



Derivatives of local times for some Gaussian fields[☆]

Minhao Hong^a, Fangjun Xu^{b,c,*}

^a School of Statistics, East China Normal University, Shanghai 200062, China

^b Key Laboratory of Advanced Theory and Application in Statistics and Data Science - MOE, School of Statistics, East China Normal University, Shanghai, 200062, China

^c NYU-ECNU Institute of Mathematical Sciences at NYU Shanghai, 3663 Zhongshan Road North, Shanghai, 200062, China



ARTICLE INFO

Article history:

Received 7 June 2019

Available online 27 November 2019

Submitted by U. Stadtmueller

Keywords:

Gaussian fields

Derivatives of local time

Local nondeterminism property

Hölder continuity

ABSTRACT

In this article, we consider derivatives of local time for a $(2, d)$ -Gaussian field

$$Z = \{Z(t, s) = X_t^{H_1} - \tilde{X}_s^{H_2}, s, t \geq 0\},$$

where X^{H_1} and \tilde{X}^{H_2} are two independent processes from a class of d -dimensional centered Gaussian processes satisfying certain local nondeterminism property. We first give a condition for existence of derivatives of the local time. Then, under this condition, we show that derivatives of the local time are Hölder continuous in both time and space variables. Moreover, under some additional assumptions, we show that this condition is also necessary for existence of derivatives of the local time at the origin.

© 2019 Elsevier Inc. All rights reserved.

1. Introduction

For $H \in (0, 1)$, let $X^H = \{X_t^H : t \geq 0\}$ be a d -dimensional centered Gaussian stochastic process whose components $X^{H,\ell} (1 \leq \ell \leq d)$ are independent and identically distributed, and satisfy the following local nondeterminism property: for any $T > 0$, there exists a positive constant $\kappa_{T,m,H}$ depending only on T , m and H , such that for any $0 = t_0 < t_1 < \dots < t_m < T$ and $x_i \in \mathbb{R} (1 \leq i \leq m)$, we have

$$\text{Var} \left(\sum_{i=1}^m x_i (X_{t_i}^{H,\ell} - X_{t_{i-1}}^{H,\ell}) \right) \geq \kappa_{T,m,H} \sum_{i=1}^m x_i^2 (t_i - t_{i-1})^{2H}. \quad (1.1)$$

[☆] M. Hong is partially supported by National Natural Science Foundation of China (Grant No. 11871219) and ECNU Academic Innovation Promotion Program for Excellent Doctoral Students (YBNLTS2019-010). F. Xu is partially supported by National Natural Science Foundation of China (Grant No. 11871219, No. 11871220).

* Corresponding author.

E-mail addresses: hongmhecnu@foxmail.com (M. Hong), fangjunxu@gmail.com, fjxu@finance.ecnu.edu.cn (F. Xu).

Let G_L^d be the class of all such d -dimensional centered Gaussian processes X^H and $G_{L,U}^d$ the class of $X^H \in G_L^d$ possessing the following property: there is a positive constant $C_{T,H}$ depending only on T and H such that $\text{Var}(X_t^{H,1}) \leq C_{T,H}|t|^{2H}$ for all $t \in [0, T]$.

From results in [9], we can easily see that the d -dimensional Gaussian processes given below are in $G_{L,U}^d$:
(i) *Bifractional Brownian motion (bi-fBm)*. The covariance function for components of this process is given by

$$\mathbb{E}(X_t^{H,\ell} X_s^{H,\ell}) = 2^{-K_0} [(t^{2H_0} + s^{2H_0})^{K_0} - |t - s|^{2H_0 K_0}],$$

where $H_0 \in (0, 1)$ and $K_0 \in (0, 1]$. Here $H = H_0 K_0$ and $K_0 = 1$ gives the classic fractional Brownian motion (fBm) case with Hurst parameter $H = H_0$.

(ii) *Subfractional Brownian motion (sub-fBm)*. The covariance function for components of this process is given by

$$\mathbb{E}(X_t^{H,\ell} X_s^{H,\ell}) = t^{2H} + s^{2H} - \frac{1}{2} [(t + s)^{2H} + |t - s|^{2H}],$$

where $H \in (0, 1)$.

It is well-known that the Dirac function $\delta(x)$ on \mathbb{R}^d can be approximated by

$$p_\varepsilon(x) = \frac{1}{(2\pi\varepsilon)^{\frac{d}{2}}} e^{-\frac{|x|^2}{2\varepsilon}} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{\iota y \cdot x} e^{-\frac{\varepsilon|y|^2}{2}} dy.$$

So, for multi-index $\mathbf{k} = (k_1, \dots, k_d)$ with all k_i being nonnegative integers, we can approximate

$$\delta^{(\mathbf{k})}(x) = \frac{\partial^{\mathbf{k}}}{\partial x_1^{k_1} \dots \partial x_d^{k_d}} \delta(x)$$

by

$$p_\varepsilon^{(\mathbf{k})}(x) = \frac{\partial^{\mathbf{k}}}{\partial x_1^{k_1} \dots \partial x_d^{k_d}} p_\varepsilon(x) = \frac{\iota^{|\mathbf{k}|}}{(2\pi)^d} \int_{\mathbb{R}^d} \left(\prod_{i=1}^d y_i^{k_i} \right) e^{\iota y \cdot x} e^{-\frac{\varepsilon|y|^2}{2}} dy,$$

where $|\mathbf{k}| = \sum_{i=1}^d k_i$.

Let X^{H_1} and \tilde{X}^{H_2} be two independent Gaussian processes in G_L^d with parameters H_1 and H_2 , respectively. Then

$$Z = \{Z(t, s) = X_t^{H_1} - \tilde{X}_s^{H_2}, s, t \geq 0\} \quad (1.2)$$

is a $(2, d)$ -Gaussian field. For any $T > 0$ and $x \in \mathbb{R}^d$, if

$$L_\varepsilon^{(\mathbf{k})}(T, x) := \int_0^T \int_0^T p_\varepsilon^{(\mathbf{k})}(X_t^{H_1} - \tilde{X}_s^{H_2} + x) ds dt \quad (1.3)$$

converges to a random variable in $L^p(p \geq 1)$ when $\varepsilon \downarrow 0$, we denote the limit by $L^{(\mathbf{k})}(T, x)$ and call it the \mathbf{k} -th derivative of local time for the $(2, d)$ -Gaussian field Z . If it exists, $L^{(\mathbf{k})}(T, x)$ admits the following L^p -representation

$$L^{(\mathbf{k})}(T, x) = \int_0^T \int_0^T \delta^{(\mathbf{k})}(X_t^{H_1} - \tilde{X}_s^{H_2} + x) ds dt. \quad (1.4)$$

When $\mathbf{k} = \mathbf{0}$, $L^{(\mathbf{0})}(T, x)$ is just the local time of the $(2, d)$ -Gaussian field Z at $-x$. Local time of Gaussian processes or Gaussian fields are important subjects in probability theory and has a long history, see, e.g., [1, 8, 2, 3] and references therein. When $x = 0$, $L^{(\mathbf{0})}(T, 0)$ is also called the intersection local time of X^{H_1} and \tilde{X}^{H_2} . Recently, intersection local time for independent fBms and its derivatives have received a lot of attention. If X^{H_1} and \tilde{X}^{H_2} are two independent d -dimensional fBms with the same Hurst parameter H , Nualart and Ortiz-Latorre in [6] proved that $L^{(\mathbf{0})}(T, 0)$ exists in L^2 if and only if $Hd < 2$. This result was later extended to (N, d) -fBms with Hurst parameters H_1 and H_2 by Wu and Xiao in [10], where they also established the regularity of the corresponding intersection local time. When $\mathbf{k} \neq \mathbf{0}$, Yan in [11] showed that $L_\varepsilon^{(\mathbf{k})}(T, x)$ converges in L^p ($p > 0$) if $\frac{1}{H_1} + \frac{1}{H_2} > 3$ for the 1-dimensional fBm case; Guo, Hu and Xiao in [4] gave a condition for existence of $L^{(\mathbf{k})}(T, 0)$ and showed the exponential integrability of $L^{(\mathbf{k})}(T, 0)$ for the d -dimensional fBm case. When the $(2, d)$ -Gaussian field Z is replaced by a 1-dimensional fBm, Jaramillo, Nourdin and Peccati in [5] gave a sharp condition for existence (in L^2) of derivatives of the local time and established their Hölder continuity in the time variable.

