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Left multipliers of reproducing kernel Hilbert C^*-modules and the Papadakis theorem

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ABSTRACT

We give a modified definition of a reproducing kernel Hilbert C^*-module (shortly, $RKHC^*M$) without using the condition of self-duality and discuss some related aspects; in particular, an interpolation theorem is presented. We investigate the exterior tensor product of $RKHC^*M$s and find their reproducing kernel. In addition, we deal with left multipliers of $RKHC^*M$s. Under some mild conditions, it is shown that one can make a new $RKHC^*M$ via a left multiplier. Moreover, we introduce the Berezin transform of an operator in the context of $RKHC^*M$s and construct a unital subalgebra of the unital C^*-algebra consisting of adjointable maps on an $RKHC^*M$ and show that it is closed with respect to a certain topology. Finally, the Papadakis theorem is extended to the setting of $RKHC^*M$, and in order for the multiplication of two specific functions to be in the Papadakis $RKHC^*M$, some conditions are explored.

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1. Introduction

Hilbert C^*-modules are generalization of Hilbert spaces by allowing the inner product to take its values in a C^*-algebra instead of the complex numbers. At the same time, they are extensions of C^*-algebras. Indeed, a C^*-algebra $\mathfrak{A}$ is a Hilbert $\mathfrak{A}$-module if we define $\langle a, b\rangle = a^*b$ $(a, b \in \mathfrak{A})$. For Hilbert C^*-modules $\mathcal{E}$ and $\mathcal{F}$, the set of all adjointable maps from $\mathcal{E}$ to $\mathcal{F}$ is denoted by $L(\mathcal{E}, \mathcal{F})$, and $L(\mathcal{E})$ stands for the unital C^*-algebra $L(\mathcal{E}, \mathcal{E})$. We assume that $\mathfrak{A} \otimes_{alg} \mathfrak{B}$ and $\mathfrak{A} \otimes_* \mathfrak{B}$ denote, respectively, the algebraic tensor product and an arbitrary fixed C^*-tensor product of the C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ with the corresponding C^*-tensor norm $\|\cdot\|_*$. For more details on the general theory of C^*-algebras, the reader is referred to [14]. Let $\mathcal{E}_1$ and

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$\mathcal{E}_2$ be Hilbert C^*-modules over C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, respectively. The algebraic tensor product and the exterior tensor product of $\mathcal{E}_1$ and $\mathcal{E}_2$ are denoted by $\mathcal{E}_1 \otimes_{alg} \mathcal{E}_2$ and $\mathcal{E}_1 \otimes \mathcal{E}_2$, respectively. Indeed, $\mathcal{E}_1 \otimes \mathcal{E}_2$ is a Hilbert $\mathfrak{A} \otimes_* \mathfrak{B}$-module equipped with the following $\mathfrak{A} \otimes_* \mathfrak{B}$-valued inner product [11]:

$$\langle x \otimes y, z \otimes w \rangle = \langle x, z \rangle \otimes \langle y, w \rangle \quad (x, z \in \mathcal{E}_1, y, w \in \mathcal{E}_2).$$

Let $\mathcal{E}$ be a Hilbert C^*-module over a C^*-algebra $\mathfrak{A}$. The set of all bounded $\mathfrak{A}$-linear maps from $\mathcal{E}$ to $\mathfrak{A}$ is denoted by $\mathcal{E}'$. The space $\mathcal{E}$ can be embedded in $\mathcal{E}'$ via $\widehat{\ }: \mathcal{E} \to \mathcal{E}'$ defined by $x \mapsto \hat{x}$, where $\hat{x}(y) = \langle x, y \rangle$ $(y \in \mathcal{E})$. A Hilbert C^*-module $\mathcal{E}$ is called self-dual if $\widehat{\mathcal{E}} = \mathcal{E}'$. We refer the reader to [11,19] for more details on the theory of Hilbert C^*-modules.

