r)^{1+3(C-c)}(1-r)^{c-C}F\left(C-a,C-b;C;r^2\right)-F(C-a,C-b;C;x)\right]$$
$$=-\frac{\Gamma(C)^2}{(C-1)\Gamma(a)\Gamma(b)}.$$

Consequently, by (5.5)–(5.7), $h(1^-)=\omega-\sigma$ for $a,b\in(0,\infty)$.

(1) If $c\le 1$, then by Lemma 3.6(1) and (5.3),

$$\overline{G}(x)\le\overline{G}_0(x)\frac{\overline{G}\left(r^2\right)}{\overline{G}_0\left(r^2\right)}=\frac{(1+r)^3}{1-r}\frac{\overline{G}\left(r^2\right)}{\overline{G}_0\left(r^2\right)}\left[F_0\left(r^2\right)+\frac{r(1+r)}{2}\overline{G}_0\left(r^2\right)\right]. \tag{5.8}$$

Let $f_6$ be as in Lemma 3.8. Then by (3.25) and (5.8), and by Lemma 3.8(1), we obtain

$$\begin{aligned}h'(r)&\ge\overline{F}\left(r^2\right)+2\rho r(1+r)\overline{G}\left(r^2\right)-4\rho\frac{\overline{G}\left(r^2\right)}{\overline{G}_0\left(r^2\right)}\left[F_0\left(r^2\right)+\frac{r(1+r)}{2}\overline{G}_0\left(r^2\right)\right]-\sigma\\&=\overline{F}\left(r^2\right)-4\rho F_0\left(r^2\right)\frac{\overline{G}\left(r^2\right)}{\overline{G}_0\left(r^2\right)}-\sigma=f_6\left(r^2\right)-\sigma\ge 0\end{aligned} \tag{5.9}$$

for $r\in(0,1)$. Hence the monotonicity of $h$ follows. The double inequality (5.1) and its equality case are clear.

(2) If $ab\ge(11-7c)/16$ and $c>1$, then by Lemmas 3.6(2) and 3.8(2), the inequalities in (5.8)–(5.9) are all reversed, and hence the result for $h$ follows. The remaining conclusions in part (2) are clear.

(3) Clearly, $h'(0)=0$, and the condition $ab<(11-7c)/16<1/4$ implies $c>1$. By (2.12) and (5.4),

$$\begin{aligned}h'(r)=&\left(1-r^2\right)^{C-c}F\left(C-a,C-b;C;r^2\right)\\&+2\rho r(1+r)\left(1-r^2\right)^{C-(c+1)}F\left(C-a,C-b;C+1;r^2\right)\\&-\frac{4\rho(1-r)}{(1+r)^3}(1-x)^{C-(c+1)}F(C-a,C-b;C+1;x)-\sigma\\=&(1+r)^{(1-c)/2}(1-r)^{(1-c)/2}F\left(C-a,C-b;C;r^2\right)\\&+2\rho r(1+r)^{1-C}(1-r)^{-C}F\left(C-a,C-b;C+1;r^2\right)\\&-4\rho(1+r)^{2C-3}(1-r)^{1-2C}F(C-a,C-b;C+1;x)-\sigma\\=&\frac{1}{(1-r)^c}\left[(1+r)^{(1-c)/2}(1-r)^{(1+c)/2}F\left(C-a,C-b;C;r^2\right)\right.\\&+2\rho r(1+r)^{(1-c)/2}(1-r)^{(c-1)/2}F\left(C-a,C-b;C+1;r^2\right)\\&\left.-4\rho(1+r)^{c-2}F(C-a,C-b;C+1;x)\right]-\sigma.\end{aligned} \tag{5.10}$$

