J. Math. Anal. Appl. 485 (2020) 123813

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Localization of real algebraic hypersurfaces with applications to the enumeration of the classes of relative equilibria of a $(5+1)$-body problem

Jean F. Barros [a], Eduardo S.G. Leandro [b,*]

[a] *Universidade Estadual de Feira de Santana, Depto de Ciências Exatas, Av. Transnor-destina, S/N, Feira de Santana, BA, 44036-900, Brasil*
[b] *Universidade Federal de Pernambuco, Depto de Matemática, Av. Jornalista Aníbal Fernandes S/N, Recife, PE, 50740-560, Brasil*

ARTICLE INFO

Article history:
Received 2 May 2019
Available online 2 January 2020
Submitted by R. Popovych

Keywords:
Algebraic hypersurfaces
N-body problem
Relative equilibria

ABSTRACT

We extend classical results on the localization of zeros of real univariate polynomials to the localization of zero sets of real multivariate polynomials P, more precisely, of real algebraic hypersurfaces (assuming 0 is a regular value). Through suitable changes of variables, we may verify whether such a hypersurface $P^{-1}(0)$ in $\mathbb{R}^n$ intersects or not a given n-dimensional box $\mathfrak{B}^n = \Pi_{l=1}^n [a_l, b_l]$, and in the affirmative case, to locate with arbitrary precision the set $P^{-1}(0) \cap \mathfrak{B}^n$. Properties of the hypersurface such as being an analytic graph may also be deduced from our results, which include a non-differentiable, non-local version of the implicit function theorem for polynomials. Next, we apply the ideas of the first part to study the bifurcations of a one-parameter family of symmetric classes of relative equilibria of the $(5+1)$-body problem. The exact numbers of classes of relative equilibria are provided, and our technique allows for the localization of all relative equilibria.

1. Introduction

Given an equation $P(x_1, \ldots, x_n) = 0$, where P is a real polynomial, a natural and often important issue is to locate its solution set $P^{-1}(0)$ in a region of $\mathbb{R}^n$. Throughout this work, we suppose that the region of interest can be confined in a box $\mathfrak{B}^n = \Pi_{l=1}^n [a_l, b_l]$, and that zero is a regular value of the restriction of P to $\mathfrak{B}^n$. A *localization* of the zero set $P^{-1}(0)$ is obtained by covering $\mathfrak{B}^n$ with smaller n-dimensional boxes $\mathfrak{B}_k^n$ such that the intersection of $P^{-1}(0)$ with each $\mathfrak{B}_k^n$ is either the empty set or the (analytic) graph of a real function of $n-1$ variables. Notice that singular points or self-intersections arising from multiple irreducible components do not occur in $P^{-1}(0) \cap \mathfrak{B}^n$.

* Corresponding author.
E-mail addresses: jfb@uefs.br (J.F. Barros), eduardo@dmat.ufpe.br (E.S.G. Leandro).

https://doi.org/10.1016/j.jmaa.2019.123813

If $n = 1$, the hypothesis of zero being a regular value of P corresponds to P having only simple roots, and localization may be viewed as the concept of root isolation. Consider Vincent's theorem, a result from the 1830s which essentially states that the number of variations of signs in a real univariate polynomial without multiple roots can be reduced to either zero or one through a succession of simple Möbius transformations [15,16]. Vincent's theorem can be seen as the culmination of the work by Lagrange [10] on the approximation of real zeros of polynomial equations using continued fractions, and of methods for root isolation by Budan and Fourier [15]. Vincent's theorem is the source of an effective root separation algorithm which surpasses algorithms based on the celebrated Sturm's method [5]. Our previous research [8,7,11] suggests that an extension of Vincent's theorem must exist for polynomials with real coefficients depending on multiple variables. Indeed, we have been able to successfully apply Möbius transformations in a way similar to Lagrange and Vincent in order to localize zero sets of real multivariate polynomials and thus prove bifurcation and enumeration results for classes of relative equilibria in Celestial Mechanics. Such results provide a more complete answer to the Chazy–Wintner–Smale finiteness conjecture, which is problem 6 in Smale's list [14] and 9 in the list [1].

The present article contains two theorems which amount to an extension of Vincent's theorem to real polynomials $P(X_1, \ldots, X_n)$, namely Theorems 2.4 and 2.5. Theorem 2.4 states that if $P = 0$ does not intersect a box $\mathfrak{B}^n = \Pi_{l=1}^{n}[a_l, b_l]$, then it is possible to find a finite covering of $\mathfrak{B}^n$ by n-dimensional boxes, and corresponding changes of variables, so that in the new variables the polynomial P assumes the form of polynomials whose coefficients all have the same sign. If $P^{-1}(0)$ is the graph of a real function $x_n = f(x_1, \ldots, x_{n-1})$ in the n-dimensional box $\mathfrak{B}^n = \Pi_{l=1}^{n}[a_l, b_l]$, so that for each x in the box $\Pi_{l=1}^{n-1}[a_l, b_l]$ the zero $(x, x_n(x))$ is simple, then Theorem 2.5 affirms the existence of a finite covering of $\mathfrak{B}^n$, and corresponding changes of variable such that, in the new variables, P assumes the form of polynomials which have either a single or no variation of signs if written as polynomials in the last variable with coefficients depending on the remaining $n - 1$ variables. Moreover, the converse of these statements also holds and corresponds to a special version of the implicit function theorem for multivariate real polynomials.

Let us briefly introduce the concept of relative equilibrium and explain why our localization results are helpful in the determination of relative equilibria. Consider N bodies of masses $m_1, \ldots, m_N$, and let $q_1, \ldots, q_N \in \mathbb{R}^2$ denote their respective positions. A *relative equilibrium* of the Newtonian N-Body Problem is a planar solution in which the bodies rotate with uniform angular velocity around the center of mass. Under the assumption that the center of mass is at the origin, the initial positions of a relative equilibrium must satisfy the so-called *relative equilibrium equations*, namely

$$\lambda m_j q_j - \sum_{j \neq k} \frac{m_j m_k}{||q_j - q_k||^3}(q_j - q_k) = 0, \quad j = 1, \ldots, N, \tag{1.1}$$

where λ is a positive constant. Notice that the left-hand sides of equations (1.1) depend on the Cartesian coordinates of the positions through polynomials and square roots of polynomials. There are several ways of writing equations (1.1) in polynomial form, for instance by using mutual distances as coordinates as in references [2,9]. Thus the relative equilibrium problem can be formulated as a problem in real algebraic geometry: it is required to determine the intersection of several real algebraic hypersurfaces depending on parameters (the masses). In section 3 we examine the $(5+1)$-body problem consisting of four unit masses placed at the vertices of a square, an arbitrary mass m placed at the center of the square, and an infinitesimal mass on the plane of the square. As it is explained in section 3, equations (1.1) reduce to a single equation for the position of the body with infinitesimal mass. It is possible to write the latter equation as a polynomial equation in two variables (sum and difference of suitable mutual distances) subject to one constraint, which is also a polynomial equation. If we pick a value for the parameter m, we have the problem of finding the intersections of two algebraic curves in an open set of $\mathbb{R}^2$. Using suitable coverings of the open set by two-dimensional boxes, we can reduce the problem to finding the intersections of the graphs of two smooth functions. If the number of intersections is already known to be finite, then we localize the zeros of the restriction of each polynomial to the sides of the boxes containing a point of intersection of the two graphs, and may use the intermediate value theorem to obtain approximations as good as desired of the coordinates of the points of intersection. Theorems 3.1 and 3.6 contain enumeration results for the classes of relative equilibria of the centered square plus a body of infinitesimal mass: depending on the value of the central mass $m \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, we may have 1, 2 or 3 classes of relative equilibria on each semi-median of the square, and 0, 1 or 2 classes of relative equilibria in each semi-diagonal of the square.

