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Large time behavior of solutions to a chemotaxis model with porous medium diffusion ☆

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ABSTRACT

In this paper, we study the large time behavior of a chemotaxis model with nonlinear diffusion and consumption

$$\begin{cases} u_t = \Delta u^m - \nabla \cdot (u\nabla v) + \mu u(1-u), \\ v_t - \Delta v = -vu, \end{cases}$$

where $m > 1$. In a previous paper [5], we have proved the existence and uniform boundedness of global weak solutions for any nonnegative initial data and any $m > 1$. In this work, we show that the weak solutions strongly converge to $(1, 0)$ in the large time limit.

1. Introduction

In this paper, we consider the following chemotaxis model

$$\begin{cases} u_t = \Delta u^m - \nabla \cdot (u\nabla v) + \mu u(1-u), \text{ in } Q, \\ v_t - \Delta v = -vu, \text{ in } Q, \\ (\nabla u^m - u\nabla v) \cdot \mathbf{n}|_{\partial\Omega} = \left.\dfrac{\partial v}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0, \\ u(x,0) = u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0, \quad x \in \Omega, \end{cases} \tag{1.1}$$

where $m > 1$, $Q = \Omega \times \mathbb{R}^+$, $\Omega \subset \mathbb{R}^3$ is a bounded domain, and the boundary $\partial\Omega$ is appropriately smooth, u, v represent the bacterial density, the chemoattractant concentration respectively, $J = -u \cdot \nabla v$ is the

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chemotactic flux, Δu^m denotes migration of the bacteria, for which, the motility depends on the bacterial density, Δv is the diffusion of chemoattractant, $\mu u(1-u)$ ($\mu > 0$) characterizes the proliferation or death of bacteria according to the logistic law, $-vu$ is the consumption of chemoattractant.

Since Keller and Segel [6] introduced the classical chemotaxis model in 1970, the chemotaxis models have been widely studied by many authors, and fruitful results have been achieved in the global existence, uniform boundedness of solutions. In addition, some significant progresses for the large time behavior of solutions also have been made for these models with linear diffusion. For example, the following chemotaxis model with signal production has been widely studied,

$$\begin{cases} u_t = \Delta u - \nabla\cdot(u\nabla v) + \mu u(1-u), \\ \tau v_t - \Delta v = -v + u, \end{cases}$$

and it is shown that for large μ, the solutions will go to the constant equilibrium $(1, 1)$ strongly in the sense of L^∞, see for example [1,4,16]. While for the following chemotaxis model with consumption

$$\begin{cases} u_t = \Delta u - \nabla\cdot(u\nabla v) + \mu u(1-u), \\ v_t - \Delta v = -vu, \end{cases} \tag{1.2}$$

it is shown that the solutions go to $(\overline{u_0}, 0)$ uniformly when $\mu = 0$ [12], where $\overline{u_0} = \frac{1}{|\Omega|}\int_\Omega u_0(x)dx$; and the solutions go to the constant equilibrium $(1, 0)$ uniformly when $\mu > 0$ [7]. However, the global existence of classical solutions is still open for small values of $\mu > 0$ or $\mu = 0$ [7,12,20].

From a physical point of view, migration of the cells should be considered as a movement in porous media [10], and the cell motility is a nonlinear function of the cell density. So in recent years, the chemotaxis systems with porous medium diffusion have been widely studied. For the following system

$$\begin{cases} u_t = \Delta u^m - \nabla\cdot(u\nabla v), \\ v_t - \Delta v = -vu, \end{cases}$$

the global bounded weak solutions were obtained for any $m > 1$ in dimension 2 [11]. While, the research is rather tortuous for the three dimensional space. In 2011, Liu and Lorz [9] established the global existence of a L^1-weak solution for $m = \frac{4}{3}$, afterwards, Duan and Xiang [3] proved the global existence of this kind of weak solutions for all adiabatic exponents $m \geq 1$. However, such kind of weak solution may be unbounded, from which, it is not possible to see whether the blow up phenomenon has occurred. Hence, the existence of bounded weak solutions is always a focus of people's concerning. In 2010, Di Francesco [2] obtained the global existence of bounded weak solutions for $m \in (m^*, 2]$ with $m^* \approx 1.81$, and also showed that $(u, v) \to (\overline{u_0}, 0)$ in $L^1(\Omega)\times L^2(\Omega)$ for $m = 2$; in 2013, a locally bounded global weak solution for $m > \frac{8}{7}$ was obtained by Tao and Winkler [13]; subsequently, Winkler [17] supplemented the boundedness for $m > \frac{7}{6}$; recently, Winkler [18] further improved the result to $m > \frac{9}{8}$, and also showed that all these global solutions will always go to $(\overline{u_0}, 0)$ in $L^p(\Omega)\times W^{1,\infty}$ for any $p \geq 1$. However, there remains a gap for $m \in (1, \frac{9}{8}]$.

In a previous paper, we established the existence and boundedness of global weak solutions for the system (1.1) for any $m > 1$ [5]. That is

Theorem 1.1 *(Global Existence and Uniform Boundedness [5]). Assume $u_0, v_0 \geq 0$, $u_0 \in L^\infty(\Omega)\cap W^{1,2}(\Omega)$, $v_0 \in C^2(\overline{\Omega})$, and $m > 1$. Then for any $\mu > 0$, the problem (1.1) admits a nonnegative weak solution $(u, v) \in \mathcal{X}_1 \times \mathcal{X}_2$ (as defined in Definition 2.1 below), such that*

$$\sup_{t\in(0,\infty)} (\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{W^{1,\infty}}) \leq M_1, \tag{1.3}$$

$$\sup_{t\in(0,+\infty)}\int_\Omega |\nabla u^m|^2dx+\sup_{t\in(0,+\infty)}\left(\|u^{\frac{m-1}{2}}u_t\|_{L^2(Q_1(t))}+\|\nabla u^{\frac{m}{2}}\|_{L^2(Q_1(t))}\right)\le M_2, \tag{1.4}$$

$$\sup_{t\in(0,+\infty)}\|v\|_{W_p^{2,1}(Q_1(t))}\le M_3, \quad \textit{for any} \;\; p>1, \tag{1.5}$$

where $Q_1(t)=\Omega\times(t,t+1)$,

$$\mathcal{X}_1=\{u\in L^\infty(\Omega\times\mathbb{R}^+);\nabla u^m\in L^\infty((0,\infty);L^2(\Omega)),\left(u^{\frac{m+1}{2}}\right)_t,\nabla u^{\frac{m}{2}}\in L^2_{loc}([0,\infty);L^2(\Omega))\},$$
$$\mathcal{X}_2=\{v\in L^\infty((0,\infty);W^{1,\infty}(\Omega);v_t,\Delta v\in L^p_{loc}([0,\infty);L^p(\Omega)),\ \textit{for any}\;\; p>1\},$$

and M_i $(i=1,2,3)$ *depend only on* μ, m, Ω, u_0, v_0.