Since $c>1$, $C-[(C-a)+(C-b)]=(c-1)/2>0$ and $(C+1)-[(C-a)+(C-b)]=(c+1)/2>1$, so that the function in the brackets in (5.10) tends to $-2^c\rho F(C-a,C-b;C+1;1)<0$ as $r\to 1$. Hence it follows from (2.10) and (5.10) that $h'(1^-)=-\infty$, so that there exists a number $r_7=r_7(a,b)\in(0,1)$ such that

$$h'(r)<0\quad\text{for}\quad r\in(r_7,1). \tag{5.11}$$

On the other hand, by (5.4), we obtain

$$\begin{aligned}\frac{(1+r)^3}{r}h'(r) =& \frac{(1+r)^3\overline{F}\left(r^2\right)-4\rho(1-r)\overline{G}(x)-\sigma(1+r)^3}{r}+2\rho(1+r)^4\overline{G}\left(r^2\right)\\ =&\frac{\overline{F}\left(r^2\right)-4\rho\overline{G}(x)-\sigma}{r}+\left(3+3r+r^2\right)\overline{F}\left(r^2\right)+2\rho(1+r)^4\overline{G}\left(r^2\right)\\ &+4\rho\overline{G}(x)-\sigma\left(3+3r+r^2\right)\\ =&f_9(r)+f_{10}(r),\end{aligned} \tag{5.12}$$

where $f_9(r)=f_{11}(r)/f_{12}(r)$, $f_{11}(r)=\overline{F}\left(r^2\right)-4\rho\overline{G}(x)-\sigma$, $f_{12}(r)=r$ and

$$f_{10}(r)=\left(3+3r+r^2\right)\overline{F}\left(r^2\right)+2\rho(1+r)^4\overline{G}\left(r^2\right)+4\rho\overline{G}(x)-\sigma\left(3+3r+r^2\right).$$

Clearly, $f_{11}(0)=f_{12}(0)=0$. Hence by (2.11), (4.3) and (5.12), and by l'Hôpital's rule, we obtain

$$\begin{aligned}\lim_{r\to 0}\frac{h'(r)}{r}&=\lim_{r\to 0}\frac{(1+r)^3h'(r)}{r}=f_{10}(0)+\lim_{r\to 0}\frac{f_{11}(r)}{f_{12}(r)}=f_{10}(0)+f_{11}'(0)\\ &=\lim_{r\to 0}\left[2\rho r\overline{G}\left(r^2\right)-\frac{16(a+1)(b+1)\rho(1-r)}{(C+1)(1+r)^3}F(a+2,b+2;C+2;x)\right]\\ &\quad+3+6\rho-3\sigma\\ &=3+6\rho-3\sigma-\frac{16(a+1)(b+1)\rho}{C+1}=\frac{32\rho}{c+3}\left(\frac{11-7c}{16}-ab\right)>0.\end{aligned}$$

This shows that there exists a number $r_8=r_8(a,b)\in(0,1)$ such that

$$h'(r)>0 \quad\text{for}\quad r\in(0,r_8). \tag{5.13}$$

Part (3) now follows from (5.11) and (5.13). $\square$

**Corollary 5.2.** *For $a,b\in(0,\infty)$ with $c=a+b$ and $C=(c+1)/2$, let $\rho$, $\delta$, $\sigma$ and $\omega$ be as in Theorem 5.1.*

*(1) If $c\le 1$, then for $r\in(0,1)$,*

$$1\le\frac{(1+r)F\left(a,b;C;r^2\right)}{F\left(a,b;C;4r/(1+r)^2\right)}\le 1+\omega, \tag{5.14}$$

$$\frac{1}{2}\le\frac{F\left(a,b;C;((1-r)/(1+r))^2\right)}{(1+r)F\left(a,b;C;r'^2\right)}\le\frac{1+\omega}{2}, \tag{5.15}$$

*with equality in each instance if and only if $a=b=1/2$.*

*(2) If $ab\ge(c+1)/8$, then for $r\in(0,1)$,*

$$\frac{(1+r)F\left(a,b;C;r^2\right)}{F\left(a,b;C;4r/(1+r)^2\right)}\le 1, \tag{5.16}$$

$$\frac{F\left(a,b;C;((1-r)/(1+r))^2\right)}{(1+r)F\left(a,b;C;r'^2\right)}\le\frac{1}{2}, \tag{5.17}$$