As a final note, we would like to comment on the possibilities of implementing our methods algorithmically. The techniques in the present paper seem reminiscent of interval arithmetic methods, such as the Krawczyk method, for which algorithmic implementations have been performed, see for instance reference [12]. We thank one of the reviewers for his comments on the algorithmic potentialities of our work, and for pointing out to us the aforementioned reference.

2. Localization of real algebraic hypersurfaces

2.1. Univariate polynomials

Let us review some definitions, notations and basic results.

Let $\mathbb{R}[X]$ denote the ring of real polynomials in the unknown X. Given $P = \sum_j a_j X^j \in \mathbb{R}[X] \setminus \{0\}$, we say that there exists a *variation of sign* between two terms $a_k X^k$ and $a_l X^l$, with $k < l$, if one of the following conditions holds:

(i) $l = k + 1$, and a_k and a_l have opposite signs; or
(ii) $l > k + 1$, and $a_{k+1}, a_{k+2}, \dots, a_{l-1}$ are all zero, while a_k and a_l have opposite signs.

The *number of variations of signs* in the polynomial P, denoted by $V(P)$, is the number of pairs k, l satisfying either condition (i) or (ii). Polynomials whose coefficients are all nonzero receive a special designation.

Definition 2.1. A polynomial $P = \sum_{j=0}^{d} a_j X^j \in \mathbb{R}[X]$ is said to be *complete* if $a_j \neq 0$ for all $j = 0, \dots, d$.

One of the earliest results on the relationship between the coefficients of a real polynomial $P \in \mathbb{R}[X]$ and the number of positive roots of P is the famous Descartes' rule of signs. Proofs can be found in [15,17].

Descartes' Rule of Signs. *Let $P = a_d X^{b_d} + \dots + a_1 X^{b_1} + a_0 X^{b_0}$ be a real polynomial whose coefficients are all nonzero, where $b_i \in \mathbb{Z}$ for $i = 0, 1, \dots, d$ and $0 \leq b_0 < b_1 < \cdots < b_d$. If $Z(P)$ denotes the number of positive zeros of P counted with multiplicities, then*

$$Z(P) \leq V(P), \quad \text{and} \quad V(P) \equiv Z(P) \pmod 2,$$

that is, $V(P)$ and $Z(P)$ have the same parity.

Albeit generally inconclusive, Descartes' rule of signs is greatly empowered by the application of suitable Möbius changes of variables. Consider Möbius transformations of the form

$$\phi_{a,b} = \frac{aX + b}{X + 1}, \tag{2.1}$$

where $a, b \in \mathbb{R}$, $a < b$. We verify that $\phi_{a,b}$ maps the interval $[0, \infty)$ to $(a, b]$. Given $P \in \mathbb{R}[X]$, let us denote by $P\big|_{(a,b]}$ the numerator of the rational function $P \circ \phi_{a,b}$, that is

$$P\big|_{(a,b]} = (X+1)^d \cdot P\left(\frac{aX+b}{X+1}\right) \qquad (d = \text{degree of } P).$$

Observe that if $V(P\big|_{(a,b]}) = 0$ or $V(P\big|_{(a,b]}) = 1$, Descartes' rule of signs respectively says that P does not have zeros or has a unique, simple zero in $(a, b]$. Thus finding transformations $\phi_{a,b}$ such that $V(P\big|_{(a,b]}) = 0$ or $V(P\big|_{(a,b]}) = 1$ becomes a problem of foremost importance in the root analysis of real polynomials.

We remark that $\big|_{(a,b]}$ is an algebra automorphism of $\mathbb{R}[X]$ (indeed, of $\mathbb{C}[X]$) whose inverse is obtained via the inverse of $\phi_{a,b}$ (viewed as a function), namely

$$\psi_{a,b} = \frac{X-b}{a-X}.$$

Every polynomial in $\mathbb{R}[X]$ can be written as a constant times a product of real monic factors of either degree one or degree two, where the latter can be further factored as $(X-\beta)(X-\bar{\beta})$, $\beta \in \mathbb{C} \setminus \mathbb{R}$. So, in order to compute $P\big|_{(a,b]}$, it suffices to consider the effect of $\big|_{(a,b]}$ on monic polynomials of degree one. If $P = X - \alpha$, where $a \neq \alpha$, we have that

$$P\big|_{(a,b]} = (a-\alpha)[X - \psi_{a,b}(\alpha)],$$

and for $P = (X-\beta)(X-\bar{\beta})$, $\beta \in \mathbb{C} \setminus \mathbb{R}$, we find

$$P\big|_{(a,b]} = |a-\beta|^2[X^2 - 2\Re(\psi_{a,b}(\beta))X + |\psi_{a,b}(\beta)|^2],$$

where $\Re$ stands for real part.

The following theorem is closely related to key results by Lagrange [10] and Vincent [16], see also [3,4]. It contains a partial converse of Descartes' rule of signs.

Theorem 2.2. *Let P be a real polynomial with zeros $x_1, \ldots, x_d \in \mathbb{C}$, not all of which are equal, and let $a, b \in \mathbb{R}$ be such that $a < b$ and $b - a < \Delta(P)$, where*

$$\Delta(P) = \min_{j<k}\{|x_j - x_k| : x_j \neq x_k\}.$$

(1) *$x_j \notin [a, b]$, for all j if and only if $V(P\big|_{(a,b]}) = 0$ and $P\big|_{(a,b]}$ is a complete polynomial.*

(2) *$x_j \in (a, b]$ is a simple zero of P, for some j and for a sufficiently close to x_j, if and only if $V(P\big|_{(a,b]}) = 1$ and the polynomial $P\big|_{(a,b]}$ is complete.*

Proof. The polynomial $P\big|_{(a,b]}$ has a factorization

$$P(a)(X - \psi_{a,b}(\alpha))^{n_\alpha} \ldots [X^2 - 2\Re(\psi_{a,b}(\beta))X + |\psi_{a,b}(\beta)|^2]^{n_\beta} \ldots, \tag{2.2}$$

where α denotes a real zero of P and β denotes a complex, nonreal zero of P. We prove item (1). Since $\alpha \notin [a, b]$ we have $\psi_{a,b}(\alpha) < 0$. If $\beta = \rho + i\sigma$, then $\Delta(P) \leq 2|\sigma|$. Consider the identity

$$|a-\beta|^2\Re(\psi_{a,b}(\beta)) = (\rho - a)(b - \rho) - \sigma^2. \tag{2.3}$$

The maximum of $(\rho - a)(b - \rho)$ is $\frac{1}{4}(a-b)^2$. Thus $b - a < \Delta(P)$ implies $(\rho - a)(b - \rho) - \sigma^2 < 0$, and from (2.3) we obtain $\Re(\psi_{a,b}(\beta)) < 0$. It is clear from expression (2.2) that $P\big|_{(a,b]}$ is complete. The reverse implication is straightforward.

