



On classical solutions to the Hartree equation

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ABSTRACT

This paper is concerned with positive classical solutions to the Hartree equation

$$-\Delta u = \left(\frac{1}{|x|^{n-\alpha}} * u^p \right) u^{p-1} \quad \text{in } \mathbb{R}^n.$$

When $n \leq 2$, we show that the equation has no positive solution. When $n \geq 3$, we prove that the equation has no positive solution if $p < \frac{n+\alpha}{n-2}$, we also classify all positive solutions to the equation in the critical case $p = \frac{n+\alpha}{n-2}$. The main novelty of this paper is that we cover the full range $0 < \alpha < n$ and $-\infty < p \leq \frac{n+\alpha}{n-2}$ in our results.

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1. Introduction and main results

We study the existence and nonexistence of positive solutions to the Hartree equation

$$-\Delta u = \left(\frac{1}{|x|^{n-\alpha}} * u^p \right) u^{p-1} \quad \text{in } \mathbb{R}^n, \quad (1)$$

where we always assume throughout this work that $0 < \alpha < n$ and $p \in \mathbb{R}$. Unless specified, positive solutions of (1) are always understood in the classical sense. That is, we call u a positive solution of (1) if $u \in C^2(\mathbb{R}^n)$, $u > 0$ and u verifies (1) pointwise.

Equation (1) is also known as the stationary Choquard or Schrödinger-Newton equation which arises naturally in a variety of applications, for instance, the physics of multiple particle systems, quantum mechanics, Hartree-Fock theory, physics of laser beams and so on, which we refer to [11,14,16].

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The existence of positive solutions for such an equation is a topic of intensive research recently, see [3,5,8,9,12,13,18] and the references therein. In [8], Lei studied positive solutions of (1) in the case $n \geq 3$ and $\alpha = 2$. He proved that the equation has no positive solution if $1 \leq p < \frac{n+2}{n-2}$. He also showed that, if $p = \frac{n+2}{n-2}$, then every positive solution u of (1) must assume the form $u(x) = c \left(\frac{\mu}{\mu^2 + |x - x_0|^2} \right)^{\frac{n-2}{2}}$ for some $c, \mu > 0$ and $x_0 \in \mathbb{R}^n$. Equation (1) with more general values of α was studied by Du-Yang in [3] and Guo-Hu-Peng-Shuai in [5], where they classified positive $D^{1,2}(\mathbb{R}^n)$ solutions of (1) when $p = \frac{n+\alpha}{n-2}$ under the additional assumption $\alpha \geq n - 4$. They required this technical restriction since their method only works for the case $p \geq 2$, which is equivalent to $\alpha \geq n - 4$ when $p = \frac{n+\alpha}{n-2}$. A Liouville type theorem was also established recently by Yang-Xu [18] for positive $H_{\text{loc}}^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ solutions of (1) when $1 \leq p < \frac{n+\alpha}{n-2}$. However, the assumption $\alpha \geq n - 4$ is still needed in [18]. On the other hand, there is no existence or nonexistence result in literature for the case $p < 0$ to the best of our knowledge.

Motivated by the above works, in this paper, we introduce some new ideas that allows us to investigate existence and uniqueness of positive solutions of (1) in the full range $0 < \alpha < n$ and $-\infty < p \leq \frac{n+\alpha}{n-2}$. Our main results are the following theorems.

Theorem 1. *If $n \leq 2$, then (1) has no positive solution for all $p \in \mathbb{R}$.*

Theorem 2. *Let $n \geq 3$. If $p < \frac{n+\alpha}{n-2}$, then (1) has no positive solution. If $p = \frac{n+\alpha}{n-2}$, then every positive solution u of (1) must assume the form*

$$u(x) = \mu^{\frac{n-\alpha}{2}} Q(\mu(x - x^0)) \quad \text{for some } \mu > 0 \text{ and } x^0 \in \mathbb{R}^n,$$

where

$$Q(x) = \left[R_n I \left(\frac{n-2}{2} \right) I \left(\frac{n-\alpha}{2} \right) \right]^{-\frac{n-2}{2(\alpha+2)}} \left(\frac{1}{1+|x|^2} \right)^{\frac{n-2}{2}}$$

with $R_n = \frac{\Gamma(\frac{n-2}{2})}{4\pi^{\frac{n}{2}}}$ and $I(s) = \frac{\pi^{\frac{n}{2}} \Gamma(\frac{n-2s}{2})}{\Gamma(n-s)}$.

