J. Math. Anal. Appl. 500 (2021) 125131

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Gromov-Hausdorff convergence of quantised intervals

Thomas Gotfredsen, Jens Kaad, David Kyed *

Department of Mathematics and Computer Science, University of Southern Denmark, Campusvej 55, DK-5230 Odense M, Denmark

ARTICLE INFO

Article history:
Received 14 September 2020
Available online 8 March 2021
Submitted by S. Eilers

Keywords:
Quantum metric spaces
Podleś sphere
Gromov-Hausdorff distance

ABSTRACT

The Podleś quantum sphere S_q^2 admits a natural commutative C^*-subalgebra I_q with spectrum $\{0\} \cup \{q^{2k} : k \in \mathbb{N}_0\}$, which may therefore be considered as a quantised version of a classical interval. We study here the compact quantum metric space structure on I_q inherited from the corresponding structure on S_q^2, and provide an explicit formula for the metric induced on the spectrum. Moreover, we show that the resulting metric spaces vary continuously in the deformation parameter q with respect to the Gromov-Hausdorff distance, and that they converge to a classical interval of length π as q tends to 1.

1. Introduction

The study of compact quantum metric spaces dates back to the work of Connes [2], in which he studied metrics on state spaces of spectral triples. This notion was later formalised in the works of Rieffel [8,9,11], in which the weak $*$-topology on the state space is metrised by the Monge-Kantorovich metric coming from a so-called Lip-norm on a C^*-algebra (see Section 2 for details). As shown by Rieffel, the classical Gromov-Hausdorff distance admits an analogue, known as quantum Gromov-Hausdorff distance, for compact quantum metric spaces, and this notion was later refined by Latrémolière through his notion of propinquity [5]. Although examples of compact quantum metric spaces are abundant, some of the most basic examples from non-commutative geometry are not well understood from this point of view, and only very recently, Aguilar and Kaad [1] showed that the Podleś standard sphere S_q^2, introduced as a homogeneous space of Woronowicz' q-deformed $SU(2)$ [7,13], admits a natural compact quantum metric space structure stemming from its non-commutative geometry. More precisely, Aguilar and Kaad show that the Lip-norm arising from the Dirac operator D_q of the Dąbrowski-Sitarz spectral triple [3], does indeed provide a quantum metric structure on S_q^2. The main question left open in [1] is that of quantum Gromov-Hausdorff convergence of S_q^2 to the classical 2-sphere S^2 as the deformation parameter tends to 1. This question seems rather

* Corresponding author.
E-mail addresses: thgot@imada.sdu.dk (T. Gotfredsen), kaad@imada.sdu.dk (J. Kaad), dkyed@imada.sdu.dk (D. Kyed).

https://doi.org/10.1016/j.jmaa.2021.125131

difficult to settle,[1] and the aim of the present paper is to show that the Podleś sphere S_q^2 contains a natural commutative C^*-algebra I_q for which the corresponding convergence question can be settled, and that the answer supports the more general conjecture that S_q^2 converges to S^2 as q tends to 1. The Podleś sphere is generated by a self-adjoint operator A and a non-normal operator B (see Section 2 for precise definitions), and the C^*-algebra I_q is simply the unital C^*-algebra generated by A inside S_q^2. Since S_q^2 admits a rather accessible representation on $B(\ell^2(\mathbb{N}_0))$ [7, Proposition 4], the spectrum of the self-adjoint generator $A \in S_q^2$ is easily derivable, and one finds that for $q \in (0,1)$ this is exactly the set

$$X_q = \{0\} \cup \{q^{2k} \colon k \in \mathbb{N}_0\},$$

which can therefore be viewed as a quantised version of a classical interval. The Lip-norm L_{D_q} coming from the Dirac operator on S_q^2 therefore, in particular, provides a metric on the state space of $I_q \cong C(X_q)$ and embedding X_q into the state space of $C(X_q)$ as point-evaluations, we obtain a metric d_q on X_q. Our first main result determines an explicit formula for this metric.

Theorem A. *For $q \in (0,1)$, the metric d_q on X_q is given by the following formula:*

$$d_q(x,y) := \begin{cases} 0 & \text{if } x = y \\ \displaystyle\sum_{k=\min\{m,n\}}^{\max\{m,n\}-1} \frac{(1-q^2)q^k}{\sqrt{1-q^{2(k+1)}}} & \text{if } x = q^{2n} \text{ and } y = q^{2m} \text{ with } n \neq m \\ \displaystyle\sum_{k=n}^{\infty} \frac{(1-q^2)q^k}{\sqrt{1-q^{2(k+1)}}} & \text{if } x = q^{2n} \text{ and } y = 0 \text{ or } x = 0 \text{ and } y = q^{2n}. \end{cases}$$

When $q = 1$, the spectrum of the operator A becomes $X_1 := [0,1]$ and in Section 3.1 we will show that when X_1 is equipped with the metric d_1 inherited from the classical 2-sphere S^2, then the space (X_1, d_1) becomes isometrically isomorphic to $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with its standard Euclidian metric. Our second main theorem therefore confirms that the quantised intervals do indeed converge to the appropriate classical interval as the deformation parameter tends to 1:

Theorem B. *The metric spaces (X_q, d_q) vary continuously with respect to the Gromov-Hausdorff distance in the deformation parameter $q \in (0,1)$ and converge to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with its standard metric as q tends to 1.*

On the class of commutative compact quantum metric spaces, convergence in both Latrémolière's propinquity [5] and Rieffel's quantum Gromov-Hausdorff distance [10] is implied by convergence in classical Gromov-Hausdorff distance (see Remark 3.7) and Theorem B therefore settles all the natural convergence question for the algebras $I_q \cong C(X_q)$.

The paper is structured as follows: In the first part we introduce the basic definitions concerning quantum metric spaces, Gromov-Hausdorff distance, $SU_q(2)$ and the standard Podleś sphere and the associated Dąbrowski-Sitarz spectral triple. In the second part we first give a description of I_q in the continuum case, i.e. when $q = 1$, followed by a thorough treatment of the quantised case, where $SU(2)$ is deformed by a parameter $q \in (0,1)$. For this we provide a detailed treatment of the metric d_q, on X_q and its Lipschitz semi-norm from which we can prove Theorem A, and finally we use this to prove Theorem B.

[1] We are currently working on this.

