J. Math. Anal. Appl. 488 (2020) 124057

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Two general families of integer–valued polynomials associated with finite trigonometric sums

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A R T I C L E   I N F O

Article history:
Received 28 September 2019
Available online 19 March 2020
Submitted by B. Bongiorno

Keywords:
Integer–valued polynomials
Trigonometric sums
Finite summation
Binomial transform
Cotangent sums
Cosecant sums

A B S T R A C T

A dozen families of integer–valued polynomials arising in finite summation of various trigonometric sums are known and all of them were deduced through numerical analysis methods. Here, using simple arguments commonly applied in work with polynomial sequences, we examined such expressions in full detail as well as in a systematic and unified manner. Two new very general integer–valued polynomial families (along with six other families derived from them, also integer–valued, including three previously studied) were obtained and they are related to each other by a binomial transform of sequences and associated with certain cosecant and cotangent sums.

1. Introduction

A polynomial $P_n(x) := a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, $n \in \mathbb{N} := \{1, 2, 3, \ldots, \}$, is said to be *integer–valued* (or integral–valued) if it takes an integer value whenever x is an integer. These polynomials, of which the most known example is the sequence of binomial coefficients $\{\binom{x}{n}\}_{n=0}^{\infty}$ with $\binom{x}{0} = 1$ and $\binom{x}{n} = x(x-1)(x-2)\cdots(x-n+1)/n!$, were introduced by Pòlya and have been extensively studied in the meantime. It is possible to characterize all integer–valued polynomials by using $\binom{x}{n}$: every integer-valued polynomial can be written as an integer linear combination (combination with coefficients from the set of integers) of binomial coefficients in exactly one way. The converse is also true (see, for instance, [15, pp. 129–133] and [5]).

In recent years, it has been shown that some finite trigonometric sums give rise to integer–valued polynomials. Byrne and Smith [4, Theorems 1 and 2], resorting to the Lagrange interpolation formula, established that evaluation of the cotangent sums,

$$p_n(q) = \sum_{p=1}^{q} (-1)^{p-1} \cot^{2n-1}\left(\frac{(2p-1)\pi}{4\,q}\right) \tag{1.1}$$

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https://doi.org/10.1016/j.jmaa.2020.124057

and

$$q_n(q) = \sum_{p=1}^{q} \cot^{2n}\left(\frac{(2p-1)\pi}{4\,q}\right), \tag{1.2}$$

with n and q non–negative integers, leads to two sequences of integral–valued polynomials whose coefficients could be determined recursively from certain relations. Six more such mutually different, and, in this instance, explicitly defined polynomial sequences were found by Hassan [13, Remark 4.5 (5), Equations 3.44, 3.46, 3.48, 3.50, 4.5, 4.22, 4.23 and 4.24] using a sampling theorem associated with second–order discrete eigenvalue problem and through considering

$$\sum_{p=1}^{q} \sec^{2n}\left(\frac{p\pi}{2\,q+1}\right) \qquad \text{and} \qquad \sum_{p=1}^{q} \tan^{2n}\left(\frac{p\pi}{2\,q+1}\right) \tag{1.3}$$

as well as

$$\sum_{p=1}^{q} \cot^{2n}\left(\frac{(2p-1)\pi}{2\,(2q+1)}\right) \qquad \text{and} \qquad \sum_{p=1}^{q} \csc^{2n}\left(\frac{(2p-1)\pi}{2\,(2q+1)}\right) \tag{1.4}$$

and another four trigonometric sums. Annaby and Hassan [2, Theorem 4.1], making use of the Hermite interpolation, deduced additional four, not necessarily distinct, recursively defined sequences of integer–valued polynomials associated with

$$\sum_{p=1}^{q} (-1)^{p-1} \sin\left(\frac{(2p-1)\pi}{q}\right) \operatorname{trig}^{2\,n+2}\left(\frac{(2p-1)\pi}{4\,q}\right) \qquad (\operatorname{trig} = \csc,\, \sec) \tag{1.5}$$

and

$$\sum_{p=1}^{q} (-1)^{p-1} \sin\left(\frac{(2p-1)\pi}{q}\right) \operatorname{trig}^{2\,n}\left(\frac{(2p-1)\pi}{4\,q}\right) \qquad (\operatorname{trig} = \cot,\, \tan). \tag{1.6}$$

Finally, interpolation was also used by Annaby and Asharabi [1] to obtain several integer–valued polynomials.

Herein, it was aimed to avoid utilizing somewhat specialized methods, examples being the Lagrange and Hermite interpolation, and consider integral–valued polynomials arising in finite summation of various trigonometric sums by making use of more familiar arguments commonly used in work with polynomials in general. By doing so, the main intention was to provide, in a general context as well as in a systematic and unified manner, more straightforward proofs for some already known and to generate and prove new results.

2. Statement of main results

Two definitions are needed before we can proceed. When two sequences, $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$, are related as

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k \quad \text{and} \quad a_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} b_k \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}) \tag{2.1}$$

we say that the sequence $\{b_n\}_{n=0}^{\infty}$ is the (direct) binomial transform of the sequence $\{a_n\}_{n=0}^{\infty}$ and that $\{a_n\}_{n=0}^{\infty}$ is the inverse binomial transform of $\{b_n\}_{n=0}^{\infty}$ [19, pp. 13 and 22]. A polynomial sequence is a sequence of polynomials indexed by the non–negative integers $0, 1, 2, 3, \ldots$, in which each index is equal to the degree of the corresponding polynomial.

Observe that, throughout the text, as usual, δ_{ij} stands for the Kronecker delta and we set an empty sum to be zero. Our main results are as follows.