Remark 1.1. It is worth noting that in [5], the result of the main theorem is that $\nabla u^{\frac{m+1}{2}}\in L^2_{loc}(\mathbb{R}^+;L^2(\Omega))$. However, it can be replaced by $\nabla u^{\frac{m}{2}}\in L^2_{loc}(\mathbb{R}^+;L^2(\Omega))$. In fact, by testing the regularized equation (11) in [5] with $1+\ln u_\varepsilon$, one can obtain

$$\sup_{t\in(0,+\infty)}\int_t^{t+1}\int_\Omega|\nabla u_\varepsilon^{\frac{m}{2}}|^2dxds\le C,$$

where C is independent of ε. Letting $\varepsilon\to 0$, we also have

$$\sup_{t\in(0,+\infty)}\int_t^{t+1}\int_\Omega|\nabla u^{\frac{m}{2}}|^2dxds\le C.$$

In the present paper, we will study the large time behavior of these solutions. We will show that the solutions we obtained above go to $(1,0)$ as $t\to\infty$ in the sense of Theorem 1.2. Although the result is not surprising, and it is similar to the linear diffusion case, but it's quite different in methods. For the linear diffusion case, there is a good Lyapunov functional with dissipative structure. That is, let

$$F(u)=\int_\Omega(u-1-\ln u)dx+\int_\Omega v^2dx,$$

by a direct calculation, it is not difficult to see that

$$\frac{dF(u)}{dt}\le-\int_\Omega\left(\frac{1}{2}\frac{|\nabla u|^2}{u^2}+\frac{1}{2}|\nabla v|^2+v^2u+\mu(u-1)^2\right)dx.$$

Using this functional, one can establish the stability of the equilibrium state $(1,0)$ directly. While, for the chemotaxis model with porous medium diffusion, although we can also construct a Lyapunov functional with dissipative structure, namely

$$\frac{dG(u)}{dt}\le-\int_\Omega\left(\frac{\alpha m}{2}u^{\alpha+m-2}|\nabla u|^2+(A-\frac{\alpha}{2m}\chi^2\|u\|_{L^\infty}^{\alpha+2-m})|\nabla v|^2+Av^2u+\mu u(u-1)(u^\alpha-1)\right)dx,$$

where $\alpha>(m-2)_+$, $A>\frac{\alpha}{2m}\chi^2\|u\|_{L^\infty}^{\alpha+2-m}$, and

$$G(u)=\int_\Omega\left(\frac{1}{\alpha+1}u^{\alpha+1}-u+\frac{A}{2}v^2\right)dx,$$

nevertheless, from this Lyapunov functional, the large time behavior of solutions is not easy to be found. In [18], the author established the large time behavior by some elaborate analytical skills, compared with this work, the main difficulty lies in the calculation of the limit for the integral mean value of $u(x,t)$ in Ω. We use five Lemmas, that is Lemma 3.1-Lemma 3.5 to show that

$$\lim_{t\to\infty}\frac{1}{|\Omega|}\int_\Omega u(x,t)dx = 1.$$

Based on this result, we finally establish the convergence of u, and the convergence of v is further established. That is for any initial datum, the solution we obtained in Theorem 1.1 goes to $(1,0)$. In fact, for the system (1.1), there is no other nonnegative and nontrivial steady state, see Remark 1.3, which is very different from the case with signal production, namely, the second equation is replaced by $v_t - \Delta v = -v + u$, for which, many different steady states have been constructed by some authors in dimension 1 [14,15].

More precisely, we have the following result

Theorem 1.2. *Assume that $u_0 \not\equiv 0$. For any initial datum, let $(u,v) \in \mathcal{X}_1 \times \mathcal{X}_2$ be a global weak solution of (1.1). Then for any $\mu > 0$, $m > 1$, we have*

$$\lim_{t\to\infty}(\|u(\cdot,t)-1\|_{L^p} + \|v(\cdot,t)\|_{L^\infty}) = 0, \quad \textit{for any } p \ge 1.$$

Furthermore, there exists $T_0 > 0$, such that

$$\|v(\cdot,t)\|_{H^1} \le Ce^{-\frac{1}{2}t}, \quad \|v(\cdot,t)\|_{L^\infty} \le Ce^{-\frac{1}{3}t}, \quad \textit{for any } t > T_0,$$

where C is a constant depending only on u_0, v_0, Ω, μ, m.

Remark 1.2. The decay rate of v obtained in Theorem 1.2 is not optimal. In fact, by a more accurate calculation, a faster decay rate will be reached.

Remark 1.3. In fact, the steady problem of (1.1) admits no nontrivial and nonnegative steady solution $(u,v) \in \tilde{\mathcal{X}}_1 \times \tilde{\mathcal{X}}_2$, where

$$\tilde{\mathcal{X}}_1 = \{u \in L^\infty(\Omega); \nabla u^m \in L^2(\Omega)\}; \qquad \tilde{\mathcal{X}}_2 = \{v \in L^\infty(\Omega) \cap H^2(\Omega)\}.$$

In fact, let $(u(x), v(x))$ be a steady solution of (1.1). We denote

$$\frac{1}{|\Omega|}\int_\Omega u^m(x)dx = A^m, \quad \frac{1}{|\Omega|}\int_\Omega v(x)dx = B,$$

and by a direct calculation, we also have

$$\int_\Omega \left(\frac{1}{2}|\nabla u^m|^2 + (M - \frac{1}{2}\chi^2\|u\|_{L^\infty}^2)|\nabla v|^2 + Av^2u + \mu u(u-1)(u^m-1)\right)dx \le 0, \tag{1.6}$$

for any constant $M > \frac{1}{2}\chi^2\|u\|_{L^\infty}^2$. Then by Poincaré inequality, we see that

$$\int_\Omega |v(x)-B|^2dx \le C\int_\Omega |\nabla v|^2dx = 0, \quad \int_\Omega |u^m(x)-A^m|^2dx \le C\int_\Omega |\nabla u^m|^2dx = 0,$$

which implies that $v = B$, $u = A$ a. e. in Ω.

2. Preliminaries

We first give the definition of weak solutions.