For item (2), firstly notice condition $b - a < \Delta(P)$ implies that at most one real zero of P, say x_1, is in the interval $(a, b]$. If x_1 is simple, we can write $P\big|_{(a,b]} = (X - \psi_{a,b}(x_1))R$ with $R \in \mathbb{R}[X]$. From item (1), $V(R) = 0$ and R is complete. Since, for b fixed, $\lim_{a\to x_1} \psi_{a,b}(x_1) = \infty$, a complete polynomial $P\big|_{(a,b]}$ can be obtained by choosing a sufficiently close to x_1, and in this case it is clear that $V(P\big|_{(a,b]}) = 1$. The reciprocal is immediate (and does not really require the completeness of $P\big|_{(a,b]}$). □

Observe that, for real polynomials whose roots are all equal, and necessarily real, we may omit the hypothesis on $b - a$ in Theorem 2.2, and item (1) still holds, while item (2) is true only if the degree of P is one. Due to the chief role played in our study by intervals $(a, b]$ as in Theorem 2.2, we introduce the following terminology.

Definition 2.3. We call an interval $(a, b]$ so that $b - a < \Delta(P)$ a *(Lagrange–Vincent) Δ-interval* for the polynomial P, or simply a *Δ-interval* if the polynomial is clear from the context.

2.2. Multivariate polynomials

Let us now consider real polynomials P in the unknowns $X_1, \dots, X_n$, $n \geq 2$. Our goal is to analyze zero sets of such polynomials in $\mathbb{R}^n$, so we view P as a polynomial function in the real variables $x_1, \dots, x_n$. We indicate by $P_{\widehat{x_j}}$, $j = 1, \dots, n$ a polynomial in x_j whose coefficients are function of the variables x_k, $k \neq j$.

Firstly we examine the case $n = 2$. Let $\mathfrak{R} = [a_1, b_1] \times [a_2, b_2]$ be such that the algebraic curve $P(x_1, x_2) = 0$ and the rectangle $\mathfrak{R}$ do not intersect, and pick $x_1 = x \in [a_1, b_1]$. According to item (1) of Theorem 2.2, it is possible to cover $[a_2, b_2]$ with finitely many Δ-intervals $J_k = (c_k, d_k]$ such that, for each k, $V(P_{\widehat{x_2}}\big|_{J_k}) = 0$ and $P_{\widehat{x_2}}\big|_{J_k}$ is a complete polynomial. For each k, let us write the polynomial $P_{\widehat{x_2}}\big|_{J_k}$ as $\sum_j Q_j^{(k)}(x) x_2^j$. Since $P_{\widehat{x_2}}\big|_{J_k}$ is complete, for every j, $Q_j^{(k)}(x)$ has the same sign. For each k and each j, let $I_j^{(k)} = (a_j^{(k)}, b_j^{(k)}]$ be a Δ-interval for $Q_j^{(k)}$ such that $x \in I_j^{(k)}$ and whose closure is as in item (1) of Theorem 2.2. Set $I_k = \cap_j I_j^{(k)}$. The rectangles $I_k \times J_k$ form a covering of the segment $\{x\} \times [a_2, b_2]$ and, for arbitrary $x_1 \in I_k$, we have that $V(P_{\widehat{x_2}}\big|_{J_k}) = 0$ and $P_{\widehat{x_2}}\big|_{J_k}$ is complete. Since P does not vanish in the rectangle $\mathfrak{R} = [a_1, b_1] \times [a_2, b_2]$, we can repeat the previous argument for arbitrary $x_1 \in [a_1, b_1]$, and, using compactness, obtain a finite covering of $\mathfrak{R}$ by rectangles $I_k \times J_k$ such that, for all k, $(P_{\widehat{x_2}}\big|_{J_k})_{\widehat{x_1}}\big|_{I_k}$ is a polynomial whose coefficients all have the same sign. Conversely, the existence of such finite covering $\{I_k \times J_k\}$ clearly implies that $P^{-1}(0) \cap \mathfrak{R} = \emptyset$.

Now consider a real algebraic hypersurface $P(x_1, \dots, x_n) = 0$, where $n > 2$, and suppose $P^{-1}(0)$ does not intersect a box $\mathfrak{B}^n = \Pi_{l=1}^n [a_l, b_l]$. Pick $x \in \mathfrak{B}^{n-1} = \Pi_{l=1}^{n-1} [a_l, b_l]$. As in the preceding paragraph, we may cover the interval $[a_n, b_n]$ with finitely many Δ-intervals J_k so that, for all k, $P_{\widehat{x_n}}\big|_{J_k}$ (with $x_1, \dots, x_{n-1}$ the coordinates of x) is a complete polynomial with zero variations of signs. By continuity, each coefficient of each $P_{\widehat{x_n}}\big|_{J_k}$ does not vanish in an $(n-1)$-dimensional box containing x, so we can apply the same procedure to the coefficients of $P_{\widehat{x_n}}\big|_{J_k}$ and each such $(n-1)$-box separately, reducing the dimension of our setting to $n - 2$. By proceeding recursively like this, we will arrive at the two-dimensional setting discussed in the above paragraph, wherefrom our analysis of Σ can be completed. The following theorem summarizes our conclusions.

Theorem 2.4. *Consider the hypersurface $P = 0$, where $P \in \mathbb{R}[X_1, \dots, X_n]$. $P^{-1}(0)$ does not intersect the box $\mathfrak{B}^n = \Pi_{l=1}^n [a_l, b_l]$ if and only if there exists a finite covering of $\mathfrak{B}^n$ by n-boxes $\Pi_{\lambda=1}^n J_\lambda$, where $J_\lambda = (c_\lambda, d_\lambda]$, for all λ, such that*

$$P\big|_{\Pi_{\lambda=1}^n J_\lambda} := (\dots(P_{\widehat{x_n}}\big|_{J_n})\dots)_{\widehat{x_1}}\big|_{J_1}$$

is a polynomial whose coefficients all have the same sign.

We remark that the order of the Möbius transformations in Theorem 2.4 is immaterial. It appears that, in principle, a covering $\{\Pi_{\lambda=1}^{n} J_\lambda\}$ as in Theorem 2.4 can be determined algorithmically. In [8,7], such coverings were found through careful inspection of numerical plots. It is noticeable that, even though we have used continuity in our argument, some of the boxes in the coverings may turn out to be not so small, as the dimensions of such boxes depend ultimately on the minimum distance between roots of polynomials in one variable.