*with equality in each instance if and only if $a=b=1/2$.*

*(3) If* $(11-7c)/16 \le ab < (c+1)/8$*, then for* $r \in (0,1)$*,*

$$\frac{(1+r)F\left(a,b;C;r^2\right)}{F\left(a,b;C;4r/(1+r)^2\right)} < 1+\sigma, \tag{5.18}$$

$$\frac{F\left(a,b;C;((1-r)/(1+r))^2\right)}{(1+r)F\left(a,b;C;r'^2\right)} < \frac{1+\sigma}{2}. \tag{5.19}$$

**Proof.** (1) Since $c \le 1$, $\omega \ge 0$ by Lemma 2.2(1), $\omega - \sigma \ge 0$ by Theorem 5.1(1), and $\sigma \ge 1 - 2c^2/(c+1) = (1-c)(2c+1)/(c+1) \ge 0$ by (1.3). Hence it follows from (5.1) that

$$\begin{aligned} 1 + \frac{\sigma r}{F\left(a,b;C;4r/(1+r)^2\right)} &\le \frac{(1+r)F\left(a,b;C;r^2\right)}{F\left(a,b;C;4r/(1+r)^2\right)} \\ &\le 1 + \frac{\omega - \sigma(1-r)}{F\left(a,b;C;4r/(1+r)^2\right)} \le 1 + \frac{\omega}{F\left(a,b;C;4r/(1+r)^2\right)}, \end{aligned} \tag{5.20}$$

yielding the double inequality (5.14).

Put $t = (1-r)/(1+r)$. Then $1+t = 2/(1+r)$, $r = (1-t)/(1+t)$, $r'^2 = 4t/(1+t)^2$ and

$$\frac{F\left(a,b;C;((1-r)/(1+r))^2\right)}{(1+r)F\left(a,b;C;r'^2\right)} = \frac{(1+t)F\left(a,b;C;t^2\right)}{2F\left(a,b;C;4t/(1+t)^2\right)}. \tag{5.21}$$

Hence (5.15) follows from (5.14) and (5.21). The equality case of (5.14) and (5.15) is clear.

(2) Since $ab \ge (c+1)/8 (\ge 1/4)$, $\sigma = 1 - 8ab/(c+1) \le 0$ and $ab \ge (c+1)/8 \ge (11-7c)/16$, so that (5.2) is valid. It follows from (5.2) that

$$\frac{(1+r)F\left(a,b;C;r^2\right)}{F\left(a,b;C;4r/(1+r)^2\right)} \le 1 + \frac{\sigma r}{F\left(a,b;C;4r/(1+r)^2\right)} \le 1, \tag{5.22}$$

with equality if and only if $a = b = 1/2$. This yields (5.16) and its equality case. (5.17) and its equality case follow from (5.16) and (5.21).

(3) If $(11-7c)/16 \le ab < (c+1)/8$, then $\sigma > 0$, and the first inequality in (5.22) also holds by (5.2). Hence for $r \in (0,1)$,

$$\frac{(1+r)F\left(a,b;C;r^2\right)}{F\left(a,b;C;4r/(1+r)^2\right)} < 1 + \frac{\sigma}{F\left(a,b;C;4r/(1+r)^2\right)} < 1+\sigma, \tag{5.23}$$

that is, (5.18) holds. The inequality (5.19) follows from (5.21) and (5.18). □

**Remark 5.3.** (1) Let $\theta$ be as in Theorem 5.1. In [31, p.524], $\theta$ was denoted and expressed by

$$H(a,b) = \frac{B((a+b+1)/2,(1-a-b)/2)}{B((1-a+b)/2,(1+a-b)/2)}, \quad c = a+b < 1. \tag{5.24}$$

By (2.1), for $C = (a+b+1)/2 = (c+1)/2 < 1$,

$$\frac{B((a+b+1)/2,(1-a-b)/2)}{B((1-a+b)/2,(1+a-b)/2)} = \frac{\Gamma(C)\Gamma(1-C)}{\Gamma((1-a+b)/2)\Gamma(1-(1-a+b)/2)}.$$

Clearly, if $c < 1$, then $1/2 < (1-a+b)/2 = (1+c-2a)/2 < (1+c)/2 < 1$ by (1.3). It follows from the third equality in (2.7) that $H(a,b) = \sin((1-a+b)\pi/2)/\sin(C\pi) = \theta$, which is simpler than the expression (5.24).