Definition 2.1. $(u, v) \in \mathcal{X}_1 \times \mathcal{X}_2$ is called a weak solution of (1.1), if $u \geq 0, v \geq 0$, and for any $t > 0$, $(\varphi, \psi) \in \mathcal{D}_1 \times \mathcal{D}_2$,

$$\int_\Omega u(x,t)\varphi(x,t)dx - \int_\Omega u_0(x)\varphi(x,0)dx - \iint_{Q_t} u\varphi_t dxdt + \iint_{Q_t} (\nabla u^m - u\nabla v)\nabla\varphi dxdt$$

$$= \mu \iint_{Q_t} u(1-u)\varphi dxdt, \tag{2.1}$$

$$\iint_{Q_t} v_t\psi dxdt - \iint_{Q_t} \Delta v\psi dxdt + \iint_{Q_t} uv\psi dxdt = 0, \tag{2.2}$$

where $Q_t = \Omega \times (0,t)$, $\mathcal{X}_1$, $\mathcal{X}_2$ are defined as in Theorem 1.1,

$$\mathcal{D}_1 = \{\varphi(x,t) \in L^\infty(\mathbb{R}^+; L^1(\Omega)); u\varphi_t, u\nabla\varphi, u^{\frac{m}{2}}\nabla\varphi \in L^2(Q_T), \text{for any } T > 0\},$$
$$\mathcal{D}_2 = \{\psi(x,t) \in L^2(Q_T) \text{ for any } T > 0\}.$$

It is worth noting that the above definition looks different from the definition given in [5], but they are equivalent when $(u, v) \in \mathcal{X}_1 \times \mathcal{X}_2$. However, to make readers more directly to figure out what kinds of test functions are admissible, we use the above definition. For example, from Definition 2.1, it is easy to see that for any $\alpha \geq \frac{m-1}{2}$, u^α can be chosen as the test function since $u \in \mathcal{X}_1$.

To show the stability of steady states, we give two preliminary lemmas, which are important in the proof of large time behavior.

Lemma 2.1. *Assume that $f \geq 0$, $f(t) \in L^1(T, \infty)$ for some constant $T > 0$, and*

$$f(t) - f(s) \leq A(t-s) \quad (\text{ or } \geq -A(t-s)), \quad \text{for all } \ t > s > T,$$

where A is a positive constant. Then

$$\lim_{t\to\infty} f(t) = 0.$$

Proof. We only prove the case

$$f(t) - f(s) \leq A(t-s), \text{ for all } \ t > s > T.$$

Suppose the contrary, then there exists a constant $\sigma_0 > 0$ and a sequence $\{t_j\}_{j=1}^\infty$ with $t_1 > T + \frac{\sigma_0}{A}$, and $t_j \nearrow \infty$ such that

$$f(t_j) > \sigma_0.$$

Noticing that $\int_T^\infty f(t)dt$ converges, then for any $\varepsilon_0 > 0$, there exists $T_0(\varepsilon_0) > 0$, such that for any $t > T_0$, we have

$$\int_t^{+\infty} f(s)ds < \varepsilon_0. \tag{2.3}$$

Noticing

$$f(t) - f(s) \le A(t-s), \quad \forall\, t > s > T,$$

when $s \in (t_j - \frac{\sigma_0}{2A}, t_j)$,

$$f(t_j) - f(s) \le A(t_j - s),$$

that is

$$f(s) \ge f(t_j) - A(t_j - s) > \sigma_0 - \frac{\sigma_0}{2} = \frac{\sigma_0}{2}, \text{ for all } s \in (t_j - \frac{\sigma_0}{2A}, t_j).$$

Then

$$\int_{t_j - \frac{\sigma_0}{2A}}^{t_j} f(t)dt > \frac{\sigma_0}{2} \cdot \frac{\sigma_0}{2A} = \frac{\sigma_0^2}{4A}.$$

Taking $\varepsilon_0 = \frac{\sigma_0^2}{4A}$ in (2.3), and noticing that $t_j \nearrow \infty$, then there exists t_{j_0} with $t_{j_0} - \frac{\sigma_0}{2A} > T_0$, thus, we have

$$\frac{\sigma_0^2}{4A} < \int_{t_{j_0} - \frac{\sigma_0}{2A}}^{t_{j_0}} f(t)dt \le \int_{t_{j_0} - \frac{\sigma_0}{2A}}^{\infty} f(t)dt < \frac{\sigma_0^2}{4A}.$$

It is a contradiction. □

Lemma 2.2. *Assume that $f(t)$, $g(t) \ge 0$, $\lim_{t\to\infty} g(t) = 0$, and $f(t) \in L^1(T, +\infty)$ with some constant $T \ge 0$. Let $F(t) = f(t) - g(t)$, and that*

$$F(t) - F(s) \ge -A(t-s), \quad \forall t > s > T \tag{2.4}$$

for some positive constant A, then

$$\lim_{t\to\infty} f(t) = 0.$$

Proof. Suppose the contrary, then there exists a constant $\sigma_0 > 0$ and a sequence $\{t_j\}$ with $t_j \nearrow \infty$ such that

$$F(t_j) > \sigma_0.$$

Then for any $s \in (t_j, t_j + \frac{\sigma_0}{2A})$

$$F(s) \ge F(t_j) - A(s - t_j) > \sigma_0 - \frac{\sigma_0}{2} = \frac{\sigma_0}{2},$$

which implies

$$f(s) > \frac{\sigma_0}{2}, \quad \forall\, s \in (t_j, t_j + \frac{\sigma_0}{2A}).$$

Then similar to the proof of Lemma 2.1, it will lead to a contradiction. The proof is complete. □

At last, we also give the following generalized Gagliardo-Nirenberg interpolation inequality [8].

Lemma 2.3. *For functions $u : \Omega \to \mathbb{R}$ defined on a bounded Lipschitz domain $\Omega \in \mathbb{R}^N$, we have*

$$\|u\|_{L^p} \le C_1 \|Du\|_{L^q}^{\alpha} \|u\|_{L^s}^{1-\alpha} + C_2 \|u\|_{L^r},$$

where $p \ge s > 0$, $q \ge 1, r > 0$, $0 \le \alpha \le 1$, and $\frac{1}{p} = (\frac{1}{q} - \frac{1}{N})\alpha + \frac{1-\alpha}{s}$.

3. Large time behavior

In this section, we study the large time asymptoticity of the weak solutions obtained in Theorem 1.1. In what follows, for simplicity, we use C, $\tilde{C}$, $\hat{C}$, C_i denote some different positive constants, which depend at most on μ, m, u_0, v_0, Ω and $M_i (i = 1, 2, 3)$ in Theorem 1.1.

We first have the following lemma.

Lemma 3.1. *Let $(u, v) \in \mathcal{X}_1 \times \mathcal{X}_2$ be a solution of (1.1). Then we have*

$$\int_0^\infty \int_\Omega (|\Delta v|^2 + |\nabla v|^2 + v^2 u) dxdt \le C_1, \tag{3.1}$$

$$\int_0^\infty \int_\Omega u^{\alpha+m-2} |\nabla u|^2 dxdt + \int_0^\infty \int_\Omega u(u-1)(u^\alpha - 1) dxdt \le C_2, \ \forall \alpha > \max\left\{(m-2)_+, \frac{m-1}{2}\right\}, \tag{3.2}$$

where C_1, C_2 depend on u_0, v_0, μ, Ω, α.