We may now consider the possibility of $P = 0$ intersecting a box $\mathfrak{B}^n$. By shrinking $\mathfrak{B}^n$ and reordering the variables $x_1, \ldots, x_n$ if necessary, we may assume $P^{-1}(0) \cap \mathfrak{B}^n$ is the graph of a function $x_n = f(x_1, \ldots, x_{n-1})$. Since 0 is a regular value of P, for each fixed $(x_1, \ldots, x_{n-1})$ in the domain of f, the corresponding zero x_n of $P_{\widehat{x_n}}$ is simple. In this case, our construction of a suitable covering $\{\Pi_{\lambda=1}^{n} J_\lambda\}$ may be performed in essentially the same way as above, employing both items of Theorem 2.2 in the first step, the only change being that in some boxes $\Pi_{\lambda=1}^{n} J_\lambda$ a single variation of signs occurs in the complete polynomial $P_{\widehat{x_n}}\big|_{J_n}$.

Theorem 2.5. *Let $P \in \mathbb{R}[X_1, \ldots, X_n]$. Suppose that $P^{-1}(0)$ is the graph of a real function $x_n = f(x_1, \ldots, x_{n-1})$ in the n-dimensional box $\mathfrak{B}^n = \Pi_{l=1}^{n}[a_l, b_l]$ in $\mathbb{R}^n$, that is, for each $x \in \mathfrak{B}^{n-1} = \Pi_{l=1}^{n-1}[a_l, b_l]$, there exists a unique $x_n(x) \in [a_n, b_n]$ such that $P(x, x_n(x)) = 0$. Moreover, suppose that $x_n(x)$ is a simple zero of the univariate polynomial $P(x, x_n)$, for all $x \in \mathfrak{B}^{n-1}$. There exists a finite covering of $\mathfrak{B}^n$ by n-boxes $\Pi_{\lambda=1}^{n} J_\lambda$, where $J_\lambda = (c_\lambda, d_\lambda]$, for all λ, such that the polynomial*

$$(\ldots(P_{\widehat{x_n}}\big|_{J_n})\ldots)_{\widehat{x_1}}\big|_{J_1}$$

has a single or no variation of signs if written as a polynomial in the last variable with coefficients depending on the remaining $n-1$ variables. The converse of these statements also holds.

The converse statement in Theorem 2.5 can be used to prove the existence of a real function $f = f(x)$ such that $P^{-1}(0)$ is the graph of f over some $(n-1)$-dimensional box $\mathfrak{B}^{n-1}$. In addition, for each fixed $x \in \mathfrak{B}^{n-1}$, the zero $f(x)$ of the polynomial $P(x, x_n)$ being simple (because of the single variation of signs) implies that zero is a regular value of P, and thus that $P^{-1}(0)$ is an analytic manifold. This result can be seen as a non-differentiable, non-local version of the implicit function theorem for real multivariate polynomials.

3. Relative equilibria of a (5 + 1)-body problem

Consider the Newtonian five-body problem from Celestial Mechanics. Among its symmetrical planar solutions one finds the relative equilibrium formed by four bodies of equal masses at the vertices of a square which uniformly rotates about a fifth body of arbitrary mass located at the center of the square. We may choose the masses of the bodies at the vertices of the square equal to one. Consider an additional particle of infinitesimal mass on the plane of the square. A theorem by Bang and Elmabsout [6] (expanding previous work by M. Lindow) asserts that, in order to form a relative equilibrium solution of the $(N+1)$-body problem where N equal masses are placed at the vertices of a regular N-gon, the infinitesimal mass particle must lie on one of the lines of symmetry of the N-gon. This theorem remains valid if a particle of mass m is put at the center of the N-gon. In this section we present an enumerative study of (classes modulo similarities of) relative equilibria in the case $N = 4$ (i.e., the regular N-gon is a square) and $m \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. Enumeration is made possible by reframing the bifurcation problem in terms of studying the intersections of two real algebraic curves on the plane. The localization results in Theorems 2.2, 2.4 and 2.5 guarantee the existence of Möbius transformations which allow us to determine the points of intersection. Due to the size of the polynomials involved, our computations are made with the help of *Mathematica*. Rational numbers are represented using decimal notation, and all calculations are exact.

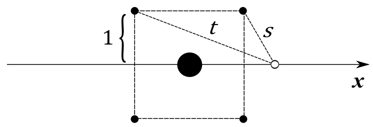

Fig. 1. A relative equilibrium formed by a centered square plus one body on one of its medians.

3.1. Relative equilibria on the medians

Consider Fig. 1, which depicts a square formed by four bodies of unit mass placed at the points $q_1 = (1, 1)$, $q_2 = (-1, 1)$, $q_3 = (-1, -1)$ and $q_4 = (1, -1)$, a fifth body of mass m located at $q_5 = (0, 0)$, and a body with infinitesimal (effectively zero) mass on the x-axis. In order to obtain a solution of the relative equilibrium equations (1.1) for the $(5 + 1)$-Body Problem, we must have, after plugging the positions and masses into the equation for the body at q_1,

$$\lambda(1,1) = \left[\frac{(1,1)-(1,-1)}{2^3} + \frac{(1,1)-(-1,1)}{2^3} + \frac{(1,1)-(-1,-1)}{(2\sqrt{2})^3} + m\frac{(1,1)}{2^{3/2}}\right],$$

whence

$$\lambda = \frac{1}{2\sqrt{2}}\left(m + \frac{1}{4} + \frac{1}{\sqrt{2}}\right). \tag{3.1}$$

Let x denote the position of the infinitesimal body on the median of the square indicated in Fig. 1 as the x-axis. Let s and t represent the distances from the infinitesimal body to the bodies at the vertices q_1 and q_2 of the square, respectively. The relative equilibrium equations (1.1) for the body with infinitesimal mass reduce to

$$\lambda x = 2\left(\frac{x-1}{s^3} + \frac{x+1}{t^3}\right) + m\frac{x}{|x|^3}, \tag{3.2}$$

where λ is given by (3.1) and

$$s^2 = (x-1)^2 + 1, \qquad t^2 = (x+1)^2 + 1. \tag{3.3}$$

See reference [7] for a detailed discussion about relative equilibria in the context of a restricted N-body problem.

Without loss of generality, from now on it will be assumed that $x > 0$, since the case $x < 0$ can be obtained by symmetry. The values $x = 0, \infty$ will be included in our analysis. From equation (3.2), we find that

$$m = \left(\frac{x^2}{x^3 - 2\sqrt{2}}\right)\left\{4\sqrt{2}[f(x-1) + f(x+1)] - \left(\frac{1}{\sqrt{2}} + \frac{1}{4}\right)x\right\}, \tag{3.4}$$

where $f(w) = w/(1 + w^2)^{3/2}$. Fig. 2 depicts the graph of m as a function of x. Notice the line $x = \sqrt{2}$ is the only vertical asymptote. Our main goal in this subsection is to demonstrate the following theorem.