Theorem 1. *Let us define two sequences of real functions in x, $\{A_{2n}(x)\}_{n=0}^{\infty}$ and $\{B_{2n}(x)\}_{n=0}^{\infty}$, by generating relations*

$$G_A(x,t) = \sum_{n=0}^{\infty} A_{2n}(x)\, t^{2n} \qquad \text{and} \qquad G_B(x,t) = \sum_{n=0}^{\infty} B_{2n}(x)\, t^{2n},$$

where

$$G_A(x,t) = \frac{x}{1+t^2}\left[1 + t\tan(x\arctan t)\right]$$

and

$$G_B(x,t) = x + tx\frac{\tan(x\arcsin t)}{\sqrt{1-t^2}}.$$

Then, the following holds:

a) *The sequences $\{A_{2n}(x)\}_{n=0}^{\infty}$ and $\{B_{2n}(x)\}_{n=0}^{\infty}$ are related to each other through the binomial transform of sequences*

$$B_{2n}(x) = \sum_{k=0}^{n} \binom{n}{k} A_{2k}(x) \quad \text{and} \quad A_{2n}(x) = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} B_{2k}(x) \quad (n \in \mathbb{N}_0),$$

while their ordinary generating functions, $G_A(x,t)$ and $G_B(x,t)$, are related to each other with

$$G_B(x,t) = \frac{1}{1-t^2}\, G_A\left(x, \frac{t}{\sqrt{1-t^2}}\right) \quad \text{and} \quad G_A(x,t) = \frac{1}{1+t^2}\, G_B\left(x, \frac{t}{\sqrt{1+t^2}}\right).$$

b) *$\{A_{2n}(x)\}_{n=0}^{\infty}$ and $\{B_{2n}(x)\}_{n=0}^{\infty}$ are polynomial sequences defined explicitly by*

$$A_{2n}(x) = (-1)^n\, x \sum_{k=0}^{n} \binom{n}{k} \sum_{l=0}^{k-1} 2^{2k-2l-2} \sum_{m=-l}^{l} (-1)^m \binom{2l}{l+m} \binom{(m-1)\,x + k - 1}{2k-1} + (-1)^n x$$

and by

$$B_{2n}(x) = (-1)^n\, x \sum_{l=0}^{n-1} 2^{2n-2l-2} \sum_{m=-l}^{l} (-1)^m \binom{2l}{l+m} \binom{(m-1)\,x + n - 1}{2n-1} + \delta_{0n} x.$$

c) *$\{A_{2n}(x)\}_{n=0}^{\infty}$ and $\{B_{2n}(x)\}_{n=0}^{\infty}$ are sequences of integer-valued polynomials with rational coefficients.*

We have that $A_0(x) = B_0(x) = x$, while, for any positive integer n, $A_{2n}(x)$ and $B_{2n}(x)$ are polynomials in a variable x of degree exactly equal to $2n$. Concerning the sequence $\{B_{2n}(x)\}_{n=1}^{\infty}$, each $B_{2n}(x)$ is even and has a root at $x = 0$. In case of $\{A_{2n}(x)\}_{n=1}^{\infty}$ each $A_{2n}(x)$ has a root at $x = 0$ and at $x = 1$.

Remark 1. We remark that the polynomial sequence $\{B_{2n}(x)\}_{n=0}^{\infty}$ could be defined by $B_0(x) = x$ and by means of $G_B(x,t) - x = t\,x \tan(x \arcsin t)/\sqrt{1-t^2}$ to generate all other polynomials in $\{B_{2n}(x)\}_{n=1}^{\infty}$. However, for the purposes of the binomial transform, it is necessary to work with $\{B_{2n}(x)\}_{n=0}^{\infty}$ and to employ the complete generating function $G_B(x,t)$.

Corollary 1. *Assume n to be any positive integer and consider the polynomial sequences $\{A_{2n}(x)\}_{n=0}^{\infty}$ and $\{B_{2n}(x)\}_{n=0}^{\infty}$ defined by Theorem 1.*

a) *Let $A_{2n}(x) = x\,a_{2n-1}(x)$ and $B_{2n}(x) = x\,b_{2n-1}(x)$.*
Then, $\{a_{2n-1}(x)\}_{n=1}^{\infty}$ and $\{b_{2n-1}(x)\}_{n=1}^{\infty}$ are integer-valued polynomial sequences.

b) *Let*

$$\left.\begin{matrix} A_{2n}^{*}(x) \\ \\ B_{2n}^{*}(x) \end{matrix}\right\} := \left\{\begin{matrix} \frac{1}{2} A_{2n}(2x) \\ \\ \frac{1}{2} B_{2n}(2x) \end{matrix}\right.$$

and

$$\left.\begin{matrix} A_{2n}^{**}(x) \\ \\ B_{2n}^{**}(x) \end{matrix}\right\} := \left\{\begin{matrix} \frac{1}{2} A_{2n}(2x+1) \\ \\ \frac{1}{2}\left[B_{2n}(2x+1) - 1\right] \end{matrix}\right. .$$

Then, $\{A_{2n}^{*}(x)\}_{n=1}^{\infty}$, $\{B_{2n}^{*}(x)\}_{n=1}^{\infty}$, $\{A_{2n}^{**}(x)\}_{n=1}^{\infty}$ and $\{B_{2n}^{**}(x)\}_{n=1}^{\infty}$ are sequences of integer-valued polynomials with rational coefficients.

Theorem 2. *Let $\{A_{2n}(x)\}_{n=0}^{\infty}$ and $\{B_{2n}(x)\}_{n=0}^{\infty}$ be the integer–valued polynomials defined as in Theorem 1.*
Then, for any positive integer n and q, we have that

$$\sum_{p=1}^{q} \cot^{2n}\left(\frac{(2p-1)\pi}{2\,q}\right) = A_{2n}(q) \quad \textit{and} \quad \sum_{p=1}^{q} \csc^{2n}\left(\frac{(2p-1)\pi}{2\,q}\right) = B_{2n}(q).$$

Corollary 2. *Under the same assumptions as in Theorem 2, for a positive integer n and q, the following summations hold*

$$\sum_{\substack{p=1 \\ q \text{ is odd}}}^{q} \tan^{2n}\left(\frac{p\pi}{q}\right) = A_{2n}(q), \qquad \sum_{\substack{p=1 \\ q \text{ is odd}}}^{q} \sec^{2n}\left(\frac{p\pi}{q}\right) = B_{2n}(q),$$

$$\sum_{p=1}^{q} \cot^{2n}\left(\frac{(2p-1)\pi}{4\,q}\right) = \sum_{p=1}^{q} \tan^{2n}\left(\frac{(2p-1)\pi}{4\,q}\right) = \frac{1}{2}\,A_{2n}(2\,q),$$

$$\sum_{p=1}^{q} \cot^{2n}\left(\frac{(2p-1)\pi}{2\,(2\,q+1)}\right) = \sum_{p=1}^{q} \tan^{2n}\left(\frac{p\pi}{2\,q+1}\right) = \sum_{p=1}^{q} \tan^{2n}\left(\frac{2\,p\pi}{2\,q+1}\right)$$
$$= \sum_{p=1}^{q} \tan^{2n}\left(\frac{(2\,p-1)\pi}{2\,q+1}\right) = \frac{1}{2}\,A_{2n}(2\,q+1),$$

$$\sum_{p=1}^{q} \csc^{2n}\left(\frac{(2p-1)\pi}{4\,q}\right) = \sum_{p=1}^{q} \sec^{2n}\left(\frac{(2p-1)\pi}{4\,q}\right) = \frac{1}{2}\,B_{2n}(2\,q),$$

$$\sum_{p=1}^{q} \csc^{2n}\left(\frac{(2p-1)\pi}{2\,(2\,q+1)}\right) = \sum_{p=1}^{q} \sec^{2n}\left(\frac{p\pi}{2\,q+1}\right) = \sum_{p=1}^{q} \sec^{2n}\left(\frac{2\,p\pi}{2\,q+1}\right)$$
$$= \sum_{p=1}^{q} \sec^{2n}\left(\frac{(2\,p-1)\pi}{2\,q+1}\right) = \frac{1}{2}\,[B_{2n}(2\,q+1)-1]\,.$$