In Section 2, we prove Proposition 3, which implies Theorem 1 and Theorem 2 in the case $p < 1$. The case $p \geq 1$ in Theorem 2 will be proved in Section 3 via the method of moving planes in integral forms introduced by Chen-Li-Ou [1]. Throughout this paper, we use C to denote various positive constants. At times, we append subscripts to C to specify its dependence on the subscript parameters. We also denote by $|\Omega|$ the Lebesgue measure of a measurable set Ω and by B_R the ball of center 0 and radius $R > 0$.

2. The singular cases

Proposition 3. *Equation (1) has no positive solution if one of the following conditions holds*

- (i) $n = 1, 2$,
- (ii) $n \geq 3$ and $p < 1$.

In order to prove Proposition 3, we utilize the following optimal nonexistence result for nonnegative nontrivial supersolutions of (1) in exterior domains with $p > 0$.

Theorem 4 (see Theorem 1 in [15]). *Let $p, \rho > 0$ and consider the problem*

$$-\Delta u(x) \geq \int_{\mathbb{R}^n \setminus \overline{B}_\rho} \frac{u^p(y)}{|x-y|^{n-\alpha}} dy \cdot u^{p-1}(x) \quad \text{for } x \in \mathbb{R}^n \setminus \overline{B}_\rho. \quad (2)$$

In dimension $n = 1, 2$, problem (2) has no nonnegative nontrivial solution. In dimension $n \geq 3$, problem (2) has a nonnegative nontrivial solution if and only if the following assumptions hold simultaneously:

$$\begin{aligned} p &> \frac{n + \alpha - 2}{n - 2} && \text{if } \alpha > n - 2, \\ p &\geq 2 && \text{if } \alpha = n - 2, \\ p &\geq \frac{n}{n - \alpha/2 - 1} && \text{if } \alpha < n - 2, \\ p &> \frac{n + \alpha/2 - 1}{n - 2}. \end{aligned}$$

Proof of Proposition 3. It suffices to consider three cases.

Case 1: $n = 1, 2$ and $p > 0$. By Theorem 4, equation (1) has no positive solution in this case.

Case 2: $n \geq 3$ and $0 < p < 1$. Clearly, all assumptions in Theorem 4 imply $p > 1$. Therefore, as a consequence of Theorem 4, equation (1) has no positive solution when $0 < p < 1$.

Case 3: $p \leq 0$. By contradiction, suppose that (1) has a positive solution u . For each $r > 0$, we define

$$\bar{u}(r) = \frac{1}{|\partial B_r|} \int_{\partial B_r} u d\sigma$$

the spherical average of u . Then we have

$$-|\partial B_r| \bar{u}'(r) = - \int_{\partial B_r} \frac{\partial u}{\partial \nu} d\sigma = - \int_{B_r} \Delta u dx = \int_{B_r} \left(\frac{1}{|x|^{n-\alpha}} * u^p \right) u^{p-1} dx > 0.$$

Therefore, $\bar{u}$ is decreasing in $(0, \infty)$. This fact and Jensen's inequality imply

$$\left(\frac{1}{|x|^{n-\alpha}} * u^p \right) (0) = \int_{\mathbb{R}^n} \frac{u^p(y)}{|y|^{n-\alpha}} dy = \int_0^{+\infty} \int_{\partial B_r} \frac{u^p(y)}{r^{n-\alpha}} d\sigma dr \geq \int_{\mathbb{R}^n} \frac{\bar{u}^p(|y|)}{|y|^{n-\alpha}} dy \geq u^p(0) \int_{\mathbb{R}^n} \frac{1}{|y|^{n-\alpha}} dy = \infty,$$

which is absurd. $\square$

3. The subcritical and critical cases

In this section, we assume $n \geq 3$ and $1 \leq p \leq \frac{n+\alpha}{n-2}$.

Lemma 5. Let $u \in C^2(\mathbb{R}^n)$ be a positive function. If u is a solution of (1), then u satisfies the integral equation

$$u(x) = R_n \int_{\mathbb{R}^n} \frac{u^{p-1}(y)}{|x-y|^{n-2}} \left(\frac{1}{|x|^{n-\alpha}} * u^p \right) (y) dy, \quad x \in \mathbb{R}^n, \quad (3)$$

and vice versa, where $R_n = \frac{\Gamma(\frac{n-2}{2})}{4\pi^{\frac{n}{2}}}$.