Theorem 3.1. *On the semi-medians (including the points at 0 and ∞) of the centered square relative equilibrium with central mass $m \in \overline{\mathbb{R}}$, the possible numbers $\mathcal{N}$ of classes relative equilibria of the planar restricted $(5 + 1)$-body problem are as follows:*

(I) *If $-\infty < m < -\frac{1}{\sqrt{2}} - \frac{1}{4}$ or $M(a_b, u_b) < m < \infty$, we have $\mathcal{N} = 1$.*

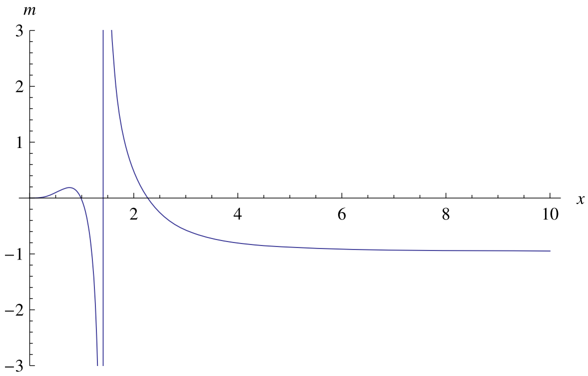

Fig. 2. The graph of m as a function of $x > 0$. The vertical line is the asymptote $x = \sqrt{2}$.

(II) *If* $-\frac{1}{\sqrt{2}} - \frac{1}{4} \leq m < 0$ *or* $m = M(a_b, u_b)$*, then* $\mathcal{N} = 2$.
(III) *If* $0 \leq m < M(a_b, u_b)$*, then* $\mathcal{N} = 3$.

In the above items, $M(a_b, u_b) \approx 0.187457$*. In item (II), the degenerate relative equilibrium associated with* $m = -\frac{1}{\sqrt{2}} - \frac{1}{4}$ *is* $x = \infty$*. In item (III), the degenerate relative equilibrium associated with* $m = 0$ *is* $x = 0$*, the center of the square.*

The proof of Theorem 3.1 is the result of a detailed analysis of m as a function of x given by equation (3.4).

In order to understand how m behaves near $x = 0$ and near $x = \infty$, we consider the Taylor expansions of the left-hand side of equation (3.4) at $x = 0$ and at $x = \infty$

$$m = \left(\frac{1}{4} + \frac{9}{8\sqrt{2}}\right) x^3 + O(x^5), \quad \text{and} \quad m = -\frac{1}{\sqrt{2}} - \frac{1}{4} + O\left(x^{-3}\right), \tag{3.5}$$

where the latter expansion is obtained by firstly replacing x with $1/x$, and then expanding at $x = 0$. The first equation in (3.5) shows that $x = 0$ is a local minimum of the left-hand side of (3.4), while the second equation in (3.5) shows that $m = 1/\sqrt{2} - 1/4$ is a horizontal asymptote of the graph of m as a function of x. Thus both $x = 0$ and $x = \infty$ can be viewed as degenerate relative equilibria associated with the bifurcation values $m = 0$ and $m = -1/\sqrt{2} - 1/4$, respectively. The bifurcating equilibria split into the positive and negative x-semiaxes, and it seems enlightening to identify $x = \infty$ with $x = -\infty$ to understand how these bifurcations work.

Fig. 2 shows the existence of one additional bifurcation value for m, namely a small positive number which is approximately 0.187457. The presence of such bifurcation is a more subtle feature of equation (3.4) and requires a deeper study.

Let us introduce the variables $a = t - s$, $u = s + t$ and $v = st$. Variables a, u, v and x satisfy the relations

$$4v = u^2 - a^2, \quad 4x = au, \tag{3.6}$$

and

$$4(4 - u^2)v + (32 - 8u^2 + u^4) = 0 \tag{3.7}$$

The latter relation is obtained from the Cayley–Menger determinant as a condition of planarity: the volume of the tetrahedron formed by the central body, the bodies at the upper vertices of the square and the infinitesimal body in Fig. 1 must be zero. Reference [13] contains the definition and additional information on the Cayley–Menger determinant.

We compute the expression of m in the variables a, u, v with the help of *Mathematica*,

$$m = -\frac{a^3u^2P(u,v)}{4v^3(128\sqrt{2} - a^3u^3)}, \tag{3.8}$$

where

$$P(u,v) = 16\sqrt{2}(4-u^2)u^2 + 16\sqrt{2}(-4+3u^2)v + (1+2\sqrt{2})uv^3.$$

Lemma 3.2. *a and u are monotonically increasing functions of x and map $x > 0$ onto $a > 0$ and $u > 2\sqrt{2}$, respectively.*

Proof. Consider the expression $t + \epsilon s$, $\epsilon = \pm 1$ and recall (3.3); $\epsilon = -1$ gives $a(x)$, and $\epsilon = +1$ gives $u(x)$. Denoting the derivative by a prime, we have that

$$(t+\epsilon s)' = \frac{x+1}{t} + \epsilon\frac{x-1}{s}.$$

If $\epsilon = +1$, the right-hand side vanishes only at $x = 0$ and is positive at $x = 1$. If $\epsilon = -1$, the right-hand side is always positive. $\square$

Lemma 3.2 tells us that either variable a or u can be used in place of x. We can solve the third equation in (3.6) for v and substitute the result into $P(u, v)$, thus producing a rational function of u whose numerator will be denoted by $Q(u)$ and whose denominator is always positive. The zeros of $Q(u)$ are the zeros of m in $x > 0$, and the sign of Q determines the sign of m.

Proposition 3.3. *Only two relative equilibria on $x > 0$ correspond to $m = 0$, namely the ones at approximately $x_0 = 0.98623$ and $x_1 = 2.26614$. The sign of m is positive on the intervals $0 < x < x_0$ and $\sqrt{2} < x < x_1$, and negative on the intervals $x_0 < x < \sqrt{2}$ and $x > x_1$.*

Proof. Consider the polynomial

$$Q(u) = -524288\sqrt{2} + \cdots - 256\sqrt{2}u^{10} - 24(1+2\sqrt{2})u^{11} + (1+2\sqrt{2})u^{13}.$$

The interval $0 < x < \sqrt{2}$ corresponds to $2\sqrt{2} < u < 2\sqrt{2+\sqrt{2}} \approx 3.6955$. We apply Möbius transformations to isolate the roots of Q.

Firstly we compute $\widetilde{Q}(u) = Q(u + 2\sqrt{2})$,

$$262144 + 589824\sqrt{2} + (425984 + 851968\sqrt{2})u - (-2949120 + 5029888\sqrt{2})u^2 - \\ \cdots + (-432128 + 905216\sqrt{2})u^7 + \cdots + (1+2\sqrt{2})u^{13},$$

and observe that the result is a complete polynomial with two variations of signs. By Descartes rule of signs, Q has either zero or two roots on $u > 2\sqrt{2}$.