Acknowledgments

The authors gratefully acknowledge the financial support from the Independent Research Fund Denmark through grant no. 7014-00145B and grant no. 9040-00107B. We are also grateful for the comments and suggestions provided by the anonymous referee.

Standing conventions

The semi-norms appearing in this text are defined everywhere on unital C^*-algebras and may take the value infinity.

2. Preliminaries

2.1. Quantum metric spaces

We begin this section by recalling some basic facts about metric spaces. Let (X, d) be a compact metric space. The *Lipschitz semi-norm*, $L_d\colon C(X) \to [0, \infty]$, on $C(X)$ is defined by the formula

$$L_d(f) := \sup\left\{\frac{|f(x) - f(y)|}{d(x, y)} : x \neq y\right\}; \quad f \in C(X).$$

A continuous function $f : X \to \mathbb{C}$ is then said to be a Lipschitz function when $L_d(f) < \infty$ and in this case $L_d(f)$ agrees with the Lipschitz constant. The Lipschitz functions on X form a $*$-subalgebra which we denote by $C_{\mathrm{Lip}}(X) \subset C(X)$. Given subsets $A, B \subset X$, their Hausdorff-distance is defined as

$$\mathrm{dist}_H^d(A, B) := \inf\{r \geqslant 0 | A \subset \mathbb{B}(B, r) \text{ and } B \subset \mathbb{B}(A, r)\},$$

where $\mathbb{B}(A, r)$ denotes the set $\{x \in X : \exists a \in A : d(x, a) < r\}$. For two metric spaces $(X, d_X), (Y, d_Y)$, their *Gromov-Hausdorff distance* is defined as

$$\mathrm{dist}_{GH}(X, Y) = \inf\{\mathrm{dist}_H^{d_Z}(\iota_X(X), \iota_Y(Y))\},$$

where the infimum ranges over all metric spaces (Z, d_Z) and all isometric embeddings $\iota_X\colon X \to Z$ and $\iota_Y\colon Y \to Z$. Next, we will recall the relevant definitions for quantum metric spaces.

Definition 2.1 *([8,9,11]).* Let A be a unital C^*-algebra, and let $L\colon A \to [0, \infty]$ be a semi-norm. We say that (A, L) is a *compact quantum metric space*, and that L is a *Lip-norm*, if the following conditions are satisfied:

(1) $\mathrm{Dom}(L) := \{a \in A : L(a) < \infty\}$ is dense in A;
(2) L is $*$-invariant and lower semi-continuous on A;
(3) $\ker(L) := \{a \in A : L(a) = 0\} = \mathbb{C}1_A$;
(4) The Monge-Kantorovich metric on the state space $S(A)$ of A, given by

$$\mathrm{mk}_L(\mu, \nu) := \sup\{|\mu(a) - \nu(a)| : a \in A, L(a) \leqslant 1\}, \quad \text{for } \mu, \nu \in S(A)$$

metrises the weak $*$-topology.

The model example for a compact quantum metric space is, unsurprisingly, $(C(X), L_d)$ where (X, d) is a compact metric space. In this case it is a well-known fact that the Monge-Kantorovich metric recaptures the metric d on X when the latter is viewed as a subset of the state space of $C(X)$:

$$d(x, y) = \sup\{|f(x) - f(y)| : f \in C(X), L_d(f) \leqslant 1\}.$$

Another interesting class of examples, which dates back to the work of Connes [2], comes from certain spectral triples: the setting is thus that of a separable Hilbert space H with a self-adjoint densely defined operator $D\colon \mathrm{Dom}(D) \to H$, and a unital C^*-algebra A represented on H via a $*$-homomorphism $\rho\colon A \to B(H)$. Then one can define the *Lipschitz algebra* $\mathrm{Lip}_D(A)$, to consist of all elements $x \in A$ which preserve $\mathrm{Dom}(D)$, and for which $[D, \rho(x)]\colon \mathrm{Dom}(D) \to H$ admits a bounded extension to H, which will be denoted by $\partial(x) \in B(H)$. Clearly, $\mathrm{Lip}_D(A) \subset A$ is a $*$-subalgebra and it follows from the definition of a spectral triple that $\mathrm{Lip}_D(A) \subset A$ is norm-dense. From the spectral triple (A, H, D), we also obtain a semi-norm as follows:

Definition 2.2. Define $L_D\colon A \to [0, \infty]$ by the formula

$$L_D(x) := \sup\left\{|\langle \xi, \rho(x^*)D\eta\rangle - \langle \rho(x)D\xi, \eta\rangle| : \xi, \eta \in \mathrm{Dom}(D), \|\xi\| = \|\eta\| = 1\right\}.$$

A first result says that $x \in \mathrm{Lip}_D(A)$ exactly when $L_D(x)$ is finite, and in this case $L_D(x) = \|\partial(x)\|$, see e.g. [1, Lemma 2.3]. Moreover, $L_D : A \to [0, \infty]$ is lower semi-continuous and $*$-invariant, see [9, Proposition 3.7]. The above construction does in general not yield a quantum metric space, but due to the work of Rieffel, there are tools available for verifying whether or not this is the case (see for instance [8, Theorem 1.8]).

Quantum analogues of the Gromov-Hausdorff distance have been defined by Rieffel and Latrémolière, and we refer the reader to [10,5] for concrete definitions. For our purposes, it suffices to know that when the compact quantum metric spaces in question are of the form $(C(X), L_d)$, then both analogues are dominated by the classical Gromov-Hausdorff distance, see Remark 3.7.

2.2. The standard Podleś sphere

The central object of interest in this paper is the standard Podleś quantum sphere, which is defined as a particular C^*-subalgebra of Woronowicz' [13] quantum group $SU_q(2)$ as given below. Fix $q \in (0, 1]$, and let $SU_q(2)$ denote the universal unital C^*-algebra with generators a and b defined such that the following relations are satisfied:

$$ba = qab, \quad b^*a = qab^*, \quad bb^* = b^*b$$
$$a^*a + q^2bb^* = 1 = aa^* + bb^*.$$

We denote the unital $*$-subalgebra generated by a and b by $\mathcal{O}(SU_q(2))$, and by $\mathcal{O}(S_q^2)$ the unital $*$-subalgebra of $\mathcal{O}(SU_q(2))$ generated by the elements

$$A := b^*b \quad \text{and} \quad B := ab^*.$$

The standard Podleś quantum sphere, S_q^2, is defined as the norm-closure of $\mathcal{O}(S_q^2) \subset SU_q(2)$ [7]. We remark that from the defining relations of $SU_q(2)$ we obtain a similar set of relations for A and B:

$$AB = q^2BA, \quad A = A^*$$
$$BB^* = q^{-2}A(1 - A), \quad B^*B = A(1 - q^2A).$$

The C^*-algebra $SU_q(2)$ comes equipped with a natural faithful state, called the Haar state, which we denote by $h : SU_q(2) \to \mathbb{C}$, see e.g. [4, Section 11.3.2]. We let $L^2(SU_q(2))$ denote the separable Hilbert space obtained by applying the GNS-construction to the C^*-algebra $SU_q(2)$ equipped with the Haar state.