(2) Theorem 5.1 and its Corollary 5.2 substantially improve [31, Theorems 2.1–2.2].

## 6. Applications to the generalized Grötzsch ring functions and the modular functions

For $a \in (0, 1/2]$, $r \in (0, 1)$ and $p \in (0, \infty)$, a Ramanujan's generalized modular equation with signature $1/a$ and order (or degree) $p$ is

$$\frac{F(a, 1-a; 1; 1-s^2)}{F(a, 1-a; 1; s^2)} = p\frac{F(a, 1-a; 1; 1-r^2)}{F(a, 1-a; 1; r^2)} \tag{6.1}$$

(cf. [16, p. 91]), which is extensively studied by S. Ramanujan (see [12–17]), who also gave numerous algebraic identities for the solution $s$ of (6.1) for some rational values of $a$. In their important paper [17], B. Berndt, S. Bhargava and F. Garvan studied generalized modular equations and gave proofs for numerous statements concerning these equations made by S. Ramanujan in his unpublished notebooks.

For $a, b, p \in (0, \infty)$ and $r \in (0, 1)$, more general modular equation can be defined as

$$\frac{F(a, b; a+b; 1-s^2)}{F(a, b; a+b; s^2)} = p\frac{F(a, b; a+b; 1-r^2)}{F(a, b; a+b; r^2)}. \tag{6.2}$$

For $a, b \in (0, \infty)$ with $c = a + b$ and for $r \in (0, 1)$, the generalized Grötzsch ring function is defined by

$$\mu_{a,\,b}(r) = \frac{B(a, b)}{2}\frac{F\left(a, b; c; 1-r^2\right)}{F\left(a, b; c; r^2\right)}, \tag{6.3}$$

which is strictly decreasing from $(0, 1)$ onto $(0, \infty)$. For $0 < a \leq 1/2$, the function $\mu_a \equiv \mu_{a,1-a}$ is also said to be the generalized Grötzsch ring function, and $\mu \equiv \mu_{1/2}$ is exactly the well-known Grötzsch ring function in the quasiconformal theory (cf. [3,5,18,21,25,27,31,36,37,41]). The function $\mu_{a,\,b}$ has applications in several fields of mathematics such as the theories of quasiconformal mappings and Ramanujan's modular equations. Many properties of $\mu(r)$ and $\mu_a(r)$ have been obtained. However, only a few properties of $\mu_{a,\,b}$ have been revealed.

For $a, b \in (0, \infty)$ with $C = (a + b + 1)/2$, let

$$\overline{\mu}_{a,\,b}(r) = \frac{B(a, b)}{2}\frac{F\left(a, b; C; 1-r^2\right)}{F\left(a, b; C; r^2\right)}, \tag{6.4}$$

which is another generalization of $\mu(r)$ and $\mu_a(r)$ since $\overline{\mu}_{a,\,b}(r) = \mu_a(r)$ if $b = 1 - a$.

It is well known that for $r \in (0, 1)$,

$$\mu(r) = 2\mu\left(2\sqrt{r}/(1+r)\right) \tag{6.5}$$

(see [5,21]). For $a, b \in (0, \infty)$ with $a + b \geq 2$ and $ab \geq a + b - 10/9$, it was proved in [31, Theorem 2.3] that

$$2\overline{\mu}_{a,\,b}\left(2\sqrt{r}/(1+r)\right) < \overline{\mu}_{a,\,b}(r),\ \ r \in (0, 1). \tag{6.6}$$