In order to show that $\widetilde{Q}(u)$ has two roots, we apply a suitable Möbius transformation. We compute

$$\begin{aligned}\widetilde{Q}\big|_{(0,1]} = &-(14470751 + 10232548\sqrt{2}) - (135839144 + 96143318\sqrt{2})u - \ldots \\ &+ (310944768 + 394997760\sqrt{2})u^8 + \cdots + (262144 + 589824\sqrt{2})u^{13},\end{aligned}$$

and verify that it has a single variation of signs. Hence Q has a root in the interval $2\sqrt{2} < u < 1 + 2\sqrt{2}$, and another root on $u > 1 + 2\sqrt{2} > 2\sqrt{2+\sqrt{2}}$.

Finally we may obtain decimal approximations as precise as desired of the roots $u_0 < u_1$ of Q using Möbius transformations. We find that

$$u_0 \approx 3.22386, \quad u_1 \approx 5.02922,$$

and these values give the approximate values x_0 and x_1 in the statement of the proposition (we express x^2 as a rational function of u^2 by inverting $u = t + s$). The changes of signs of m follow easily from the fact that $u_0 < 2\sqrt{2+\sqrt{2}} < u_1$ and the fact that, while Q goes from positive to negative and then back to positive when u increases through u_0 and then through u_1, the denominator of m in (3.8) changes signs from positive to negative at $u = 2\sqrt{2+\sqrt{2}}$. □

Remark 3.4. The relative equilibria corresponding to $m = 0$ are indeed relative equilibria of the (4+1)-body problem. As far as we know, they have not been previously determined. Notice there is an additional relative equilibrium at $x = 0$.

Our next goal is to prove that, in addition to the bifurcations at $m = -1/4 - 1/\sqrt{2}$ and $m = 0$, there exists only one additional bifurcation, namely when $m \approx 0.187457$ (and at $x \approx 0.763729$). We will work with the variables a and u. The planarity condition (3.7) becomes

$$F(a,u) := (4-u^2)a^2 + 4(u^2-8) = 0, \tag{3.9}$$

and, using the first equation in (3.6), the mass function (3.8) can be written as the rational function

$$M(a,u) := \frac{a^3u^2G(a,u)}{4(128\sqrt{2} - a^3u^3)(u^2-a^2)^3}, \tag{3.10}$$

where

$$\begin{aligned} G(a,u) :=& 1024\sqrt{2}a^2 - (1+2\sqrt{2})a^6u + 768\sqrt{2}(4-a^2)u^2 + 3(1+2\sqrt{2}) \\ & a^4u^3 - 256\sqrt{2}u^4 - 3(1+2\sqrt{2})a^2u^5 + (1+2\sqrt{2})u^7. \end{aligned}$$

We must determine the critical points of M subject to the constaint $F = 0$. One way of formulating this problem is to look for points (a,u) on $F = 0$ such that the restriction of the derivative of M to the tangent line of $F = 0$ at (a,u) is the zero functional. Thus we compute the expression

$$\begin{bmatrix} \partial M/\partial a & \partial M/\partial u \end{bmatrix} \begin{bmatrix} -\partial F/\partial u \\ \partial F/\partial a \end{bmatrix},$$

determine its fators, and observe that the above expression vanishes precisely at the zero set of the polynomial

$$\begin{aligned} H(a,u) :=& -8192a^6 + 6(4+\sqrt{2})a^{10}u - 1024(68-15a^2)a^4u^2 - 2[60 + (15+8a)\sqrt{2}] \\ & a^8u^3 + \cdots - 3[8 + (2-a^5)\sqrt{2}]u^{11}. \end{aligned}$$

So the degenerate relative equilibria are the points of intersection of the algebraic curves $H = 0$ and $F = 0$ which satisfy $a > 0$ and $u > 2\sqrt{2}$.

Proposition 3.5. *The curves $F = 0$ and $H = 0$ intersect at a unique point in the quadrant $\{a > 0\} \times \{u > 2\sqrt{2}\}$.*

Proof. We apply a classical argument using resultants of polynomials.

The resultant of F and H with respect to the variable a is $8 \cdot R(u)$, where

$$R(u) = 2305843009213693952 + \cdots + 720(183 - 4\sqrt{2})u^{40} - 9(247 - 4\sqrt{2})u^{42}$$

is a polynomial of degree 42. The translation $u \mapsto u + 3.5$ produces a polynomial

$$\begin{aligned} R\big|_{[3.5,\infty)} =& 3(118114773819483400795053945724191659 - 266036665958879919908 0 \\ & 3574930712500\sqrt{2}) + \cdots + 39582418599936(-247 + 4\sqrt{2})u^{42}, \end{aligned}$$

which has a single variation of signs. Thus R has a single root at some $u > 3.5$. We compute the polynomial $R\big|_{(3.7,4]}$ and find that it has one variation of signs, so this root is actually greater than $2\sqrt{2+\sqrt{2}} \approx 3.6955$, which forces $a > 2\sqrt{2-\sqrt{2}} \approx 1.53073$. However,

$$\begin{aligned} H\big|_{[1.5,\infty)\times(3.7,4]} =& 75 \cdot 10^9(1727116708 - 250989463\sqrt{2}) + \cdots + 7104 \cdot 10^{10}(4 + \sqrt{2}) \\ & a^{10}u^{11} \end{aligned}$$

is a polynomial whose coefficients are all positive. Thus the only root of R greater than $2\sqrt{2+\sqrt{2}}$ does not extend to a common zero (a, u) of H and F corresponding to $x > 0$.

Next we analyze the restriction of R to the interval $2\sqrt{2} < u < 3.5$. The polynomial

$$\begin{aligned} R\big|_{(2\sqrt{2},3.5]} =& 3(118114773819483400795053945724191659 - 266036665958879919 90 \\ & 803574930712500\sqrt{2}) + \cdots - 36028797018963968(15201472938466873 \\ & 359987709085415 - 1074895706418422732997689346040 4\sqrt{2})u^{24} - \ldots \\ & - 236118324143482260 6848(-122293027610686738247 17 + 86550165 \\ & 72253186170668\sqrt{2})u^{30} + \cdots + 356526731314189519170947776512 \\ & (89 + 36\sqrt{2})u^{42} \end{aligned}$$

has exactly two variations of signs. Hence R has either zero or two roots in the interval $\left(2\sqrt{2}, 3.5\right]$ of the u-axis. The latter turns out to be the case.