Proof. Noticing that $v \in \mathcal{X}_2$ by Theorem 1.1, by the definition of weak solutions, and taking the test function $\psi = v$ in (2.2), we obtain

$$\frac{1}{2} \int_\Omega v^2(x,t) dx + \int_0^t \int_\Omega |\nabla v|^2 dxds + \int_0^t \int_\Omega v^2 u dxds = \frac{1}{2} \int_\Omega v_0^2(x) dx, \text{for any } t > 0.$$

Letting $t \to \infty$, we arrive at

$$\int_0^\infty \int_\Omega (|\nabla v|^2 + v^2 u) dxdt \le \int_\Omega |v_0|^2. \tag{3.3}$$

Choosing the test function $\psi = \Delta v$ in (2.2), and using Young's inequality, we see that

$$\begin{aligned} &\int_\Omega |\nabla v(x,t)|^2 dx + \int_0^t \int_\Omega |\Delta v|^2 dxds \le \int_\Omega |\nabla v_0|^2 dx + \int_0^t \int_\Omega v^2 u^2 dxds \\ &\le \int_\Omega |\nabla v_0|^2 dx + \|u\|_{L^\infty} \int_0^t \int_\Omega v^2 u dxds, \end{aligned}$$

letting $t \to \infty$, we obtain

$$\int_0^\infty \int_\Omega |\Delta v|^2 dxdt \le \int_\Omega |\nabla v_0|^2 + \sup_{t>0} \|u\|_{L^\infty} \int_0^\infty \int_\Omega v^2 u dxdt. \tag{3.4}$$

Combining (1.3), (3.3) and (3.4), we obtain (3.1).

Noticing that $u^{\frac{m-1}{2}}u_t \in L^2_{loc}[0,\infty); L^2(\Omega))$, taking the test function $\varphi = \chi_{[0,T]}(u^\alpha - 1)$, for any $\alpha > \max\{(m-2)_+, \frac{m-1}{2}\}$ and $T > 0$, where $\chi_{[0,T]}$ is the characteristic function of the segment $[0,T]$, we see that

$$\begin{aligned}
&\int\limits_\Omega \left(\frac{1}{\alpha+1}u^{\alpha+1}(x,T) - u(x,T)\right)dx + \alpha m \iint\limits_{Q_T} u^{\alpha+m-2}|\nabla u|^2 dxdt + \mu \iint\limits_{Q_T} u(u-1)(u^\alpha - 1)dxdt \\
&= \alpha \iint\limits_{Q_T} u^\alpha \nabla v \nabla u dxdt + \int\limits_\Omega \left(\frac{1}{\alpha+1}u_0^{\alpha+1}(x) - u_0(x)\right)dx \\
&\le \frac{\alpha m}{2}\iint\limits_{Q_T} u^{\alpha+m-2}|\nabla u|^2 dxdt + \frac{\alpha}{2m}\iint\limits_{Q_T} u^{\alpha+2-m}|\nabla v|^2 dxdt + \int\limits_\Omega \left(\frac{1}{\alpha+1}u_0^{\alpha+1}(x) - u_0(x)\right)dx \\
&\le \frac{\alpha m}{2}\iint\limits_{Q_T} u^{\alpha+m-2}|\nabla u|^2 dxdt + \frac{\alpha}{2m}\sup_t \|u(\cdot,t)\|_{L^\infty}^{\alpha+2-m}\iint\limits_{Q_T} |\nabla v|^2 dxdt + \int\limits_\Omega \left(\frac{1}{\alpha+1}u_0^{\alpha+1}(x) - u_0(x)\right)dx.
\end{aligned}$$

Using (1.3), (3.1), and letting $T \to \infty$, we obtain (3.2). $\square$

Using Lemma 3.1, we can further prove that

Lemma 3.2. *Let $(u,v) \in \mathcal{X}_1 \times \mathcal{X}_2$ be a solution of (1.1). Then we have*

$$\int\limits_\Omega |\nabla v(x,t)|^2 dx - \int\limits_\Omega |\nabla v(x,s)|^2 dx + \int_s^t \int\limits_\Omega |\Delta v|^2 dxd\tau \le 2C(t-s), \quad \text{for any } t > s > 0, \tag{3.5}$$

and we further have

$$\lim_{t\to\infty}\int\limits_\Omega |\nabla v(x,t)|^2 dx = 0, \tag{3.6}$$

$$\lim_{t\to\infty}\int\limits_\Omega u(t,x)(u(t,x)-1)^2 dx = 0. \tag{3.7}$$

Proof. Choosing the test function $\psi = 2\Delta v \chi_{[s,t]}$ in (2.2) and using Young's inequality, we obtain

$$\int\limits_\Omega |\nabla v(x,t)|^2 dx - \int\limits_\Omega |\nabla v(x,s)|^2 dx + \int_s^t \int\limits_\Omega |\Delta v|^2 dxd\tau \le \int_s^t \int\limits_\Omega v^2 u^2 dxd\tau$$

for any $t > s > 0$. By (1.3), we have (3.5). Combining (3.5), Lemma 2.1 and the inequality (3.1), (3.6) is achieved.

Next, we show (3.7). Recalling (1.3), and noticing that when $\alpha > 1$, $|u-1| \le |u^\alpha - 1| \le C|u-1|$, then by (3.2), we have

$$\int_0^\infty \int\limits_\Omega u(u^\alpha - 1)^2 dxdt \le C_2, \ \forall \alpha > \max\left\{(m-2)_+, \frac{m-1}{2}, 1\right\}. \tag{3.8}$$

By (1.3), we see that for any $\alpha > \max\left\{(m-2)_+, \frac{m-1}{2}, 1\right\}$, $t > s > 0$, choosing $u^\alpha \chi_{[s,t]}$, $u^{2\alpha}\chi_{[s,t]}$, $\chi_{[s,t]}$ as the test functions, we conclude that

$$\begin{aligned}
&\int_\Omega u(x,t)(u^\alpha(x,t)-1)^2dx - \int_\Omega u(x,s)(u^\alpha(x,s)-1)^2dx \\
&= \int_\Omega (u^{2\alpha+1} - 2u^{\alpha+1} + u)(x,t)dx - \int_\Omega (u^{2\alpha+1} - 2u^{\alpha+1} + u)(x,s)dx \\
&= -2\alpha m(2\alpha+1)\int_s^t\int_\Omega u^{2\alpha+m-2}|\nabla u|^2dxd\tau + 2\alpha m(\alpha+1)\int_s^t\int_\Omega u^{m+\alpha-2}|\nabla u|^2dxd\tau \\
&+ 2\alpha\int_s^t\int_\Omega (u^{\alpha+1} - u^{2\alpha+1})\Delta v dxd\tau + \mu\int_s^t\int_\Omega ((2\alpha+1)u^{2\alpha} - 2(\alpha+1)u^\alpha + 1)u(1-u)dxd\tau \\
&\geq -2\alpha m(2\alpha+1)\int_s^t\int_\Omega u^{2\alpha+m-2}|\nabla u|^2dxd\tau + 2\alpha m(\alpha+1)\int_s^t\int_\Omega u^{m+\alpha-2}|\nabla u|^2dxd\tau \\
&- C(t-s) - \int_s^t\int_\Omega |\Delta v|^2dxd\tau.
\end{aligned}$$