In this paper we consider existence of $L^{(\mathbf{k})}(T, x)$ for independent processes from a large class of d -dimensional centered Gaussian processes including fBms, bi-fBms and sub-fBms. In addition, we do not require that X^{H_1} and \tilde{X}^{H_2} are both fBms. They can be different, for example, X^{H_1} can be a bi-fBm while \tilde{X}^{H_2} a sub-fBm. We give a mild condition for existence of $L^{(\mathbf{k})}(T, x)$ and then establish its Hölder continuity in both time and space variables. Our condition is sharp since it is also necessary for existence of $L^{(\mathbf{k})}(T, 0)$ if X^{H_1} and \tilde{X}^{H_2} satisfy certain additional property, which is possessed by Gaussian processes, say fBms, bi-fBms and sub-fBms. A reason for focusing on two independent Gaussian processes is that the methodology developed here can be easily used to obtain the corresponding results for one Gaussian process or k ($k \geq 3$) independent Gaussian processes. Moreover, this paper can be viewed as an extension of [9, 7] where central limit theorems for functionals of $X_t^H - \tilde{X}_s^H$ are not available for $H \leq \frac{2}{d+2}$, see [5] for this phenomenon in the one 1-dimensional fBm case. Here the main difficulty comes from the second independent Gaussian process. Especially in the proof of Theorem 1.2, we need some kind of chaining argument to get the main ingredient in $\mathbb{E}[|L_\varepsilon^{(\mathbf{k})}(T, 0)|^2]$ as $\varepsilon \downarrow 0$.

The following are main results of this paper.

Theorem 1.1. Assume that $X^{H_1} = \{X_t^{H_1} : t \geq 0\}$ and $\tilde{X}^{H_2} = \{\tilde{X}_t^{H_2} : t \geq 0\}$ are two independent Gaussian processes in G_L^d with parameters $H_1, H_2 \in (0, 1)$, respectively. If $\frac{H_1 H_2}{H_1 + H_2} (2|\mathbf{k}| + d) < 1$, then the \mathbf{k} -th derivative of local time $L^{(\mathbf{k})}(T, x)$ exists in L^p for any $p \in [1, \infty)$. Moreover, for any $M > 0$, on $[0, M] \times \mathbb{R}^d$, $L^{(\mathbf{k})}(T, x)$ has a modification which is θ_1 -Hölder continuous in space for all $\theta_1 \in (0, 1 \wedge (\frac{1}{H_1} + \frac{1}{H_2} - 2|\mathbf{k}| - d))$ and θ_2 -Hölder continuous in time for all $\theta_2 \in (0, 1 - \frac{H_1 H_2}{H_1 + H_2} (|\mathbf{k}| + d))$.

Theorem 1.2. Assume that $X^{H_1} = \{X_t^{H_1} : t \geq 0\}$ and $\tilde{X}^{H_2} = \{\tilde{X}_t^{H_2} : t \geq 0\}$ are two independent Gaussian processes in $G_{L,U}^d$ with parameters $H_1, H_2 \in (0, 1)$, respectively. The following statements are true:

- (i) if $\frac{H_1 H_2}{H_1 + H_2} d \geq 1$, then $L_\varepsilon^{(\mathbf{0})}(T, 0)$ diverges in L^2 as $\varepsilon \downarrow 0$;
- (ii) if $\frac{H_1 H_2}{H_1 + H_2} d \leq 1$ and $\frac{H_1 H_2}{H_1 + H_2} (2|\mathbf{k}| + d) \geq 1$, then $L_\varepsilon^{(\mathbf{k})}(T, 0)$ diverges in L^2 as $\varepsilon \downarrow 0$.

Remark 1.3. For independent d -dimensional centered Gaussian processes X^{H_1} and \tilde{X}^{H_2} in $G_{L,U}^d$, Theorem 1.1 and statement (i) in Theorem 1.2 imply that $L^{(\mathbf{0})}(T, 0)$ exists in L^2 if and only if $\frac{H_1 H_2}{H_1 + H_2} d < 1$. Theorem 1.1 and statement (ii) in Theorem 1.2 say that, if $L^{(\mathbf{0})}(T, 0)$ exists in L^2 , then $L^{(\mathbf{k})}(T, 0)$ ($\mathbf{k} \neq \mathbf{0}$) exists in L^2 if and only if $\frac{H_1 H_2}{H_1 + H_2} (2|\mathbf{k}| + d) < 1$. In the proof of Theorem 1.2 below, we also give concrete

divergence rates of $\mathbb{E} |L_\varepsilon^{(\mathbf{k})}(T, 0)|^2$ as $\varepsilon \downarrow 0$ in different cases. The asymptotic behavior of $L_\varepsilon^{(\mathbf{k})}(T, 0)$ as $\varepsilon \downarrow 0$ in these cases will be studied in a future paper.

Remark 1.4. For $N \in \mathbb{N}$, define the (N, d) -Gaussian field

$$Z^N = \left\{ \sum_{j=1}^N X_{t_j}^{j, H_j} : t_j \geq 0, j = 1, \dots, N \right\},$$

where $X_{t_j}^{j, H_j}$ are independent Gaussian processes in G_L^d . Replace Z in $L^{(\mathbf{k})}(T, x)$ and $L^{(\mathbf{k})}(T, x)$ by Z^N and denote the new terms as $L_N^{(\mathbf{k})}(T, x)$ and $L_{N, \varepsilon}^{(\mathbf{k})}(T, x)$, respectively. Using similar arguments as in the proof of Theorems 1.1 and 1.2, we can also obtain that

- (1) If $2|\mathbf{k}| + d < \sum_{j=1}^N H_j^{-1}$, $L_N^{(\mathbf{k})}(T, x)$ exists in L^p for any $p \in [1, \infty)$. Moreover, for any $M > 0$, on $[0, M] \times \mathbb{R}^d$, $L_N^{(\mathbf{k})}(T, x)$ has a modification which is θ_1 -Hölder continuous in space for all $\theta_1 \in (0, 1 \wedge (\sum_{j=1}^N \frac{1}{H_j} - 2|\mathbf{k}| - d))$ and θ_2 -Hölder continuous in time for all $\theta_2 \in (0, 1 - \frac{|\mathbf{k}| + d}{\sum_{j=1}^N H_j})$.
- (2) If $d \geq \sum_{j=1}^N H_j^{-1}$ and $X_{t_j}^{j, H_j} \in G_{L, U}^d$ for each $j = 1, 2, \dots, N$, then $L_{N, \varepsilon}^{(\mathbf{0})}(T, 0)$ diverges in L^2 as $\varepsilon \downarrow 0$.
- (3) If $(2|\mathbf{k}| + d) \geq \sum_{j=1}^N H_j^{-1} \geq d$ and $X_{t_j}^{j, H_j} \in G_{L, U}^d$ for each $j = 1, 2, \dots, N$, then $L_{N, \varepsilon}^{(\mathbf{k})}(T, 0)$ diverges in L^2 as $\varepsilon \downarrow 0$.

After some preliminaries in Section 2, Section 3 is devoted to the proofs of Theorems 1.1 and 1.2. Throughout this paper, if not mentioned otherwise, the letter c , with or without a subscript, denotes a generic positive finite constant whose exact value may change from line to line. For any $x, y \in \mathbb{R}^d$, we use $x \cdot y$ to denote the usual inner product and $|x| = (\sum_{i=1}^d |x_i|^2)^{1/2}$. Moreover, we use ι to denote $\sqrt{-1}$.

2. Preliminaries

In this section, we give two lemmas for Gaussian processes X^H in G_L^d . Lemma 2.1 is needed in the proof of Theorem 1.1, while Lemma 2.2 plays an important role in the proof of Theorem 1.2.

Lemma 2.1. For $0 < s_1 < s_2 < \dots < s_m < T$, $k \in \mathbb{N} \cup \{0\}$ and $\varepsilon \geq 0$, there exists a constant $C_{T, k, m, H}$ depending only on T , k , m and H such that

$$\begin{aligned} & \int_{\mathbb{R}^{md}} \exp \left(-\frac{1}{2} \text{Var} \left(\sum_{j=1}^m y_j \cdot X_{s_j}^H \right) - \frac{\varepsilon}{2} \sum_{j=1}^m |y_j|^2 \right) \prod_{j=1}^m |y_j|^k dy \\ & \leq C_{T, k, m, H} \sum_{\mathcal{S}} \prod_{j=1}^m \left[(s_j - s_{j-1})^{2H} + \varepsilon \right]^{-\frac{1+k(p_j + \bar{p}_{j-1})}{2}}, \end{aligned}$$

where $\mathcal{S} = \{p_i, \bar{p}_i : p_i \in \{0, 1\}, p_i + \bar{p}_i = 1, i = 1, \dots, m-1, p_m = 1\}$.