From now on, we assume that $q \neq 1$. Define an automorphism ∂_k on $\mathcal{O}(SU_q(2))$ by $\partial_k(x) = q^{\frac{1}{2}}x$ if $x \in \{a, b\}$, and $\partial_k(x) = q^{-\frac{1}{2}}x$ if $x \in \{a^*, b^*\}$, and for each $n \in \mathbb{Z}$, define the vector subspaces

$$\mathcal{A}_n := \{x \in \mathcal{O}(SU_q(2)) : \partial_k(x) = q^{n/2}x\} \subset \mathcal{O}(SU_q(2)).$$

It turns out that $\mathcal{A}_0 = \mathcal{O}(S_q^2)$ and that the algebra structure on $\mathcal{O}(SU_q(2))$ allows us to consider each $\mathcal{A}_n$ as a left module over $\mathcal{O}(S_q^2)$. We let H_+ and H_- denote the separable Hilbert spaces obtained by taking the Hilbert space closures of $\mathcal{A}_1$ and $\mathcal{A}_{-1}$ (respectively) when considered as subspaces of $L^2(SU_q(2))$. The GNS-representation of $SU_q(2)$ on $L^2(SU_q(2))$ (when properly restricted) then provides us with two unital $*$-homomorphisms $\rho_+ : S_q^2 \to B(H_+)$ and $\rho_- : S_q^2 \to B(H_-)$.

By [3] there exists an even spectral triple, $(S_q^2, H_+ \oplus H_-, D_q)$, where the representation in question is given by the direct sum $\rho : \rho_+ \oplus \rho_- : S_q^2 \to B(H_+ \oplus H_-)$. For an explicit construction of the Dirac operator $D_q : \mathrm{Dom}(D_q) \to H_+ \oplus H_-$, we refer to [3,6] or [1].

For $x \in \mathrm{Lip}_{D_q}(S_q^2)$, the associated operator $\partial(x)$ (obtained as the closure of $[D_q, \rho(x)]$) takes the form

$$\begin{pmatrix} 0 & \partial_2(x) \\ \partial_1(x) & 0 \end{pmatrix} : H_+ \oplus H_- \to H_+ \oplus H_-,$$

where $\partial_1\colon \mathrm{Lip}_{D_q}(S_q^2) \to B(H_+, H_-)$ and $\partial_2\colon \mathrm{Lip}_{D_q}(S_q^2) \to B(H_-, H_+)$ are derivations satisfying $\partial_2(x^*) = -\partial_1(x)^*$ (remark in this respect that $B(H_+, H_-)$ and $B(H_-, H_+)$ can be considered as bimodules over S_q^2 via the representations ρ_+ and ρ_-). Consequently the Lip-norm is, for $x \in \mathrm{Lip}_{D_q}(S_q^2)$, given by

$$L_{D_q}(x) = \max\{\|\partial_1(x)\|, \|\partial_1(x^*)\|\}.$$

By [7, Proposition 4], S_q^2 admits a faithful representation, $\pi\colon S_q^2 \to B(\ell^2(\mathbb{N}_0))$, defined by

$$\pi(A)(e_k) := q^{2k}e_k, \quad \pi(B)(e_k) = q^k\sqrt{1 - q^{2(k+1)}}e_{k+1}, \tag{1}$$

where e_k denotes the characteristic function on the point-set $\{k\} \subset \mathbb{N}_0$. In fact, this representation even provides a $*$-isomorphism to the unitisation of the compact operators on $\ell^2(\mathbb{N}_0)$. Using this representation it is easy to see that the spectrum of the operator A for a specific $q \in (0, 1)$ is given by

$$X_q := \{0\} \cup \{q^{2k} : k \in \mathbb{N}_0\}.$$

Hence the indicator functions $\chi_{\{q^{2k}\}}\colon X_q \to \{0, 1\}$ are continuous for all k. In fact, these indicator functions and the unit generate $C(X_q)$, since any continuous function, $f\colon X_q \to \mathbb{C}$, can be written as $f(0) + \sum_{k=0}^{\infty}(f(q^{2k}) - f(0)) \cdot \chi_{\{q^{2k}\}}$, where $\lim_{k\to\infty} f(q^{2k}) = f(0)$. By [1, Theorem 8.3], (S_q^2, L_{D_q}) is a compact quantum metric space, and consequently so is $I_q := C^*(A, 1) \cong C(X_q)$ with the restricted Lip-norm. The compact quantum metric space (I_q, L_{D_q}) is our main object of interest in the present paper. As I_q is commutative, the Lip-norm L_{D_q} defines a genuine metric d_q on X_q when the latter is considered as a subset of the state space $S(S_q^2)$. In order to describe d_q explicitly, the following lemma will be key:

Lemma 2.3 *([1, Lemma 5.3]). Let $k \in \mathbb{N}_0$ and let $q \in (0, 1)$. We have that $\chi_{\{q^{2k}\}}(A) \in \mathrm{Lip}_{D_q}(S_q^2)$ and the derivative is given by*

$$\partial_1(\chi_{\{q^{2k}\}}(A)) = \frac{1}{q^{2k}(1 - q^2)}\chi_{\{q^{2k}\}}(A) \cdot b^*a^* - \frac{1}{q^{2(k-1)}(1 - q^2)}\chi_{\{q^{2(k-1)}\}}(A) \cdot b^*a^*.$$

In particular, we obtain that

$$\partial_1(f(A)) = \sum_{k=0}^{\infty} \frac{f(q^{2k}) - f(q^{2(k+1)})}{q^{2k}(1 - q^2)}\chi_{\{q^{2k}\}}(A) \cdot b^*a^* \tag{2}$$

for every $f \in \mathrm{span}_{\mathbb{C}}\{\chi_{\{q^{2k}\}} : k \in \mathbb{N}_0\}$.