Noticing that $2\alpha m(2\alpha+1)u^{2\alpha+m-2} \leq Cu^{2(m-1)}$ when $\alpha \geq \frac{m}{2}$ since u is bounded, and $(2\alpha+1)u^{2\alpha+m-2} \leq (\alpha+1)u^{\alpha+m-2} + Cu^{2(m-1)}$ when $\alpha < \frac{m}{2}$ for some sufficiently large constant C, combining with (1.4) and the above inequality, we get

$$\begin{aligned}
&\int_\Omega u(x,t)(u^\alpha(x,t)-1)^2dx - \int_\Omega u(x,s)(u^\alpha(x,s)-1)^2dx \\
&\geq -C\int_s^t\int_\Omega |\nabla u^m|^2dxd\tau - \int_s^t\int_\Omega |\Delta v|^2dxd\tau \\
&\geq -\tilde{C}(t-s) - \int_s^t\int_\Omega |\Delta v|^2dxd\tau, \text{ for any } t > s > 0.
\end{aligned} \tag{3.9}$$

Letting $F(t) = \int_\Omega \left(u(t,x)(u^\alpha(t,x)-1)^2 - |\nabla v(t,x)|^2\right)dx$, and using (3.5), (3.9), we have

$$F(t) - F(s) \geq -\hat{C}(t-s), \quad \text{for any } t > s > 0,$$

using Lemma 2.2, and combining with (3.2), (3.6), we arrive at

$$\lim_{t\to\infty}\int_\Omega u(t,x)(u^\alpha(t,x)-1)^2dx = 0, \forall \alpha > \max\left\{(m-2)_+, \frac{m-1}{2}, 1\right\}.$$

Noticing that $|u-1| \leq |u^\alpha - 1|$ for any $\alpha > 1$, then (3.7) is a direct result of the above inequality. □

In what follows, we pay our attention to show that

$$\lim_{t\to\infty} \frac{1}{|\Omega|} \int_\Omega u(x,t)dx = 1.$$

We first show the following lemma.

Lemma 3.3. *Let $(u,v) \in \mathcal{X}_1 \times \mathcal{X}_2$ be a solution of (1.1). Denote $a(t) = \frac{1}{|\Omega|}\int_\Omega u(x,t)dx$, $b^m(t) = \frac{1}{|\Omega|}\int_\Omega u^m dx$. Then it is easy to obtain that*

$$a(t) \le b(t),$$

and we have

$$\lim_{t\to\infty} \int_\Omega |u^m(x,t) - b^m(x,t)|^2 dx = 0. \tag{3.10}$$

And if there exists a sequence $\{t_j\}$ with $t_j \nearrow \infty$ and a positive constant σ_0, such that

$$\lim_{j\to\infty} a(t_j) = \sigma_0,$$

then $\sigma_0 = 1$, which implies

$$\lim_{j\to\infty} a(t_j) = \lim_{j\to\infty} b(t_j) = 1,$$

and

$$\lim_{j\to\infty} \int_\Omega |u(x,t_j) - 1|^2 dx = 0. \tag{3.11}$$

Proof. Notice that

$$b^m(t) = \frac{1}{|\Omega|}\int_\Omega u^m dx \ge \left(\frac{1}{|\Omega|}\int_\Omega u dx\right)^m = a(t)^m,$$

which implies that $a(t) \le b(t)$.

By a direct calculation, and using (1.3), (1.4), we obtain

$$\frac{d}{dt}\int_\Omega |u^m - b^m|^2 dx = \frac{d}{dt}\int_\Omega u^{2m} dx - 2|\Omega| b^m (b^m)' = \int_\Omega 2m u^{2m-1} u_t dx - 2|\Omega| b^m (b^m)',$$

then for any $t > s > 0$, we have

$$\begin{aligned} &\int_\Omega |u^m(x,t) - b^m(t)|^2 dx - \int_\Omega |u^m(x,s) - b^m(s)|^2 dx \\ &= 2m^2(m-1)\int_s^t b^m(\tau)\int_\Omega u^{2m-3}|\nabla u|^2 dx d\tau - 2(2m-1)\int_s^t\int_\Omega u^{m-1}|\nabla u^m|^2 dx d\tau \end{aligned}$$

$$- (2m-1)\int_s^t\int_\Omega u^{2m}\Delta v dxd\tau + 2(m-1)\int_s^t b^m(\tau)\int_\Omega u^m\Delta v dxd\tau$$

$$+ 2\ \mu m\int_s^t\int_\Omega u^{2m}(1-u)dxd\tau - 2m\mu\int_s^t b^m(\tau)\int_\Omega u^m(1-u)dxd\tau$$

$$\geq -C(t-s) - \int_s^t\int_\Omega |\Delta v|^2 dxd\tau,\ \text{for any } t>0.$$

Combining with (3.5), for any $t > s > 0$ we obtain

$$\int_\Omega (|u^m(x,t)-b^m(t)|^2 - |\nabla v(x,t)|^2)dx - \int_\Omega (|u^m(x,s)-b^m(s)|^2 - |\nabla v(x,s)|^2)dx \geq -C(t-s).$$

Furthermore, by Poincaré inequality, and using (3.2) with $\alpha = m$, we see that

$$\int_0^\infty\int_\Omega |u^m - b^m|^2 dxdt \leq C\int_0^\infty\int_\Omega |\nabla u^m|^2 dxdt \leq \tilde{C}. \tag{3.12}$$

Combining with Lemma 2.2, (3.6), we obtain (3.10).

Noticing that $|u^m - b^m| \geq b^{m-1}|u-b|$, then we have

$$a(t)^{2m-2}\int_\Omega |u-b|^2 dx \leq b^{2m-2}(t)\int_\Omega |u-b|^2 dx \leq \int_\Omega |u^m - b^m|^2 dx. \tag{3.13}$$