Remark 2. Five (among eight) sequences of integral–valued polynomials given in Theorem 1 and Corollary 1 are new: $\{A_{2n}(x)\}_{n=0}^{\infty}$, $\{B_{2n}(x)\}_{n=0}^{\infty}$, $\{a_{2n-1}(x)\}_{n=1}^{\infty}$, $\{b_{2n-1}(x)\}_{n=1}^{\infty}$, and $\{B^{*}_{2n}(x)\}_{n=0}^{\infty}$. On the other hand, $\{A^{*}_{2n}(x)\}_{n=0}^{\infty}$, the sequence associated with the sum (1.2), was deduced by Byrne and Smith [4, Theorem 2], while the polynomial sequences $\{A^{**}_{2n}(x)\}_{n=0}^{\infty}$ and $\{B^{**}_{2n}(x)\}_{n=0}^{\infty}$ associated with the sums in (1.4) were obtained by Hassan (see, respectively, Theorem 4.3 and Eq. (3.18) together with Remark 4.5 (2) in [13, pp. 822 and 817]. Clearly, all these three known sequences along with the new one $\{B^{*}_{2n}(x)\}_{n=0}^{\infty}$ are straightforward special cases of more general, and novel, results stated by Theorem 1.

Remark 3. Some of the above sums were previously evaluated in closed form by Hassan: $\sum_{p=1}^{q}\csc^{2n}((2p-1)\pi/(4\,q))$ [13, p. 817, Eq. (3.7)], $\sum_{p=1}^{q}\cot^{2n}((2p-1)\pi/(4\,q))$ [13, p. 817, Eq. (3.8)], $\sum_{p=1}^{q}\csc^{2n}((2p-1)\pi/(2\,(2\,q+1))$ [13, p. 818, Eq. (3.18)] and $\sum_{p=1}^{q}\cot^{2n}((2p-1)\pi/(2\,(2\,q+1)))$ [13, p. 818, Eq. (3.19)].

However, although no recorded summation of $c_{2\,n}(q) = \sum_{p=1}^{q}\cot^{2n}((2p-1)\pi/(2\,q))$ and $s_{2\,n}(q) = \sum_{p=1}^{q}\csc^{2n}((2p-1)\pi/(2\,q))$ may be found in the literature, in fact, these, and, moreover, all sums given by Corollary 2 can be summed by means of already known results. Namely, by $c_{2\,n}(q) = C_{2n}(2\,q) - C_{2n}(q)$ and $s_{2\,n}(q) = S_{2n}(2\,q) - S_{2n}(q)$, the sums $c_{2\,n}(q)$ and $s_{2\,n}(q)$ are related to $C_{2n}(q) = \sum_{p=1}^{q-1}\cot^{2n}(p\pi/q)$ and $S_{2n}(q) = \sum_{p=1}^{q-1}\csc^{2n}(p\pi/q)$ whose summations are known. $C_{2n}(q)$ and $S_{2n}(q)$ were first closed–form summed by Chu and Marini [6, p. 137 and p. 126]. Furthermore, $C_{2n}(q)$ was differently evaluated by Cvijović and Klinowski [8, p. 156], Cvijović, Klinowski and Srivastava [9, p. 251, Corollary 1], Berndt and Yeap [3, p. 364] and Cvijović [7, p. 1137], while summations for $S_{2n}(q)$ were reported by Cvijović, Klinowski and Srivastava [9, p. 251, Corollary 1] as well as Cvijović and Srivastava [10, Eq. (2.8)]. A general review of such series can be found in [11].

To summarize, two explicit formulae for the polynomial sequences $\{A_{2n}(x)\}_{n=0}^{\infty}$ and $\{B_{2n}(x)\}_{n=0}^{\infty}$ given by Theorem 1(b), provide, in a relatively simple manner, closed-form summations of 16 families of finite trigonometric sums (which, in essence, amounts to eight distinct families) specified in Theorem 2 and Corollary 2. These summations, in comparison with existing ones, apart from being deduced through a systematic and unified approach, are also in a more compact and simple form.

3. Proof of the results

The Gauss (or ordinary) hypergeometric ${}_2F_1$ is, as usual, defined by the following hypergeometric series [14, Ch. 15, p. 384, Eq. (15.2.1)]

$$ {}_2F_1\left[\begin{matrix}\alpha,\,\beta\\ \gamma\end{matrix};z\right] := \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!} \tag{3.1}$$

on the disk $|z|<1$, and by analytic continuation elsewhere. Here, a variable z and, respectively, numerator and denominator parameters α,β and γ are, in general, complex numbers, provided that γ is not zero or a negative integer, while $(\cdot)_n$ denotes the Pochhammer symbol [14, Ch. 5, p. 136, Eqs. (5.2.4) and (5.2.5)] given by

$$(\lambda)_n = (-1)^n n!\binom{-\lambda}{n} = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases}1 & (n=0)\\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n\in\mathbb{N})\end{cases} \tag{3.2}$$

in terms of the familiar (generalized) binomial coefficient $\binom{}{}$ or classical gamma function Γ (when $\lambda \neq 0, -1, -2, \ldots$). On the circle of convergence, $|z| = 1$, the series converges absolutely if $\Re(\gamma - \alpha - \beta) > 0$.

In order to establish the main results we need several (known) auxiliary, rather heterogeneous results collected separately here for a better clarity of the proofs given below and proved for the sake of a self-contained presentation.