Proof. Make the change of variables $x_j = \sum_{k=j}^m y_k$ for $j = 1, 2, \dots, m$ with the convention $x_{m+1} = 0$. Then, using the local nondeterminism property (1.1) and Lemma A.4, we can obtain

$$\begin{aligned} & \int_{\mathbb{R}^{md}} \exp \left(-\frac{1}{2} \text{Var} \left(\sum_{j=1}^m y_j \cdot X_{s_j}^H \right) - \frac{\varepsilon}{2} \sum_{j=1}^m |y_j|^2 \right) \prod_{j=1}^m |y_j|^k dy \\ & \leq \int_{\mathbb{R}^{md}} \exp \left(-\frac{\kappa_{T,m,H}}{2} \sum_{j=1}^m |x_j|^2 \left[(s_j - s_{j-1})^{2H} + \frac{2\varepsilon}{m(m+1)\kappa_{T,m,H}} \right] \right) \prod_{j=1}^m |x_j - x_{j+1}|^k dx. \end{aligned}$$

Let $c = \kappa_{T,m,H} \wedge \frac{2}{m(m+1)}$. Then

$$\begin{aligned} & \int_{\mathbb{R}^{md}} \exp \left(-\frac{1}{2} \text{Var} \left(\sum_{j=1}^m y_j \cdot X_{s_j}^H \right) - \frac{\varepsilon}{2} \sum_{j=1}^m |y_j|^2 \right) \prod_{j=1}^m |y_j|^k dy \\ & \leq 2^{km} \int_{\mathbb{R}^{md}} \exp \left(-\frac{c}{2} \sum_{j=1}^m |x_j|^2 \left[(s_j - s_{j-1})^{2H} + \varepsilon \right] \right) \prod_{j=1}^m (|x_j|^k + |x_{j+1}|^k) dx \\ & = 2^{km} \sum_S \int_{\mathbb{R}^{md}} \exp \left(-\frac{c}{2} \sum_{j=1}^m |x_j|^2 \left[(s_j - s_{j-1})^{2H} + \varepsilon \right] \right) \prod_{j=1}^m |x_j|^{k(p_j + \bar{p}_{j-1})} dx \\ & \leq C_{T,k,m,H} \sum_S \prod_{j=1}^m \left[(s_j - s_{j-1})^{2H} + \varepsilon \right]^{-\frac{1+k(p_j + \bar{p}_{j-1})}{2}}. \end{aligned}$$

This gives the desired result. \square

Lemma 2.2. For any $p \geq 1$ and $m \in \mathbb{N}$, there exists a positive constant $C_{T,m,H,p}$ depending only on T , m , H and p such that

$$\begin{aligned} & \left(\int_{[0,T]_{<}^m} \exp \left(-\frac{1}{2} \text{Var} \left(\sum_{j=1}^m y_j \cdot X_{u_j}^H \right) \right) du \right)^p \\ & \leq C_{T,m,H,p}^m \int_{[0,T]_{<}^m} \exp \left(-\frac{\kappa_{T,m,H}}{2} \sum_{j=1}^m \left| \sum_{i=j}^m y_i \right|^2 (\Delta u_j)^{\frac{2H}{p}} \right) du, \end{aligned}$$

where

$$[0, T]_{<}^m = \{0 = u_0 < u_1 < u_2 < \dots < u_m < T\}$$

and $\Delta u_j = u_j - u_{j-1}$ for $j = 1, 2, \dots, m$.

Proof. For any $a \geq 0$ and $T > 0$, let $f_T(a) = \frac{(\int_0^T e^{-a} v^{2H} dv)^p}{\int_0^{T/m} e^{-a} v^{\frac{2H}{p}} dv}$. Clearly, $f_T(\cdot)$ is continuous on $[0, \infty)$, $\lim_{a \rightarrow \infty} f_T(a) = \frac{\Gamma^p(\frac{1}{2H})}{p(2H)^{p-1}\Gamma(\frac{p}{2H})}$ and $f_T(0) = mT^{p-1}$. Therefore, there exists a positive constant $C_{T,m,H,p}$ depending only on T , m , H and p such that

$$\left(\int_0^T e^{-a} v^{2H} dv \right)^p \leq C_{T,m,H,p} \int_0^{T/m} e^{-a} v^{\frac{2H}{p}} dv, \quad \text{for all } a \geq 0.$$

Using local nondeterminism property (1.1) and the above inequality,

$$\begin{aligned}
 & \left(\int_{[0,T]^m} \exp \left(-\frac{1}{2} \text{Var} \left(\sum_{j=1}^m y_j \cdot X_{u_j}^H \right) \right) du \right)^p \\
 & \leq \left(\int_{[0,T]^m} \exp \left(-\frac{\kappa_{T,m,H}}{2} \sum_{j=1}^m \left| \sum_{i=j}^m y_i \right|^2 (\Delta u_j)^{2H} \right) du \right)^p \\
 & \leq \prod_{j=1}^m \left(\int_0^T \exp \left(-\frac{\kappa_{T,m,H}}{2} \left| \sum_{i=j}^m y_i \right|^2 v^{2H} \right) dv \right)^p \\
 & \leq C_{T,m,H,p}^m \prod_{j=1}^m \int_0^{T/m} \exp \left(-\frac{\kappa_{T,m,H}}{2} \left| \sum_{i=j}^m y_i \right|^2 v^{\frac{2H}{p}} \right) dv \\
 & \leq C_{T,m,H,p}^m \int_{[0,T]^m} \exp \left(-\frac{\kappa_{T,m,H}}{2} \sum_{j=1}^m \left| \sum_{i=j}^m y_i \right|^2 (\Delta u_j)^{\frac{2H}{p}} \right) du.
 \end{aligned}$$

This completes the proof. \square

3. Proofs of main results

In this section, we will give proofs of Theorems 1.1 and 1.2.

3.1. Proof of Theorem 1.1

Proof. Recall the definition of $L_\varepsilon^{(\mathbf{k})}(T, x)$ in (1.3). Using Fourier transform,

$$L_\varepsilon^{(\mathbf{k})}(T, x) = \frac{\iota^{|\mathbf{k}|}}{(2\pi)^d} \int_0^T \int_0^T \int_{\mathbb{R}^d} \prod_{i=1}^d y_i^{k_i} e^{\iota y \cdot (X_u^{H_1} - \tilde{X}_v^{H_2} - x)} e^{-\frac{\varepsilon |y|^2}{2}} dy du dv.$$

Fix an even integer $m \geq 1$ and denote $T_m = [0, T]^{2m}$. We have

$$\begin{aligned}
 \mathbb{E} \left[|L_\varepsilon^{(\mathbf{k})}(T, x)|^m \right] & \leq \frac{1}{(2\pi)^{md}} \int_{T_m} \int_{\mathbb{R}^{md}} \exp \left\{ -\frac{1}{2} \mathbb{E} \left[\sum_{j=1}^m y_j \cdot (X_{s_j}^{H_1} - \tilde{X}_{t_j}^{H_2}) \right]^2 \right\} \\
 & \quad \times \exp \left\{ -\frac{\varepsilon}{2} \sum_{j=1}^m |y_j|^2 \right\} \prod_{j=1}^m \left(\prod_{i=1}^d |y_{j,i}|^{k_i} \right) dy dt ds.
 \end{aligned}$$

Let \mathcal{P}_m be the set of all permutations of $\{1, 2, \dots, m\}$ and

$$D_T^m = \{u \in [0, T]^m : 0 < u_1 < \dots < u_m < T\}.$$

Then

$$\begin{aligned}
 & \mathbb{E} \left[|L_\varepsilon^{(\mathbf{k})}(T, x)|^m \right] \\
 & \leq \frac{m!}{(2\pi)^{md}} \sum_{\sigma \in \mathcal{P}_m} \int_{D_T^m} \int_{\mathbb{R}^{md}} \exp \left\{ -\frac{1}{2} \mathbb{E} \left[\sum_{j=1}^m y_j \cdot (X_{s_j}^{H_1} - \tilde{X}_{t_{\sigma(j)}}^{H_2}) \right]^2 \right\} \prod_{j=1}^m |y_j|^{|\mathbf{k}|} dy dt ds.
 \end{aligned}$$

