J. Math. Anal. Appl. 486 (2020) 123889

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Infinitely many nodal solutions for nonlinear elliptic equations with combined nonlinearities and lack of symmetry ☆

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A R T I C L E   I N F O

Article history:
Received 26 November 2019
Available online 21 January 2020
Submitted by D. Repovš

Keywords:
Nodal solutions
Variational approach
Gradient flow
Strong comparison principle
Concave-convex nonlinearities

A B S T R A C T

We present a new approach to studying a parametric nonlinear Dirichlet problem driven by a nonhomogeneous differential operator and with a reaction which is concave (i.e., $(p-1)$-sublinear) near zero and convex (i.e., $(p-1)$-superlinear) near $\pm\infty$. The reaction term is not assumed to be odd. We show that for all small values of the parameter $\lambda > 0$, the problem has infinitely many nodal solutions.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$ and let $1 < p < \infty$. In this paper we study the nonlinear nonhomogeneous Dirichlet boundary value problem

$$-\mathrm{div}\, a(z, Du(z)) = f(z, u(z), \lambda) \quad \text{in } \Omega, \qquad u\,|_{\partial\Omega} = 0. \tag{P_λ}$$

Here $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous map such that, for every $z \in \overline{\Omega}$, $a(z, \cdot)$ is strictly monotone on $\mathbb{R}^N$, while $(z, y) \to a(z, y)$ is C^1 on $\overline{\Omega} \times \mathbb{R}^N \setminus \{0\}$. Also, the reaction term $f : \Omega \times \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ is assumed to be a Carathéodory function in the $(z, x) \in \Omega \times \mathbb{R}$ variables (i.e., for all $x \in \mathbb{R}$ and all $\lambda > 0$, $z \to f(z, x, \lambda)$ is measurable and for almost all $z \in \Omega$ and all $\lambda > 0$, $x \to f(z, x, \lambda)$ is continuous), which exhibits $(p-1)$-superlinear growth near $\pm\infty$ but without satisfying the usual in such cases Ambrosetti-Rabinowitz condition. Instead, we employ an alternative weaker condition, which incorporates in our framework functions with

☆ Supported by the Natural Science Foundation of Guangdong Province (No. 1914050001946) and by Social Science-Philosophy plan project of Guangzhou (No. 2019GZGJ125).
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https://doi.org/10.1016/j.jmaa.2020.123889

"slower" growth near $\pm\infty$. In addition, our hypotheses on $x \to f(z, x, \lambda)$ imply the presence of concave ($(p-1)$-sublinear) terms in the reaction. So, we have in problem (P_λ) the competing effects of convex and concave nonlinearities. A very special case is the classical convex-concave nonlinearity of the form

$$\lambda |x|^{q-2} x + |x|^{r-2} x \quad \text{for all } x \in \mathbb{R}, \text{ with } 1 < q < p < r < p^\star,$$

where

$$p^\star = \begin{cases} \frac{Np}{N-p} & \text{if} \quad p < N, \\ +\infty & \text{if} \quad p \geq N. \end{cases}$$

This particular reaction can be found in the semilinear works (i.e., equations driven by the Laplacian) of Ambrosetti-Brezis-Cerami [2], Bartsch-Willem [9], Li-Wu-Zhou [23]. Extensions to equations driven by the p-Laplacian, can be found in the works of Filippakis-Kristaly-Papageorgiou [12], Garcia Azorero-Manfredi-Peral Alonso [14], Guo-Zhang [16] and Hu-Papageorgiou [20]. With the exception of [12], the other works do not provide sign information for all the solutions. In [12] the concave term is perturbed by a $(p-1)$-superlinear nonlinearity. We should also mention the very recent works of Aizicovici-Papageorgiou-Staicu [1] and Filippakis-O'Regan-Papageorgiou [13], which are concerned with equations driven by a nonhomogeneous differential operator and include concave terms. In [1,12,13], one nodal (that is, sign changing) solution was obtained.

Our work is closely related to the papers of Aizicovici-Papageorgiou-Staicu [1] and Filippakis-O'Regan-Papageorgiou [13] and in fact it complements them. More precisely, by using constant sign solutions obtained in [1, Theorem 2]; [13, Theorem 4.6], we show that problem (P_λ), with no odd symmetry on reaction, admits infinitely many nodal solutions. We mention the related recent works of Amster [3], Papageorgiou-Rădulescu-Repovš [28,29], which deal with Robin problems and that of Papageorgiou-Rădulescu-Repovš [31], Rolando [36], which deal with singular problems, and that of Papageorgiou-Rădulescu [27] concerning resonant (p,2)-equations.

If the reaction term is odd in a neighborhood of zero and satisfies certain conditions, by using a variant of the symmetric mountain pass theorem due to Kajikiya [21], problem (P_λ) admits infinitely many nodal solutions converging to zero in $C_0^1(\overline{\Omega})$ (see He-Yan-Sun-Zhang [18] and Papageorgiou-Rădulescu-Repovš [30]). We also mention the work of Bartsch-Liu-Weth [8], where under symmetry conditions (namely that the reaction term is odd), the authors produce infinitely many nodal solutions. If the reaction term is not odd in x, the symmetry of the corresponding functional is completely broken. The idea of using perturbation methods for solving such broken symmetry problems was introduced by Bahri-Berestycki [5], Bolle [10], Rabinowitz [35] and Struwe [37]. These perturbative arguments have been used by several authors (see, for instance, Bartolo [6], Candela-Palmieri-Salvatore [11], Lancelotti-Musesti-Squassina [22] and Zhang-Tang-Chen [38]). It is worth noticing that the authors in [5,6,10,11,35,37,38] produce infinitely many solutions but do not show that they are nodal.

Our approach is new in finding infinitely many nodal solutions. The main idea of this work consists in the construction of infinitely many closed convex sets which contain the biggest positive solution and the smallest negative solution in their interior, via upper-lower solutions and the strong monotonicity property of solutions. Moreover, these closed convex sets are invariant for the flow associated to a suitably constructed pseudo-gradient vector field. Refined variational arguments will be developed to find nodal critical points inside of closed convex sets and outside of smaller closed convex sets.

In the next section, for easy reference, we recall some of the main mathematical tools that will be used in the sequel. We also present the hypotheses on the map $a(\cdot, \cdot)$ and state some useful consequences of them.

2. Mathematical background

In the analysis of problem (P_λ), in addition to the Sobolev space $W_0^{1,p}(\Omega)$, we will also use the ordered Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u\mid_{\partial\Omega}= 0\}$. The order cone of $C_0^1(\overline{\Omega})$ is $C_+ := \{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}$. This cone has a nonempty interior given by

$$\mathrm{int}C_+ := \{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n} < 0 \text{ for all } z \in \partial\Omega\},$$

where $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$. By $\|\cdot\|$ we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$, and by $\|\cdot\|_s$ we denote the norm of $L^s(\Omega)$ $(1 \leq s \leq +\infty)$. Thanks to the Poincaré inequality we have

$$\|u\| = \|Du\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We mention that $\|\cdot\|$ also denotes the norm of $\mathbb{R}^N$. However, no confusion will come up since the context will make it clear. By $(\cdot,\cdot)_{\mathbb{R}^N}$ we denote the inner product in $\mathbb{R}^N$ and by $|\cdot|_N$ the Lebesgue measure on $\mathbb{R}^N$.