Remark 2.4. The formula in (2) for $\partial_1(f(A))$ is related to the notion of q-differentiation from q-calculus. Indeed, the q^2-differentiation of $f \in \mathrm{span}_{\mathbb{C}}\{\chi_{\{q^{2k}\}} : k \in \mathbb{N}_0\}$ would be given by

$$\mathcal{D}_{q^2}(f) = \sum_{k=0}^{\infty} \frac{f(q^{2k}) - f(q^{2(k+1)})}{q^{2k}(1-q^2)} \chi_{\{q^{2k}\}},$$

see for example [4, Chapter 2.2]. The extra term b^*a^* appearing in (2) comes from the geometry of the quantised 2-sphere as it operates between the Hilbert space completions H_+ and H_- of the quantised spinor bundles $\mathcal{A}_1$ and $\mathcal{A}_{-1}$.

3. Metric properties of the quantised interval

In this section we first provide the explicit descriptions of the compact metric spaces (X_q, d_q) which encode the compact quantum metric space structure of (I_q, L_{D_q}). More precisely, the algebra of Lipschitz functions of the metric space (X_q, d_q) must agree with the Lipschitz algebra $\mathrm{Lip}_{D_q}(S_q^2) \cap I_q$ and the two semi-norms must agree, in the sense that $L_{D_q}(f(A)) = L_{d_q}(f)$ whenever f is a Lipschitz function on (X_q, d_q). This analysis is separated into the case $q = 1$, referred to as the continuum case, and the case $q < 1$, referred to as the quantised case.

3.1. The continuum case

We consider the 2-sphere $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ whereas $S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$ both equipped with the subspace topology coming from the usual topology on $\mathbb{R}^3$ and $\mathbb{C}^2$.

In the situation where $q = 1$ we have a homeomorphism between the characters of $SU_q(2)$ and the 3-sphere S^3, which sends $(z, w) \in S^3 \subset \mathbb{C}^2$ to the unique character $\chi_{z,w}$ satisfying that $\chi_{z,w}(a) = z$ and $\chi_{z,w}(b) = w$ (see [13]). Consequently, we can identify $SU_q(2)$ with $C(S^3)$ such that $a(z, w) = z$ and $b(z, w) = w$. We may moreover view the 2-sphere S^2 as the quotient space of S^3 under the circle action $\lambda \cdot (z, w) := (\lambda \cdot z, \lambda \cdot w)$ and this identification happens via the Hopf-fibration

$$S^3 \ni (z, w) \longmapsto \big(2\,\mathrm{Re}(z\bar{w}), 2\,\mathrm{Im}(z\bar{w}), |z|^2 - |w|^2\big) \in S^2.$$

Since both $A(z, w) = (b^*b)(w) = |w|^2$ and $B(z, w) = z\bar{w}$ are invariant under the circle action we may consider them as continuous function on S^2 and as such they are given by

$$A(x_1, x_2, x_3) = \frac{1 - x_3}{2} \qquad \text{and} \qquad B(x_1, x_2, x_3) = \frac{x_1 + ix_2}{2}.$$

It is now clear that A has range $[0, 1]$ and so we have a $*$-isomorphism $C([0, 1]) \cong I_1$. Let d_1 be the metric on $[0, 1]$ obtained from the standard round metric on S^2 so that

$$d_1(s, t) := \inf\big\{d_{S^2}\big((x_1, x_2, 1 - 2s), (y_1, y_2, 1 - 2t)\big) : x_1^2 + x_2^2 + (1 - 2s)^2 = 1 = y_1^2 + y_2^2 + (1 - 2t)^2\big\}$$

for all $s, t \in [0, 1]$. We record the following elementary result:

Proposition 3.1. *The map* $\phi\colon [-\frac{\pi}{2}, \frac{\pi}{2}] \to [0, 1]$ *given by* $\phi(t) = \frac{1}{2} + \frac{1}{2}\sin(t)$ *is an isometric isomorphism when* $[-\frac{\pi}{2}, \frac{\pi}{2}]$ *is equipped with the standard Euclidean metric* d *and* $[0, 1]$ *is equipped with the metric* d_1. *In particular, we have a* $*$*-isomorphism* $\beta\colon C([-\frac{\pi}{2}, \frac{\pi}{2}]) \to I_1$, $\beta(f) = (f \circ \phi^{-1})(A)$, *which maps* $C_{\mathrm{Lip}}([-\frac{\pi}{2}, \frac{\pi}{2}])$ *onto* $I_1 \cap C_{\mathrm{Lip}}(S^2)$ *and satisfies* $L_{d_{S^2}}(\beta(f)) = L_d(f)$.

Remark 3.2. For completeness, we note that when $q = 1$, the standard Podleś sphere is of course isomorphic to $C(S^2)$. Indeed, the continuous maps corresponding to A and B separate points in S^2 and the Stone-Weierstrass Theorem then shows that $S_1^2 = C^*(1, A, B) \cong C(S^2)$.

3.2. The quantised case

We will now address the case of a fixed $q \in (0, 1)$. We let X_q denote the spectrum of $A \in S_q^2$, and, as we already saw, $X_q = \{0\} \cup \{q^{2k} : k \in \mathbb{N}_0\}$. As explained in the introduction, the Lip-norm L_{D_q} gives rise to a metric on the state space of $C^*(A, 1) \cong C(X_q)$, which therefore, in particular, determines a metric d_q on X_q when the latter is viewed as a subset of the state space via point evaluations. The aim of the current section is to find an explicit formula for this metric, and show that the metric spaces (X_q, d_q) converge in the Gromov-Hausdorff distance to the Euclidean interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ as q tends to 1.