For $a, b, K \in (0, \infty)$ and $r \in (0, 1)$, let

$$\varphi_K(a, b, r) = \mu_{a,\,b}^{-1}\left(\mu_{a,\,b}(r)/K\right), \varphi_K(a, b, 0) = \varphi_K(a, b, 1) - 1 = 0, \tag{6.7}$$

$$\overline{\varphi}_K(a, b, r) = \overline{\mu}_{a,\,b}^{-1}\left(\overline{\mu}_{a,\,b}(r)/K\right), \overline{\varphi}_K(a, b, 0) = \overline{\varphi}_K(a, b, 1) - 1 = 0, \tag{6.8}$$

$$\varphi_K(a, r) = \varphi_K(a, 1-a, r) = \overline{\varphi}_K(a, 1-a, r),\ \ \varphi_K(a, 0) = \varphi_K(a, 1) - 1 = 0. \tag{6.9}$$

Clearly, as a function of $r$, $\varphi_K$ is an increasing homeomorphism from $[0, 1]$ onto itself. $\varphi_K(a, r)$ is called the modular function with signature $1/a$ and degree $1/K$ (cf. [3]). We also call $\varphi_K(a, b, r)$ the modular function.

In the case when $a = 1/2$, $\varphi_K(r) \equiv \varphi_K(1/2, r)$ is exactly the well-known Hersch–Pfluger distortion function in the theory of quasiconformal mappings (see [5,21]). Many properties of $\varphi_K(r)$ and $\varphi_K(a, r)$ have been obtained (see, for example, [3,5,12–16]). For instance, $\varphi_K(r)$ satisfies the following well-known identities

$$\varphi_K(r)^2 + \varphi_{1/K}(r')^2 = 1, \tag{6.10}$$

$$\varphi_{2K}(r) = \varphi_K\left(\frac{2\sqrt{r}}{1+r}\right), \ \varphi_{2K}\left(\frac{1-r}{1+r}\right) = \varphi_K(r') \tag{6.11}$$

for $r \in [0, 1]$ (see [5, Theorem 10.5]).

By (6.3), (6.1) and (6.2) can be rewritten as

$$\mu_a(s) = p\mu_a(r) \ \text{ and } \ \mu_{a,\,b}(s) = p\mu_{a,\,b}(r), \tag{6.12}$$

respectively, so that the solution of (6.1) ((6.2)) can be given by

$$s = \varphi_{1/p}(a, r) \ \left(s = \varphi_{1/p}(a, b, r), \ \text{ respectively}\right). \tag{6.13}$$

In this section, we shall apply the results proved in Sections 4–5 to present several inequalities for $\mu_{a,\,b}(r)$, $\overline{\mu}_{a,\,b}(r)$ and $\varphi_K(a, b, r)$, which extend the well-known Landen transformation identity (6.5) to the functions $\mu_{a,\,b}$ and $\overline{\mu}_{a,\,b}$ (see Theorems 6.1–6.2 below), and extend (6.11) to $\varphi_K(a, b, r)$ (see Corollaries 6.3–6.4 below). In particular, (6.6) is improved in Theorem 6.2.

**Theorem 6.1.** *For $a, b \in (0, \infty)$ and $c = a + b$, let $\alpha$ and $\beta$ be as in Theorem 4.1, and $\delta_1$, $\delta_2$ and $\delta_3$ as in Corollary 4.2.*

*(1) If $ab \le 1/4$, then for $r \in (0, 1)$,*

$$\mu_{a,\,b}(r) \le 2\mu_{a,\,b}\left(\frac{2\sqrt{r}}{1+r}\right) \le \delta_1^2\mu_{a,\,b}(r), \tag{6.14}$$

*with equality in each instance if and only if $a = b = 1/2$.*

*(2) If $ab \ge c/4$, then for $r \in (0, 1)$,*

$$\delta_2^2\mu_{a,\,b}(r) \le 2\mu_{a,\,b}\left(\frac{2\sqrt{r}}{1+r}\right) \le \mu_{a,\,b}(r), \tag{6.15}$$