In what follows $\eta \in C^1(0,\infty)$ is such that

$$0 < \hat{c} \leq \frac{t\eta'(t)}{\eta(t)} \leq c_0 \quad \text{for all } t > 0 \text{ and some } \hat{c}, c_0 > 0$$

and

$$c_1 t^{p-1} \leq \eta(t) \leq c_2(t^{q_0-1} + t^{p-1}) \quad \text{for all } t > 0 \text{ and some } c_1, c_2 > 0,$$

with $1 < q_0 \leq p < \infty$. The map $a(\cdot,\cdot)$ entering the left-hand side of problem (P_λ) is assumed to satisfy the following conditions:

H_1: $a(z,y) = a_0(z,\|y\|)y$, where $a_0 : \overline{\Omega} \times (0,+\infty) \to (0,+\infty)$ is such that $\lim_{t\to 0^+} a_0(z,t)t = 0$ for every $z \in \overline{\Omega}$; moreover,

(i) $a \in C(\overline{\Omega}\times\mathbb{R}^N, \mathbb{R}^N) \cap C^1(\overline{\Omega} \times \mathbb{R}^N \setminus \{0\}, \mathbb{R}^N)$ and for every $K \subseteq \mathbb{R}^N \setminus \{0\}$ compact, there exists $\beta = \beta(K) \in (0,1)$ such that $a \in C^{0,\beta}(\overline{\Omega}\times K, \mathbb{R}^N)$;

(ii) for all $(z,y) \in \overline{\Omega} \times \mathbb{R}^N \setminus \{0\}$, we have

$$\frac{\eta(\|y\|)}{\|y\|}\|\xi\|^2 \leq (D_y a(z,y)\xi,\xi)_{\mathbb{R}^N} \quad \text{for all } \xi \in \mathbb{R}^N;$$

(iii) for all $(z,y) \in \overline{\Omega} \times \mathbb{R}^N \setminus \{0\}$ we have

$$\|D_y a(z,y)\| \leq c_3\frac{\eta(\|y\|)}{\|y\|} \quad \text{for some } c_3 > 0;$$

(iv) the primitive $G(z,y)$ determined by

$$D_y G(z,y) = a(z,y) \quad \text{for all } (z,y) \in \overline{\Omega}\times\mathbb{R}^N \text{ and } G(z.0) = 0 \text{ for all } z \in \overline{\Omega}$$

satisfies

$$k(z) \leq pG(z,y) - (a(z,y),y)_{\mathbb{R}^N} \quad \text{for all } z \in \Omega \text{ and all } y \in \mathbb{R}^N,$$

with $k \in L^1(\Omega)$;

(v) there exists $q \in (1,p)$ such that

$$\lim_{y \to 0} \frac{G(z,y)}{\|y\|^q} = 0 \text{ uniformly for all } z \in \overline{\Omega}$$

and if $G_0(z,t) = \int_0^t a_0(z,s) s \mathrm{d}s$ for all $t > 0$, then for some $\tau \in (q,p)$, the function $t \mapsto G_0(z, t^{\frac{1}{\tau}})$ is convex.

Example 1. The following maps satisfy hypotheses (H_1):

(a) $a_1(z,y) = \theta(z) \|y\|^{p-2} y$ with $1 < p < +\infty, \theta \in C^1(\overline{\Omega}), \theta(z) > 0$ for all $z \in \overline{\Omega}$. This map corresponds to the weighted p-Laplacian differential operator.

(b) $a_2(z,y) = \theta_1(z) \|y\|^{p-2} y + \theta_2(z) \|y\|^{q-2} y$ with $1 < q < p < +\infty, \theta_1, \theta_2 \in C^1(\overline{\Omega}), \theta_1(z) > 0, \theta_2(z) > 0$ for all $z \in \overline{\Omega}$. This map corresponds to the weighted (p,q)-Laplacian differential operator.

(c) $a_3(z,y) = (1 + \|y\|^2)^{\frac{p-2}{2}} y$ with $1 < p < +\infty$. This map corresponds to the generalized p-mean curvature operator.

The following properties of the map $a(\cdot,\cdot)$ follow by straightforward arguments.

Lemma 2. *If hypotheses* (H_1) *hold, then*
(i) *for all* $z \in \overline{\Omega}$, $y \mapsto a(z,y)$ *is strictly monotone;*
(ii) $\|a(z,y)\| \le c_4(1 + \|y\|^{p-1})$ *for all* $(z,y) \in \overline{\Omega} \times \mathbb{R}^N$ *and some* $c_4 > 0$*;*
(iii) $(a(z,y), y)_{\mathbb{R}^N} \ge \frac{c_1}{p-1} \|y\|^p$ *for all* $(z,y) \in \overline{\Omega} \times \mathbb{R}^N$*;*
(iv) $\frac{c_1}{p(p-1)} \|y\|^p \le G(z,y) \le c_5(1 + \|y\|^p)$ *for all* $(z,y) \in \overline{\Omega} \times \mathbb{R}^N$ *and some* $c_5 > 0$.

In the sequel $\langle \cdot, \cdot \rangle$ denotes the duality brackets for the pair $((W_0^{1,p}(\Omega))^\star, W_0^{1,p}(\Omega))$. Let $A : W_0^{1,p}(\Omega) \mapsto W^{-1,p'}(\Omega)) = (W_0^{1,p}(\Omega))^\star$ ($\frac{1}{p} + \frac{1}{p'} = 1$) be the nonlinear map defined by

$$\langle A(u), y \rangle = \int_\Omega (a(z, Du), Dy)_{\mathbb{R}^N} \mathrm{d}z \quad \text{for all } u, y \in W_0^{1,p}(\Omega).$$

We know that A is bounded (maps bounded sets to bounded ones), continuous, strictly monotone and of type $(S)_+$, i.e., if $u_n \to u$ weakly in $W_0^{1,p}(\Omega)$ and $\limsup_{n \to \infty} \langle A(u_n), u_n - u \rangle \le 0$, then $u_n \to u$ in $W_0^{1,p}(\Omega)$ (see, for example, Gasiński-Papageorgiou [15]).

The following simple fact about ordered Banach spaces can be found in Filippakis-Kristály-Papageorgiou [12, Lemma 3.3] and will be used in the paper.

Proposition 3. *Let X be an ordered Banach space, K is an order cone of X, with* $\mathrm{int}K \neq \emptyset$ *and* $x_0 \in \mathrm{int}K$*. Then, for every $y \in X$ there exists $t = t(y) > 0$ such that* $t x_0 - y \in \mathrm{int}K$*.*

Now, let $f_0 : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ be a Carathéodory function such that

$$|f_0(z,x)| \le a_0(z)(1 + |x|^{r-1}) \quad \text{for a. a. } z \in \Omega, \text{ all } x \in \mathbb{R}$$

with $a_0 \in L^\infty(\Omega)_+$ and $1 < r < p^\star$. We set $F_0(z,x) = \int_0^x f_0(z,t) \mathrm{d}t$ and consider the C^1-functional $\varphi_0 : W_0^{1,p}(\Omega) \mapsto \mathbb{R}$ defined by

$$\varphi_0(u) = \int_\Omega G(z, Du(z)) \mathrm{d}z - \int_\Omega F_0(z, u(z)) \mathrm{d}z.$$

From Aizicovici-Papageorgiou-Staicu [1, Proposition 2] we know that if $u_0 \in W_0^{1,p}(\Omega)$ is a local $C_0^1(\overline{\Omega})$-minimizer of φ_0, then it is also a local $W_0^{1,p}(\Omega)$-minimizer of φ_0.