Using Hölder's inequality,

$$\begin{aligned}
& \int_{\mathbb{R}^{md}} \exp \left\{ -\frac{1}{2} \mathbb{E} \left[\sum_{j=1}^m y_j \cdot (X_{s_j}^{H_1} - \tilde{X}_{t_{\sigma(j)}}^{H_2}) \right]^2 \right\} \prod_{j=1}^m |y_j|^{|k|} dy \\
& \leq \left(\int_{\mathbb{R}^{md}} \exp \left\{ -\frac{H_1+H_2}{2H_2} \text{Var} \left(\sum_{j=1}^m y_j \cdot X_{s_j}^{H_1} \right) \right\} \prod_{j=1}^m |y_j|^{|k|} dy \right)^{\frac{H_2}{H_1+H_2}} \\
& \quad \times \left(\int_{\mathbb{R}^{md}} \exp \left\{ -\frac{H_1+H_2}{2H_1} \text{Var} \left(\sum_{j=1}^m y_j \cdot X_{t_{\sigma(j)}}^{H_2} \right) \right\} \prod_{j=1}^m |y_j|^{|k|} dy \right)^{\frac{H_1}{H_1+H_2}} \\
& \leq \left(\int_{\mathbb{R}^{md}} \exp \left\{ -\frac{1}{2} \text{Var} \left(\sum_{j=1}^m y_j \cdot X_{s_j}^{H_1} \right) \right\} \prod_{j=1}^m |y_j|^{|k|} dy \right)^{\frac{H_2}{H_1+H_2}} \\
& \quad \times \left(\int_{\mathbb{R}^{md}} \exp \left\{ -\frac{1}{2} \text{Var} \left(\sum_{j=1}^m y_j \cdot X_{t_{\sigma(j)}}^{H_2} \right) \right\} \prod_{j=1}^m |y_j|^{|k|} dy \right)^{\frac{H_1}{H_1+H_2}}.
\end{aligned}$$

Now, by Lemma 2.1 and the inequality $(\sum_{i=1}^n a_i)^\alpha \leq \sum_{i=1}^n a_i^\alpha$ for any $n \in \mathbb{N}$, $a_i \geq 0$ and $\alpha \in [0, 1]$, we have

$$\mathbb{E} \left[|L_\varepsilon^{(\mathbf{k})}(T, x)|^m \right] \leq c_1 \left(\sum_{\mathcal{S}} \int_{D_T^m} \prod_{j=1}^m (s_j - s_{j-1})^{-\frac{H_1 H_2}{H_1+H_2} d - \frac{H_1 H_2}{H_1+H_2} |\mathbf{k}| (p_j + \bar{p}_{j-1})} ds \right)^2, \quad (3.1)$$

where $\mathcal{S} = \{p_j, \bar{p}_j : p_j \in \{0, 1\}, p_j + \bar{p}_j = 1, j = 1, \dots, m-1, p_m = 1\}$.

Note that $p_j + \bar{p}_{j-1} \in \{0, 1, 2\}$ for $j = 1, \dots, m$. Hence, when $\frac{H_1 H_2}{H_1+H_2} (d + 2|\mathbf{k}|) < 1$,

$$\mathbb{E} \left[|L_\varepsilon^{(\mathbf{k})}(T, x)|^m \right] < \infty$$

for all $\varepsilon > 0$.

Observe that

$$\begin{aligned}
& \mathbb{E} \left[|L_\varepsilon^{(\mathbf{k})}(T, x) - L_\eta^{(\mathbf{k})}(T, x)|^m \right] \\
& \leq \frac{1}{(2\pi)^{md}} \int_{T_m} \int_{\mathbb{R}^{md}} \exp \left\{ -\frac{1}{2} \mathbb{E} \left[\sum_{j=1}^m y_j \cdot (X_{s_j}^{H_1} - \tilde{X}_{t_j}^{H_2}) \right]^2 \right\} \\
& \quad \times \prod_{j=1}^m \left| \exp \left\{ -\frac{\varepsilon}{2} |y_j|^2 \right\} - \exp \left\{ -\frac{\eta}{2} |y_j|^2 \right\} \right| \prod_{j=1}^m \left(\prod_{i=1}^d |y_{j,i}|^{k_i} \right) dy dt ds
\end{aligned}$$

and

$$\int_{T_m} \int_{\mathbb{R}^{md}} \exp \left\{ -\frac{1}{2} \mathbb{E} \left[\sum_{j=1}^m y_j \cdot (X_{s_j}^{H_1} - \tilde{X}_{t_j}^{H_2}) \right]^2 \right\} \prod_{j=1}^m \left(\prod_{i=1}^d |y_{j,i}|^{k_i} \right) dy dt ds < \infty$$

when $\frac{H_1 H_2}{H_1+H_2} (d + 2|\mathbf{k}|) < 1$. Now, by the dominated convergence theorem, we can easily obtain that the \mathbf{k} -th derivative of the local time $L^{(\mathbf{k})}(T, x)$ exists in L^p for any $p \in [1, \infty)$ if $\frac{H_1 H_2}{H_1+H_2} (d + 2|\mathbf{k}|) < 1$.

In the sequel, we show the Hölder continuity of $L^{(\mathbf{k})}(T, x)$ in time and space variables. For Hölder continuity in the space variable, using Fourier transform,

$$\begin{aligned} & L_{\varepsilon}^{(\mathbf{k})}(T, z+x) - L_{\varepsilon}^{(\mathbf{k})}(T, x) \\ &= \frac{\iota^{|\mathbf{k}|}}{(2\pi)^d} \int_0^T \int_0^T \int_{\mathbb{R}^d} \prod_{i=1}^d y_i^{k_i} e^{\iota y \cdot (X_u^{H_1} - \tilde{X}_v^{H_2})} (e^{-\iota y \cdot (z+x)} - e^{-\iota y \cdot x}) e^{-\frac{\varepsilon|y|^2}{2}} dy du dv. \end{aligned}$$

Then, for any even integer m ,

$$\begin{aligned} & \mathbb{E} \left[|L_{\varepsilon}^{(\mathbf{k})}(T, z+x) - L_{\varepsilon}^{(\mathbf{k})}(T, x)|^m \right] \\ & \leq \int_{T_m} \int_{\mathbb{R}^{md}} \exp \left\{ -\frac{1}{2} \mathbb{E} \left[\sum_{j=1}^m y_j \cdot (X_{s_j}^{H_1} - \tilde{X}_{t_j}^{H_2}) \right]^2 \right\} \\ & \quad \times \exp \left\{ -\frac{\varepsilon}{2} \sum_{j=1}^m |y_j|^2 \right\} \prod_{j=1}^m (|e^{-\iota y_j \cdot z} - 1| |y_j|^{|\mathbf{k}|}) dy dt ds. \end{aligned}$$

Note that $|e^{-\iota y_j \cdot z} - 1| \leq c_{\alpha} |z|^{\alpha} |y_j|^{\alpha}$ for any $\alpha \in [0, 1]$. Hence

$$\prod_{j=1}^m (|e^{-\iota y_j \cdot z} - 1| |y_j|^{|\mathbf{k}|}) \leq c_{\alpha}^m |z|^{m\alpha} \prod_{j=1}^m |y_j|^{|\mathbf{k}|+\alpha}.$$

Similarly, for any $\alpha \in [0, 1 \wedge (\frac{1}{H_1} + \frac{1}{H_2} - 2|\mathbf{k}| - d))$,

$$\mathbb{E} \left[|L_{\varepsilon}^{(\mathbf{k})}(T, z+x) - L_{\varepsilon}^{(\mathbf{k})}(T, x)|^m \right] \leq c_2 |z|^{m\alpha}.$$

Therefore,

$$\begin{aligned} & \mathbb{E} \left[|L^{(\mathbf{k})}(T, z+x) - L^{(\mathbf{k})}(T, x)|^m \right] \\ & \leq 3^m \lim_{\varepsilon \downarrow 0} \mathbb{E} \left[|L^{(\mathbf{k})}(T, z+x) - L_{\varepsilon}^{(\mathbf{k})}(T, z+x)|^m \right] + 3^m \limsup_{\varepsilon \downarrow 0} \mathbb{E} \left[|L_{\varepsilon}^{(\mathbf{k})}(T, z+x) - L_{\varepsilon}^{(\mathbf{k})}(T, x)|^m \right] \\ & \quad + 3^m \lim_{\varepsilon \downarrow 0} \mathbb{E} \left[|L_{\varepsilon}^{(\mathbf{k})}(T, x) - L^{(\mathbf{k})}(T, x)|^m \right] \\ & \leq c_3 |z|^{m\alpha}. \end{aligned}$$

The desired θ_1 -Hölder continuity in the space variable follows from the Kolmogorov continuity criterion.