To this end, we consider the function $\rho_q \colon [-1, \infty) \to \mathbb{R}$ by

$$\rho_q(x) := \frac{\sqrt{1 - q^{2(x+1)}}}{(1 - q^2)q^x}.$$

Definition 3.3. Define the metric $d_q \colon X_q \times X_q \to [0, \infty)$ by

$$d_q(x, y) := \begin{cases} 0 & \text{if } x = y \\ \displaystyle\sum_{\min\{m,n\}}^{\max\{m,n\}-1} \frac{1}{\rho_q(k)} & \text{if } x = q^{2n} \text{ and } y = q^{2m} \text{ with } n \neq m \\ \displaystyle\sum_{k=n}^{\infty} \frac{1}{\rho_q(k)} & \text{if } x = q^{2n} \text{ and } y = 0 \text{ or } x = 0 \text{ and } y = q^{2n}. \end{cases}$$

Remark that the series $\sum_{k=0}^{\infty} \frac{1}{\rho_q(k)}$ is convergent as can be seen from the estimate

$$\frac{1}{\rho_q(k)} = \frac{q^k(1 - q^2)}{\sqrt{1 - q^{2(k+1)}}} \leqslant q^k \qquad \text{for all } k \in \mathbb{N}_0. \tag{3}$$

In order to prove Theorem A, we need several lemmas, the first of which shows that the Lipschitz semi-norm on $C(X_q)$ defined by the metric d_q and the Lip-norm L_{D_q} on I_q agree on all finite linear combinations of characteristic functions on X_q:

Lemma 3.4. *For any $f \in \operatorname{span}_{\mathbb{C}}\{\chi_{\{q^{2k}\}} : k \in \mathbb{N}_0\} \subset C(X_q)$, it holds that $f(A) \in \operatorname{Lip}_{D_q}(S_q^2) \cap I_q$. Moreover, we have the identities*

$$L_{D_q}(f(A)) = \max\{\rho_q(k) \cdot |f(q^{2k}) - f(q^{2(k+1)})| : k \in \mathbb{N}_0\} = L_{d_q}(f).$$

In particular, f is also Lipschitz with respect to the metric d_q.

Note that the maximum is indeed well-defined, since f is non-zero at no more than finitely many elements from X_q.

Proof. Let $f \in \operatorname{span}_{\mathbb{C}}\{\chi_{\{q^{2k}\}} : k \in \mathbb{N}_0\}$ be given. The fact that $f(A) \in \operatorname{Lip}_{D_q}(S_q^2) \cap I_q$ is a consequence of Lemma 2.3. Moreover, from Lemma 2.3 and the defining identities for $SU_q(2)$ we obtain that

$$\begin{aligned}\partial_1(f(A))\partial_1(f(A))^* &= A(1-q^2A)\sum_{k=0}^{\infty}\frac{|f(q^{2k})-f(q^{2(k+1)})|^2}{q^{4k}(1-q^2)^2}\chi_{\{q^{2k}\}}(A)\\ &= \sum_{k=0}^{\infty}\rho_q(k)^2\cdot|f(q^{2k})-f(q^{2(k+1)})|^2\chi_{\{q^{2k}\}}(A).\end{aligned}$$

The continuous functional calculus applied to $A \in I_q$ then implies that

$$\left\|\partial_1(f(A))\right\|^2 = \max\{\rho_q(k)\cdot|f(q^{2k})-f(q^{2(k+1)})| : k \in \mathbb{N}_0\}. \tag{4}$$

The identity

$$L_{D_q}(f(A)) = \max\{\rho_q(k)\cdot|f(q^{2k})-f(q^{2(k+1)})| : k \in \mathbb{N}_0\}$$

now follows since the formula in (4) implies that $\|\partial_2(f(A))\| = \|\partial_1(\bar{f}(A))\| = \|\partial_1(f(A))\|$.

For the second equality, choose $l \in \mathbb{N}_0$ such that

$$\rho_q(l)\cdot|f(q^{2l})-f(q^{2(l+1)})| = \max\{\rho_q(k)\cdot|f(q^{2k})-f(q^{2(k+1)})| : k \in \mathbb{N}_0\}.$$

This choice of $l \in \mathbb{N}_0$ implies that

$$|f(q^{2k})-f(q^{2(k+1)})| \leqslant |f(q^{2l})-f(q^{2(l+1)})|\cdot\frac{\rho_q(l)}{\rho_q(k)}$$

for all $k \in \mathbb{N}_0$. Thus, for every $m < n$ we may now estimate as follows:

$$\begin{aligned}|f(q^{2m})-f(q^{2n})| &\leqslant \sum_{k=m}^{n-1}|f(q^{2k})-f(q^{2(k+1)})| \leqslant \sum_{k=m}^{n-1}|f(q^{2l})-f(q^{2(l+1)})|\cdot\frac{\rho_q(l)}{\rho_q(k)}\\ &= |f(q^{2l})-f(q^{2(l+1)})|\cdot\rho_q(l)\cdot d_q(q^{2m},q^{2n}).\end{aligned} \tag{5}$$

This shows that $f : X_q \to \mathbb{C}$ is Lipschitz with $L_{d_q}(f) \leqslant L_{D_q}(f(A))$. The fact that equality is achieved is then a consequence of Definition 3.3. Indeed, we obtain that

$$L_{D_q}(f(A)) = |f(q^{2l})-f(q^{2(l+1)})|\cdot\rho_q(l) = \frac{|f(q^{2l})-f(q^{2(l+1)})|}{d_q(q^{2l},q^{2(l+1)})} \leqslant L_{d_q}(f). \quad \square$$

The next lemma computes the Lipschitz semi-norms of general continuous functions on X_q and provides information on the behaviour of the Lipschitz constants of a particularly interesting approximation.

Lemma 3.5. *For any $f \in C(X_q)$ one has*

$$L_{d_q}(f) = \sup\{|f(q^{2k})-f(q^{2(k+1)})|\cdot\rho_q(k) : k \in \mathbb{N}_0\}.$$

Moreover, if $f(0) = 0$ and f is Lipschitz with respect to the metric d_q, then the sequence $\big\{L_{d_q}(f\cdot\chi_{\{q^{2k}:k\leqslant n\}})\big\}_{n=0}^{\infty}$ is bounded.

Proof. We first notice that Definition 3.3 implies the inequality

$$\sup\{|f(q^{2k})-f(q^{2(k+1)})|\cdot\rho_q(k) : k \in \mathbb{N}_0\} \leqslant L_{d_q}(f)$$

(see also the proof of Lemma 3.4 for more details).

We then claim that

$$\frac{|f(x) - f(y)|}{d_q(x, y)} \leqslant \sup\{|f(q^{2k}) - f(q^{2(k+1)})| \cdot \rho_q(k) : k \in \mathbb{N}_0\} \tag{6}$$

whenever $x, y \in X_q \setminus \{0\}$ satisfy $x \neq y$. We have to be careful at this point since the inequality in (6) is not an immediate consequence of Definition 3.3: the right hand side of our inequality only uses successive elements as exponents (i.e. k and $k+1$) whereas $x = q^{2n}$ and $y = q^{2m}$ for some $n, m \in \mathbb{N}_0$ without any further constraints (except for $n \neq m$). The inequality in (6) does however follow by an application of Lemma 3.4 to a suitable restriction of f.