Let $g, h \in L^\infty(\Omega)$. We write $g \prec h$ if for every compact subset $K \subseteq \Omega$ there exists $\varepsilon = \varepsilon(K) > 0$, such that

$$g(z) + \varepsilon \leq h(z) \quad \text{for a. a. } z \in K.$$

Clearly, if $g, h \in C(\Omega)$ and $g(z) < h(z)$ for all $z \in \Omega$, then $g \prec h$. Using this order $\prec$ we state the following strong comparison theorem (see [13, Proposition 2.4]; [33, Proposition 2.9]) which extends Proposition 2.6 of Arcoya-Ruiz [4] where the case of the p-Laplacian (i.e., $G(z, y) = \frac{1}{p}\|y\|^p$ for all $(z, y) \in \overline{\Omega} \times \mathbb{R}^N$) is considered.

Proposition 4. *If hypotheses* (H_1) *hold,* $\xi \geq 0, g, h \in L^\infty(\Omega)$ *satisfy* $g \prec h$ *and* $u, v \in W_0^{1,p}(\Omega)$ *are solutions of the Dirichlet problems*

$$-\mathrm{div}\, a(z, Du) + \xi\, |u|^{p-2}\, u = g \quad \text{in } \Omega, \qquad u\,|_{\partial\Omega}= 0$$

and

$$-\mathrm{div}\, a(z, Dv) + \xi\, |v|^{p-2}\, v = h \quad \text{in } \Omega, \qquad v\,|_{\partial\Omega}= 0$$

and $v \in \mathrm{int} C_+$, then $v - u \in \mathrm{int} C_+$.

Let us recall what we mean by upper and lower solutions for problem (P_λ).

Definition 5. (a) An upper solution for problem (P_λ) is a function $\overline{u} \in W_0^{1,p}(\Omega)$ such that $\overline{u}\,|_{\partial\Omega} \geq 0$ and

$$\int_\Omega (a(z, D\overline{u}), Dh)_{\mathbb{R}^N} \mathrm{d}z \geq \int_\Omega f(x, \overline{u}, \lambda) h \mathrm{d}z \tag{1}$$

for all $h \in W_0^{1,p}(\Omega)$ with $h \geq 0$. We say that $\overline{u}$ is a strict upper solution if it is an upper solution but not a solution for (P_λ).

(b) A lower solution for problem (P_λ) is a function $\underline{u} \in W_0^{1,p}(\Omega)$ such that $\underline{u}\,|_{\partial\Omega} \leq 0$ and the inequality in (1) is reversed. Analogously, we define a strict lower solution.

Finally, we conclude this section by introducing some notation. Let $z \in \mathbb{R}$. We set $z^\pm = \max\{\pm z, 0\}$ and for $u \in W_0^{1,p}(\Omega)$, we define $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in W_0^{1,p}(\Omega), \quad |u| = u^+ + u^- \quad \text{and} \quad u = u^+ - u^-.$$

The Nemytskii map corresponding to a measurable function $h : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is indicated as

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Furthermore, for $u, v \in W_0^{1,p}(\Omega)$ and $v \leq u$ a.e. in Ω, we define by $[v, u]$ the ordered interval given by

$$[v, u] = \{w \in W_0^{1,p}(\Omega) : v(z) \leq w(z) \leq u(z) \quad \text{a. e. in } \Omega\}.$$

The sets $\mathrm{int}_{C_0^1(\overline{\Omega})}[v, u]$ and $\partial_{C_0^1(\overline{\Omega})}[v, u]$ are the interior and boundary of the order interval $[v, u]$ in $C_0^1(\overline{\Omega})$, respectively. If $\varphi \in C^1(X, \mathbb{R})$ (X a Banach space), then by K_φ we denote the critical set of φ, that is,

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}.$$

By S_λ^+ (respectively S_λ^-), we denote the set of nontrivial positive (respectively negative) solutions of problem (P_λ).

3. Infinitely many nodal solutions

In this section, we use truncation techniques and flow invariance arguments to prove the existence of infinitely many nodal solutions for problem (P_λ), with no odd symmetry on reaction, whenever λ belongs to a small interval of the form $(0, \lambda^\star)$. Our hypotheses on the reaction $f : \Omega \times \mathbb{R} \times (0, +\infty) \mapsto \mathbb{R}$ are the following.

H_2 : $f : \Omega \times \mathbb{R} \times (0, +\infty) \mapsto \mathbb{R}$ is a function, such that for all $\lambda > 0, (z, x) \mapsto f(z, x, \lambda)$ is a Carathéodory function, $f(z, 0, \lambda) = 0$ for a.a. $z \in \Omega$, for all $\lambda > 0$, and

(i) for every $\rho > 0$ and $\lambda > 0$, there exists $a_\rho(\cdot, \lambda) \in L^\infty(\Omega)_+$ such that $|f(z, x, \lambda)| \le a_\rho(z, \lambda)$ for a.a. $z \in \Omega$ and all $|x| \le \rho$;

(ii) if $F(z, x, \lambda) = \int_0^x f(z, s, \lambda)\mathrm{d}s$, then for all $\lambda > 0$, $\lim_{x\to\pm\infty} \frac{F(z,x,\lambda)}{|x|^p} = +\infty$ uniformly for a.a. $z \in \Omega$ and there exists $r \in (p, p^\star)$ and $\eta_1, \eta_1 \in L^\infty(\Omega)$ such that for every $\lambda > 0$, we have

$$\eta_1(z) \le \liminf_{x\to\pm\infty} \frac{f(z, x, \lambda)}{|x|^{r-2}\, x} \le \limsup_{x\to\pm\infty} \frac{f(z, x, \lambda)}{|x|^{r-2}\, x} \le \eta_2(z) \quad \text{uniformly for a. a. } z \in \Omega;$$

(iii) for every $\lambda > 0$, there exist $\tau_0 = \tau_0(\lambda) \in ((r - p)\max\{\frac{N}{p}, 1\}, p^\star)$ and $\beta_0 = \beta_0(\lambda) > 0$ such that

$$\liminf_{x\to\pm\infty} \frac{f(z, x, \lambda)x - pF(z, x, \lambda)}{|x|^{\tau_0}} \ge \beta_0 \quad \text{uniformly for a. a. } z \in \Omega;$$

(iv) if $q \in (1, p)$ is as in hypothesis H_1(v), then for all $\lambda > 0$ we have $\hat{c}_0 |x|^q \le f(z, x, \lambda)x$ for a.a. $z \in \Omega$ and all $x \in \mathbb{R}$, with $\hat{c}_0 = \hat{c}_0(\lambda) > 0$, there exists $\delta_0 = \delta_0(\lambda) > 0$ such that

$$0 < f(z, x, \lambda) \le qF(z, x, \lambda) \quad \text{for a. a. } z \in \Omega,\ \text{all } |x| \le \delta_0,$$
$$\operatorname{ess\,inf}_\Omega F(\cdot, \delta_0, \lambda) > 0,$$

and there exists $\eta_0(\cdot, \lambda) \in L^\infty(\Omega)_+$ with $\|\eta_0(\cdot, \lambda)\|_\infty \to 0$ as $\lambda \to 0^+$ and

$$\limsup_{x\to 0} \frac{F(z, x, \lambda)}{|x|^q} \le \eta_0(z, \lambda) \quad \text{uniformly for a. a. } z \in \Omega.$$

Remark 6. Hypothesis H_2(ii) implies that the primitive $F(z, \cdot, \lambda)$ $(\lambda > 0)$ is p-superlinear near $\pm\infty$. However, we stress that we do not employ the usual AR-condition when dealing with superlinear problems. Instead, we use hypotheses H_2(ii), (iii) which are weaker and incorporates in our framework superlinear nonlinearities with “slower” growth near $\pm\infty$ which fail to satisfy the AR-condition.