Throughout this article, S and X stand for nonempty sets. We denote C^*-algebras by $\mathfrak{A}$ and $\mathfrak{B}$. By $\mathcal{Z}(\mathfrak{A})$ and $\mathfrak{A}^+$ we mean the center of the C^*-algebra $\mathfrak{A}$ and the set of all positive elements of $\mathfrak{A}$, respectively. When $\mathfrak{A}$ is unital, $\mathrm{Inv}(\mathfrak{A})$ stands for the set of all invertible elements of $\mathfrak{A}$. The C^*-algebra of all $n \times n$ matrices with entries in $\mathfrak{A}$ is presented by $\mathbb{M}_n(\mathfrak{A})$.

Aronszajn [2] defined the concept of reproducing kernel Hilbert space (shortly, $RKHS$), and Schwartz [16] developed the concept. This theory has many applications in integral equations, complex analysis, and so on; see [10]. Indeed, an $RKHS$ $\mathcal{H}$ is a Hilbert space of $\mathbb{C}$-valued functions on a set S such that, for all $s \in S$, the evaluation map $\delta_s : \mathcal{H} \to \mathbb{C}$ defined by $\delta_s(f) = f(s)$ is bounded. It follows from the Riesz representation theorem that, for every $s \in S$, there exists a unique element $k_s \in \mathcal{H}$ such that

$$\delta_s(f) = f(s) = \langle f, k_s \rangle \quad (f \in \mathcal{H}).$$

Furthermore, the two-variable function $K : S \times S \to \mathbb{C}$ defined by $K(s,t) = k_t(s)(s,t \in S)$ is called a reproducing kernel for $\mathcal{H}$. A theorem due to Moore [15, Theorem 2.14] states that for a scalar-valued positive definite kernel, there is a unique $RKHS$ such that K is its reproducing kernel. Indeed, there is a two-sided relation between scalar-valued positive definite kernels and $RKHS$s. For more information about reproducing kernel spaces we refer the interested reader to [1,3,15] and references therein.

The Papadakis theorem [15, Theorem 2.10] shows that $\{f_s : s \in S\}$ is a Parseval frame for an $RKHS$ if and only if $K(x,y) = \sum_{s \in S} \overline{f_s(x)} f_s(y)$, where the series converges pointwise. In general, finding k_s for every $s \in S$, and so K, is not easy, but the Papadakis theorem provides a useful benchmark.

Although Hilbert C^*-modules generalize Hilbert spaces, some fundamental properties of Hilbert spaces are no longer valid in Hilbert C^*-modules in their full generality. For instance, not every bounded $\mathfrak{A}$-linear operator is adjointable. Thus in the theory of Hilbert C^*-modules, it is interesting to ask which results, similar to those for Hilbert spaces, can be proved probably under some conditions. Inspiring by some ideas in the Hilbert space setting [15], we extend some significant classical results to the setting of Hilbert C^*-modules.

The paper is organized as follows.

In the next section, we use some ideas of Szafraniec [18] to give a modified definition of a reproducing kernel Hilbert C^*-module (shortly, $RKHC^*M$) due to Heo [9] without using the condition of self-duality and discuss some related aspects. Such a lack of self-duality shows that our investigation is nontrivial and is not a straightforward generalization of the classical case of $RKHS$s. In the same section, the exterior tensor product of $RKHC^*M$s is investigated and an interpolation theorem is presented. Section 3 deals with left multipliers of $RKHC^*M$s. Under some mild conditions, we show that one can make a new $RKHC^*M$ by a left multiplier. In addition, we introduce the Berezin transform of an operator in the context of $RKHC^*M$s and construct a unital subalgebra of the unital C^*-algebra consisting of adjointable maps on an $RKHC^*M$ and show that it is closed with respect to a certain topology. In section 4, we extend the Papadakis theorem to the setting of $RKHC^*M$ and find some conditions, in order for the multiplication of two specific functions to be in the Papadakis $RKHC^*M$.

2. A modified definition of $RKHC^*M$

We denote by $\mathbb{F}(S,\mathfrak{A})$ the set of all $\mathfrak{A}$-valued functions on S. It is clear that $\mathbb{F}(S,\mathfrak{A})$ is a right $\mathfrak{A}$-module equipped with the ordinary operations.