Proposition 1. *Let two even-indexed sequences of real functions of a single real variable, $\{a_{2n}(x)\}_{n=0}^{\infty}$ and $\{b_{2n}(x)\}_{n=0}^{\infty}$, be defined by generating relations*

$$\mathcal{A}(x,t) = \sum_{n=0}^{\infty} a_{2n}(x)\, t^{2n} \qquad \text{and} \qquad \mathcal{B}(x,t) = \sum_{n=0}^{\infty} b_{2n}(x)\, t^{2n}. \tag{3.3}$$

Then, these sequences are a binomial transform pair, as defined in (2.1), if and only if their ordinary generating functions, $\mathcal{A}(x,t)$ and $\mathcal{B}(x,t)$, are related as

$$\mathcal{B}(x,t) = \frac{1}{1-t^2}\, \mathcal{A}\left(x, \frac{t}{\sqrt{1-t^2}}\right) \quad \text{and} \quad \mathcal{A}(x,t) = \frac{1}{1+t^2}\, \mathcal{B}\left(x, \frac{t}{\sqrt{1+t^2}}\right). \tag{3.4}$$

Proof. Suppose that $\{a_{2n}(x)\}_{n=0}^{\infty}$ and $\{b_{2n}(x)\}_{n=0}^{\infty}$ are the pair, then the first relation between generating functions in (3.4) could be deduced like this

$$\begin{aligned}\frac{1}{1-t^2}\, \mathcal{A}\left(x, \frac{t}{\sqrt{1-t^2}}\right) &= \sum_{m=0}^{\infty} a_{2m}(x)\, t^{2m} \frac{1}{(1-t^2)^{m+1}} \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{2m}(x)\, t^{2\,m} \binom{n+m}{m} t^{2\,n} \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{m} a_{2(m-n)}(x)\, t^{2\,(m-n)} \binom{m}{m-n} t^{2\,n} \\ &= \sum_{m=0}^{\infty} t^{2\,m} \sum_{n=0}^{m} \binom{m}{n} a_{2n}(x) = \sum_{m=0}^{\infty} b_{2m}(x)\, t^{2\,m} = \mathcal{B}(x,t),\end{aligned}$$

while the second one (3.4) follows in the exact same manner.

Two summations are employed here, $1/(1-t)^{m+1} = \sum_{n=0}^{\infty} \binom{n+m}{m} t^n$, $|t| < 1$, [16, p. 564, Eq. (5.2.11.3)] and the elementary double series identity $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k, n-k)$ [18, p. 57, Eq. (2)]. The former is available upon mth derivation of the both sides of the well–known $1/(1-t) = \sum_{n=0}^{\infty} t^n$, $|t| < 1$, with respect to t, whereas the simple short proof of the latter can be found in [18, p. 57].

Conversely, suppose that $\mathcal{A}(x,t)$ and $\mathcal{B}(x,t)$ are related as proposed, then, by an analogous argument, it is easy to conclude that $\{a_{2n}(x)\}_{n=0}^{\infty}$ and $\{b_{2n}(x)\}_{n=0}^{\infty}$ form a binomial transform pair. □

Proposition 2. *For any real θ and m a non-negative integer, we have that*

$$\sin^{2m+1}\theta \cos\theta = \frac{1}{2^{2m+1}} \sum_{k=-m}^{m} (-1)^{k-1} \binom{2\,m}{m+k} \sin\left[2\,(k-1)\theta\right]. \tag{3.5}$$

Proof. To derive the formula (3.5), it suffices to use the following identity

$$(-1)^m 2^{2\,m+2}\imath \sin^{2\,m+1}\theta\cos\theta = \left(e^{\imath\theta} - e^{-\imath\theta}\right)^{2m+1}\left(e^{\imath\theta} + e^{-\imath\theta}\right)$$

in conjunction with the last line of this expression

$$\begin{aligned}
RHS := &\left(e^{\imath\theta} - e^{-\imath\theta}\right)^{2\,m+1}\left(e^{\imath\theta} + e^{-\imath\theta}\right) = \left(e^{\imath\theta} - e^{-\imath\theta}\right)^{2\,m}\left(e^{2\,\imath\theta} - e^{-2\imath\theta}\right) \\
&= \left(e^{2\imath\theta} - e^{-2\imath\theta}\right)\sum_{k=0}^{2\,m}(-1)^{2m-k}\binom{2m}{k}e^{-2\imath\theta(m-k)} \\
&= \sum_{k=0}^{2\,m}(-1)^{2m-k}\binom{2\,m}{k}\left[e^{-2\imath\theta(m-k-1)} - e^{-2\imath\theta(m-k+1)}\right] \\
&= \sum_{k=0}^{2\,m}(-1)^{2m-k}\binom{2\,m}{k}(-1)\imath\left[\sin\left(2(m-k-1)\theta\right) - \sin\left(2(m-k+1)\theta\right)\right] \\
&= 2\,(-1)^m\imath\sum_{k=0}^{2\,m}(-1)^{m-k-1}\binom{2\,m}{k}\sin\left[2(m-k-1)\theta\right]. \quad \square
\end{aligned}$$

Proposition 3. *(See, for instance, [17, p. 389, Eq. (7.3.1.94)].) If $\theta \neq (2\,n+1)\frac{\pi}{2}$ for any integer n, then*

$$\frac{2\sin(a\theta)}{a\sin(2\,\theta)} = {}_2F_1\left[\begin{matrix}1+\frac{a}{2},\, 1-\frac{a}{2} \\ \frac{3}{2}\end{matrix};\sin^2\theta\right]. \tag{3.6}$$

Proof. Making use of the familiar binomial expansion $(u+v)^\lambda = \sum_{k=0}^{\infty}\binom{\lambda}{k}u^k v^{\lambda-k}$, the Euler identity $e^{\imath\theta} = \cos\theta + \imath\sin\theta$ and the Moivre identity $\left(e^{\imath\theta}\right)^\lambda = e^{\imath\lambda\theta}$, there is no difficulty in showing that

$$\frac{1}{\cos^{a-1}\theta}\,\frac{\sin(a\theta)}{\sin\theta} = \sum_{k=0}^{\infty}\binom{a}{2\,k+1}(-\tan^2\theta)^k.$$