Example 7. The following functions satisfy hypotheses H_2 (for the sake of simplicity we drop the z-dependence):

(i) $f_1(x, \lambda) = \lambda |x|^{q-2} x + |x|^{r-2} x$ for all $x \in \mathbb{R}$ and with $1 < q < p < r < p^\star$. This is the nonlinearity considered in Ambrosetti-Brezis-Cerami [2] where $p = 2$ (semilinear equations driven by the Laplacian) and in Garcia Azorero-Manfredi-Peral Alonso [14], Guo-Zhang [16] where $1 < p < +\infty$ (nonlinear equations driven by the p-Laplacian).

(ii) $f_2(x, \lambda) = \lambda |x|^{q-2} x + |x|^{p-2} x \ln(|x| + 1)$ for all $x \in \mathbb{R}$ and with $1 < q < p < +\infty$. This function does not satisfy the AR-condition.

By using variational methods coupled with truncation techniques and comparison principles, Aizicovici-Papageorgiou-Staicu [1] have established the following multiplicity result.

Proposition 8. *[1, Theorem 2] If hypotheses* H_1 *and* H_2 *hold, then there exists* $\lambda^\star > 0$ *such that for all* $\lambda \in (0, \lambda^\star)$*, problem* (P_λ) *has at least five nontrivial smooth solutions* $u, \hat{u} \in \mathrm{int}C_+$*,* $v, \hat{v} \in -\mathrm{int}C_+$ *and* $y \in C_0^1(\overline{\Omega}) \setminus \{0\}$ *nodal.*

In the above theorem, the authors only produce one nodal solution. Next will produce infinitely many nodal solutions for problem (P_λ). To do this, we need to strengthen the hypotheses on the reaction $f(z, x, \lambda)$. H_2' : $f : \Omega \times \mathbb{R} \times (0, +\infty) \mapsto \mathbb{R}$ is a function, such that for all $\lambda > 0, (z, x) \mapsto f(z, x, \lambda)$ is a Carathéodory function, $\lambda \mapsto f(z, x, \lambda)x$ is nondecreasing, $f(z, 0, \lambda) = 0$ for a.a. $z \in \Omega$, for all $\lambda > 0$, hypotheses H_2'(i)-(iv) are the same as the corresponding hypotheses H_2(i)-(iv) and

(v) for every $\lambda > 0$ and every $\rho > 0$, there exists $\xi_\rho^\lambda > 0$ such that for a.a. $z \in \Omega$, the function $x \mapsto f(z, x, \lambda) + \xi_\rho^\lambda |x|^{p-2} x$ is nondecreasing on $[-\rho, \rho]$; and for every $s > 0$, there exists $m_s > 0$ such that

$$|f(z, x, \mu) - f(z, x, \lambda)| \geq m_s$$

for a.a. $z \in \Omega$, all $|x| \geq s$ and all $\mu > \lambda > 0$.

Remark 9. The examples of functions presented after hypotheses H_2 still satisfy the new conditions stated in hypotheses H_2'.

Proposition 10. *If hypotheses* H_1 *and* H_2' *hold,* $0 < \mu < \lambda < \lambda^\star$ *and* $u_\lambda \in S_\lambda^+$ *(resp.* $v_\lambda \in S_\lambda^-$*), then there exists* $u_\mu \in S_\mu^+$ *(resp.* $v_\mu \in S_\mu^-$*) such that* $u_\lambda - u_\mu \in \mathrm{int}C_+$ *(resp.* $v_\lambda - v_\mu \in -\mathrm{int}C_+$*).*

Proof. Let $0 < \mu < \lambda < \lambda^\star$ and let $u_\lambda \in S_\lambda^+$ (see Proposition 8). We have

$$-\mathrm{div}\, a(z, Du_\lambda(z)) = f(z, u_\lambda(z), \lambda) \geq f(z, u_\lambda(z), \mu) \quad \text{a. e. in } \Omega, \tag{2}$$

because $\mu < \lambda$ and the fact that $\lambda \mapsto f(z, x, \lambda)$ is nondecreasing (see hypotheses H_2'). Then, u_λ is an upper solution of problem (P_μ). We introduce the following Carathéodory function

$$\hat{f}_\mu(z, x) = \begin{cases} 0 & \text{if} \quad x < 0, \\ f(z, x, \mu) & \text{if} \quad 0 \leq x \leq u_\lambda(z), \\ f(z, u_\lambda, \mu) & \text{if} \quad u_\lambda(z) < x. \end{cases} \tag{3}$$

Setting $\hat{F}_\mu(z, x) = \int_0^x \hat{f}_\mu(z, s)\mathrm{d}s$ we defined the C^1-functional $\hat{\varphi}_\mu : W_0^{1,p}(\Omega) \mapsto \mathbb{R}$ through

$$\hat{\varphi}_\mu(u) = \int_\Omega G(z, Du(z))\mathrm{d}z - \int_\Omega \hat{F}_\mu(z, u(z))\mathrm{d}z.$$

By virtue of Lemma 2 (iv) and the truncation defined in (3), we conclude that $\hat{\varphi}_\mu$ is coercive. Moreover, using the Sobolev embedding theorem, we see that $\hat{\varphi}_\mu$ is sequentially weakly lower semicontinuous. So, by the Weierstrss theorem, there exists $u_\mu \in W_0^{1,p}(\Omega)$ such that

$$\hat{\varphi}_\mu(u_\mu) = \inf_{u \in W_0^{1,p}(\Omega)} \hat{\varphi}_\mu(u). \tag{4}$$

Given $\varepsilon > 0$, by virtue of hypothesis H_1(v), there exists $\delta = \delta(\varepsilon) > 0$ such that

$$G(z,y) \leq \varepsilon \|y\|^q \quad \text{for all } z \in \overline{\Omega}, \text{ all } \|y\| \leq \delta. \tag{5}$$

Let $\tilde{u} \in \mathrm{int}C_+$ and recall that $u_\lambda \in \mathrm{int}C_+$. By Proposition 3 there exists a number $t \in (0,1)$ small enough such that

$$\|D(t\tilde{u})(z)\| \leq \delta,\ t\tilde{u}(z) \leq u_\lambda(z) \quad \text{for all } z \in \overline{\Omega}. \tag{6}$$

Then, due to (5), (6) and H_2'(iv), we obtain

$$\begin{aligned} \hat{\varphi}_\mu(t\tilde{u}) &= \int_\Omega G(z, D(t\tilde{u}))\mathrm{d}z - \int_\Omega \hat{F}_\mu(z, t\tilde{u})\mathrm{d}z \\ &\leq \varepsilon t^q \|D\tilde{u}\|_q^q - \hat{c}_0 t^q \|\tilde{u}\|_q^q . \end{aligned} \tag{7}$$