For Hölder continuity in the time variable, we see that

$$\begin{aligned} & L_{\varepsilon}^{(\mathbf{k})}(T+h, x) - L_{\varepsilon}^{(\mathbf{k})}(T, x) \\ &= \int_0^{T+h} \int_0^T p_{\varepsilon}^{(\mathbf{k})}(X_t^{H_1} - \tilde{X}_s^{H_2} + x) ds dt + \int_0^T \int_T^{T+h} p_{\varepsilon}^{(\mathbf{k})}(X_t^{H_1} - \tilde{X}_s^{H_2} + x) ds dt \\ & \quad + \int_T^{T+h} \int_T^{T+h} p_{\varepsilon}^{(\mathbf{k})}(X_t^{H_1} - \tilde{X}_s^{H_2} + x) ds dt \\ &=: I_{\varepsilon,1} + I_{\varepsilon,2} + I_{\varepsilon,3}. \end{aligned}$$

It suffices to show that, for any even integer m , $\mathbb{E} |I_{\varepsilon,1}|^m$ is less than a constant multiple of $h^{m(1-\frac{H_1 H_2}{H_1+H_2}(|\mathbf{k}|+d))}$. Let $T_m^h = [T, T+h]^m \times [0, T]^m$. By Fourier transform,

$$\begin{aligned} \mathbb{E} |I_{\varepsilon,1}|^m &\leq \frac{1}{(2\pi)^{md}} \int_{T_m^h} \int_{\mathbb{R}^{md}} \exp \left\{ -\frac{1}{2} \mathbb{E} \left[\sum_{j=1}^m y_j \cdot (X_{s_j}^{H_1} - \tilde{X}_{t_j}^{H_2}) \right]^2 \right\} \\ &\quad \times \exp \left\{ -\frac{\varepsilon}{2} \sum_{j=1}^m |y_j|^2 \right\} \prod_{j=1}^m |y_j|^{|\mathbf{k}|} dy dt ds. \end{aligned}$$

Let $D_{T,h}^m = \{u \in [T, T+h]^m : T < u_1 < \dots < u_m < T+h\}$. Now using Hölder's inequality as in obtaining (3.1) and then Lemmas A.1 and A.2,

$$\begin{aligned} \mathbb{E} |I_{\varepsilon,1}|^m &\leq c_3 \sum_{\mathcal{S}} \int_{D_{T,h}^m} \prod_{j=1}^m (s_j - s_{j-1})^{-\frac{H_1 H_2}{H_1+H_2}d - \frac{H_1 H_2}{H_1+H_2}|\mathbf{k}|(p_j + \bar{p}_{j-1})} ds \\ &\leq c_4 \sum_{\mathcal{S}} h^{\sum_{j=1}^m \left[1 - \frac{H_1 H_2}{H_1+H_2}d - \frac{H_1 H_2}{H_1+H_2}|\mathbf{k}|(p_j + \bar{p}_{j-1}) \right]} \\ &\leq c_5 h^{m(1-\frac{H_1 H_2}{H_1+H_2}(|\mathbf{k}|+d))}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\mathbb{E} \left[|L^{(\mathbf{k})}(T+h, x) - L^{(\mathbf{k})}(T, x)|^m \right] \\ &\leq 3^m \lim_{\varepsilon \downarrow 0} \mathbb{E} \left[|L^{(\mathbf{k})}(T+h, x) - L_{\varepsilon}^{(\mathbf{k})}(T+h, x)|^m \right] + 3^m \limsup_{\varepsilon \downarrow 0} \mathbb{E} \left[|L_{\varepsilon}^{(\mathbf{k})}(T+h, x) - L_{\varepsilon}^{(\mathbf{k})}(T, x)|^m \right] \\ &\quad + 3^m \lim_{\varepsilon \downarrow 0} \mathbb{E} \left[|L_{\varepsilon}^{(\mathbf{k})}(T, x) - L^{(\mathbf{k})}(T, x)|^m \right] \\ &\leq c_6 h^{m(1-\frac{H_1 H_2}{H_1+H_2}(|\mathbf{k}|+d))}. \end{aligned}$$

By Kolmogorov continuity criterion, we get the θ_2 -Hölder continuity in the time variable.

This completes the proof. \square

3.2. Proof of Theorem 1.2

Proof. We divide the proof into several steps.

Step 1. Recall that

$$\begin{aligned} \mathbb{E} \left[|L_{\varepsilon}^{(\mathbf{k})}(T, 0)|^2 \right] &= \frac{(-1)^{|\mathbf{k}|}}{(2\pi)^{2d}} \int_{[0,T]^4} \int_{\mathbb{R}^{2d}} I_2(H_1, s, x) \tilde{I}_2(H_2, t, x) \\ &\quad \times \exp \left\{ -\frac{\varepsilon}{2} (|x_1|^2 + |x_2|^2) \right\} \prod_{i=1}^d x_{2,i}^{k_i} \prod_{i=1}^d x_{1,i}^{k_i} dx dt ds, \end{aligned} \quad (3.2)$$

where $I_2(H, t, x) = \exp \left\{ -\frac{1}{2} \mathbb{E} [x_2 \cdot X_{t_2}^H + x_1 \cdot X_{t_1}^H]^2 \right\}$ and

$$\tilde{I}_2(H, t, x) = \exp \left\{ -\frac{1}{2} \mathbb{E} [x_2 \cdot \tilde{X}_{t_2}^H + x_1 \cdot \tilde{X}_{t_1}^H]^2 \right\}.$$

We define $F_{T,\varepsilon}^{(\mathbf{k})}$ by replacing $\prod_{i=1}^d x_{1,i}^{k_i}$ in (3.2) with $\prod_{i=1}^d (-x_{2,i})^{k_i}$. That is,

$$F_{T,\varepsilon}^{(\mathbf{k})} = \frac{1}{(2\pi)^{2d}} \int_{[0,T]^4} \int_{\mathbb{R}^{2d}} I_2(H_1, s, x) \tilde{I}_2(H_2, t, x) \\ \times \exp \left\{ -\frac{\varepsilon}{2}(|x_1|^2 + |x_2|^2) \right\} \prod_{i=1}^d x_{2,i}^{2k_i} dx dt ds.$$

Note that $\min\{I_2(H, t, y), \tilde{I}_2(H, t, y)\} \geq \exp\{-c_1(|x_2|^2 t_2^{2H} + |x_1|^2 t_1^{2H})\}$. Hence,

$$F_{T,\varepsilon}^{(\mathbf{k})} \geq c_2 \int_{[0,T]^4} \int_{\mathbb{R}^{2d}} \exp \left\{ -|x_2|^2(t_2^{2H_2} + s_2^{2H_1} + \varepsilon) - |x_1|^2(t_1^{2H_2} + s_1^{2H_1} + \varepsilon) \right\} \prod_{i=1}^d x_{2,i}^{2k_i} dx dt ds \\ = c_3 \int_{[0,T]^4} (t_2^{2H_2} + s_2^{2H_1} + \varepsilon)^{-\frac{d}{2}} (t_1^{2H_2} + s_1^{2H_1} + \varepsilon)^{-\frac{d}{2} - |\mathbf{k}|} dt ds.$$