Next, we need the following identities involving the Pochhammer symbol, $(a)_{2m} = 2^{2m}(a/2)_m(1/2+a/2)_m$, $(1+a)_m = (a)_m(a+m)/a$ as well as its special value $\left(\frac{3}{2}\right)_m = 2^{-2m}(2m+1)!/m!$, which are all valid for each non-negative integer m and can be derived (or, use identities tabulated in [17, pp. 647–648]) starting from the definition in (3.2) by applying known properties of the gamma function, for instance, the Legendre duplication formula. Now, since we have

$$\binom{a}{2\,k+1} = \frac{a-2\,k}{2\,k+1}\binom{a}{2\,k} = \frac{a-2\,k}{(2\,k+1)!}\,(-a)_{2k} = \frac{a(1-a)_{2k}}{(2\,k+1)!} = \frac{\left(1-\frac{a}{2}\right)_k\left(\frac{1}{2}-\frac{a}{2}\right)_k}{\left(\frac{3}{2}\right)_k}\,\frac{a}{k!},$$

and, on taking into account (3.1), we get

$$\begin{aligned}
\frac{1}{\cos^{a-1}\theta}\,\frac{\sin(a\theta)}{a\,\sin\theta} &= \sum_{k=0}^{\infty}\frac{\left(1-\frac{a}{2}\right)_k\left(\frac{1}{2}-\frac{a}{2}\right)_k}{\left(\frac{3}{2}\right)_k}\,\frac{(-\tan^2\theta)^k}{k!} \\
&= {}_2F_1\left[\begin{matrix}1-\frac{a}{2},\, \frac{1}{2}-\frac{a}{2} \\ \frac{3}{2}\end{matrix};\frac{\sin^2\theta}{\sin^2\theta-1}\right].
\end{aligned}$$

Finally, the expansion (3.6) is deduced upon transforming the obtained Gauss hypergeometric function by the Pfaff transformation [14, p. 390, Eq. (15.8.1)]

$${}_2F_1\left[\begin{matrix}\beta,\, \gamma-\alpha \\ \gamma\end{matrix};\frac{z}{z-1}\right] = (1-z)^\beta\,{}_2F_1\left[\begin{matrix}\alpha,\, \beta \\ \gamma\end{matrix};z\right]. \quad \square$$

Proposition 4. *We have*

$$
{}_2F_1\left[\begin{matrix}1-a,\,1+a\\ \frac{3}{2}\end{matrix};z^2\right]=\frac{1}{4\,a\,z^2}\sum_{m=1}^{\infty}(-1)^{m-1}\binom{a+m-1}{2\,m-1}(2z)^{2m}. \tag{3.7}
$$

Proof. Like in the proof of Proposition 3, we need $(1+a)_m=(a)_m(a+m)/a$ and $\left(\frac{3}{2}\right)_m=2^{-2m}(2m+1)!/m!$, while $(1-a)_m=(-1)^m(a-m)_{2m}/(a)_m$ follows from the definition in (3.2) in conjunction with the gamma function properties (or, use tabulated identities in [17, pp. 647–648]). Then

$$
\begin{aligned}
a\,\frac{(1+a)_m(1-a)_m}{\left(\frac{3}{2}\right)_m m!}&=(-1)^m\frac{(a-m)_{2m}(a+m)}{(2m+1)!}=(-1)^m\frac{(a-m)_{2m+1}}{(2m+1)!}\\
&=(-1)^{m-1}\frac{(-a-m)_{2m+1}}{(2m+1)!}=(-1)^m\binom{a+m}{2\,m+1}.
\end{aligned}
$$

In other words, the next holds

$$
4\,a\,z^2{}_2F_1\left[\begin{matrix}1-a,\,1+a\\ \frac{3}{2}\end{matrix};z^2\right]=\sum_{m=0}^{\infty}(-1)^m\binom{a+m}{2\,m+1}(2z)^{2m+2},
$$

hence the sought formula (3.7). □

Proposition 5. *For any real θ and δ, $\theta-\delta\neq 2\,k\,\pi$, k being an integer, the following summation holds true*

$$
\sum_{k=0}^{n-1}\frac{\sin\theta\cos\theta}{\cos^2\theta-\cos^2\left(\delta+\frac{2\,k\pi}{n}\right)}=\frac{n\sin(2\,n\theta)}{\cos(2\,n\theta)-\cos(2\,n\delta)}\qquad(n\in\mathbb{N}). \tag{3.8}
$$

Proof. Note that the factorization of the polynomial $P_{2\,n}(x)=x^{2\,n}-2\,x^n\cos(n\delta)+1$

$$
x^{2\,n}-2\,x^n\cos(n\delta)+1=\prod_{k=0}^{n-1}\left[x^2-2\,x\cos\left(\delta+\frac{2\,k\pi}{n}\right)+1\right]
$$

(see, for instance, [12, p. 41, Eq. (1.395)], or deduce the factorization upon solving the equation $P_{2\,n}(x)=0$) yields the partial–fraction decomposition

$$
\sum_{k=0}^{n-1}\frac{1-x^2}{x^2-2\,x\cos\left(\delta+\frac{2\,k\pi}{n}\right)+1}=\frac{n(1-x^{2\,n})}{x^{2\,n}-2\,x^n\cos(n\delta)+1},
$$

which enables straightforward deduction of the summation (3.8) upon putting $x=e^{2\,\imath\theta}$ and $\cos^2\theta=(1+x)^2/(4\,x)$ in the expression Lhs = Rhs, with

$$
\mathrm{Lhs}:=\sum_{k=0}^{n-1}\frac{(1+x)^2}{1+x^2-2\,x\cos\left(2\,\delta+\frac{2\,k\pi}{n}\right)}=\sum_{k=0}^{n-1}\frac{(1+x)^2/(4x)}{(1+x)^2/(4x)-\cos^2\left(\delta+\frac{k\pi}{n}\right)}
$$

and

$$
\mathrm{Rhs}:=\frac{n(1-x^{2\,n})(1+x)^2}{(x^{2\,n}-2\,x^n\cos(2\,n\delta)+1)}=-\frac{n(x^n-x^{-n})}{(x^n+x^{-n}-2\,\cos(2\,n\delta))}\frac{1+x}{1-x}.\qquad\square
$$

Now we are ready to prove our main results. The fact that $\{A_{2n}(x)\}_{n=0}^{\infty}$ and $\{B_{2n}(x)\}_{n=0}^{\infty}$ are a transform pair (see Theorem 1(a)) much simplifies the proof of results stated in Section 2. We choose to consider $\{B_{2n}(x)\}_{n=0}^{\infty}$ in detail, assertions for $\{A_{2n}(x)\}_{n=0}^{\infty}$ then follow by the binomial transform. The formal series method used by Wang and Zheng [20] is provided here with necessary proofs and all details.