If we choose ε small enough such that $\varepsilon \|D\tilde{u}\|_q^q < \hat{c}_0 \|\tilde{u}\|_q^q$, then from (7) we infer that $\hat{\varphi}_\mu(t\tilde{u}) < 0$, that means, $\hat{\varphi}_\mu(u_\mu) < 0 = \hat{\varphi}_\mu(0)$, and thus, $u_\mu \neq 0$. From (4), there holds $(\hat{\varphi}_\mu)'(u_\mu) = 0$ which gives

$$A(u_\mu) = N_{\hat{f}_\mu}(u_\mu). \tag{8}$$

Taking $-(u_\mu)^- \in W_0^{1,p}(\Omega)$ as test function in (8) and applying Lemma 2 (iii) combined with the truncation in (3) gives

$$\frac{c_1}{p-1} \left\|D(u_\mu)^-\right\|_p^p \leq 0$$

ensuring that $u_\mu \geq 0$, $u_\mu \neq 0$. Now, taking $(u_\mu - u_\lambda)^+ \in W_0^{1,p}(\Omega)$ as test function in (8) results in, due to (3) and (2),

$$\begin{aligned} \left\langle A(u_\mu), (u_\mu - u_\lambda)^+ \right\rangle &= \int_\Omega \hat{f}_\mu(z, u_\mu)(u_\mu - u_\lambda)^+ \mathrm{d}z \\ &= \int_\Omega \hat{f}_\mu(z, u_\lambda)(u_\mu - u_\lambda)^+ \mathrm{d}z \\ &\leq \left\langle A(u_\lambda), (u_\mu - u_\lambda)^+ \right\rangle . \end{aligned}$$

Therefore,

$$\int_{\{u_\mu > u_\lambda\}} (a(z, Du_\mu) - a(z, Du_\lambda), Du_\mu - Du_\lambda)_{\mathbb{R}^N} \mathrm{d}z \leq 0,$$

which implies $|\{u_\mu > u_\lambda\}|_N = 0$, so $u_\mu \leq u_\lambda$. Hence, we have proved that $u_\mu \in [0, u_\lambda]$ with $u_\mu \neq 0$. Then, according to (3), relation (8) becomes

$$A(u_\mu) = N_{f_\mu}(u_\mu) \ \text{ with } \quad f_\mu(z,x) = f(z,x,\mu),$$

which means that

$$-\mathrm{div}\, a(z, Du_\mu(z)) = f(z, u_\mu(z), \mu) \quad \text{a.e. in } \Omega, \qquad u\,|_{\partial\Omega} = 0. \tag{9}$$

From the regularity result of Lieberman [24, p. 320] we infer that $u_\mu \in C_+ \setminus \{0\}$.

Let $\varrho = \|u_\mu\|_\infty$ and let $\xi_\varrho^\mu > 0$ be as postulated by hypothesis H_2'(v). Then,

$$-\mathrm{div}\, a(z, Du_\mu(z)) + \xi_\varrho^\mu u_\mu(z)^{p-1} = f(z, u_\mu(z), \mu) + \xi_\varrho^\mu u_\mu(z)^{p-1} \geq 0 \ \text{ a.e. in } \Omega$$

(see (9) hypothesis H_2'(v)). We know that

$$\mathrm{div}\, a(z, Du_\mu(z)) \leq \xi_\varrho^\mu u_\mu(z)^{p-1} \ \text{ a.e. in } \Omega.$$

Then, the strong maximum principle of Pucci-Serrin [34, p. 111] implies that $u_\mu(z) > 0$ for all $z \in \Omega$ (recall that $u_\mu \neq 0$). Thus, by the boundary point theorem of Pucci-Serrin [34, p. 120] we conclude that $u_\mu \in \mathrm{int}C_+$.

Let $\rho = \|u_\lambda\|_\infty$ and let $\xi_\rho^\mu > 0$ be as postulated by hypothesis H_2'(v). Using hypothesis H_2'(v), $\lambda > \mu, u_\lambda \geq u_\mu$ and the fact $u_\mu \in S_\mu^+$, we have

$$\begin{aligned}
-\mathrm{div}\, a(z, Du_\mu) + \xi_\rho^\mu {u_\mu}^{p-1} &= f(z, u_\mu, \mu) + \xi_\rho^\mu {u_\mu}^{p-1} \\
&\leq f(z, u_\lambda, \mu) + \xi_\rho^\mu {u_\lambda}^{p-1} \\
&= f(z, u_\lambda, \lambda) + \xi_\rho^\mu {u_\lambda}^{p-1} - (f(z, u_\lambda, \lambda) - f(z, u_\lambda, \mu)) \\
&\leq f(z, u_\lambda, \lambda) + \xi_\rho^\mu {u_\lambda}^{p-1} \\
&= -\mathrm{div}\, a(z, Du_\lambda) + \xi_\rho^\mu {u_\lambda}^{p-1}.
\end{aligned}$$

Note that, if $g(z) = f(z, u_\lambda, \lambda) - f(z, u_\lambda, \mu)$, then since $u_\lambda \in \mathrm{int}C_+$ and owing to hypothesis H_2'(v) we have $0 \prec g$ and so we may apply the strong maximum principle (see Proposition 4) to conclude that $u_\lambda - u_\mu \in \mathrm{int}C_+$.

An similar reasoning shows that there exists $v_\mu \in S_\mu^-$ such that $v_\mu - v_\lambda \in -\mathrm{int}C_+$. $\square$

Recall that a nonempty set $S \subseteq W_0^{1,p}(\Omega)$ is upward (respectively downward) directed if for all $u, v \in S$ we can find $w \in S$ such that $w \geq \max\{u, v\}$ (respectively $w \leq \min\{u, v\}$).

Proposition 11. *Let $\overline{u} \in \mathrm{int}C_+$ and $\underline{v} \in -\mathrm{int}C_+$ be, respectively, an upper solution and a lower solution of problem (P_λ). If hypotheses H_1 and H_2' hold and $\lambda \in (0, \lambda^\star)$, then problem ($P_\lambda$) has a biggest positive solution $u^\star \in [\underline{v}, \overline{u}] \cap \mathrm{int}C_+$ and a smallest negative solution $v^\star \in [\underline{v}, \overline{u}] \cap (-\mathrm{int}C_+)$.*

Proof. As before, we deal only with existence of the biggest positive solution in $[\underline{v}, \overline{u}]$; the other part goes in a similar way.

Let S_0^+ be the set of positive solutions of problem (P_λ) belonging to the order interval $[\underline{v}, \overline{u}]$. In Proposition 10 we have seen that $S_0^+ \neq \emptyset$. We will show that the set S_0^+ is upward directed. For this purpose, we fix $w_1, w_2 \in S_0^+$. These are also lower solutions for problem (P_λ). Hence, Lemma 4.12 of Papageorgiou-Rocha-Staicu [32] implies that $w_0 = \max\{w_1, w_2\}$ is a lower solution too (see also Lemma 4.2 of Filippakis-Kristaly-Papageorgiou [12]). Then we truncate the reaction term of problem (P_λ) at $\{w_0, \overline{u}\}$ and proceed as in the proof of Proposition 10. We obtain a solution $\hat{w}_0 \in [w_0, \overline{u}]$ of problem (P_λ). Hence $\hat{w}_0 \in S_0^+$ and so S_0^+ is upward directed.