If $\lim\limits_{j\to\infty} a(t_j) = \sigma_0 > 0$, and noticing that

$$|a(t)-b(t)|^2 = \left|\frac{1}{|\Omega|}\int_\Omega (u-b)dx\right|^2 \leq \frac{1}{|\Omega|}\int_\Omega |u-b|^2 dx, \tag{3.14}$$

combining with (3.10), (3.13) and (3.14), we obtain

$$\lim_{j\to\infty}(b(t_j) - a(t_j)) = 0. \tag{3.15}$$

We note that

$$\int_\Omega |u(x,t_j)-\sigma_0|^2 dx \leq 3\int_\Omega |u(x,t_j)-b(t_j)|^2 dx + 3\int_\Omega |b(t_j)-a(t_j)|^2 dx + 3\int_\Omega |a(t_j)-\sigma_0|^2 dx,$$

by (3.10), (3.13) and (3.15), we obtain

$$\lim_{j\to\infty}\int_\Omega |u(x,t_j)-\sigma_0|^2 dx = 0. \tag{3.16}$$

On the other hands, we see that for any $\alpha > (m-2)_+$,

$$\int_\Omega b(t)(u-1)^2dx = \int_\Omega u(u-1)^2dx - \int_\Omega (u-b(t))(u-1)^2dx$$
$$\leq \int_\Omega u(u-1)^2dx + \frac{1}{2}\int_\Omega \frac{(u-b(t))^2}{b(t)}(u-1)^2dx + \frac{1}{2}\int_\Omega b(t)(u-1)^2dx,$$

which implies that

$$\int_\Omega b(t)(u-1)^2dx \leq 2\int_\Omega u(u-1)^2dx + C\int_\Omega \frac{(u-b(t))^2}{b(t)}dx.$$

By (3.7), (3.10) and (3.13), we get

$$\lim_{j\to\infty}\int_\Omega (u(x,t_j)-1)^2dx = 0, \tag{3.17}$$

since $\lim_{j\to\infty} b(t_j) = \sigma_0 > 0$.

By (3.16) and (3.17), and note that

$$|\Omega||\sigma_0-1|^2 \leq 2\int_\Omega (u(x,t_j)-1)^2dx + 2\int_\Omega (u(x,t_j)-\sigma_0)^2dx,$$

then we have $\sigma_0 = 1$. The proof is complete. □

From the above lemma, if $\lim_{t_j\to\infty} a(t_j)$ exists, it is either 1 or 0. To show that $\lim_{t_j\to\infty} a(t_j) = 0$ is impossible, we need the following lemma.

Lemma 3.4. *Let $(u,v) \in \mathcal{X}_1 \times \mathcal{X}_2$ be a solution of (1.1). If $\lim_{t\to\infty}\|u(\cdot,t)\|_{L^1} = 0$, then*

$$\lim_{t\to\infty}\|u(\cdot,t)\|_{L^\infty} = 0.$$

Proof. By (1.3), and notice that

$$\|u(\cdot,t)\|_{L^r} \leq \|u(\cdot,t)\|_{L^1}^{\frac{1}{r}}\|u(\cdot,t)\|_{L^\infty}^{\frac{r-1}{r}},$$

then for any fixed $r > 1$, we have

$$\lim_{t\to\infty}\|u(\cdot,t)\|_{L^r} = 0.$$

Therefore, for any sufficiently small constant $\varepsilon_0 > 0$, there exists a constant $T > 0$, such that

$$\|u(\cdot,t)\|_{L^{3m}} < \varepsilon_0, \quad \text{for any } t > T \tag{3.18}$$

Using smoothing operator, we can construct a cut off function $\eta(t) \in C^1[T,+\infty)$ with $\eta(T) = 0$, $\eta(t) = 1$ for $t > T+1$, and $|\eta'(t)| \leq C$ for some positive constant C, see for example [19].

Testing the first equation of (1.1) with $\varphi = r\chi_{[T,s]}\eta^r u^{r-1}e^t$ for any $r > 4m$, $s > T$, and using (1.3), we obtain

$$
\begin{aligned}
&\int_\Omega \eta^r(s)u^r(x,s)e^s dx + mr(r-1)\int_T^s\int_\Omega e^t\eta^r u^{m+r-3}|\nabla u|^2 dxdt \\
&\le r(r-1)\int_T^s\int_\Omega e^t\eta^r u^{r-1}\nabla v\nabla u dxdt + (\mu r+1)\int_T^s\int_\Omega e^t\eta^r u^r dt + r\int_T^s\int_\Omega e^t\eta'\eta^{r-1}u^r dxdt \\
&\le \frac{mr(r-1)}{4}\int_T^s\int_\Omega e^t\eta^r u^{m+r-3}|\nabla u|^2 dxdt + Cr^2\int_T^s\int_\Omega e^t\eta^r u^{r+1-m}dxdt + Cr\int_T^s\int_\Omega e^t\eta^{r-1}u^r dxdt. \quad (3.19)
\end{aligned}
$$

Using Lemma 2.3 (choosing $p = \frac{2(r+1-m)}{r+m-1}$, $q = 2$, $s = \frac{r}{m+r-1}$) and Young's inequality, for any sufficiently small $\sigma > 0$, and noticing that u is bounded, we see that

$$
\begin{aligned}
Cr^2\eta^r\|u\|_{L^{r+1-m}}^{r+1-m} =& Cr^2\eta^r\|u^{\frac{r+m-1}{2}}\|_{L^{\frac{2(r+1-m)}{r+m-1}}}^{\frac{2(r+1-m)}{r+m-1}} \\
\le& C_1r^2\eta^r\|\nabla u^{\frac{r+m-1}{2}}\|_{L^2}^{\frac{6(r+2-2m)}{6(m-1)+5r}}\|u^{\frac{r+m-1}{2}}\|_{L^{\frac{r}{m+r-1}}}^{\frac{4r(r+2(m-1))}{(6(m-1)+5r)(r+m-1)}} + C_2r^2\eta^r\|u\|_{\frac{r}{2}}^{r+1-m} \\
\le& \sigma\eta^r\|\nabla u^{\frac{r+m-1}{2}}\|_{L^2}^2 + C_\sigma r^{\frac{6(m-1)+5r}{6(m-1)+r}}\eta^r\|u\|_{\frac{r}{2}}^{\frac{r(2(m-1)+r)}{6(m-1)+r}} + C_2r^2\eta^r\|u\|_{\frac{r}{2}}^{r+1-m} \\
\le& \sigma\eta^r\|\nabla u^{\frac{r+m-1}{2}}\|_{L^2}^2 + C_\sigma r^5\eta^r\|u\|_{\frac{r}{2}}^{\frac{r(2(m-1)+r)}{6(m-1)+r}} + C_2r^2\eta^r\|u\|_{\frac{r}{2}}^{r+1-m} \\
\le& \sigma\eta^r\|\nabla u^{\frac{r+m-1}{2}}\|_{L^2}^2 + \hat{C}_\sigma r^5\|\eta u\|_{\frac{r}{2}}^{r-4m}
\end{aligned}
$$

since $\frac{2(r+1-m)}{r+m-1} < 6$, $\frac{6(r+2-2m)}{6(m-1)+5r} < 2$, $\frac{6(m-1)+5r}{6(m-1)+r} < 5$, $r > 1$, and noticing that $\frac{2r}{r+m-1} < 6$, $\frac{6r}{6(m-1)+5r} < 2$, then by Lemma 2.3 (choosing $p = \frac{2r}{r+m-1}$, $q = 2$, $s = \frac{r}{m+r-1}$), we also have