Thus, to establish the claimed identity, it only remains to be shown that the supremum in (6) is still an upper bound when $x = q^{2n}$ for some $n \in \mathbb{N}_0$ and $y = 0$. However, this follows immediately from the estimate in (6) together with continuity of the function f and the metric d_q.

For the second part, we assume that f is Lipschitz and that $f(0) = 0$. By Lemma 3.4 it suffices to show that the sequence $\{|f(q^{2n})| \cdot \rho_q(n)\}_{n=0}^{\infty}$ is bounded. To this end, we first note that since f is Lipschitz we may find a constant C such that $|f(q^{2n})| \leqslant C \cdot d_q(q^{2n}, 0)$ for all $n \in \mathbb{N}_0$. It follows that

$$\rho_q(n) \cdot |f(q^{2n})| \leqslant C \cdot \sum_{k=n}^{\infty} \frac{\rho_q(n)}{\rho_q(k)} = C \cdot \sum_{k=0}^{\infty} q^k \frac{\sqrt{1 - q^{2(n+1)}}}{\sqrt{1 - q^{2(k+n+1)}}} \leqslant C \cdot \sum_{k=0}^{\infty} q^k = \frac{C}{1 - q}$$

for all $n \in \mathbb{N}_0$. This ends the proof of the lemma. □

The metric $d_q \colon X_q \times X_q \to [0, \infty)$ yields a Lipschitz algebra $C_{\mathrm{Lip}}(X_q) \subset C(X_q)$ and the semi-norm $L_{D_q} \colon I_q \to [0, \infty]$ yields an *a priori different* Lipschitz algebra $\mathrm{Lip}_{D_q}(I_q) \subset I_q$. The Lipschitz algebras $C_{\mathrm{Lip}}(X_q)$ and $\mathrm{Lip}_{D_q}(I_q)$ agree with the domains of the semi-norms L_{d_q} and L_{D_q}, respectively (recall that the domain consists of the elements where a semi-norm is finite). Moreover, the two unital commutative C^*-algebras $C(X_q)$ and I_q are related by the $*$-isomorphisms $f \mapsto f(A)$. We are going to show that the $*$-isomorphism $f \mapsto f(A)$ restricts to a $*$-isomorphism $C_{\mathrm{Lip}}(X_q) \to \mathrm{Lip}_{D_q}(I_q)$ which is moreover isometric with respect to the semi-norms L_{d_q} and L_{D_q}.

Suppressing the identification $C(X_q) \cong I_q$ we have by now proved that the two semi-norms L_{d_q} and L_{D_q} agree on *finite* linear combinations of the indicator functions $\chi_{\{q^{2k}\}}$, $k \in \mathbb{N}_0$ (Lemma 3.4) and we have moreover succeeded in computing the semi-norm $L_{d_q} \colon C(X_q) \to [0, \infty]$ (Lemma 3.5).

The passage from finite linear combinations of indicator functions to general Lipschitz elements is however quite subtle. To explain a bit what the subtle point is, we let $\mathcal{I}_q \subset I_q$ denote the smallest unital $*$-subalgebra containing all the projections $\chi_{\{q^{2k}\}}(A)$. Then even though $\mathcal{I}_q \subset I_q$ is norm-dense and the derivation $\partial : \mathrm{Lip}_{D_q}(I_q) \to B(H_+ \oplus H_-)$ is closed, it is not true that $\mathrm{Lip}_{D_q}(I_q)$ can be recovered by taking the closure of the restriction $\partial \colon \mathcal{I}_q \to B(H_+ \oplus H_-)$. In particular, for a general element $f(A) = f(0) + \sum_{k=0}^{\infty} (f(q^{2k}) - f(0)) \cdot \chi_{\{q^{2k}\}}(A) \in \mathrm{Lip}_{D_q}(I_q)$ we cannot a priori compute $\partial(f(A)) \in B(H_+ \oplus H_-)$ by using Lemma 2.3 and applying the derivation ∂ term by term.

After these clarifications we are ready to state and prove the first main result of this section:

Theorem 3.1. *The Lip-algebra of I_q associated with the Dąbrowski-Sitarz spectral triple $(S_q^2, H_+ \oplus H_-, D_q)$ agrees with $\{f(A) : f \in C_{\mathrm{Lip}}(X_q)\}$, and for $f \in C_{\mathrm{Lip}}(X_q)$, we have $L_{D_q}(f(A)) = L_{d_q}(f)$.*

Proof. Let $f \in C(X_q)$ be given.

Suppose first that $L_{D_q}(f(A)) < \infty$. For each $n \in \mathbb{N}_0$ we define the projection $Q_n := \sum_{k=0}^{n} \chi_{\{q^{2k}\}}(A)$. Since ∂_1 is a derivation, we obtain from Lemma 2.3 that

$$\begin{aligned}\partial_1(f(A))Q_n &= \partial_1(f(A)Q_n) - f(A)\partial_1(Q_n)\\ &= \sum_{k=0}^{n-1}\big(f(q^{2k}) - f(q^{2(k+1)})\big)\frac{1}{q^{2k}(1-q^2)}\chi_{\{q^{2k}\}}(A)\cdot b^*a^*.\end{aligned}$$

Following the proof of Lemma 3.4 we then get that

$$\|\partial_1(f(A))Q_n\| = \max\{|f(q^{2k}) - f(q^{2(k+1)})|\cdot\rho_q(k) : k\in\{0,1,\dots,n-1\}\} \tag{7}$$

and hence (using that Q_n is an orthogonal projection) we obtain the estimate

$$\begin{aligned}\sup\{|f(q^{2k}) - f(q^{2(k+1)})|\cdot\rho_q(k) : k\in\mathbb{N}_0\} &= \sup\{\|\partial_1(f(A))Q_n\| : n\in\mathbb{N}_0\}\\ &\leqslant \|\partial_1(f(A))\|.\end{aligned} \tag{8}$$