Using Lemma 3.10 of Hu-Papageorgiou [19], we can find a sequence $\{u_n\} \subseteq S_0^+$, nondecreasing, such that $\sup S_0^+ = \sup_{n\geq 1} u_n$. We have

$$A(u_n) = N_{\hat{f}_\lambda}(u_n) \ \text{ for all } n \geq 1, \tag{10}$$

where $\hat{f}_\lambda$ is the truncation of f_λ at $\{0, \overline{u}\}$, where $f_\lambda(z, x) = f(z, x, \lambda)$. Invoking the regularity results of Lieberman [24] and recalling that $0 \leq u_n \leq \bar{u}$ for all $n \geq 1$, we can find $\alpha \in (0, 1)$ and $c_6 > 0$ such that

$$u_n \in C_0^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \le c_6 \quad \text{for all } n \ge 1.$$

Exploiting the compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ and passing to a suitable subsequence if necessary, we have

$$u_n \to u^\star \quad \text{in} \quad C_0^1(\overline{\Omega}) \quad \text{as} \quad n \to \infty.$$

Then, we directly obtain from (10) that

$$A(u^\star) = N_{\hat{f}_\lambda}(u^\star).$$

Since $u_1 \le u_n \le \bar{u}$ for all $n \ge 1$, we see that $u^\star \ne 0$ and $u^\star \le \bar{u}$. Therefore, $u^\star \in S_0^+ \subseteq \text{int}C_+$ (again we apply nonlinear regularity theory) and $u^\star = \sup S_0^+$. $\square$

Now we are ready to produce the first nodal solution of problem (P_λ). We fix $\lambda \in (0, \lambda^\star)$. Let $\lambda^\star > \lambda_1 > \lambda_2 > \lambda > 0$ and let $u_1 = u_{\lambda_1} \in S_{\lambda_1}^+, v_1 = v_{\lambda_1} \in S_{\lambda_1}^-$. It is clear from hypotheses H_2' that u_1 and v_1 are an upper solution and a lower solution of problem (P_λ), respectively. Proposition 10 implies that there exist $u_0 \in S_\lambda^+, v_0 \in S_\lambda^-$ such that $u_1 - u_0 \in \text{int}C_+$ and $v_1 - v_0 \in -\text{int}C_+$. Invoking Proposition 11, we infer that problem (P_λ) has a biggest positive solution $u_1^\star \in S_\lambda^+ \cap [v_1, u_1] \cap \text{int}C_+$ and a smallest negative solution $v_1^\star \in S_\lambda^- \cap [v_1, u_1] \cap (-\text{int}C_+)$.

Note that u_1 and $u_1^\star$ are an upper solution and a lower solution of problem (P_{λ_2}), respectively. We truncate the reaction term of problem (P_{λ_2}) at $\{u_1^\star, u_1\}$ and proceed as in the proof of Proposition 10. We can find $u_2 \in W_0^{1,p}(\Omega)$ such that

$$u_2 \in S_{\lambda_2}^+ \cap [u_1^\star, u_1] \cap \text{int}C_+,\ u_1 - u_2 \in \text{int}C_+ \text{ and } u_2 - u_1^\star \in \text{int}C_+. \tag{11}$$

Similarly, we can find $v_2 \in W_0^{1,p}(\Omega)$ such that

$$v_2 \in S_{\lambda_2}^- \cap [v_1, v_1^\star] \cap (-\text{int}C_+),\ v_1 - v_2 \in -\text{int}C_+ \text{ and } v_2 - v_1^\star \in -\text{int}C_+. \tag{12}$$

Let $\rho_1 = \max\{\|u_1\|_\infty, \|v_1\|_\infty\}$ and let $\xi_{\rho_1}^\lambda$ be as postulated by hypothesis H_2'(v). Consider the map $T_1 : W_0^{1,p}(\Omega) \mapsto (W_0^{1,p}(\Omega))^\star$ defined for all $u, v \in W_0^{1,p}(\Omega)$ by

$$\langle T_1(u), v\rangle = \int_\Omega (a(z, Du), Dv)_{\mathbb{R}^N} \mathrm{d}z + \int_\Omega \xi_{\rho_1}^\lambda |u|^{p-2} uv \mathrm{d}z.$$

Then, the inverse $T_1^{-1} : (W_0^{1,p}(\Omega))^\star \mapsto W_0^{1,p}(\Omega)$ of T_1 exists and it is continuous (see Motreanu-Tanaka [26, Proposition 9]). Let $\hat{h}_1$ be the truncation of h_1 at $\{v_1, u_1\}$, where $h_1(z, x) = f(z, x, \lambda) + \xi_{\rho_1}^\lambda |x|^{p-2} x$. We set $\hat{H}_1(z, x) = \int_0^x \hat{h}_1(z, t)\mathrm{d}t$ and consider the C^1-functional $\psi_1 : W_0^{1,p}(\Omega) \mapsto \mathbb{R}$ defined by

$$\psi_1(u) = \int_\Omega G(z, Du(z))\mathrm{d}z + \frac{\xi_{\rho_1}^\lambda}{p} \|u\|_p^p - \int_\Omega \hat{H}_1(z, u(z))\mathrm{d}z \quad \text{for } u \in W_0^{1,p}(\Omega).$$

From the proof of Proposition 10 we know that $K_{\psi_1} \subseteq [v_1, u_1]$. Let us define a map $B_1 : W_0^{1,p}(\Omega) \mapsto W_0^{1,p}(\Omega)$ by

$$B_1(u) = T_1^{-1}(\hat{h}_1(\cdot, u)) \quad \text{for } u \in W_0^{1,p}(\Omega).$$

By the Sobolev Embedding Theorem and the definition of $\hat{h}_1$, we see that B_1 is a compact operator (continuous and maps bounded sets into relatively compact sets) from $W_0^{1,p}(\Omega) \mapsto W_0^{1,p}(\Omega)$. Moreover, critical points of the functional ψ_1 correspond to fixed points of B_1. Invoking the regularity result in Lieberman [24, p. 320], we infer that $B_1(C_0^1(\overline{\Omega})) \subseteq C_0^1(\overline{\Omega})$.

Proposition 12. *If hypotheses* H_1 *and* H_2' *hold and* $\lambda \in (0, \lambda^\star)$, *then* $B_1(\pm C_+ \setminus \{0\}) \subseteq (\pm \mathrm{int} C_+)$ *and* $B_1([v_2, u_2]) \subseteq [v_2, u_2]$.

Proof. We first do the proof for $u \in C_+ \setminus \{0\}$. The proof is similar for $u \in -C_+ \setminus \{0\}$. Let $v = B_1(u)$. Then we have $v \in C_0^1(\overline{\Omega})$ and

$$-\mathrm{div} a(z, Dv) + \xi_{\rho_1}^\lambda |v|^{p-2} v = \hat{h}_1(z, u) \quad \text{for a. a. } z \in \Omega.$$

As before (see the proof of Proposition 10), taking $-v^- \in W_0^{1,p}(\Omega)$ as a test function, one gets, thanks to Lemma 2(iii) and $\mathrm{H}_2'(\mathrm{v})$,

$$\begin{aligned}\frac{c_1}{p-1} \|Dv^-\|^p &\le \int_\Omega (a(z, Dv), -Dv^-)_{\mathbb{R}^N} \mathrm{d}z + \int_\Omega \xi_{\rho_1}^\lambda |v^-|^p \mathrm{d}z \\ &= -\int_\Omega \hat{h}_1(z, u) v^- \mathrm{d}z \le 0.\end{aligned}$$

So, $v^- = 0$ a.e. in Ω. Evidently $\hat{h}_1(\cdot, u) \ne 0$ in $(W_0^{1,p}(\Omega))^\star$. Then $v \ne 0$ due to $v = T_1^{-1}(\hat{h}_1(\cdot, u))$. Hence $v \in C_+ \setminus \{0\}$ (see Lieberman [24, p. 320]). Note that

$$-\mathrm{div} a(z, Dv) + \xi_{\rho_1}^\lambda |v|^{p-2} v = \hat{h}_1(z, u) \ge 0 \quad \text{for a. a. } z \in \Omega.$$

Then, from the maximum principle of Pucci-Serrin [34, p. 111, 120], we infer that $v = B_1(u) \in \mathrm{int} C_+$.