Proof. We recall that $G(x, y) = \frac{R_n}{|x-y|^{n-2}}$ is the Green's function for $-\Delta$ on $\mathbb{R}^n$. If $u \in C^2(\mathbb{R}^n)$ is a positive solution of (3), then

$$\begin{aligned}
-\Delta u(x) &= \int_{\mathbb{R}^n} -\Delta \left(\frac{R_n}{|x-y|^{n-2}} \right) u^{p-1}(y) \left(\frac{1}{|x|^{n-\alpha}} * u^p \right) (y) dy \\
&= \int_{\mathbb{R}^n} \delta_x(y) u^{p-1}(y) \left(\frac{1}{|x|^{n-\alpha}} * u^p \right) (y) dy = \left(\frac{1}{|x|^{n-\alpha}} * u^p \right) (x) u^{p-1}(x).
\end{aligned}$$

That is, u satisfies (1).

Conversely, assume that u is a positive solution of (1). For arbitrary $R > 0$ and $x \in \overline{B_R}$, let

$$v_R(x) = \int_{B_R} G_R(x, y) \left(\frac{1}{|x|^{n-\alpha}} * u^p \right) (y) u^{p-1}(y) dy,$$

where G_R is the Green's function for $-\Delta$ on B_R , which is given by

$$G_R(x, y) = R_n \left(\frac{1}{|x-y|^{n-2}} - \frac{1}{\left| \frac{R}{|x|}x - \frac{|x|}{R}y \right|^{n-2}} \right)$$

(see [4, Chapter 2]). Then we have

$$\begin{cases} -\Delta v_R = \left(\frac{1}{|x|^{n-\alpha}} * u^p \right) u^{p-1} & \text{in } B_R, \\ v_R = 0 & \text{on } \partial B_R. \end{cases}$$

Hence

$$\begin{cases} -\Delta(u - v_R) = 0 & \text{in } B_R, \\ u - v_R \geq 0 & \text{on } \partial B_R. \end{cases}$$

This implies $u - v_R \geq 0$ in B_R for all $R > 0$ thanks to the maximum principle. For arbitrary x fixed, by letting $R \rightarrow \infty$, we deduce

$$u(x) \geq v(x) := \int_{\mathbb{R}^n} \frac{R_n}{|x-y|^{n-2}} \left(\frac{1}{|x|^{n-\alpha}} * u^p \right) (y) u^{p-1}(y) dy.$$

On the other hand,

$$-\Delta v(x) = \left(\frac{1}{|x|^{n-\alpha}} * u^p \right) (x) u^{p-1}(x), \quad x \in \mathbb{R}^n.$$

Therefore,

$$\begin{cases} -\Delta(u - v) = 0 & \text{in } \mathbb{R}^n, \\ u - v \geq 0 & \text{in } \mathbb{R}^n. \end{cases}$$

From the Liouville theorem for entire harmonic functions, we must have $u - v \equiv C_0 \geq 0$. Hence

$$u(x) = \int_{\mathbb{R}^n} \frac{R_n}{|x-y|^{n-2}} \left(\frac{1}{|x|^{n-\alpha}} * u^p \right) (y) u^{p-1}(y) dy + C_0 \geq C_0.$$

Consequently,

$$\begin{aligned} \infty > u(0) &\geq v(0) = \int_{\mathbb{R}^n} \frac{R_n}{|y|^{n-2}} \left(\int_{\mathbb{R}^n} \frac{u^p(z)}{|y-z|^{n-\alpha}} dz \right) u^{p-1}(y) dy \\ &\geq C_0^{2p-1} \int_{\mathbb{R}^n} \frac{R_n}{|y|^{n-2}} \int_{\mathbb{R}^n} \frac{1}{|y-z|^{n-\alpha}} dz dy. \end{aligned}$$

Since the last integral is infinite, we deduce that $C_0 = 0$. Therefore, u satisfies (3). $\square$

Let u be a positive solution of (1). By setting $v = \frac{1}{|x|^{n-\alpha}} * u^p$ and using Lemma 5, we may transform (1) to the equivalent integral system

$$\begin{cases} u(x) = R_n \int_{\mathbb{R}^n} \frac{u^{p-1}(y)v(y)}{|x-y|^{n-2}} dy, & x \in \mathbb{R}^n, \\ v(x) = \int_{\mathbb{R}^n} \frac{u^p(y)}{|x-y|^{n-\alpha}} dy, & x \in \mathbb{R}^n. \end{cases} \quad (4)$$

Let $\bar{u}$ and $\bar{v}$ be the Kelvin transform of u and v centered at the origin, respectively. That is,

$$\bar{u}(x) = \frac{1}{|x|^{n-2}} u\left(\frac{x}{|x|^2}\right) \quad \text{and} \quad \bar{v}(x) = \frac{1}{|x|^{n-\alpha}} v\left(\frac{x}{|x|^2}\right) \quad \text{for } x \in \mathbb{R}^n \setminus \{0\}.$$