By Lemma A.3,

$$F_{T,\varepsilon}^{(\mathbf{k})} \geq \begin{cases} c_4 \varepsilon^{\frac{H_1+H_2}{H_1+H_2}d-d} & \text{if } \frac{H_1H_2}{H_1+H_2}d > 1, |\mathbf{k}| = 0 \\ c_4 \ln^2(1 + \varepsilon^{-\frac{1}{2}}) & \text{if } \frac{H_1H_2}{H_1+H_2}d = 1, |\mathbf{k}| = 0 \\ c_4 \ln(1 + \varepsilon^{-\frac{1}{2}}) \varepsilon^{\frac{H_1+H_2}{2H_1H_2}d - \frac{d}{2} - |\mathbf{k}|} & \text{if } \frac{H_1H_2}{H_1+H_2}d = 1, \frac{H_1H_2}{H_1+H_2}(2|\mathbf{k}| + d) > 1 \\ c_4 \varepsilon^{\frac{H_1+H_2}{2H_1H_2}d - \frac{d}{2} - |\mathbf{k}|} & \text{if } \frac{H_1H_2}{H_1+H_2}d < 1, \frac{H_1H_2}{H_1+H_2}(2|\mathbf{k}| + d) > 1 \\ c_4 \ln(1 + \varepsilon^{-\frac{1}{2}}) & \text{if } \frac{H_1H_2}{H_1+H_2}d < 1, \frac{H_1H_2}{H_1+H_2}(2|\mathbf{k}| + d) = 1. \end{cases} \quad (3.3)$$

Step 2. We estimate $\left| \mathbb{E} \left[|L_\varepsilon^{(\mathbf{k})}(T, 0)|^2 \right] - F_{T,\varepsilon}^{(\mathbf{k})} \right|$. It is easy to see that $\left| \mathbb{E} \left[|L_\varepsilon^{(0)}(T, 0)|^2 \right] - F_{T,\varepsilon}^{(0)} \right| = 0$. So it suffices to consider the case $|\mathbf{k}| \geq 1$ in the sequel. Recall that \mathcal{P}_2 is the set of all permutations of $\{1, 2\}$. That is,

$$\mathcal{P}_2 = \left\{ \sigma_1, \sigma_2 : \sigma_1(1) = \sigma_2(2) = 1, \sigma_1(2) = \sigma_2(1) = 2 \right\}.$$

For $\sigma \in \mathcal{P}_2$, define

$$\tilde{J}_2^\sigma(H, t, y) = \exp \left\{ -\frac{1}{2} \text{Var}[(y_{\sigma(2)} - y_{\sigma(2)+1}) \cdot (\tilde{X}_{t_2}^H - \tilde{X}_{t_1}^H) + y_1 \cdot \tilde{X}_{t_1}^H] \right\}$$

with the convention $y_3 = 0$ and

$$J_2(H, t, y) = \exp \left\{ -\frac{1}{2} \text{Var}[y_2 \cdot (X_{t_2}^H - X_{t_1}^H) + y_1 \cdot X_{t_1}^H] \right\}.$$

Making the change of variables $y_2 = x_2$ and $y_1 = x_2 + x_1$ gives

$$\mathbb{E} \left[|L_\varepsilon^{(\mathbf{k})}(T, 0)|^2 \right] - F_{T,\varepsilon}^{(\mathbf{k})} \\ = 2 \frac{(-1)^{|\mathbf{k}|}}{(2\pi)^{2d}} \sum_{\sigma \in \mathcal{P}_2} \int_{D_T^2} \int_{D_T^2} J_2(H_1, s, y) \tilde{J}_2^\sigma(H_2, t, y) \exp \left\{ -\frac{\varepsilon}{2}(|y_1 - y_2|^2 + |y_2|^2) \right\} \\ \times \prod_{i=1}^d y_{2,i}^{k_i} \left(\prod_{i=1}^d (y_{1,i} - y_{2,i})^{k_i} - \prod_{i=1}^d (-y_{2,i})^{k_i} \right) dy dt ds,$$

where $D_T^2 = \{0 < t_1 < t_2 < T\}$.

Set

$$\Delta^{\mathbf{k},d}(y) = \prod_{i=1}^d y_{2,i}^{k_i} \left(\prod_{i=1}^d (y_{1,i} - y_{2,i})^{k_i} - \prod_{i=1}^d (-y_{2,i})^{k_i} \right).$$

It is easy to see that

$$|\Delta^{\mathbf{k},d}(y)| \leq c_5 \sum_{\ell=1}^{|\mathbf{k}|} |y_2|^{2|\mathbf{k}|-\ell} |y_1|^\ell.$$

For $\ell = 1, \dots, |\mathbf{k}|$, define

$$F^\ell(y) = \exp \left\{ -\frac{\varepsilon}{2} (|y_1 - y_2|^2 + |y_2|^2) \right\} |y_2|^{2|\mathbf{k}|-\ell} |y_1|^\ell.$$

Then,

$$\left| \mathbb{E} \left[|L_\varepsilon^{(\mathbf{k})}(T, 0)|^2 \right] - F_{T,\varepsilon}^{(\mathbf{k})} \right| \leq c_5 \sum_{\ell=1}^{|\mathbf{k}|} (K_{\varepsilon,1}^\ell + K_{\varepsilon,2}^\ell), \quad (3.4)$$

where

$$K_{\varepsilon,i}^\ell = \int_{D_T^2 \times D_T^2} \int_{\mathbb{R}^{2d}} J_2(H_1, s, y) \tilde{J}_2^{\sigma_i}(H_2, t, y) F^\ell(y) dy dt ds, \quad i = 1, 2.$$

Step 3. We estimate $K_{\varepsilon,1}^\ell$ and $K_{\varepsilon,2}^\ell$ for $\ell = 1, \dots, |\mathbf{k}|$. Using the local nondeterminism property (1.1) and Lemma A.4,

$$\begin{aligned} K_{\varepsilon,1}^\ell &\leq c_6 \int_{D_T^2 \times D_T^2} \int_{\mathbb{R}^{2d}} \exp \left\{ -|y_2|^2 \{ (t_2 - t_1)^{2H_2} + (s_2 - s_1)^{2H_1} + \varepsilon \} \right\} \\ &\quad \times \exp \left\{ -|y_1|^2 \{ t_1^{2H_2} + s_1^{2H_1} + \varepsilon \} \right\} |y_2|^{2|\mathbf{k}|-\ell} |y_1|^\ell dy dt ds \\ &\leq c_7 \int_{D_T^2 \times D_T^2} \{ (t_2 - t_1)^{2H_2} + (s_2 - s_1)^{2H_1} + \varepsilon \}^{-\frac{2|\mathbf{k}|-\ell+d}{2}} \{ t_1^{2H_2} + s_1^{2H_1} + \varepsilon \}^{-\frac{\ell+d}{2}} dt ds \\ &\leq c_7 \int_{[0,T]^2} (u^{2H_2} + v^{2H_1} + \varepsilon)^{-\frac{2|\mathbf{k}|-\ell+d}{2}} du dv \int_{[0,T]^2} (u^{2H_2} + v^{2H_1} + \varepsilon)^{-\frac{\ell+d}{2}} du dv. \end{aligned}$$

For $\ell = 1, \dots, |\mathbf{k}|$, by Lemma A.3, we could obtain that

$$K_{\varepsilon,1}^\ell \leq c_8 h_{H_1, H_2}^{d, |\mathbf{k}|, 1}(\varepsilon), \quad (3.5)$$

where

$$h_{H_1, H_2}^{d, |\mathbf{k}|, 1}(\varepsilon) = \begin{cases} \varepsilon^{\frac{H_1+H_2}{2H_1H_2} - \frac{d}{2} - |\mathbf{k}|} & \text{if } \frac{H_1H_2}{H_1+H_2}d = 1, \frac{H_1H_2}{H_1+H_2}(2|\mathbf{k}|+d) > 1 \\ \varepsilon^{\frac{H_1+H_2}{2H_1H_2} - \frac{d}{2} - |\mathbf{k}|+\beta} & \text{if } \frac{H_1H_2}{H_1+H_2}d < 1, \frac{H_1H_2}{H_1+H_2}(2|\mathbf{k}|+d) > 1 \\ 1 & \text{if } \frac{H_1H_2}{H_1+H_2}d < 1, \frac{H_1H_2}{H_1+H_2}(2|\mathbf{k}|+d) = 1, \end{cases} \quad (3.6)$$

where $\beta = \frac{1}{4} \{1 \wedge (\frac{H_1+H_2}{H_1H_2} - d) \wedge (d + 2|\mathbf{k}| - \frac{H_1+H_2}{H_1H_2})\}$.