By Lemma 3.5 this shows that f is Lipschitz with respect to the metric d_q and that

$$L_{d_q}(f) \leqslant \|\partial_1(f(A))\|.$$

To prove that equality holds, we observe that by [12, Theorem 6.2.17],

$$h(Q_n) = (1-q^2)\sum_{k=0}^{n} q^{2k} \underset{n\to\infty}{\longrightarrow} 1,$$

where h denotes the Haar state on $SU_q(2)$. Since h is faithful and $\{Q_n\}_{n=0}^{\infty}$ is an increasing sequence of projections, Q_n converges to the identity in the strong operator topology on $B(L^2(SU_q(2)))$, and hence also on $B(H_+)$. It now follows from (8) and Lemma 3.5 that for any ξ in the unit ball of H_+, we have

$$\begin{aligned}\|\partial_1(f(A))\xi\| &= \lim_{n\to\infty}\|\partial_1(f(A))Q_n\xi\| \leqslant \sup\{\|\partial_1(f(A))Q_n\| : n\in\mathbb{N}_0\}\\ &= L_{d_q}(f),\end{aligned}$$

and hence that $\|\partial_1(f(A))\| = L_{d_q}(f)$. Since we moreover have the identities

$$\|\partial_2(f(A))\| = \|\partial_1(\bar{f}(A))\| = L_{d_q}(f)$$

we may conclude that $L_{D_q}(f(A)) = L_{d_q}(f)$.

Suppose next that $f\in C(X_q)$ is Lipschitz with respect to the metric d_q. Since subtracting a constant changes neither the Lipschitz constant of f nor $L_{D_q}(f(A))$, we may, without loss of generality, assume that $f(0)=0$. For each $n\in\mathbb{N}_0$ define the function $f_n := f\cdot\chi_{\{q^{2k}:k\leqslant n\}}$. By Lemma 3.5, the sequence $\{L_{d_q}(f_n)\}_{n=0}^{\infty}$ is then bounded and moreover $f_n(A)$ converges to $f(A)$ in operator norm.

Hence, since $L_{D_q}(f_n(A)) = L_{d_q}(f_n)$ by Lemma 3.4, we obtain by lower semi-continuity of $L_{D_q} : I_q \to [0,\infty]$ that

$$L_{D_q}(f(A)) \leqslant \sup\{L_{D_q}(f_n(A)) : n\in\mathbb{N}_0\} < \infty.$$

This shows that $f(A)\in \mathrm{Lip}_{D_q}(I_q)$ and this ends the proof of the theorem. □

Theorem A now follows easily:

Proof of Theorem A. The metric d_q' on X_q induced by L_{D_q} is by definition given by

$$d_q'(x,y) := \sup\{|f(x)-f(y)| : f \in C(X_q), L_{D_q}(f(A)) \leqslant 1\}.$$

However, by Theorem 3.1 we have

$$\begin{aligned} d_q(x,y) &= \sup\{|f(x)-f(y)| : f \in C(X_q), L_{d_q}(f) \leqslant 1\} \\ &= \sup\{|f(x)-f(y)| : f \in C(X_q), L_{D_q}(f(A)) \leqslant 1\}, \end{aligned}$$

and hence the two metrics agree. □

In the following, we will consider the behaviour of (X_q, d_q) with respect to the Gromov-Hausdorff metric, and provide a proof of Theorem B. To this end, we first establish a preliminary result about the diameter of X_q:

Lemma 3.6. *It holds that* $\lim_{q\to 1} d_q(0,1) = \pi$.

Proof. Observe that the function $\frac{1}{\rho_q} : x \mapsto (1-q^2)\frac{q^x}{\sqrt{1-q^{2(x+1)}}}$ is positive and decreasing on $(-1,\infty)$. This yields the estimates

$$\int_1^\infty \frac{1}{\rho_q(x)}\,dx \leqslant \sum_{k=0}^\infty \frac{1}{\rho_q(k)} \leqslant \int_0^\infty \frac{1}{\rho_q(x)}\,dx. \tag{9}$$

Furthermore, it can be verified that $F(x) := \frac{1-q^2}{q\ln(q)}\arcsin(q^{x+1})$ is an antiderivative of $\frac{1}{\rho_q(x)}$ and $\lim_{x\to\infty} F(x) = 0$. We therefore obtain the inequalities

$$-\frac{1-q^2}{q\ln(q)}\arcsin(q^2) \leqslant d_q(0,1) \leqslant -\frac{1-q^2}{q\ln(q)}\arcsin(q).$$

Since $\lim_{q\to1}\frac{1-q^2}{q\ln(q)} = -2$ and $\arcsin(1) = \frac{\pi}{2}$ we may conclude that $\lim_{q\to1} d_q(0,1) = \pi$. □

Proof of Theorem B. For each $q \in (0,1)$, we consider the isometric embedding $\iota_q : X_q \to \mathbb{R}$ given by $\iota_q(x) = d_q(1,x) - \frac{\pi}{2}$.

We start by proving continuity at a fixed $q_0 \in (0,1)$. Let $\varepsilon > 0$ be given. Choose a $\delta_0 > 0$ such that $J := [q_0-\delta_0, q_0+\delta_0] \subset (0,1)$. From the estimate in (3) we obtain that

$$\sum_{k=0}^\infty \sup\left\{\frac{1}{\rho_q(k)} : q \in J\right\} \leqslant \sum_{k=0}^\infty \sup\{q^k\sqrt{1-q^2} : q \in J\} \leqslant \sum_{k=0}^\infty (q_0+\delta_0)^k < \infty.$$

We may therefore choose an $n_0 \in \mathbb{N}_0$ such that

$$\sum_{k=n_0}^\infty \frac{1}{\rho_q(k)} < \frac{\varepsilon}{3} \tag{10}$$

for all $q \in J = [q_0-\delta_0, q_0+\delta_0]$. Now, for each $k \in \mathbb{N}_0$, the function $q \mapsto \sum_{k=0}^{n_0-1}\frac{1}{\rho_q(k)}$ is continuous and we may thus choose a $\delta \in (0,\delta_0)$ such that

$$\left|\sum_{k=0}^{m-1}\frac{1}{\rho_q(k)}-\sum_{k=0}^{m-1}\frac{1}{\rho_{q_0}(k)}\right|<\frac{\varepsilon}{3} \tag{11}$$

for all $m\in\{1,\dots,n_0\}$ and all $q\in(q_0-\delta,q_0+\delta)$.