Next, we show that $B_1(\hat{u}) \in [v_2, u_2]$ for every $\hat{u} \in [v_2, u_2]$. Indeed, we have $\hat{v} := B_1(\hat{u}) \in C_0^1(\overline{\Omega})$ and

$$-\mathrm{div} a(z, D\hat{v}) + \xi_{\rho_1}^\lambda |\hat{v}|^{p-2} \hat{v} = \hat{h}_1(z, \hat{u}) \quad \text{for a. a. } z \in \Omega. \tag{13}$$

On (13) we act with $w := (\hat{v} - u_2)^+ \in W_0^{1,p}(\Omega)$. Then, using hypothesis $\mathrm{H}_2'(\mathrm{v})$ and recalling that u_2 is an upper solution of problem (P_λ), we have

$$\begin{aligned}\langle A(\hat{v}), w\rangle + \int_\Omega \xi_{\rho_1}^\lambda |\hat{v}|^{p-2} \hat{v} w \mathrm{d}z &= \int_\Omega \hat{h}_1(z, \hat{u}) w \mathrm{d}z \le \int_\Omega \hat{h}_1(z, u_2) w \mathrm{d}z \\ &\le \langle A(u_2), w\rangle + \int_\Omega \xi_{\rho_1}^\lambda |u_2|^{p-2} u_2 w \mathrm{d}z,\end{aligned}$$

so

$$\langle A(\hat{v}) - A(u_2), (\hat{v} - u_2)^+\rangle + \int_\Omega \xi_{\rho_1}^\lambda (|\hat{v}|^{p-2}\hat{v} - |u_2|^{p-2} u_2)(\hat{v} - u_2)^+ \mathrm{d}z \le 0$$

and thus $|\{\hat{v} > u_2\}|_N = 0$, i.e., $\hat{v} \le u_2$. Similarly, acting on (13) with $(v_2 - \hat{v})^+ \in W_0^{1,p}(\Omega)$, we obtain $v_2 \le \hat{v}$. Therefore, $\hat{v} \in [v_2, u_2]$ and $B_1([v_2, u_2]) \subseteq [v_2, u_2]$. □

The proof of the following proposition can be shown by the argument in [7, Lemmas 3.7 and 3.8]. Thus, we omit the proof.

Proposition 13. *There exist $a_1, a_2 > 0$ such that, if $1 < p \leq 2$ then*

$$\langle \psi_1'(u), u - B_1(u) \rangle \geq a_1 \|u - B_1(u)\|^2 (\|u\| + \|B_1(u)\|)^{p-2}, \quad \|\psi_1'(u)\| \leq a_2 \|u - B_1(u)\|^{p-1}$$

and *if $p \geq 2$ then*

$$\langle \psi_1'(u), u - B_1(u) \rangle \geq a_1 \|u - B_1(u)\|^p, \quad \|\psi_1'(u)\| \leq a_2 \|u - B_1(u)\| (\|u\| + \|B_1(u)\|)^{p-2}$$

hold for all $u \in W_0^{1,p}(\Omega)$.

We need to construct a special descending flow of ψ_1. Since it is not assumed that B_1 is locally Lipschitz continuous, we first construct a locally Lipschitz continuous operator A_1 on $X_1 = W_0^{1,p}(\Omega) \setminus K_{\psi_1}$, which inherits the properties of B_1. The next result follows from a similar argument as in [7, Lemma 4.1] and [26, Lemma 17] using the properties of B_1 described in Propositions 12 and 13.

Proposition 14. *There exists a locally Lipschitz continuous operator $A_1 : X_1 \mapsto W_0^{1,p}(\Omega)$ with the following properties:*
(i) *$A_1(\pm C_+ \setminus \{0\}) \subseteq (\pm \mathrm{int} C_+)$ and $A_1([v_2, u_2]) \subseteq [v_2, u_2]$;*
(ii) *$\frac{1}{2} \|u - B_1(u)\| \leq \|u - A_1(u)\| \leq 2 \|u - B_1(u)\|$ for all $u \in X_1$;*
(iii) *for all $u \in X_1$ and a_1 as in Proposition 13, we have*

$$\langle \psi_1'(u), u - A_1(u) \rangle \geq \frac{a_1}{2} \|u - B_1(u)\|^2 (\|u\| + \|B_1(u)\|)^{p-2} \quad \text{if } 1 < p \leq 2,$$
$$\langle \psi_1'(u), u - A_1(u) \rangle \geq \frac{a_1}{2} \|u - B_1(u)\|^p \quad \text{if } p \geq 2.$$

Using A_1, we define a flow as follows. For $u \in X_1$, we consider the following initial value problem in X_1:

$$\begin{cases} \frac{\mathrm{d}\sigma(t,u)}{\mathrm{d}t} = -\sigma(t,u) + A_1(\sigma(t,u)), \\ \sigma(0,u) = u. \end{cases} \tag{14}$$

By the theory of ordinary differential equations in Banach spaces, (14) has a unique solution in X_1, still denoted by $\sigma(t,u)$, with right maximal interval of existence $[0, \tau(u))$. Note that $\psi_1(\sigma(t,u))$ is strictly decreasing in $t \in [0, \tau(u))$ and therefore $\sigma(t,u) (0 \leq t < \tau(u))$ is called a descending flow curve. The flow is given by

$$\sigma(t,u) = e^{-t} u + \int_0^t e^{-(t-s)} A_1(\sigma(s,u)) \mathrm{d}s \quad \text{for } 0 \leq t < \tau(u).$$

Definition 15. (See [25]) A nonempty subset M of $W_0^{1,p}(\Omega)$ is said to be invariant for the descending flow σ, or M is simply said to be invariant, if

$$\{\sigma(t,u) : 0 \leq t < \tau(u)\} \subseteq M \quad \text{for all } u \in M \setminus K_{\psi_1}.$$

Theorem 16. *If hypotheses H_1 and H_2' hold and $\lambda \in (0, \lambda^\star)$, then problem ($P_\lambda$) has a sequence of nodal solutions $\{w_i\}_{i \geq 1} \subseteq C_0^1(\overline{\Omega}) \setminus \{0\}$.*

Proof. We introduce the following set:

$$D_1 = \left\{ u \in C_0^1(\overline{\Omega}) \setminus K_{\psi_1} : \sigma(t,u) \in \mathrm{int}_{C_0^1(\overline{\Omega})} [v_2, u_2] \text{ for some } t \in [0, \tau(u)) \right\} \cup \mathrm{int}_{C_0^1(\overline{\Omega})} [v_2, u_2].$$