Then

$$\bar{u}(x) \sim \frac{1}{|x|^{n-2}} \quad \text{and} \quad \bar{v}(x) \sim \frac{1}{|x|^{n-\alpha}} \quad \text{as } |x| \rightarrow \infty \quad (5)$$

and $(\bar{u}, \bar{v})$ satisfies the integral system

$$\begin{cases} \bar{u}(x) = R_n \int_{\mathbb{R}^n} \frac{\bar{u}^{p-1}(y)\bar{v}(y)}{|x-y|^{n-2}|y|^\gamma} dy, & x \in \mathbb{R}^n, \\ \bar{v}(x) = \int_{\mathbb{R}^n} \frac{\bar{u}^p(y)}{|x-y|^{n-\alpha}|y|^\gamma} dy, & x \in \mathbb{R}^n, \end{cases} \quad (6)$$

where

$$\gamma = n + \alpha - p(n-2) \geq 0.$$

For any $\lambda \in \mathbb{R}$, let $T_\lambda = \{x \in \mathbb{R}^n \mid x_1 = \lambda\}$ be the moving plane, $\Sigma_\lambda = \{x \in \mathbb{R}^n \mid x_1 < \lambda\}$ be the region to the left of the plane and $x^\lambda = (2\lambda - x_1, x_2, \dots, x_n)$ be the reflection of the point $x = (x_1, x_2, \dots, x_n)$ about the plane T_λ . We also define

$$\bar{u}_\lambda(x) := \bar{u}(x^\lambda), \quad U_\lambda(x) := \bar{u}_\lambda(x) - \bar{u}(x), \quad \bar{v}_\lambda(x) := \bar{v}(x^\lambda), \quad V_\lambda(x) := \bar{v}_\lambda(x) - \bar{v}(x).$$

Moreover, for $\lambda < 0$, we denote

$$\Sigma_\lambda^* = \Sigma_\lambda \setminus \{0^\lambda\}, \quad \Sigma_\lambda^u = \{x \in \Sigma_\lambda^* \mid U_\lambda(x) < 0\}, \quad \Sigma_\lambda^v = \{x \in \Sigma_\lambda^* \mid V_\lambda(x) < 0\}.$$

From (6), we deduce that, for any $x \in \Sigma_\lambda^*$ with $\lambda < 0$,

$$\bar{u}(x) = R_n \int_{\Sigma_\lambda} \frac{\bar{u}^{p-1}(y)\bar{v}(y)}{|x-y|^{n-2}|y|^\gamma} dy + R_n \int_{\Sigma_\lambda} \frac{\bar{u}_\lambda^{p-1}(y)\bar{v}_\lambda(y)}{|x-y^\lambda|^{n-2}|y^\lambda|^\gamma} dy \quad (7)$$

and

$$\bar{u}_\lambda(x) = R_n \int_{\Sigma_\lambda} \frac{\bar{u}^{p-1}(y)\bar{v}(y)}{|x^\lambda-y|^{n-2}|y|^\gamma} dy + R_n \int_{\Sigma_\lambda} \frac{\bar{u}_\lambda^{p-1}(y)\bar{v}_\lambda(y)}{|x^\lambda-y^\lambda|^{n-2}|y^\lambda|^\gamma} dy. \quad (8)$$

Combining (7), (8) and using the fact that $|x-y^\lambda| = |x^\lambda-y|$ and $|x-y| = |x^\lambda-y^\lambda|$, we have

$$\begin{aligned} U_\lambda(x) &= R_n \int_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-2}} - \frac{1}{|x^\lambda-y|^{n-2}} \right) \left(\frac{\bar{u}_\lambda^{p-1}(y)\bar{v}_\lambda(y)}{|y^\lambda|^\gamma} - \frac{\bar{u}^{p-1}(y)\bar{v}(y)}{|y|^\gamma} \right) dy \\ &\geq R_n \int_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-2}} - \frac{1}{|x^\lambda-y|^{n-2}} \right) \frac{\bar{u}_\lambda^{p-1}(y)\bar{v}_\lambda(y) - \bar{u}^{p-1}(y)\bar{v}(y)}{|y|^\gamma} dy. \end{aligned} \quad (9)$$