$$
\begin{aligned}
Cr\eta^{r-1}\|u\|_{L^r}^r =& Cr\eta^{r-1}\|u^{\frac{r+m-1}{2}}\|_{L^{\frac{2r}{r+m-1}}}^{\frac{2r}{r+m-1}} \\
\le& C_1r\eta^{r-1}\|\nabla u^{\frac{r+m-1}{2}}\|_{L^2}^{\frac{6r}{6(m-1)+5r}}\|u^{\frac{r+m-1}{2}}\|_{L^{\frac{r}{m+r-1}}}^{\frac{4r^2+6r(m-1)}{(6(m-1)+5r)(r+m-1)}} + C_2r\eta^{r-1}\|u\|_{\frac{r}{2}}^{r} \\
\le& \sigma\eta^r\|\nabla u^{\frac{r+m-1}{2}}\|_{L^2}^2 + C_\sigma\eta^{r-\frac{6(m-1)+5r}{6(m-1)+2r}}r^{\frac{6(m-1)+5r}{6(m-1)+2r}}\|u\|_{\frac{r}{2}}^{\frac{r(3(m-1)+2r)}{6(m-1)+2r}} + C_2r\eta^{r-1}\|u\|_{\frac{r}{2}}^{r} \\
\le& \sigma\eta^r\|\nabla u^{\frac{r+m-1}{2}}\|_{L^2}^2 + C_\sigma r^{5/2}\|\eta u\|_{\frac{r}{2}}^{r-3m} + C_2r\|\eta u\|_{\frac{r}{2}}^{r-1} \\
\le& \sigma\eta^r\|\nabla u^{\frac{r+m-1}{2}}\|_{L^2}^2 + \tilde{C}_\sigma r^5\|\eta u\|_{\frac{r}{2}}^{r-4m}.
\end{aligned}
$$

Taking σ appropriately small, and substituting the above two inequalities into (3.19), we obtain

$$
e^s\int_\Omega \eta^r(s)u^r(x,s)dx \le C_3\int_T^s r^5\|\eta u\|_{\frac{r}{2}}^{r-4m}e^t dt. \quad (3.20)
$$

Letting $r_j = 2r_{j-1} = 2^j r_0$, $r_0 = 5m$, $M_j = \sup_{t\in(T,\infty)}\|\eta u\|_{L^{r_j}}$, then by a direct calculation, we conclude that

$$
M_j^{r_j} \le C_3 r_j^5 M_{j-1}^{r_j-4m},
$$

by an iteration process, we see that

$$\begin{aligned} M_j \leq & C_3^{\frac{1}{r_j}} r_j^{\frac{5}{r_j}} M_{j-1}^{1-\frac{4m}{r_0 2^j}} \\ \leq & C_3^{\sum_{k=1}^{j} \frac{1}{r_0 2^k}} r_0^{\sum_{k=1}^{j} \frac{5}{r_0 2^k}} M_0^{\prod_{k=1}^{j}(1-\frac{4m}{r_0 2^k})} \leq C_4 M_0^{\prod_{k=1}^{j} \frac{r_0 2^k - 4m}{r_0 2^k}}. \end{aligned}$$

Next, we show that $S = \prod_{k=1}^{\infty} \frac{r_0 2^k - 4m}{r_0 2^k} > 0$. Denote $S_n = \prod_{k=1}^{n} \frac{r_0 2^k - 4m}{r_0 2^k}$, then $S_n \searrow S$. We see that $\ln \frac{1}{S_n} = \sum_{k=1}^{n} \ln(1 + \frac{4m}{r_0 2^k - 4m})$, it is easy to see that $0 \leq \ln(1 + \frac{4m}{r_0 2^k - 4m}) \leq \frac{4m}{r_0 2^k - 4m}$, it implies that $\sum_{k=1}^{\infty} \ln(1 + \frac{4m}{r_0 2^k - 4m})$ converges since $\sum_{k=1}^{\infty} \frac{4m}{r_0 2^k - 4m}$ converges. Then there exists a positive constant $A > 0$ such that $\ln \frac{1}{S_n} \nearrow A$, namely $S = e^{-A} > 0$. Hence,

$$M_j \leq C_4 M_0^S, \text{ for any } \ j \geq 1.$$

Letting $j \to \infty$, we obtain

$$\sup_{t>T} \|\eta u(x,t)\|_{L^\infty} \leq C_4 \sup_{t>T} \|\eta u(x,t)\|_{L^{5m}}^S,$$

and this lemma is proved. □

Using the above lemmas, we finally prove that

Lemma 3.5. *Let $(u, v) \in \mathcal{X}_1 \times \mathcal{X}_2$ be a solution of (1.1). Then*

$$\lim_{t\to\infty} a(t) = 1,$$

and

$$\lim_{t\to\infty} \int_\Omega |u(x,t) - 1|^2 dx = 0, \tag{3.21}$$

where $a(t)$ is defined in Lemma 3.3.

Proof. We first show that

$$\lim_{t\to\infty} a(t) \neq 0. \tag{3.22}$$

Otherwise, by Lemma 3.4, we have

$$\lim_{t\to\infty} \|u(\cdot,t)\|_{L^\infty} = 0.$$

Then there exists $T > 0$, such that or any $t > T$, $u(x,t) \leq \frac{1}{2}$ a.e. in Ω. Choosing the test function $\varphi = e^{-\frac{\mu}{2}s}\chi_{[T,t]}$ in (2.1), with $u(x,0), \varphi(x,0)$ being replaced by $u(x,T)$ and $\varphi(x,T)$, we obtain

$$\int_\Omega ue^{-\frac{\mu}{2}t}dx - \int_\Omega ue^{-\frac{\mu}{2}T}dx = \mu \int_T^t \int_\Omega u(1-u)e^{-\frac{\mu}{2}s}dxds - \frac{\mu}{2} \int_T^t \int_\Omega ue^{-\frac{\mu}{2}s}dxds \geq 0, \text{ for } t > T,$$

which implies

$$\int_\Omega u(x,t)dx \geq e^{\frac{\mu}{2}(t-T)} \int_\Omega u(x,T)dx \to \infty, \text{ as } t \to \infty.$$

It is a contradiction. Thus, we have (3.22). Next, we show

$$\lim_{t\to\infty} a(t) = 1.$$

Suppose the contrary, by (3.22) and Lemma 3.3, there exists two sequences $\{t_j^{(1)}\}_j$ and $\{t_j^{(2)}\}_j$ with $t_j^{(1)}, t_j^{(2)} \to \infty$ as $j \to \infty$, such that

$$\lim_{j\to\infty} a(t_j^{(1)}) = 0, \qquad \lim_{j\to\infty} a(t_j^{(2)}) = 1, \tag{3.23}$$

since a is bounded. Note that

$$|a'(t)| = \frac{\mu}{|\Omega|}\left|\int_\Omega u(1-u)dx\right| \le C,$$

which implies that a is Lipschitz continuous. Then by (3.23) and intermediate value theorem, there exists $\{t_j^{(3)}\}_j$ with $t_j^{(3)} \to \infty$ as $j \to \infty$, such that

$$\lim_{j\to\infty} a(t_j^{(3)}) = \frac{1}{2}.$$

It contradicts with Lemma 3.3, and (3.21) is a direct result of Lemma 3.3. $\square$

Next, we show that $v \to 0$ as $t \to \infty$.