Proof of Theorem 1. To prove **Part a**, recall that Proposition 1 provides a necessary and sufficient condition in terms of generating functions for two even-indexed sequences related by the binomial transform. It is therefore enough to simply verify that there exist such relations between $G_A(x,t)$ and $G_B(x,t)$, which is straightforward, so as to conclude that the sequences of functions, $\{A_{2n}(x)\}_{n=0}^{\infty}$ and $\{B_{2n}(x)\}_{n=0}^{\infty}$, are related to each other through the binomial transform of sequences.

Part b. To deduce the required explicit formulae, it is sufficient to consider only the sequence $\{B_{2n}(x)\}_{n=0}^{\infty}$ using $G_B(x,t)-x$ (see Remark 1) since the formula for $\{A_{2n}(x)\}_{n=0}^{\infty}$ follows at once by Theorem 1(a) from the formula for $\{B_{2n}(x)\}_{n=0}^{\infty}$.

First, observe that the formal expansion

$$\frac{\sin(2\,x\theta)}{\cos(2\,x\theta)-\cos(2\,x\delta)} = \sum_{k=0}^{\infty} \csc^{2\,k+2}(x\delta)\sin^{2\,k+1}(x\theta)\cos(x\theta) \tag{3.9}$$

is readily available from

$$\frac{\sin(2\,x\theta)}{\cos(2\,x\theta)-\cos(2\,x\delta)} = \frac{\sin(x\theta)\cos(x\theta)}{\sin^2(x\delta)}\cdot\frac{\sin^2(x\delta)}{\sin^2(x\delta)-\sin^2(x\theta)}$$
$$= \frac{\sin(x\theta)\cos(x\theta)}{\sin^2(x\delta)}\sum_{k=0}^{\infty}\left(\frac{\sin(x\theta)}{\sin(x\delta)}\right)^{2\,k}.$$

Next, upon substituting $t=\sin\theta$ along with using

$$\sin(2\,x\theta)/\big(\cos(2\,x\theta)-\cos(2\,x\delta)\big)\big|_{\delta=-\frac{\pi}{2\,x}} = \tan(x\theta)$$

and Proposition 2, the expansion (3.9) becomes

$$\tan(x\arcsin t) = \sum_{k=0}^{\infty}\sin^{2\,k+1}(x\arcsin t)\cos(x\arcsin t)$$
$$= \sum_{k=0}^{\infty}2^{-(2\,k+1)}\sum_{l=-k}^{k}(-1)^{l-1}\binom{2\,k}{k+l}\sin\big[2\,(l-1)x\arcsin t\big].$$

Last, by utilizing the relationship

$$\frac{\sin(a\arcsin t)}{\sqrt{1-t^2}} = a\,t\;{}_2F_1\left[\begin{matrix}1+\frac{a}{2},\,1-\frac{a}{2}\\ \frac{3}{2}\end{matrix};t^2\right]$$

implied by Proposition 3, we obtain

$$\frac{t\,x}{\sqrt{1-t^2}}\tan(x\arcsin t) = x\sum_{k=0}^{\infty}2^{-(2\,k+2)}\sum_{l=-k}^{k}(-1)^{l-1}\binom{2\,k}{k+l}$$
$$\times\,4\,(l-1)x\,t^2{}_2F_1\left[\begin{matrix}1+(l-1)x,\,1-(l-1)x\\ \frac{3}{2}\end{matrix};t^2\right],$$

which, in view of Proposition 4, results in

$$\frac{t\,x}{\sqrt{1-t^2}}\tan(x\arcsin t)=\sum_{n=1}^{\infty}t^{2n}\,x\sum_{k=0}^{\infty}2^{2\,n-2\,k-2}\sum_{l=-k}^{k}(-1)^{l+n}\binom{2\,k}{k+l}$$
$$\times\binom{(l-1)x+n-1}{2\,n-1}. \tag{3.10}$$

Here, the sum with the index k is finite and k runs from 0 to $n-1$ because the binomial coefficients with x vanish for $k\geq n$. Finally, by extracting the coefficient of $t^{2\,n}$ in (3.10), the derivation of the explicit formula for $\{B_{2n}(x)\}_{n=1}^{\infty}$ is concluded. The term $\delta_{0n}x$ needs to be added into the formula to take into an account the case $B_0(x)=x$.

What remains is to show that $\{B_{2n}(x)\}_{n=0}^{\infty}$ is a polynomial sequence. Indeed, $B_0(x)=x$, whereas each $B_{2n}(x)$ for a fixed n, $n\geq 1$, is a linear combination of polynomials in x of degree $2\,n-1$

$$\binom{(m-1)\,x+n-1}{2\,n-1}=x\,\frac{(m-1)^{2\,n-1}}{(2\,n-1)!}\prod_{k=1}^{n-1}\left(x^2-\frac{k^2}{(m-1)^2}\right),$$

where m varies from $-l$ to l and l varies from 0 to $n-1$. In addition, $\{B_{2n}(x)\}_{n=0}^{\infty}$, which is the binomial transform of $\{B_{2n}(x)\}_{n=0}^{\infty}$, is a polynomial sequence too.

Part c. In proving the assertions of this part, we need the explicit formulae for $\{A_{2n}(x)\}_{n=0}^{\infty}$ and $\{B_{2n}(x)\}_{n=0}^{\infty}$ given in Theorem 1(b).

Recall basic facts about integer–valued polynomials. Note, that $\binom{x}{n}$, defined as $\binom{x}{0}=1$ and $\binom{x}{n}=x(x-1)(x-2)\cdots(x-n+1)/n!$, $n=1,2,\ldots$, is an integer–valued polynomial in x of degree n. Also, any linear combination of integer-valued polynomials with integer coefficients is integer-valued. Hence, we can infer that $\binom{(m-1)\,x+n-1}{2\,n-1}$ in Theorem 1(b) is a polynomial in $(m-1)x+n-1$ of degree $2\,n-1$, which in turn means a polynomial in x of degree $2\,n-1$, and it takes integer values whenever x is an integer. To conclude, in essence, the formulae for $\{A_{2n}(x)\}_{n=0}^{\infty}$ and $\{B_{2n}(x)\}_{n=0}^{\infty}$ amount to linear combinations of $\binom{(m-1)\,x+n-1}{2\,n-1}$ with integer coefficients and therefore yield two sequences of integral–valued polynomials of degree $2\,n$.