*and if $(c + 1)/8 \le ab \le c/4$, then for $r \in (0, 1)$,*

$$\delta_3^2\mu_{a,\,b}(r) \le 2\mu_{a,\,b}\left(\frac{2\sqrt{r}}{1+r}\right) \le (1+\beta)^2\mu_{a,\,b}(r), \tag{6.16}$$

*with equality in each instance if and only if $a = b = 1/2$.*

**Proof.** (1) Set $t = 2\sqrt{r}/(1 + r)$ for $r \in (0, 1)$. Then

$$1 - t^2 = \left(\frac{1-r}{1+r}\right)^2, \ t^2 = \frac{4r}{(1+r)^2}, \tag{6.17}$$

and by (6.3),

$$\begin{aligned}2\mu_{a,b}(t) =&\frac{B(a,b)F\left(a,b;c;t'^2\right)}{F\left(a,b;c;t^2\right)} = \frac{B(a,b)F\left(a,b;c;((1-r)/(1+r))^2\right)}{F\left(a,b;c;4r/(1+r)^2\right)}\\ =&\mu_{a,b}(r)\frac{(1+r)F\left(a,b;c;r^2\right)}{F\left(a,b;c;4r/(1+r)^2\right)}\cdot\frac{2F\left(a,b;c;((1-r)/(1+r))^2\right)}{(1+r)F\left(a,b;c;r'^2\right)}.\end{aligned} \tag{6.18}$$

Hence the double inequality (6.14) follows from (4.11), (4.30) and (6.18). The equality case of (6.14) is clear.

(2) Similarly, the double inequality (6.15) ((6.16)) follows from (4.13), (4.31) and (6.18) ((4.15), (4.32) and (6.18), respectively). The equality case of (6.15) and (6.16) is clear. □

**Theorem 6.2.** *For $a, b \in (0,\infty)$ with $c = a+b$ and $C = (a+b+1)/2$, let $\rho$, $\sigma$ and $\omega$ be as in Theorem 5.1.*

*(1) If $c \le 1$, then for $r \in (0,1)$,*

$$\overline{\mu}_{a,b}(r) \le 2\overline{\mu}_{a,b}\left(\frac{2\sqrt{r}}{1+r}\right) \le (1+\omega)^2\overline{\mu}_{a,b}(r), \tag{6.19}$$

*with equality if and only if $a = b = 1/2$.*

*(2) If $ab \ge (c+1)/8$, then for $r \in (0,1)$,*

$$2\overline{\mu}_{a,b}\left(\frac{2\sqrt{r}}{1+r}\right) \le \overline{\mu}_{a,b}(r), \tag{6.20}$$

*and if $(11-7c)/16 \le ab \le (c+1)/8$, then for $r \in (0,1)$,*

$$2\overline{\mu}_{a,b}\left(\frac{2\sqrt{r}}{1+r}\right) \le (1+\sigma)^2\overline{\mu}_{a,b}(r), \tag{6.21}$$

*with equality in each instance if and only if $a = b = 1/2$.*

**Proof.** Let $t = 2\sqrt{r}/(1+r)$ for $r \in (0,1)$. Then (6.17) is valid.

(1) By (6.4), we can write

$$2\overline{\mu}_{a,b}(t) = \overline{\mu}_{a,b}(r)\frac{(1+r)F\left(a,b;C;r^2\right)}{F\left(a,b;C;4r/(1+r)^2\right)}\cdot\frac{2F\left(a,b;C;((1-r)/(1+r))^2\right)}{(1+r)F\left(a,b;C;r'^2\right)}. \tag{6.22}$$

Hence part (1) follows from Corollary 5.2(1).