We next estimate $K_{\varepsilon,2}^\ell$. Using Hölder's inequality,

$$\begin{aligned} K_{\varepsilon,2}^\ell &\leq \left(\int_{\mathbb{R}^{2d}} F^\ell(y) \left(\int_{D_T^2} J_2(H_1, s, y) ds \right)^{\frac{H_1+H_2}{H_2}} dy \right)^{\frac{H_2}{H_1+H_2}} \\ &\quad \times \left(\int_{\mathbb{R}^{2d}} F^\ell(y) \left(\int_{D_T^2} \tilde{J}_2^{\sigma_2}(H_2, t, y) dt \right)^{\frac{H_1+H_2}{H_1}} dy \right)^{\frac{H_1}{H_1+H_2}}. \end{aligned}$$

Then, by Lemmas 2.2 and A.4,

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} F^\ell(y) \left(\int_{D_T^2} J_2(H_1, s, y) ds \right)^{\frac{H_1+H_2}{H_2}} dy \\ &\leq \int_{\mathbb{R}^{2d}} \int_{D_T^2} \exp \left\{ -c_9(|y_2|^2 \{(s_2 - s_1)^{\frac{2H_1H_2}{H_1+H_2}} + \varepsilon\} + |y_1|^2 \{s_1^{\frac{2H_1H_2}{H_1+H_2}} + \varepsilon\}) \right\} |y_2|^{2|\mathbf{k}|-\ell} |y_1|^\ell ds dy \\ &\leq c_{10} \int_0^T (u^{\frac{2H_1H_2}{H_1+H_2}} + \varepsilon)^{-\frac{2|\mathbf{k}|-\ell+d}{2}} du \int_0^T (u^{\frac{2H_1H_2}{H_1+H_2}} + \varepsilon)^{-\frac{\ell+d}{2}} du \\ &\leq c_{11} h_{H_1, H_2}^{d, |\mathbf{k}|, 1}(\varepsilon). \end{aligned}$$

Making the change of variables $y_1 - y_2 = x_2$ and $y_1 = x_1$ gives

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} F^\ell(y) \left(\int_{D_T^2} \tilde{J}_2^{\sigma_2}(H_2, t, y) dt \right)^{\frac{H_1+H_2}{H_1}} dy \\ &= \int_{\mathbb{R}^{2d}} \exp \left\{ -\frac{\varepsilon}{2}(|x_1 - x_2|^2 + |x_2|^2) \right\} |x_1 - x_2|^{2|\mathbf{k}|-\ell} |x_1|^\ell \left(\int_{D_T^2} \tilde{J}_2^{\sigma_1}(H_2, t, x) dt \right)^{\frac{H_1+H_2}{H_1}} dx \\ &\leq \int_{\mathbb{R}^{2d}} \int_{D_T^2} \exp \left\{ -c_{12}(|x_2|^2 \{(t_2 - t_1)^{\frac{2H_1H_2}{H_1+H_2}} + \varepsilon\} + |x_1|^2 \{t_1^{\frac{2H_1H_2}{H_1+H_2}} + \varepsilon\}) \right\} \\ &\quad \times \sum_{j=0}^{2|\mathbf{k}|-\ell} |x_1|^{j+\ell} |x_2|^{2|\mathbf{k}|-\ell-j} dt dx \\ &\leq c_{13} \sum_{j=0}^{2|\mathbf{k}|-\ell} \int_0^T (u^{\frac{2H_1H_2}{H_1+H_2}} + \varepsilon)^{-\frac{2|\mathbf{k}|-\ell-j+d}{2}} du \int_0^T (u^{\frac{2H_1H_2}{H_1+H_2}} + \varepsilon)^{-\frac{\ell+j+d}{2}} du \\ &\leq c_{14} h_{H_1, H_2}^{d, |\mathbf{k}|, 2}(\varepsilon), \end{aligned}$$

where in the last inequality we used Lemma A.3 and

$$h_{H_1, H_2}^{d, |\mathbf{k}|, 2}(\varepsilon) = \begin{cases} \ln(1 + \varepsilon^{-\frac{1}{2}}) \varepsilon^{\frac{H_1+H_2}{2H_1H_2} - \frac{d}{2} - |\mathbf{k}|} & \text{if } \frac{H_1H_2}{H_1+H_2}d = 1, \frac{H_1H_2}{H_1+H_2}(2|\mathbf{k}| + d) > 1 \\ \varepsilon^{\frac{H_1+H_2}{2H_1H_2} - \frac{d}{2} - |\mathbf{k}|} & \text{if } \frac{H_1H_2}{H_1+H_2}d < 1, \frac{H_1H_2}{H_1+H_2}(2|\mathbf{k}| + d) > 1 \\ \ln(1 + \varepsilon^{-\frac{1}{2}}) & \text{if } \frac{H_1H_2}{H_1+H_2}d < 1, \frac{H_1H_2}{H_1+H_2}(2|\mathbf{k}| + d) = 1. \end{cases} \quad (3.7)$$

Therefore,

$$K_{\varepsilon,2}^{\ell} \leq c_{15} \left(h_{H_1,H_2}^{d,|\mathbf{k}|,1}(\varepsilon) \right)^{\frac{H_1}{H_1+H_2}} \left(h_{H_1,H_2}^{d,|\mathbf{k}|,2}(\varepsilon) \right)^{\frac{H_2}{H_1+H_2}}. \quad (3.8)$$

Step 4. We show the divergence of $\mathbb{E} \left[|L_{\varepsilon}^{(\mathbf{k})}(T, 0)|^2 \right]$ as ε tends to 0. Combining inequalities (3.4), (3.5) and (3.8) gives

$$\left| \mathbb{E} \left[|L_{\varepsilon}^{(\mathbf{k})}(T, 0)|^2 \right] - F_{T,\varepsilon}^{(\mathbf{k})} \right| \leq c_{16} 1_{\mathbb{N}}(|\mathbf{k}|) \left(h_{H_1,H_2}^{d,|\mathbf{k}|,1}(\varepsilon) \right)^{\frac{H_1}{H_1+H_2}} \left(h_{H_1,H_2}^{d,|\mathbf{k}|,2}(\varepsilon) \right)^{\frac{H_2}{H_1+H_2}}.$$

Recall the inequality (3.3), definition of $h_{H_1,H_2}^{d,|\mathbf{k}|,1}(\varepsilon)$ in (3.6) and definition of $h_{H_1,H_2}^{d,|\mathbf{k}|,2}(\varepsilon)$ in (3.7). We finally have

$$\mathbb{E} \left[|L_{\varepsilon}^{(\mathbf{k})}(T, 0)|^2 \right] \geq \begin{cases} c_{17} \varepsilon^{\frac{H_1+H_2}{H_1H_2}-d} & \text{if } \frac{H_1H_2}{H_1+H_2}d > 1, |\mathbf{k}| = 0 \\ c_{17} \ln^2(1 + \varepsilon^{-\frac{1}{2}}) & \text{if } \frac{H_1H_2}{H_1+H_2}d = 1, |\mathbf{k}| = 0 \\ c_{17} \ln(1 + \varepsilon^{-\frac{1}{2}}) \varepsilon^{\frac{H_1+H_2}{2H_1H_2}-\frac{d}{2}-|\mathbf{k}|} & \text{if } \frac{H_1H_2}{H_1+H_2}d = 1, \frac{H_1H_2}{H_1+H_2}(2|\mathbf{k}| + d) > 1 \\ c_{17} \varepsilon^{\frac{H_1+H_2}{2H_1H_2}-\frac{d}{2}-|\mathbf{k}|} & \text{if } \frac{H_1H_2}{H_1+H_2}d < 1, \frac{H_1H_2}{H_1+H_2}(2|\mathbf{k}| + d) > 1 \\ c_{17} \ln(1 + \varepsilon^{-\frac{1}{2}}) & \text{if } \frac{H_1H_2}{H_1+H_2}d < 1, \frac{H_1H_2}{H_1+H_2}(2|\mathbf{k}| + d) = 1. \end{cases}$$

This completes the proof. \square

Appendix A

In this section, we give some known results that are used in this paper.

Lemma A.1. For any $T > 0$ and $a_i \in (0, 1)$ with $i = 1, 2, \dots, m$,

$$\int_{D_T^m} \prod_{j=1}^m u_j^{-a_i} du = \frac{\prod_{j=1}^m \Gamma(1 - a_j)}{\Gamma(m + 1 - \sum_{i=1}^m a_i)} T^{\sum_{i=1}^m (1 - a_i)},$$

where $D_T^m = \left\{ 0 < u_1 + u_2 + \dots + u_m < T : u_i > 0, i = 1, 2, \dots, m \right\}$.