Lemma 3.6. *Let $(u, v) \in \mathcal{X}_1 \times \mathcal{X}_2$ be a solution of (1.1). Then there exists a constant T_0, such that*

$$\|v\|_{H^1} \le Ce^{-\frac{1}{2}t}, \qquad \|v(\cdot,t)\|_{L^\infty} \le Ce^{-\frac{1}{3}t}$$

for any $t > T_0$.

Proof. Choosing the test function $\phi = ve^t\chi_{[\tau,s]}$ in (2.2), and using Gagliardo-Nirenberg interpolation inequality and Cauchy's inequality, for any $s > \tau > 0$, we have

$$\begin{aligned}
&\frac{1}{2}\left[e^t\int_\Omega v^2dx\right]_\tau^s + \int_\tau^s e^t\int_\Omega |\nabla v|^2dxdt + \frac{1}{2}\int_\tau^s e^t\int_\Omega v^2dxdt = \int_\tau^s e^t\int_\Omega v^2(1-u)dxdt \\
&\le \int_\tau^s e^t\|v\|_{L^4}^2\|u-1\|_{L^2}dt \\
&\le C_1\int_\tau^s e^t(\|\nabla v\|_{L^2}^{\frac{3}{2}}\|v\|_{L^2}^{\frac{1}{2}}\|u-1\|_{L^2})dt + C_2\int_\tau^s e^t\|v\|_{L^2}^2\|u-1\|_{L^2}dt \\
&\le C_3\int_\tau^s e^t\left(\|\nabla v\|_{L^2}^2\|u-1\|_{L^2} + \|v\|_{L^2}^2\|u-1\|_{L^2}\right)dt.
\end{aligned}$$

By (3.21), there exists a large constant $T > 0$, such that

$$C_3\|u(\cdot,t) - 1\|_{L^2} < \frac{1}{4}, \quad \text{for any} \quad t \ge T.$$

Then we obtain

$$\left[\int_\Omega e^t v^2 dx\right]_\tau^s + \frac{3}{2}\int_\tau^s\int_\Omega |\nabla v|^2 e^t dxdt + \frac{1}{2}\int_\tau^s\int_\Omega v^2 e^t dxdt \le 0, \quad \text{for any} \quad s > \tau \ge T. \tag{3.24}$$

Choosing the test function $\psi = -e^t\chi_{[\tau,s]}\Delta v$ in (2.2), and using Gagliardo-Nirenberg interpolation inequality and Cauchy's inequality, we have

$$\begin{aligned}
&\left[\frac{1}{2}\int_\Omega |\nabla v|^2 e^t dx\right]_\tau^s + \int_\tau^s\int_\Omega |\Delta v|^2 e^t dxdt + \frac{1}{2}\int_\tau^s\int_\Omega |\nabla v|^2 e^t dxdt = \int_\tau^s\int_\Omega v(u-1)\Delta v e^t dxdt \\
&\le \frac{1}{2}\int_\tau^s\int_\Omega |\Delta v|^2 e^t dxdt + \frac{1}{2}\int_\tau^s\int_\Omega v^2(u-1)^2 e^t dxdt \\
&\le \frac{1}{2}\int_\tau^s\int_\Omega |\Delta v|^2 e^t dxdt + \frac{1}{2}\int_\tau^s \|v\|_{L^4}^2\|u-1\|_{L^2}\|u-1\|_{L^\infty} e^t dt \\
&\le \frac{1}{2}\int_\tau^s\int_\Omega |\Delta v|^2 e^t dxdt + C_4\int_\tau^s \|\nabla v\|_{L^2}^{\frac{3}{2}}\|v\|_{L^2}^{\frac{1}{2}}\|u-1\|_{L^2} e^t dt + C_5\int_\tau^s \|v\|_{L^2}^2\|u-1\|_{L^2} e^t dt \\
&\le \frac{1}{2}\int_\tau^s\int_\Omega |\Delta v|^2 e^t dxdt + C_6\int_\tau^s \left(\|\nabla v\|_{L^2}^2\|u-1\|_{L^2} + \|v\|_{L^2}^2\|u-1\|_{L^2}\right) e^t dt, \quad \text{for any } s > \tau > 0.
\end{aligned}$$

By (3.21), there exists a large constant $T' > 0$, such that

$$C_6\|u(\cdot,t)-1\|_{L^2} < \frac{1}{4}, \quad \text{for any} \quad t \ge T'.$$

Then for any $s > \tau \ge T_0 = \max\{T, T'\}$, we obtain

$$\left[\int_\Omega |\nabla v|^2 e^t dx\right]_\tau^s + \int_\tau^s\int_\Omega |\Delta v|^2 e^t dxdt + \frac{1}{2}\int_\tau^s\int_\Omega |\nabla v|^2 e^t dxdt \le \frac{1}{2}\int_\tau^s\int_\Omega v^2 e^t dxdt. \tag{3.25}$$

Combining with (3.24), we obtain

$$\left[\|v(\cdot,t)\|_{H^1}^2 e^t\right]_\tau^s \le 0, \quad \text{for any } s > \tau \ge T_0,$$

that is

$$\|v(\cdot,s)\|_{H^1}^2 \le \|v(\cdot,T_0)\|_{H^1}^2 e^{-(t-T_0)}, \quad \text{for any} \quad s > T_0. \tag{3.26}$$

By Gagliardo-Nirenberg interpolation inequality and Sobolev inequality, we get

$$\|v\|_{L^\infty} \le C_1\|\nabla v\|_{L^\infty}^{\frac{1}{3}}\|v\|_{L^6}^{\frac{2}{3}} + C_2\|v\|_{L^2} \le C_3\|\nabla v\|_{L^\infty}^{\frac{1}{3}}\|v\|_{H^1}^{\frac{2}{3}} + C_2\|v\|_{L^2},$$

combining with (3.26), we obtain

$$\|v(\cdot,t)\|_{L^\infty} \le Ce^{-\frac{1}{3}t}, \quad \text{for any} \quad t > T_0.$$

The proof is complete. □

Proof of Theorem 1.2. If $p < 2$, then

$$\|u-1\|_{L^p} \leq C\|u-1\|_{L^2} \to 0.$$

Noticing that

$$\|u-1\|_{L^p} \leq \|u-1\|_{L^2}^{\frac{2}{p}} \|u-1\|_{L^\infty}^{\frac{p-2}{p}}, \quad \text{for any} \quad p \geq 2,$$

and using Lemma 3.5 and Lemma 3.6, we complete the proof of Theorem 1.2. □

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