(2) If $ab \ge (c+1)/8$ ($(11-7c)/16 \le ab \le (c+1)/8$), then (6.20) ((6.21)) follows from (5.16)–(5.17) and (6.22) ((5.18)–(5.19) and (6.22), respectively). □

**Corollary 6.3.** *For $a, b \in (0,\infty)$ and $c = a+b$, let $\alpha$ and $\beta$ be as in Theorem 4.1, and $\delta_1$, $\delta_2$ and $\delta_3$ as in Corollary 4.2, and let $K_1 = 2K/\delta_1^2$, $K_2 = 2K/\delta_2^2$, $K_3 = 2K/(1+\beta)^2$ and $K_4 = 2K/\delta_3^2$.*

*(1) If $ab \le 1/4$, then for $r \in (0,1)$,*

$$\varphi_{K_1}(a,b,r) \le \varphi_K\left(a,b,\frac{2\sqrt{r}}{1+r}\right) \le \varphi_{2K}(a,b,r), \tag{6.23}$$

*with equality in each instance if and only if $a = b = 1/2$.*

*(2) If $ab \ge c/4$, then for $r \in (0,1)$,*

$$\varphi_{2K}(a,b,r) \le \varphi_K\left(a,b,\frac{2\sqrt{r}}{1+r}\right) \le \varphi_{K_2}(a,b,r), \tag{6.24}$$

*and if* $(c+1)/8 \le ab \le c/4$, *then for* $r \in (0,1)$,

$$\varphi_{K_3}(a,b,r) \le \varphi_K\left(a,b,\frac{2\sqrt{r}}{1+r}\right) \le \varphi_{K_4}(a,b,r), \tag{6.25}$$

*with equality in each instance if and only if* $a = b = 1/2$.

**Proof.** (1) It follows from Theorem 6.1(1) that

$$\mu_{a,\,b}^{-1}\left(\frac{\delta_1^2}{2K}\mu_{a,\,b}(r)\right) \le \mu_{a,\,b}^{-1}\left(\frac{1}{K}\mu_{a,\,b}\left(\frac{2\sqrt{r}}{1+r}\right)\right) \le \mu_{a,\,b}^{-1}\left(\frac{1}{2K}\mu_{a,\,b}(r)\right),$$

with equality in each instance if and only if $a = b = 1/2$. This yields part (1).

(2) Similarly, part (2) follows from Theorem 6.1(2). □

**Corollary 6.4.** *For* $a, b \in (0,\infty)$ *with* $c = a + b$ *and* $C = (a + b + 1)/2$, *let* $\rho$, $\sigma$ *and* $\omega$ *be as in Theorem 5.1, and let* $K_5 = 2K/(1+\omega)^2$ *and* $K_6 = 2K/(1+\sigma)^2$.

*(1) If* $c \le 1$, *then for* $r \in (0,1)$,

$$\overline{\varphi}_{K_5}(a,b,r) \le \overline{\varphi}_K\left(a,b,\frac{2\sqrt{r}}{1+r}\right) \le \overline{\varphi}_{2K}(a,b,r), \tag{6.26}$$

*with equality if and only if* $a = b = 1/2$.

*(2) If* $ab \ge (c+1)/8$, *then for* $r \in (0,1)$,

$$\overline{\varphi}_K\left(a,b,\frac{2\sqrt{r}}{1+r}\right) \ge \overline{\varphi}_{2K}(a,b,r), \tag{6.27}$$

*and if* $(11-7c)/16 \le ab < (c+1)/8$, *then for* $r \in (0,1)$,

$$\overline{\varphi}_K\left(a,b,\frac{2\sqrt{r}}{1+r}\right) \ge \overline{\varphi}_{K_6}(a,b,r), \tag{6.28}$$

*with equality in each instance if and only if* $a = b = 1/2$.

**Proof.** (1) By Theorem 6.2(1), we obtain

$$\overline{\mu}_{a,\,b}^{-1}\left(\frac{(1+\omega)^2}{2K}\overline{\mu}_{a,\,b}(r)\right) \le \overline{\mu}_{a,\,b}^{-1}\left(\frac{1}{K}\overline{\mu}_{a,\,b}\left(\frac{2\sqrt{r}}{1+r}\right)\right) \le \overline{\mu}_{a,\,b}^{-1}\left(\frac{1}{2K}\overline{\mu}_{a,\,b}(r)\right),$$

with equality in each instance if and only if $a = b = 1/2$, which yields part (1).

(2) Similarly, part (2) follows from Theorem 6.2(2). □

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