By the mean value theorem, one may show that, for any $0 < a \leq b$ and $q \geq 0$,

$$a^q - b^q \geq \max\{q, 1\}b^{q-1}(a-b). \quad (10)$$

With the aid of (10), for each $y \in \Sigma_\lambda^*$, we have four possible cases:

- If $\bar{u}_\lambda(y) \geq \bar{u}(y)$ and $\bar{v}_\lambda(y) \geq \bar{v}(y)$, then $\bar{u}_\lambda^{p-1}(y)\bar{v}_\lambda(y) - \bar{u}^{p-1}(y)\bar{v}(y) \geq 0$.
- If $\bar{u}_\lambda(y) \geq \bar{u}(y)$ and $\bar{v}_\lambda(y) < \bar{v}(y)$, then $\bar{u}_\lambda^{p-1}(y)\bar{v}_\lambda(y) - \bar{u}^{p-1}(y)\bar{v}(y) \geq \bar{u}^{p-1}(y)V_\lambda(y)$.
- If $\bar{u}_\lambda(y) < \bar{u}(y)$ and $\bar{v}_\lambda(y) \geq \bar{v}(y)$, then

$$\bar{u}_\lambda^{p-1}(y)\bar{v}_\lambda(y) - \bar{u}^{p-1}(y)\bar{v}(y) \geq \left(\bar{u}_\lambda^{p-1}(y) - \bar{u}^{p-1}(y) \right) \bar{v}(y) \geq \max\{p-1, 1\} \bar{u}^{p-2}(y)\bar{v}(y)U_\lambda(y).$$

- If $\bar{u}_\lambda(y) < \bar{u}(y)$ and $\bar{v}_\lambda(y) < \bar{v}(y)$, then

$$\begin{aligned} \bar{u}_\lambda^{p-1}(y)\bar{v}_\lambda(y) - \bar{u}^{p-1}(y)\bar{v}(y) &= \left(\bar{u}_\lambda^{p-1}(y) - \bar{u}^{p-1}(y) \right) \bar{v}_\lambda(y) + \bar{u}^{p-1}(y) (\bar{v}_\lambda(y) - \bar{v}(y)) \\ &\geq \max\{p-1, 1\} \bar{u}^{p-2}(y)\bar{v}(y)U_\lambda(y) + \bar{u}^{p-1}(y)V_\lambda(y). \end{aligned}$$

Therefore, in all cases, we have

$$\bar{u}_\lambda^{p-1}(y)\bar{v}_\lambda(y) - \bar{u}^{p-1}(y)\bar{v}(y) \geq \max\{p-1, 1\} \bar{u}^{p-2}(y)\bar{v}(y)U_\lambda^-(y) + \bar{u}^{p-1}(y)V_\lambda^-(y), \quad (11)$$

where, as usual, we denote $a^- = \min\{0, a\}$ for any $a \in \mathbb{R}$.

Substituting (11) into (9), we obtain

$$U_\lambda(x) \geq C \int_{\Sigma_\lambda^u} \frac{\bar{u}^{p-2}(y)\bar{v}(y)U_\lambda(y)}{|x-y|^{n-2}|y|^\gamma} dy + C \int_{\Sigma_\lambda^v} \frac{\bar{u}^{p-1}(y)V_\lambda(y)}{|x-y|^{n-2}|y|^\gamma} dy. \quad (12)$$

By a similar reasoning, we have

$$\begin{aligned} V_\lambda(x) &= \int_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda-y|^{n-\alpha}} \right) \left(\frac{\bar{u}_\lambda^p(y)}{|y^\lambda|^\gamma} - \frac{\bar{u}^p(y)}{|y|^\gamma} \right) dy \\ &\geq \int_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda-y|^{n-\alpha}} \right) \frac{\bar{u}_\lambda^p(y) - \bar{u}^p(y)}{|y|^\gamma} dy, \end{aligned} \quad (13)$$

which yields

$$V_\lambda(x) \geq p \int_{\Sigma_\lambda^u} \frac{\bar{u}^{p-1}(y) U_\lambda(y)}{|x-y|^{n-\alpha} |y|^\gamma} dy. \quad (14)$$