Proof. For $i = 1, 2, \dots, m$, let $d\bar{u}_i = \prod_{j=i}^m du_j$. Then

$$\begin{aligned} \int_{D_T^m} \prod_{j=1}^m u_j^{-a_i} du &= \frac{T^{\sum_{i=1}^m (1 - a_i)}}{1 - a_1} \int_{D_1^{m-1}} \left(1 - \sum_{j=2}^m u_j \right)^{1 - a_1} \prod_{j=2}^m u_j^{-a_i} d\bar{u}_2 \\ &= \frac{T^{\sum_{i=1}^m (1 - a_i)}}{1 - a_1} \int_{D_1^{m-2}} \int_0^{1 - \sum_{j=3}^m u_j} u_2^{-a_2} \left(1 - \sum_{j=3}^m u_j - u_2 \right)^{1 - a_1} du_2 \prod_{j=3}^m u_j^{-a_i} d\bar{u}_3 \\ &\vdots \\ &= \frac{T^{\sum_{i=1}^m (1 - a_i)}}{1 - a_1} \prod_{j=2}^m B(1 - a_j, j - \sum_{i=1}^{j-1} a_i) \end{aligned}$$

$$= \frac{\prod_{j=1}^m \Gamma(1-a_j)}{\Gamma(\sum_{i=1}^m (1-a_i) + 1)} T^{\sum_{i=1}^m (1-a_i)}. \quad \square$$

Lemma A.2. For any $T > 0$ and $a_i \in (0, 1)$ with $i = 1, 2, \dots, m$,

$$\int_{D_{T,h}^m} \prod_{j=1}^m u_j^{-a_i} du \leq \frac{\prod_{j=1}^m \Gamma(1-a_j)}{\Gamma(m+1 - \sum_{i=1}^m a_i)} h^{\sum_{i=1}^m (1-a_i)},$$

where $D_{T,h}^m = \{T < u_1, T < u_1 + u_2 + \dots + u_m < T + h : u_i > 0, i = 2, \dots, m\}$.

Proof. For $i = 1, 2, \dots, m$, let $d\bar{u}_i = \prod_{j=i}^m du_j$. Then

$$\begin{aligned} \int_{D_{T,h}^m} \prod_{j=1}^m u_j^{-a_i} du &= \frac{1}{1-a_1} \int_{D_h^{m-1}} [(T+h - \sum_{j=2}^m u_j)^{1-a_1} - T^{1-a_1}] \prod_{j=2}^m u_j^{-a_i} d\bar{u}_2 \\ &\leq \frac{1}{1-a_1} \int_{D_h^{m-1}} (h - \sum_{j=2}^m u_j)^{1-a_1} \prod_{j=2}^m u_j^{-a_i} d\bar{u}_2 \\ &= \int_{D_h^m} \prod_{j=1}^m u_j^{-a_i} du \\ &= \frac{\prod_{j=1}^m \Gamma(1-a_j)}{\Gamma(\sum_{i=1}^m (1-a_i) + 1)} h^{\sum_{i=1}^m (1-a_i)}, \end{aligned}$$

where we used Lemma A.1 in the last equality. \square

Lemma A.3. Assume that $T > 0$ and $\alpha \geq 0$. Then for all $\varepsilon \in (0, T/2)$,

$$\int_0^T (u^{\frac{2H_1+H_2}{H_1+H_2}} + \varepsilon)^{-\frac{d}{2}-\alpha} du \asymp \begin{cases} \varepsilon^{\frac{H_1+H_2}{2H_1H_2}-\frac{d}{2}-\alpha} & \text{if } \frac{H_1+H_2}{H_1H_2} < d+2\alpha \\ \ln(1+\varepsilon^{-\frac{1}{2}}) & \text{if } \frac{H_1+H_2}{H_1H_2} = d+2\alpha \\ 1 & \text{if } \frac{H_1+H_2}{H_1H_2} > d+2\alpha \end{cases}$$

and

$$\int_{[0,T]^2} (u^{2H_1} + v^{2H_2} + \varepsilon)^{-\frac{d}{2}-\alpha} du dv \asymp \begin{cases} \varepsilon^{\frac{H_1+H_2}{2H_1H_2}-\frac{d}{2}-\alpha} & \text{if } \frac{H_1+H_2}{H_1H_2} < d+2\alpha \\ \ln(1+\varepsilon^{-\frac{1}{2}}) & \text{if } \frac{H_1+H_2}{H_1H_2} = d+2\alpha \\ 1 & \text{if } \frac{H_1+H_2}{H_1H_2} > d+2\alpha \end{cases},$$

where $f(\varepsilon) \asymp g(\varepsilon)$ means that the ratio $f(\varepsilon)/g(\varepsilon)$ is bounded from below and above by positive constants not depending on $\varepsilon \in (0, T/2)$.

Proof. The first result follows from Lemma 2.2 in [10]. To get the second one, we make the change of variables $u^{H_1} = r \cos \theta$ and $v^{H_2} = r \sin \theta$. It is easy to see that $\int_{[0,T]^2} (u^{2H_1} + v^{2H_2} + 2\varepsilon)^{-\frac{d}{2}-\alpha} du dv$ is less than a constant multiple of $\int_0^{2T^{H_1} \vee 2T^{H_2}} \frac{r^{\frac{H_1+H_2}{H_1H_2}-1}}{(r^2+\varepsilon)^{\frac{d}{2}+\alpha}} dr$ and greater than a constant multiple of $\int_0^{T^{H_1} \wedge T^{H_2}} \frac{r^{\frac{H_1+H_2}{H_1H_2}-1}}{(r^2+\varepsilon)^{\frac{d}{2}+\alpha}} dr$. The desired result now follows from Lemma 2.2 in [10]. \square

Lemma A.4. For any $m \in \mathbb{N}$, $x_j \in \mathbb{R}^d$ with $j = 1, \dots, m$ and $x_{m+1} = 0$,

$$\sum_{j=1}^m |x_j - x_{j+1}|^2 \geq \frac{2}{m(m+1)} \sum_{j=1}^m |x_j|^2.$$

Proof. Make the change of variables $y_j = x_j - x_{j+1}$ for $j = 1, \dots, m$. Then

$$\sum_{j=1}^m |x_j|^2 = \sum_{j=1}^m \left| \sum_{k=j}^m y_k \right|^2 \leq \sum_{j=1}^m (m-j+1) \sum_{k=j}^m |y_k|^2 \leq \frac{m(m+1)}{2} \sum_{k=1}^m |y_k|^2.$$

Substituting the original variables back gives the desired inequality. \square

References

- [1] S.M. Berman, Local nondeterminism and local times of Gaussian processes, *Indiana Univ. Math. J.* 23 (1973) 69–94.
- [2] D. Geman, J. Horowitz, Occupation densities, *Ann. Probab.* 8 (1980) 1–67.
- [3] D. Geman, J. Horowitz, J. Rosen, A local time analysis of intersections of Brownian paths in the plane, *Ann. Probab.* 12 (1984) 86–107.
- [4] J. Guo, Y. Hu, Y. Xiao, High-order derivative of intersection local time for two independent fractional Brownian motions, *J. Theoret. Probab.* 32 (3) (2019) 1190–1201.
- [5] A. Jaramillo, I. Nourdin, G. Peccati, Approximation of fractional local times zero energy and weak derivatives, *arXiv*: 1903.08683.
- [6] D. Nualart, S. Ortiz-Latorre, Intersection local time for two independent fractional Brownian motions, *J. Theoret. Probab.* 20 (4) (2007) 759–767.
- [7] D. Nualart, F. Xu, Asymptotic behavior for an additive functional of two independent self-similar Gaussian processes, *Stochastic Process. Appl.* 129 (10) (2019) 3981–4008.
- [8] L. Pitt, Local times for Gaussian vector fields, *Indiana Univ. Math. J.* 27 (2) (1978) 309–330.
- [9] J. Song, F. Xu, Q. Yu, Limit theorems for functionals of two independent Gaussian processes, *Stochastic Process. Appl.* 129 (11) (2019) 4791–4836.
- [10] D. Wu, Y. Xiao, Regularity of intersection local times of fractional Brownian motions, *J. Theoret. Probab.* 23 (4) (2010) 972–1001.
- [11] L. Yan, Derivative for the intersection local time of fractional Brownian motions, *arXiv*:1403.4102.