Let now $q\in(q_0-\delta,q_0+\delta)\subset J$ be given. It then follows immediately from (11) that

$$|\iota_q(q^{2m})-\iota_{q_0}(q_0^{2m})|<\frac{\varepsilon}{3}<\varepsilon$$

for all $m\in\{1,\dots,n_0\}$. Moreover, for $m>n_0$ we apply (10) and (11) to estimate that

$$\begin{aligned}|\iota_q(q^{2m})-\iota_{q_0}(q_0^{2m})|&=\left|\sum_{k=0}^{n_0-1}\frac{1}{\rho_q(k)}+\sum_{k=n_0}^{m-1}\frac{1}{\rho_q(k)}-\sum_{k=0}^{n_0-1}\frac{1}{\rho_{q_0}(k)}-\sum_{k=n_0}^{m-1}\frac{1}{\rho_{q_0}(k)}\right|\\&\leqslant|\iota_q(q^{2n_0})-\iota_{q_0}(q_0^{2n_0})|+\sum_{k=n_0}^{\infty}\frac{1}{\rho_q(k)}+\sum_{k=n_0}^{\infty}\frac{1}{\rho_{q_0}(k)}\\&<\varepsilon.\end{aligned}$$

A similar argument also shows that $|\iota_q(0)-\iota_{q_0}(0)|<\varepsilon$. We conclude that

$$\mathrm{dist}_H(\iota_q(X_q),\iota_{q_0}(X_{q_0}))\leqslant\varepsilon$$

and hence that $(0,1)\ni q\mapsto(X_q,d_q)$ varies continuously in Gromov-Hausdorff distance.

For convergence, it suffices to show that the Hausdorff distance between $\iota_q(X_q)$ and $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ converges to 0 as $q\to1$. To this end, let $\varepsilon>0$ be arbitrary. By Lemma 3.6, we may find a $q_1\in(0,1)$ such that for any $q\in(q_1,1)$, we have $|\iota_q(0)-\frac{\pi}{2}|<\varepsilon$. Moreover, since $-\frac{\pi}{2}\leqslant\iota_q(x)\leqslant\iota_q(0)$ for all $x\in X_q$, it follows that for every $x\in X_q$ there exists a $y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ with $|\iota_q(x)-y|<\varepsilon$. It remains to be shown that we can find a $q_2\in(0,1)$ such that given any $y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and any $q\in(q_2,1)$, we can find $x\in X_q$ such that $|y-\iota_q(x)|<\varepsilon$. Since $\frac{1}{\rho_q(0)}=\sqrt{1-q^2}\xrightarrow[q\to1]{}0$ and $d_q(0,1)\xrightarrow[q\to1]{}\pi$ by Lemma 3.6 we can find a $q_2\in(0,1)$ such that $\frac{1}{\rho_q(0)}<\varepsilon$ and $|\iota_q(0)-\frac{\pi}{2}|<\frac{\varepsilon}{2}$ for all $q\in(q_2,1)$. Let now $q\in(q_2,1)$ be given. It follows that $|y-\iota_q(0)|<\varepsilon$ for $y\in\left(\frac{\pi}{2}-\frac{\varepsilon}{2},\frac{\pi}{2}\right]$. On the other hand, we may for each $y\in\left[-\frac{\pi}{2},\frac{\pi}{2}-\frac{\varepsilon}{2}\right]$ find an $n\in\mathbb{N}_0$ such that $y\in[\iota_q(q^{2n}),\iota_q(q^{2(n+1)})]$ and consequently

$$\left|y-\iota_q(q^{2n})\right|\leqslant\left|\iota_q(q^{2n})-\iota_q(q^{2(n+1)})\right|=\frac{1}{\rho_q(n)}\leqslant\frac{1}{\rho_q(0)}<\varepsilon.\quad\square$$

Remark 3.7. As stated in the introduction, Theorem B also applies if we replace the classical Gromov-Hausdorff distance with respectively the quantum Gromov-Hausdorff distance of Rieffel [10] or Latrémolière's propinquity. To see this, note that by [5, Corollary 6.4] the former is dominated by two times the latter and by [5, Theorem 6.6], propinquity is dominated by the classical Gromov-Hausdorff distance on the class of compact metric spaces, and hence the convergence and continuity are also obtained for these distances.

References

[1] Konrad Aguilar, Jens Kaad, The Podleś sphere as a spectral metric space, J. Geom. Phys. 133 (2018) 260–278.
[2] Alain Connes, Compact metric spaces, Fredholm modules, and hyperfiniteness, Ergod. Theory Dyn. Syst. 9 (2) (1989) 207–220.
[3] Ludwik Dąbrowski, Andrzej Sitarz, Dirac operator on the standard Podleś quantum sphere, in: Noncommutative Geometry and Quantum Groups, Warsaw, 2001, in: Banach Center Publ., vol. 61, Polish Acad. Sci. Inst. Math, Warsaw, 2003, pp. 49–58.

[4] Anatoli Klimyk, Konrad Schmüdgen, Quantum Groups and Their Representations, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1997.

[5] Frédéric Latrémolière, The quantum Gromov-Hausdorff propinquity, Trans. Am. Math. Soc. 368 (1) (2016) 365–411.

[6] Sergey Neshveyev, Lars Tuset, A local index formula for the quantum sphere, Commun. Math. Phys. 254 (2) (2005) 323–341.

[7] Piotr Podleś, Quantum spheres, Lett. Math. Phys. 14 (3) (1987) 193–202.

[8] Marc Rieffel, Metrics on states from actions of compact groups, Doc. Math. 3 (1998) 215–229.

[9] Marc Rieffel, Metrics on state spaces, Doc. Math. 4 (1999) 559–600.

[10] Marc Rieffel, Gromov-Hausdorff distance for quantum metric spaces, Mem. Am. Math. Soc. 168 (796) (2004) 1–65.

[11] Marc Rieffel, Compact quantum metric spaces, in: Operator Algebras, Quantization, and Noncommutative Geometry, in: Contemp. Math., vol. 365, Amer. Math. Soc., Providence, RI, 2004, pp. 315–330.

[12] Thomas Timmermann, An Invitation to Quantum Groups and Duality. From Hopf Algebras to Multiplicative Unitaries and Beyond, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2008.

[13] Stanisław L. Woronowicz, Twisted $SU(2)$ group. An example of a noncommutative differential calculus, Publ. Res. Inst. Math. Sci. 23 (1) (1987) 117–181.