Applying Hölder's and the Hardy-Littlewood-Sobolev inequality (see [17,10]), from (12), we have

$$\begin{aligned} \|U_\lambda\|_{L^{\frac{2n}{n-2}}(\Sigma_\lambda^u)} &\leq C \left\| \frac{\bar{u}^{p-2} \bar{v} U_\lambda}{|x|^\gamma} \right\|_{L^{\frac{2n}{n+2}}(\Sigma_\lambda^u)} + C \left\| \frac{\bar{u}^{p-1} V_\lambda}{|x|^\gamma} \right\|_{L^{\frac{2n}{n+2}}(\Sigma_\lambda^v)} \\ &\leq C \left\| \frac{\bar{u}^{p-2} \bar{v}}{|x|^\gamma} \right\|_{L^{\frac{n}{2}}(\Sigma_\lambda^u)} \|U_\lambda\|_{L^{\frac{2n}{n-2}}(\Sigma_\lambda^u)} + C \left\| \frac{\bar{u}^{p-1}}{|x|^\gamma} \right\|_{L^{\frac{2n}{n+2}}(\Sigma_\lambda^v)} \|V_\lambda\|_{L^{\frac{2n}{n-2}}(\Sigma_\lambda^v)}. \end{aligned} \quad (15)$$

Similarly, from (14), we have

$$\|V_\lambda\|_{L^{\frac{2n}{n-\alpha}}(\Sigma_\lambda^v)} \leq C \left\| \frac{\bar{u}^{p-1} U_\lambda}{|x|^\gamma} \right\|_{L^{\frac{2n}{n+\alpha}}(\Sigma_\lambda^u)} \leq C \left\| \frac{\bar{u}^{p-1}}{|x|^\gamma} \right\|_{L^{\frac{2n}{n+\alpha}}(\Sigma_\lambda^u)} \|U_\lambda\|_{L^{\frac{2n}{n-2}}(\Sigma_\lambda^u)}. \quad (16)$$

Combining (15) and (16), we derive

$$\|U_\lambda\|_{L^{\frac{2n}{n-2}}(\Sigma_\lambda^u)} \leq C \left(\left\| \frac{\bar{u}^{p-2} \bar{v}}{|x|^\gamma} \right\|_{L^{\frac{n}{2}}(\Sigma_\lambda^u)} + \left\| \frac{\bar{u}^{p-1}}{|x|^\gamma} \right\|_{L^{\frac{2n}{n+2}}(\Sigma_\lambda^u)} \left\| \frac{\bar{u}^{p-1}}{|x|^\gamma} \right\|_{L^{\frac{2n}{n+2}}(\Sigma_\lambda^v)} \right) \|U_\lambda\|_{L^{\frac{2n}{n-2}}(\Sigma_\lambda^u)}, \quad (17)$$

where the constant C is independent of λ . Note that, with the aid of (5), one may check that

$$\frac{\bar{u}^{p-2} \bar{v}}{|x|^\gamma} \in L^{\frac{n}{2}}(\mathbb{R}^n \setminus B_\varepsilon) \quad \text{and} \quad \frac{\bar{u}^{p-1}}{|x|^\gamma} \in L^{\frac{2n}{n+2}}(\mathbb{R}^n \setminus B_\varepsilon) \quad \text{for all } \varepsilon > 0. \quad (18)$$

Proof of Theorem 2. Due to Proposition 3, it suffices to consider two cases.

Case 1: The subcritical case $1 \leq p < \frac{n+\alpha}{n-2}$. In this case, $\gamma > 0$.

We start moving the plane T_λ from near $\lambda = -\infty$ to the right along the x_1 direction until it reaches the limiting position in order to derive symmetry. This procedure contains two steps.

Step 1. We show that, for λ sufficiently negative,

$$U_\lambda \geq 0 \quad \text{in } \Sigma_\lambda^*. \quad (19)$$

Indeed, from (18), we can choose $R_0 > 0$ sufficiently large, such that for $\lambda \leq -R_0$, we have

$$\left\| \frac{\bar{u}^{p-2} \bar{v}}{|x|^\gamma} \right\|_{L^{\frac{n}{2}}(\Sigma_\lambda^u)} + \left\| \frac{\bar{u}^{p-1}}{|x|^\gamma} \right\|_{L^{\frac{2n}{n+2}}(\Sigma_\lambda^u)} \left\| \frac{\bar{u}^{p-1}}{|x|^\gamma} \right\|_{L^{\frac{2n}{n+2}}(\Sigma_\lambda^v)} \leq \frac{1}{2C}, \quad (20)$$

where the constant C is the same as in (17). Therefore, (17) and (20) imply $\|U_\lambda\|_{L^{\frac{2n}{n-2}}(\Sigma_\lambda^u)} = 0$ and hence $|\Sigma_\lambda^u| = 0$ for $\lambda \leq -R_0$. Thus, (19) holds for $\lambda \leq -R_0$. This completes Step 1.