With regard to roots of the polynomials, upon inspection of the formulae for $\{A_{2n}(x)\}_{n=0}^{\infty}$ and $\{B_{2n}(x)\}_{n=0}^{\infty}$, it is obvious that $x=0$ annuls all the polynomials $A_{2n}(x)$ and $B_{2n}(x)$, $n\geq 1$. Furthermore, the formula for $\{A_{2n}(x)\}_{n=0}^{\infty}$ at once gives $A_{2n}(1)=0$, $n\geq 1$, upon resorting to the identity

$$\sum_{l=0}^{k-1}2^{2\,k-2\,l-2}\sum_{m=-l}^{l}(-1)^m\binom{2\,l}{l+m}\binom{m-1+k-1}{2\,k-1}=(-1)^k\qquad(k\in\mathbb{N}),$$

which is easily derivable, one of the ways being by induction on k.

In order to show that all $B_{2\,n}(x)$ are even, it suffices to apply $\binom{\lambda}{k}=(-1)^k\binom{-\lambda+k-1}{k}$ [17, p. 647] and verify the identity $\binom{-(m-1)\,x+n-1}{2\,n-1}=-\binom{(m-1)\,x+n-1}{2\,n-1}$. Clearly, in view of this identity, when changing x to $-x$ the formula for $B_{2\,n}(x)$ does not change, in other words we have $B_{2\,n}(-x)=B_{2\,n}(x)$.

Proof of Theorem 2. Note that, by the transform given by Theorem 1(a), the summation in terms of $\{A_{2n}(x)\}_{n=1}^{\infty}$ follows at once from the sum with $\{B_{2n}(x)\}_{n=1}^{\infty}$.

As a result, we consider only a cosecant sum (for any fixed $n\geq 1$) by means of the generating function $G_B(x,t)-x=t\,x\tan(x\arcsin t)/\sqrt{1-t^2}$ (see Remark 1). The main ingredient of the proof is Proposition 5, and, upon using $t=\sin\theta$ and $\tan(x\theta)=\sin(2\,x\theta)/\big(\cos(2\,x\theta)-\cos(2\,x\delta)\big)\big|_{\delta=-\frac{\pi}{2\,x}}$, we get

$$\begin{aligned} G_B(x,\sin\theta) - x\big|_{x=q} &= \frac{q\,\sin\theta}{\cos\theta}\,\tan(q\theta) = \frac{q\,\sin\theta}{\cos\theta}\,\frac{\sin(2\,q\theta)}{\cos(2\,q\theta)-\cos\big(2\,q\,(-\pi/(2q))\big)} \\ &= \sum_{p=1}^{q}\frac{\sin^2\theta}{\sin^2\left(\frac{(2p-1)\pi}{2\,q}\right)-\sin^2\theta} = \sum_{n=1}^{\infty}\sum_{p=1}^{q}\frac{\sin^{2\,n}\theta}{\sin^{2\,n}\left(\frac{(2p-1)\pi}{2\,q}\right)} \\ &= \sum_{n=1}^{\infty}\big(\sin\theta\big)^{2\,n}\sum_{p=1}^{q}\csc^{2\,n}\left(\frac{(2p-1)\pi}{2\,q}\right). \end{aligned}$$

In other words, for any positive integer n and q, it is shown that we have

$$G_B(x,t) - x\big|_{x=q} = \sum_{n=1}^{\infty} t^{2\,n}\sum_{p=1}^{q}\csc^{2n}\left(\frac{(2p-1)\pi}{2\,q}\right) = \sum_{n=1}^{\infty} t^{2\,n} B_{2\,n}(q),$$

which leads to the claimed summation formula by extracting the coefficient of $t^{2\,n}$.

Proof of Corollary 1. As to prove **Part a**, recall that $x=0$ annuls all the polynomials $A_{2n}(x)$ and $B_{2n}(x)$, $n \geq 1$ (see Theorem 1(c)). Hence, factorizations $A_{2n}(x) = x\,a_{2n-1}(x)$ and $B_{2n}(x) = x\,b_{2n-1}(x)$, where, because of the form of the explicit formulae (linear combinations of integral–valued $\binom{(m-1)\,x+n-1}{2\,n-1}$ with integer coefficients), all $a_{2n-1}(x)$ and $b_{2n-1}(x)$ must take integer values at any integer x. This completes the proof.

Part b. For any even integer $2\,q$ and any positive integer n, by Corollary 1(a), we have $A_{2n}(2\,q) = 2\,q\,a_{2n-1}(2\,q)$ and $B_{2n}(2\,q) = 2\,q\,b_{2n-1}(2\,q)$, where $a_{2n-1}(2\,q)$ and $a_{2n-1}(2\,q)$ are necessarily integer-valued. That is, all $A_{2n}(2\,q)$ and $B_{2n}(2\,q)$ are even integers, therefore all $A_{2n}(x)/2$ and $B_{2n}(x)/2$ are integer–valued polynomials.

Regarding the case of any odd integer $2\,q+1$, recall that it was shown by Hassan that the sequences $\{A^{**}_{2n}(x)\}_{n=1}^{\infty}$ and $\{B^{**}_{2n}(x)\}_{n=1}^{\infty}$, which are associated with the sums $\sum_{p=1}^{q}\cot^{2\,n}\big((2\,p-1)\pi)/(2\,(2\,q+1))\big)$ and $\sum_{p=1}^{q}\csc^{2\,n}\big((2\,p-1)\pi)/(2\,(2\,q+1))\big)$, are integral valued (see, respectively, Theorem 4.3 and Eq. (3.18) together with Remark 4.5 (2) in [13, pp. 822 and 817]). On the other hand, by making use of elementary series and trigonometric identities, it was shown here that $A_{2\,n}(2\,q+1) = 2\sum_{p=1}^{q}\cot^{2\,n}\big((2\,p-1)\pi)/(2\,(2\,q+1))\big)$ as well as $B_{2\,n}(2\,q+1) - 1 = 2\sum_{p=1}^{q}\csc^{2\,n}\big((2\,p-1)\pi)/(2\,(2\,q+1))\big)$ (see Corollary 1 and the proof of Corollary 2). In that way $A^{**}_{2n} = \frac{1}{2}A_{2n}(2x+1)$ and $B^{**}_{2n} = \frac{1}{2}\big(B_{2n}(2x+1)-1\big)$ were obtained.