We compute the polynomial $R\big|_{(3,3.03]}$ and verify that it has a single variation of signs. Thus R vanishes at one point in the interval $3 < u \leq 3.03$. For u in the latter interval, we have $0 < a \leq 2\sqrt{2-\sqrt{2}}$. So we examine the curves $F = 0$ and $H = 0$ in the rectangle $(0, 1.6] \times (3, 3.03] \supset (0, 2\sqrt{2-\sqrt{2}}] \times (3, 3.03]$ of the au-plane. The polynomial

$$F\big|_{(1,1.6]\times(3,3.03]} = 2134876 + 1782920a + 114325a^2 + \cdots + 50000(44 + 40a + 5a^2)u^2$$

has no variations of signs, while

$$\begin{aligned} H\big|_{(0,1]\times(3,3.03]} =& 153282565783531516118622 67672 - 117458196989066254297724 66 \\ & 541\sqrt{2} - \cdots - 13122 \cdot 10^{22}(846 + 81\sqrt{2})a^{10}u^{11} \end{aligned}$$

has all of its coefficients negative. Therefore the root of R in $3 < u \leq 3.03$ does not extend to an intersection point of $F = 0$ and $H = 0$ such that $x > 0$.

Finally we look at the remaining root u_b of R in the interval $\left(2\sqrt{2}, 3.5\right]$. The polynomial $R\big|_{(3.04,3.06]}$ has a single variation of signs, hence $3.04 < u_b \leq 3.06$, and $0 < a < 1.6$. However, the coefficients of the polynomials

$$\begin{aligned} F\big|_{(0,0.9]\times(3.04,3.06]} =& -277471 - 2727200a - 1363600a^2 - \cdots - 1241600a^2u^2, \\ F\big|_{[1.1,1.6)\times(3.04,3.06]} =& 258889 + 1992768a + 2069104a^2 + \cdots + 2113024a^2u^2 \end{aligned}$$

are all negative and all positive, respectively. Thus the possible (real) intersection points (a_b, u_b) of $F = 0$ and $H = 0$ necessarily satisfy $0.9 < a_b < 1.1$, and also $3.04 < u_b \leq 3.06$, as we have just asserted. In the next paragraph we prove that such (a_b, u_b) exists and is unique.

The resultant of F and H with respect to u can be expressed as $-32 \cdot S(a)$, where

$$S(a) = 81064793292668928(55 - 4\sqrt{2}) + \cdots + 1536(1 + 2\sqrt{2})a^{41} + 9(9 + 4\sqrt{2})a^{42}.$$

We compute

$$\begin{aligned} S\big|_{(0.9,1.1]} =& 9(1357540062305471157878601431157054528424270738687857260332 9 \\ & - 3433287277117546887544014043317132159762099521091303262236 \\ & \sqrt{2}) + \cdots + 9(563458115104322845762189274616317872868679600278 \\ & 6082698311 + 624023894348285348083972509153411564059082642860 \\ & 6164733756\sqrt{2})a^{42} \end{aligned}$$

and verify that $S\big|_{(0.9,1.1]}$ has one variation of signs. Thus $F = 0$ and $H = 0$ intersect at points whose a-coordinate is unique and lies on the interval $0.9 < a \leq 1.1$. From our previous discussion, the u-coordinate of the points of intersection, if real, is unique and belongs to the interval $3.04 < u \leq 3.06$. Thus the possible intersection point (a_b, u_b) is unique. To prove that an intersection at (a_b, u_b) actually occurs, we claim that in the rectangle $(0.9, 1.1] \times (3.04, 3.06]$ the curves $F = 0$ and $H = 0$ are graphs which intersect at least once. Indeed, on one hand we have

$$\begin{aligned} F\big|_{(0.9,1.1]\times(3.04,3.06]} =& 258889 + 603152u + 343984u^2 - 2(36109 - 19888u - 55696 \\ & u^2)a - (277471 + 457328u + 180176u^2)a^2, \end{aligned}$$

so, for each $u \geq 0$ fixed, the above polynomial has a single variation of signs. On the other hand,

$$\begin{aligned} H\big|_{(0.9,1.1]\times(3.04,3.06]} =& 16(116311541062214746426419200000 - 454060380157873209 \\ & 81179923599\sqrt{2}) + \cdots - 1536(101250397213744566832262500 \\ & + 3529134440482873521438567 29\sqrt{2})a^{10}u^{11} \end{aligned}$$

is a polynomial which can be written in the form $\sum_{i=0}^{10} h_i(u)a^i$ where each h_i is a nonzero polynomial of degree 14 whose coefficients are all positive if $i < 6$ and all negative if $i \geq 6$. Thus the restrictions of $F = 0$ and $H = 0$ to $(0.9, 1.1] \times (3.04, 3.06]$ are graphs of functions $a_F(u)$ and $a_H(u)$, respectively. Using Möbius transformations, we verify at once that $a_F(0) < a_H(0)$ and $a_F(1) > a_H(1)$, hence $F = 0$ and $H = 0$ must intersect at least once in $(0.9, 1.1] \times (3.04, 3.06]$. $\square$

Through suitable Möbius transformations, we obtain the approximate values of a and u at the intersection of the curves $F = 0$ and $H = 0$, namely

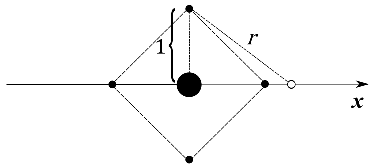

Fig. 3. A relative equilibrium formed by a centered square plus a body of infinitesimal mass on one of its diagonals.

$$a_b \approx 0.999963, \quad u_b \approx 3.055029.$$

These values give $x \approx 0.763729$, from the second equation in (3.6), and $M(a_b, u_b) \approx 0.187457$, using equation (3.10).

The above result completes our study of bifurcations and the proof of Theorem 3.1.

3.2. Relative equilibria on the diagonals

For the sake of completeness, we now consider the relative equilibria at points on one of the diagonals of the centered square, see Fig. 3.

The single relative equilibrium equation is

$$\lambda x = \frac{x-1}{|x-1|^3} + \frac{x+1}{|x+1|^3} + m\frac{x}{|x|^3} + 2\frac{x}{r^3}, \tag{3.11}$$

where

$$\lambda = m + \frac{1}{\sqrt{2}} + \frac{1}{4}, \qquad r^2 = x^2 + 1.$$

As in the previous subsection, without loss of generality, from now on it will be assumed that $x > 0$, since the case $x < 0$ can be obtained by symmetry. The values $x = 0, \infty$ will be included in our analysis. We solve equation (3.11) for m, thus obtaining

$$m = \left(\frac{x^2}{x^3-1}\right)\left\{\left[\frac{x-1}{|x-1|^3} + \frac{1}{(x+1)^2} + 2\frac{x}{r^3}\right] - \left(\frac{1}{\sqrt{2}} + \frac{1}{4}\right)x\right\}. \tag{3.12}$$

Our goal is this subsection is to prove the theorem below.