Step 2. Let

$$\lambda_0 = \sup\{\lambda \leq 0 \mid U_\mu \geq 0 \text{ in } \Sigma_\mu^* \text{ for all } \mu \leq \lambda\}. \quad (21)$$

In this step, we show that

$$\lambda_0 = 0. \quad (22)$$

Suppose on contrary that $\lambda_0 < 0$. We already have $U_{\lambda_0} \geq 0$ in $\Sigma_{\lambda_0}^*$. Hence, it follows from (13) that

$$V_{\lambda_0}(x) \geq \int_{\Sigma_{\lambda_0}} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^{\lambda_0}-y|^{n-\alpha}} \right) \left(\frac{1}{|y^{\lambda_0}|^\gamma} - \frac{1}{|y|^\gamma} \right) \bar{u}^p(y) dy > 0. \quad (23)$$

Then (9) implies

$$U_{\lambda_0}(x) \geq R_n \int_{\Sigma_{\lambda_0}} \left(\frac{1}{|x-y|^{n-2}} - \frac{1}{|x^{\lambda_0}-y|^{n-2}} \right) \left(\frac{1}{|y^{\lambda_0}|^\gamma} - \frac{1}{|y|^\gamma} \right) \bar{u}^{p-1}(y) \bar{v}(y) dy > 0. \quad (24)$$

That is, $U_{\lambda_0} > 0$ in $\Sigma_{\lambda_0}^*$. We will obtain a contradiction with (21) by showing there exists $\varepsilon > 0$ small enough such that $U_\lambda \geq 0$ in Σ_λ^* for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon]$.

It can be clearly seen from (17) that, our primary task is to prove that, one can choose $\varepsilon > 0$ sufficiently small such that

$$\left\| \frac{\bar{u}^{p-2}\bar{v}}{|x|^\gamma} \right\|_{L^{\frac{n}{2}}(\Sigma_\lambda^u)} + \left\| \frac{\bar{u}^{p-1}}{|x|^\gamma} \right\|_{L^{\frac{2n}{\alpha+2}}(\Sigma_\lambda^u)} \left\| \frac{\bar{u}^{p-1}}{|x|^\gamma} \right\|_{L^{\frac{2n}{\alpha+2}}(\Sigma_\lambda^v)} \leq \frac{1}{2C} \quad (25)$$

for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon]$, where the constant C is the same as in (17).

From (18), there exists $R > 0$ large enough such that

$$\left\| \frac{\bar{u}^{p-2}\bar{v}}{|x|^\gamma} \right\|_{L^{\frac{n}{2}}(\Sigma_\lambda^u \setminus B_R)} + \left\| \frac{\bar{u}^{p-1}}{|x|^\gamma} \right\|_{L^{\frac{2n}{\alpha+2}}(\Sigma_\lambda^u \setminus B_R)} \left\| \frac{\bar{u}^{p-1}}{|x|^\gamma} \right\|_{L^{\frac{2n}{\alpha+2}}(\Sigma_\lambda^v)} \leq \frac{1}{4C} \quad (26)$$

for all $\lambda \in [\lambda_0, \lambda_0/2]$.

Since $U_{\lambda_0} > 0$ in $\Sigma_{\lambda_0}^*$, which means $\Sigma_{\lambda_0}^u = \emptyset$, we have $\lim_{\lambda \rightarrow \lambda_0^+} |\Sigma_\lambda^u \cap B_R| = 0$. Therefore,

$$\left\| \frac{\bar{u}^{p-2}\bar{v}}{|x|^\gamma} \right\|_{L^{\frac{n}{2}}(\Sigma_\lambda^u \cap B_R)} + \left\| \frac{\bar{u}^{p-1}}{|x|^\gamma} \right\|_{L^{\frac{2n}{\alpha+2}}(\Sigma_\lambda^u \cap B_R)} \left\| \frac{\bar{u}^{p-1}}{|x|^\gamma} \right\|_{L^{\frac{2n}{\alpha+2}}(\Sigma_\lambda^v)} \leq \frac{1}{4C} \quad (27)$$

for λ sufficiently close to λ_0 .

Combining (26) and (27), we arrive at (25).

We deduce from (17) and (25) that, there exists an $\varepsilon \in (0, |\lambda_0|/2)$ sufficiently small such that $|\Sigma_\lambda^u| = 0$ for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon]$. Hence $U_\lambda \geq 0$ in Σ_λ^* for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon]$. This contradicts with the definition of λ_0 in (21). Therefore, (22) must hold and hence $U_0 \geq 0$ in Σ_0 . This completes Step 2.