Proof of Corollary 2. The required summations are readily derivable by making use of either such elementary series identities as (for example)

$$\sum_{p=1}^{q}\tan^{2\,n}\left(\frac{p\pi}{2\,q+1}\right) = \sum_{p=1}^{q}\cot^{2\,n}\left(\frac{\pi}{2}-\frac{p\pi}{2\,q+1}\right) = \sum_{p=1}^{q}\cot^{2\,n}\left(\frac{(2\,p-1)\pi}{2\,(2\,q+1)}\right),$$

or by simple arguments, for instance to show $\sum_{p=1}^{q}\cot^{2\,n}\big((2\,p-1)\pi)/(2\,(2\,q+1))\big) = A_{2\,n}(2\,q+1)/2$ as follows: In view of

$$\begin{aligned} X_{2\,n}(2\,q+1) = \sum_{p=q+1}^{2\,q+1}\cot^{2\,n}\left(\frac{(2\,p-1)\pi}{2\,(2\,q+1)}\right) &= \sum_{p=1}^{q}\cot^{2\,n}\left(\frac{2\,(2\,q+1)\pi-(2\,p+1)\pi}{2\,(2\,q+1)}\right) \\ &= \sum_{p=1}^{q}\cot^{2\,n}\left(\frac{(2\,p-1)\pi}{2\,(2\,q+1)}\right) \end{aligned}$$

we have

$$A_{2\,n}(2\,q+1)=\sum_{p=1}^{q}\cot^{2\,n}\left(\frac{(2\,p-1)\pi}{2\,(2\,q+1)}\right)+X_{2\,n}(2\,q+1)=2\sum_{p=1}^{q}\cot^{2\,n}\left(\frac{(2\,p-1)\pi}{2\,(2\,q+1)}\right).$$

4. Concluding remarks

By Theorem 1, two new very general families of integer–valued polynomials with rational coefficients and associated with finite trigonometric summation, $A_{2n}(x)$ and $B_{2n}(x)$, were introduced. As illustrative examples for reference, we list a few of them, respectively generated by

$$A_{2n}(x)=\frac{1}{(2n)!}\,\frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}}\,G_A(t,x)\bigg|_{t=0}\qquad(n\in\mathbb{N})$$

and

$$B_{2n}(x)=\frac{1}{(2n)!}\,\frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}}\,G_B(t,x)\bigg|_{t=0}\qquad(n\in\mathbb{N}),$$

or by means of the corresponding explicit formula given in Theorem 1(b). In general, $A_{2\,n}(x)$ is of degree $2\,n$ in x and these first several polynomials are

$$\begin{aligned}
A_2(x)&=x^2-x,\\
A_4(x)&=\tfrac{1}{3}\,x^4-\tfrac{4}{3}\,x^2+x,\\
A_6(x)&=\tfrac{2}{15}\,x^6-\tfrac{2}{3}\,x^4+\tfrac{23}{15}\,x^2-x,\\
A_8(x)&=\tfrac{17}{315}\,x^8-\tfrac{16}{45}\,x^6+\tfrac{44}{45}\,x^4-\tfrac{176}{105}\,x^2+x,\\
A_{10}(x)&=\tfrac{62}{2835}\,x^{10}-\tfrac{34}{189}\,x^8+\tfrac{86}{135}\,x^6-\tfrac{718}{567}\,x^4+\tfrac{563}{315}\,x^2-x,\\
A_{12}(x)&=\tfrac{1382}{155\,925}\,x^{12}-\tfrac{248}{2835}\,x^{10}+\tfrac{1802}{4725}\,x^8-\tfrac{544}{567}\,x^6+\tfrac{21\,757}{14\,175}\,x^4-\tfrac{6508}{3465}\,x^2+x.
\end{aligned}$$

Similarly

$$\begin{aligned}
B_2(x)&=x^2,\\
B_4(x)&=\tfrac{1}{3}\,x^4+\tfrac{2}{3}\,x^2,\\
B_6(x)&=\tfrac{2}{15}\,x^6+\tfrac{1}{3}\,x^4+\tfrac{8}{15}\,x^2,\\
B_8(x)&=\tfrac{17}{315}\,x^8+\tfrac{8}{45}\,x^6+\tfrac{14}{45}\,x^4+\tfrac{16}{35}\,x^2,\\
B_{10}(x)&=\tfrac{62}{2835}\,x^{10}+\tfrac{17}{189}\,x^8+\tfrac{26}{135}\,x^6+\tfrac{164}{567}\,x^4+\tfrac{128}{315}\,x^2,\\
B_{12}(x)&=\tfrac{1382}{155\,925}\,x^{12}+\tfrac{124}{2835}\,x^{10}+\tfrac{527}{4725}\,x^8+\tfrac{566}{2835}\,x^6+\tfrac{3832}{14\,175}\,x^4+\tfrac{256}{693}\,x^2.
\end{aligned}$$

In addition, by Corollary 1, six more example sets for $a_{2n-1}(x)$ and $b_{2n-1}(x)$ as well as $A^{*}_{2n}(x)$, $B^{*}_{2n}(x)$, $A^{**}_{2n}(x)$ and $B^{**}_{2n}(x)$ are readily available, since $A_{2n}(x)$ and $B_{2n}(x)$ include these integer–valued polynomials as special cases. It is noteworthy that five of these polynomial sequences were not previously studied, $\{A_{2n}(x)\}_{n=0}^{\infty}$, $\{B_{2n}(x)\}_{n=0}^{\infty}$, $\{a_{2n-1}(x)\}_{n=1}^{\infty}$, $\{b_{2n-1}(x)\}_{n=1}^{\infty}$, and $\{B^{*}_{2n}(x)\}_{n=0}^{\infty}$.

In conclusion, the polynomials $A_{2n}(x)$ and $B_{2n}(x)$, and their special cases, enable closed-form summation of a great deal of general families of finite sums involving even-powered trigonometric functions, which generalize the identities like

$$\sum_{p=1}^{q} \csc^2\left(\frac{(2p-1)\pi}{2\,q}\right) = q^2 - q \quad \text{and} \quad \sum_{p=1}^{q} \cot^2\left(\frac{(2p-1)\pi}{2\,(2\,q+1)}\right) = 2\,q^2 + q,$$

which are valid for any positive integer q, and that somewhat surprisingly such sums are positive integers for each value of q.

Acknowledgments

The author thankfully acknowledges constructive comments and suggestions from Editor Steven Krantz and an anonymous reviewer. The present work was financially supported by the *Ministry of Education, Science and Technological Development of the Republic of Serbia* under Research project 172015.

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