Theorem 3.6. *On the semi-diagonals (including the points at 0 and ∞) of the centered square relative equilibrium with central mass $m \in \overline{\mathbb{R}}$, the possible numbers $\mathcal{N}$ of classes of relative equilibria of the planar restricted $(5+1)$-body problem are as follows:*

(I) *If $-\infty < m < -\frac{1}{\sqrt{2}} - \frac{1}{4}$, we have $\mathcal{N} = 0$.*
(II) *If $-\frac{1}{\sqrt{2}} - \frac{1}{4} \leq m < 0$, then $\mathcal{N} = 1$.*
(III) *If $m \geq 0$, then $\mathcal{N} = 2$.*

In item (II), the degenerate relative equilibrium associated with $m = -\frac{1}{\sqrt{2}} - \frac{1}{4}$ is $x = \infty$. In item (III), the degenerate relative equilibrium associated with $m = 0$ is $x = 0$, the center of the square.

Proof. The assertions on degenerate relative equilibria follow from the asymptotic developments of (3.12) at $x = 0$ and $x = \infty$, namely

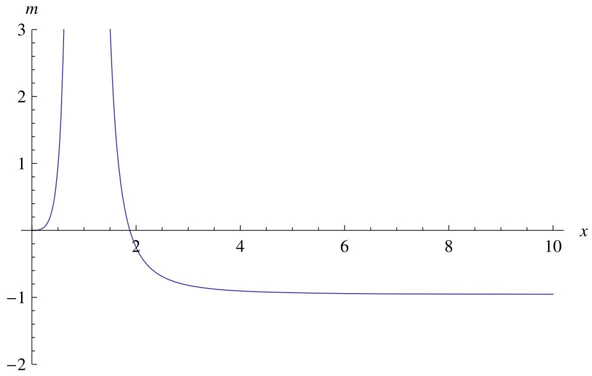

Fig. 4. The graph of m as a function of $x > 0$ along one of the diagonals of the square.

$$m = \left(\frac{9}{4} + \frac{1}{\sqrt{2}}\right) x^3 + O(x^5), \qquad m = -\frac{1}{\sqrt{2}} - \frac{1}{4} + O(x^{-3}).$$

The remaining claims in the statement of the theorem are direct consequences of the next proposition. □

Proposition 3.7. *Let $m(x)$ be the expression on the right-hand side of equation (3.12), $x > 0$. As x increases from 0 to 1, $m(x)$ increases from 0 to ∞, and as x increases from 1 to ∞, $m(x)$ decreases from ∞ to $-1/\sqrt{2} - 1/4$. The vertical line $x = 1$ and the horizontal line $m = -1/\sqrt{2} - 1/4$ are asymptotes of the graph of $m(x)$.*

Proof. For $0 < x < 1$, simple manipulations transform equation (3.12) into

$$m(x) = \left(\frac{2x^3}{1 - x^3}\right) \left\{ \left[\frac{1}{(x^2 - 1)^2} - \frac{1}{r^3}\right] + \left(\frac{1}{2\sqrt{2}} + \frac{1}{8}\right) \right\}.$$

By comparing the growth of the numerators and denominators of the rational functions as x grows from 0 to 1, we realize that the above expression is the product of two increasing functions on the interval $0 < x < 1$. We also see that $m(0) = 0$ and $\lim_{x \to 1^-} m(x) = \infty$.

If $x > 1$, equation (3.12) can be written as

$$m(x) = \left(\frac{x^2}{x^3 - 1}\right) \left\{ 2\left[\frac{x^2 + 1}{(x^2 - 1)^2} + \frac{x}{r^3}\right] - \left(\frac{1}{\sqrt{2}} + \frac{1}{4}\right) x \right\}.$$

We have that

$$\frac{x^2}{x^3 - 1} = \frac{1}{x}\left(1 + \frac{1}{x^3 - 1}\right), \qquad \frac{x^2 + 1}{(x^2 - 1)^2} = \frac{1}{x^2 - 1} + \frac{2}{(x^2 - 1)^2},$$

and, using the identity $r^2 = 1 + x^2$, we compute

$$\frac{d}{dx}\left(\frac{x}{r^3}\right) = \frac{1 - 2x^2}{r^5} < 0 \quad \Leftrightarrow \quad x^2 > \frac{1}{2}.$$

Thus $m(x)$ is a decreasing function if $x > 1$. Cleraly $\lim_{x \to 1^+} m(x) = \infty$ and $\lim_{x \to \infty} m(x) = -1/4 - 1/\sqrt{2}$. □

Fig. 4 displays the main features of the graph of $m = m(x)$ as stated in the last proposition.

References

[1] A. Albouy, H.E. Cabral, A.A. Santos, Some problems on the classical n-body problem, Celestial Mech. Dynam. Astronom. 113 (2012) 369–375.
[2] A. Albouy, A. Chenciner, Le problème des n corps et les distances mutuelles, Invent. Math. 131 (1998) 151–184.
[3] A. Alesina, M. Galuzzi, A new proof of Vincent's theorem, Enseign. Math. 44 (1998) 219–256.
[4] A. Alesina, M. Galuzzi, Vincent's theorem from a modern point of view, in: Categorical Studies in Italy, Rend. Circ. Mat. Palermo (2) 64 (2000) 179–191.
[5] A.G. Akritas, Elements of Computer Algebra with Applications, John Wiley & Sons, New York, 1989.
[6] D. Bang, B. Elmabsout, Representations of complex functions, means on the regular n-gon and applications to gravitational potential, J. Phys. A: Math. Gen. 36 (2003) 11435–11450.
[7] J.F. Barros, E.S.G. Leandro, The set of degenerate central configurations in the planar restricted four-body problem, SIAM J. Math. Anal. 43 (2011) 634–661.
[8] J.F. Barros, E.S.G. Leandro, Bifurcations and enumeration of classes of relative equilibria in the planar restricted four-body problem, SIAM J. Math. Anal. 46 (2014) 1185–1203.
[9] O. Dziobek, Über einen merkwürdigen Fall des Vielkörperproblems, Astronaut. Nach. 152 (1900) 33–46.
[10] J.-L. Lagrange, Traité de la résolution des équations numériques de tous les degrés, avec des notes sur plusieurs points de la théorie des équations algébriques, Troisième Édition, Chez Bachelier, Paris, 1826.
[11] E.S.G. Leandro, Finiteness and bifurcations of some symmetrical classes of central configurations, Arch. Ration. Mech. Anal. 176 (2003) 147–177.
[12] M. Malgorzata, P. Zgliczynski, Central configurations in planar n-body problem with equal masses for $n = 5, 6, 7$, Celestial Mech. Dynam. Astronom. 131 (2019) 46, https://doi.org/10.1007/s10569-019-9920-6.
[13] R. Moeckel, Generic finiteness for Dziobek configurations, Trans. Amer. Math. Soc. 353 (2001) 4673–4686.
[14] S. Smale, Mathematical problems for the next century, Math. Intelligencer 20 (2) (1998) 7–15.
[15] J.V. Uspensky, Theory of Equations, McGraw-Hill, New York, 1948.
[16] A.J.H. Vincent, Sur la résolution des equations numériques, J. Math. Pures Appl. 1 (1836) 341–372.
[17] X. Wang, A simple proof of Descartes's rule of signs, Amer. Math. Monthly 111 (2004) 525–526.