Similar to the previous steps, one can move the plane T_λ from $+\infty$ to the left to get that $U_0 \leq 0$ in Σ_0 . Hence, $U_0 \equiv 0$ and $\bar{u}$ is symmetric about T_0 . We can repeat the previous reasoning to any direction in order to deduce that $\bar{u}$ is radially symmetric about 0. So is u .

Since equation (1) is invariant under translations, we deduce that u is radially symmetric about any point in $\mathbb{R}^n$. This only occurs if u is constant. However, it is absurd since a positive constant function does not satisfy (1).

Case 2: The critical case $p = \frac{n+\alpha}{n-2}$. In this case, $\gamma = 0$.

Since a positive constant function does not satisfy (1), there exists a hyperplane H such that u is not symmetric about H . Without loss of generality, we may assume $H = T_0$.

Step 1 is entirely the same as that in the subcritical case, that is, we can show that for λ sufficiently negative,

$$U_\lambda \geq 0 \quad \text{in } \Sigma_\lambda^*. \quad (28)$$

Let

$$\lambda_0 = \sup\{\lambda \leq 0 \mid U_\mu \geq 0 \text{ in } \Sigma_\mu^* \text{ for all } \mu \leq \lambda\}.$$

Unlike the subcritical case, we are not able to rule out the possibility $\lambda_0 < 0$ since the integrals in (23) and (24) are zero when $\gamma = 0$. Indeed, we have to consider two possibilities.

Possibility (i): $\lambda_0 < 0$. We show that $U_{\lambda_0} = 0$ in $\Sigma_{\lambda_0}^*$. Otherwise, assume $U_{\lambda_0} \geq 0$, but $U_{\lambda_0} \not\equiv 0$ in $\Sigma_{\lambda_0}^*$. It follows from (13) that

$$V_{\lambda_0}(x) = \int_{\Sigma_{\lambda_0}} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^{\lambda_0}-y|^{n-\alpha}} \right) (\bar{u}_{\lambda_0}^p(y) - \bar{u}^p(y)) dy > 0$$

and hence (9) implies

$$U_{\lambda_0}(x) = R_n \int_{\Sigma_{\lambda_0}} \left(\frac{1}{|x-y|^{n-2}} - \frac{1}{|x^{\lambda_0}-y|^{n-2}} \right) (\bar{u}_{\lambda_0}^{p-1}(y)\bar{v}_{\lambda_0}(y) - \bar{u}^{p-1}(y)\bar{v}(y)) dy > 0.$$

Similar to the subcritical case, one can show that the plane T_λ can move a little bit to the right such that (28) still holds. This contradicts the definition of λ_0 . Hence, we must have $U_{\lambda_0} = 0$ in $\Sigma_{\lambda_0}^*$.

Possibility (ii): $\lambda_0 = 0$. In this case, we move the plane T_λ from near $+\infty$ to the left and let

$$\lambda_1 = \inf\{\lambda \geq 0 \mid U_\mu \leq 0 \text{ in } \Sigma_\mu \setminus \{0\} \text{ for all } \mu \geq \lambda\}.$$

If $\lambda_1 = 0$, then we derive that $\bar{u}$ is symmetric about T_0 . Hence, u is also symmetric about T_0 . This contradicts the previous assumption. Therefore, $\lambda_1 > 0$ and we may argue as in Possibility (i) to conclude that $\bar{u}$ is symmetric about T_{λ_1} .

In any case, $\bar{u}$ is symmetric about T_{μ_0} for some $\mu_0 \neq 0$. Hence,

$$\lim_{x \rightarrow 0} \bar{u}(x) = \bar{u}(0^{\mu_0}) \quad \text{and} \quad \lim_{x \rightarrow 0} \bar{v}(x) = \bar{v}(0^{\mu_0}),$$

which yield

$$u(x) \sim \frac{1}{|x|^{n-2}} \quad \text{and} \quad v(x) \sim \frac{1}{|x|^{n-\alpha}} \quad \text{as } |x| \rightarrow \infty.$$

This enables us to apply the method of moving plane to system (4) directly and show that u is symmetric and monotone decreasing about T_{μ_1} for some $\mu_1 \in \mathbb{R}$.

Since (1) is invariant under rotations and translations, the above arguments show that for any direction l , the solution u is symmetric about some hyperplane H_l which is orthogonal to l . This implies that u is radially symmetric about some point. Hence, u must assume the form $u(x) = \mu^{\frac{n-2}{2}} Q(\mu(x-x^0))$ due to a recent result by Guo-Hu-Peng-Shuai [5] (see also [2,6,7] for the detail computation of coefficient of Q). $\square$

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