Evidently $0 \in D_1$, and by virtue of the continuity of $u \to \sigma(t,u)$, we see that D_1 is an open subset of $C_0^1(\overline{\Omega})$. As in He et al. [17], we can show that D_1 and ∂D_1 are invariant for the flow σ. Since ψ_1 is bounded from below on $[v_2, u_2]$, we infer that

$$d_1 := \inf_{u \in [v_2,u_2]} \psi_1(u) > -\infty.$$

Note that for $u \in D_1$, $t \mapsto \psi_1(\sigma(t,u))$ is decreasing. From the definition of D_1, we obtain that $\sigma(t,u) \in \mathrm{int}_{C_0^1(\overline{\Omega})}[v_2,u_2]$ for some $t \in [0,\tau(u))$, where $u \in D_1$. Then $\psi_1(u) \geq \psi_1(\sigma(t,u)) \geq d_1$ for all $u \in D_1$. This implies that $\psi_1(u)$ is bounded from below on D_1. Let $v \in \partial D_1$. Then we can find a sequence $\{v_n\}_{n\geq 1} \subseteq D_1$ such that $v_n \to v$ and $\psi_1(v_n) \to \psi_1(v)$. It follows that there exists $n_0 \geq 1$ such that $\psi_1(v_n) < \psi_1(v) + 1$ for all $n \geq n_0$. But $\psi_1(v_n) \geq d_1$ for all $n \geq 1$. Then

$$\inf_{v \in \partial D_1} \psi_1(v) \geq d_1 - 1.$$

Recalling that ψ_1 is coercive, it satisfies the Palais-Smale condition on $W_0^{1,p}(\Omega)$. By Theorem 2.1 of Liu-Sun [25], we can find $w_1 \in W_0^{1,p}(\Omega)$ such that $w_1 \in \partial D_1 \cap K_{\psi_1}$. Note that $w_1 \in C_0^1(\overline{\Omega})$ (see Lieberman [24, p. 320]). Evidently, $w_1 \neq 0$ and $w_1 \notin \mathrm{int}_{C_0^1(\overline{\Omega})}[v_2,u_2]$. Note that $K_{\psi_1} \subseteq [v_1,u_1]$. As before (see the proof of Proposition 10), we obtain $w_1 \in \mathrm{int}_{C_0^1(\overline{\Omega})}[v_1,u_1]$. To sum up we have that

$$w_1 \in \mathrm{int}_{C_0^1(\overline{\Omega})}[v_1,u_1] \quad \text{and} \quad w_1 \notin \mathrm{int}_{C_0^1(\overline{\Omega})}[v_2,u_2]. \tag{15}$$

From (11) and (12), we have $\{v_1^\star, u_1^\star\} \subseteq \mathrm{int}_{C_0^1(\overline{\Omega})}[v_2,u_2]$. By virtue of extremality of the solutions $v_1^\star$ and $u_1^\star$ in $[v_1,u_1]$, we conclude that w_1 is nodal.

We now produce the ith nodal solution for problem (P_λ) with $i = 2,3,\cdots$. Let $\lambda_i > \lambda_{i+1} > \lambda$, $i = 2,3,\cdots$. Invoking Proposition 11, we infer that problem (P_λ) has a biggest positive solution $u_i^\star \in S_\lambda^+ \cap [v_i,u_i] \cap \mathrm{int}C_+$ and a smallest negative solution $v_i^\star \in S_\lambda^- \cap [v_i,u_i] \cap (-\mathrm{int}C_+)$. Then u_i and $u_i^\star$ are an upper solution and a lower solution of problem ($P_{\lambda_{i+1}}$), respectively. As before (see (11)), we can find $u_{i+1} \in W_0^{1,p}(\Omega)$ such that

$$u_{i+1} \in S_{\lambda_{i+1}}^+ \cap [u_i^\star, u_i] \cap \mathrm{int}C_+,\ u_i - u_{i+1} \in \mathrm{int}C_+ \text{ and } u_{i+1} - u_i^\star \in \mathrm{int}C_+.$$

Similarly, we can find $v_{i+1} \in W_0^{1,p}(\Omega)$ such that

$$v_{i+1} \in S_{\lambda_{i+1}}^- \cap [v_i, v_i^\star] \cap (-\mathrm{int}C_+),\ v_i - v_{i+1} \in -\mathrm{int}C_+ \text{ and } v_{i+1} - v_i^\star \in -\mathrm{int}C_+.$$

Let $\rho_i = \max\{\|u_i\|_\infty, \|v_i\|_\infty\}$ and let $\xi_{\rho_i}^\lambda$ be as postulated by hypothesis $\mathrm{H}_2'(\mathrm{v})$. Consider the map $T_i : W_0^{1,p}(\Omega) \mapsto (W_0^{1,p}(\Omega))^\star$ defined for all $u, v \in W_0^{1,p}(\Omega)$ by

$$\langle T_i(u), v\rangle = \int_\Omega (a(z,Du),Dv)_{\mathbb{R}^N} \mathrm{d}z + \int_\Omega \xi_{\rho_i}^\lambda |u|^{p-2} uv \mathrm{d}z.$$

Then, the inverse $T_i^{-1} : (W_0^{1,p}(\Omega))^\star \mapsto W_0^{1,p}(\Omega)$ of T_i exists and it is continuous. Let $\hat{h}_i$ be the truncation of h_i at $\{v_i,u_i\}$, where $h_i(z.x) = f(z.x,\lambda) + \xi_{\rho_i}^\lambda |x|^{p-2} x$. We set $\hat{H}_i(z,x) = \int_0^x \hat{h}_i(z,t)\mathrm{d}t$ and consider the C^1-functional $\psi_i : W_0^{1,p}(\Omega) \mapsto \mathbb{R}$ defined by

$$\psi_i(u) = \int_\Omega G(z,Du(z))\mathrm{d}z + \frac{\xi_{\rho_i}^\lambda}{p}\|u\|_p^p - \int_\Omega \hat{H}_i(z,u(z))\mathrm{d}z \quad \text{for } u \in W_0^{1,p}(\Omega).$$

Then $K_{\psi_i} \subseteq [v_i,u_i]$. Let us define a map $B_i : W_0^{1,p}(\Omega) \mapsto W_0^{1,p}(\Omega)$ by

$$B_i(u) = T_i^{-1}(\hat{h}_i(\cdot, u)) \quad \text{for } u \in W_0^{1,p}(\Omega).$$

Note that B_i may merely be continuous. As before (see Proposition 14), we can show that there exists a locally Lipschitz continuous operator A_i on $X_i = W_0^{1,p}(\Omega) \setminus K_{\psi_i}$ such that $A_i(\pm C_+ \setminus \{0\}) \subseteq (\pm\mathrm{int}C_+)$ and $A_i([v_{i+1}, u_{i+1}]) \subseteq [v_{i+1}, u_{i+1}]$. For $u \in X_i$, we consider the following initial value problem in X_i:

$$\begin{cases} \frac{\mathrm{d}\sigma(t,u)}{\mathrm{d}t} = -\sigma(t,u) + A_i(\sigma(t,u)), \\ \sigma(0,u) = u. \end{cases} \tag{16}$$

By ODE theory, (16) has a unique solution in X_i, still denoted by $\sigma(t,u)$, with right maximal interval of existence $t \in [0, \tau(u))$. Let

$$D_i = \left\{u \in C_0^1(\overline{\Omega}) \setminus K_{\psi_i} : \sigma(t,u) \in \mathrm{int}_{C_0^1(\overline{\Omega})}\,[v_{i+1}, u_{i+1}] \text{ for some } t \in [0, \tau(u))\right\} \cup \mathrm{int}_{C_0^1(\overline{\Omega})}\,[v_{i+1}, u_{i+1}].$$

Reasoning as above, we can find $w_i \in \partial D_i \cap K_{\psi_i}$ such that w_i is nodal and

$$w_i \in \mathrm{int}_{C_0^1(\overline{\Omega})}\,[v_i, u_i] \quad \text{and} \quad w_i \notin \mathrm{int}_{C_0^1(\overline{\Omega})}\,[v_{i+1}, u_{i+1}]. \tag{17}$$

By (15) and (17), we have $w_i \neq w_k$ for $i = 2, 3, \cdots, k = 1, 2, \cdots, i-1$. Hence, problem ($P_\lambda$) admits a sequence of nodal solutions $\{w_i\}_{i \geq 1} \subseteq C_0^1(\overline{\Omega}) \setminus \{0\}$. This proves the Theorem 16. $\square$

Acknowledgments

The authors thank the very knowledgeable referee very much for his/her valuable comments and helpful suggestions.

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