Definition 2.1 *(see [9]).* By a kernel on S we mean a map $K : S \times S \to \mathfrak{A}$. A kernel K is called positive definite whenever the matrix $\big(K(s_i,s_j)\big)_{i,j=1}^n \in \mathbb{M}_n(\mathfrak{A})$ is positive or, equivalently,

$$\sum_{i,j=1}^{n} a_i^* K(s_i,s_j)a_j \geq 0,$$

for all $n \in \mathbb{N}$, $s_1,s_2,\ldots,s_n \in S$, and $a_1,a_2,\ldots,a_n \in \mathfrak{A}$. Then, $K(s,t) = K(t,s)^*$ for all $s,t \in S$. We say that a kernel K is strictly positive whenever $\big(K(s_i,s_j)\big)_{i,j=1}^n$ is positive and invertible in $\mathbb{M}_n(\mathfrak{A})$.

The following definition of an $RKHC^*M$ differs from that of [9] and is inspired by [18].

Definition 2.2. A right $\mathfrak{A}$-submodule $\mathcal{E}$ of $\mathbb{F}(S,\mathfrak{A})$ is called a reproducing kernel Hilbert C^*-module structure on S (or simply, $RKHC^*M$ on S) if it satisfies the following conditions:

(i) $\mathcal{E}$ is a Hilbert C^*-module over $\mathfrak{A}$.
(ii) For every $s \in S$, there exists $k_s \in \mathcal{E}$ such that the evaluation map $\delta_s : \mathcal{E} \to \mathfrak{A}$ at $s \in S$ satisfies $\delta_s(f) = f(s) = \langle k_s, f\rangle$ for all $f \in \mathcal{E}$.
(iii) The $\mathfrak{A}$-linear span of $\{k_s : s \in S\}$ is dense in $\mathcal{E}$.

The element k_s is called the reproducing kernel for the point $s \in S$.

The corresponding reproducing kernel $K : S \times S \to \mathfrak{A}$ is given by $K(s,t) = \langle k_s, k_t\rangle$ for every $s,t \in S$; see [4]. In what follows, we use the following theorems.

Theorem 2.3. *[9, proposition 3.1] Let S, $\mathfrak{A}$, and $\mathcal{E}$ be as above. Then*

(i) *the kernel K is positive definite;*
(ii) *$K(s,s) \in \mathfrak{A}^+$ for every $s \in S$;*
(iii) *$\|K(s,t)\|^2 \leq \|K(s,s)\|\|K(t,t)\|$ for all $s,t \in S$.*

Now let $K : S \times S \to \mathfrak{A}$ be a positive definite kernel. For every $s \in S$, consider the function $k_s : S \to \mathfrak{A}$ by $k_s(t) = K(t,s)$. Assume that $\mathcal{E}_0$ is the right $\mathfrak{A}$-module of $\mathfrak{A}$-valued functions on S generated by $\{k_s : s \in S\}$. Now, setting

$$\left\langle \sum_{i=1}^{m} k_{s_i}a_i, \sum_{j=1}^{n} k_{t_j}b_j \right\rangle := \sum_{i=1}^{m}\sum_{j=1}^{n} a_i^* K(s_i,t_j)b_j,$$

where $m,n \in \mathbb{N}$, $a_1,a_2,\ldots,a_n,b_1,b_2,\ldots,b_n \in \mathfrak{A}$, and $s_1,s_2,\ldots,s_m,t_1,t_2,\ldots,t_n \in S$, we make $\mathcal{E}_0$ into a pre-Hilbert $\mathfrak{A}$-module. Suppose that $\mathcal{E}$ denotes its completion. In addition, $\mathcal{E}$ can be considered as a Hilbert $\mathfrak{A}$-module of $\mathfrak{A}$-valued functions on S, and evidently $\langle k_s, k_t\rangle = K(s,t)$ for all $s,t \in S$, which means that K is its reproducing kernel. The above construction is given by Heo [9] and holds true for the Definition 2.2. It entails the following result.

Theorem 2.4. *[9, Theorem 3.2] If $K : S \times S \to \mathfrak{A}$ is positive definite, then there exists a unique Hilbert $\mathfrak{A}$-module consisting of $\mathfrak{A}$-valued functions on S such that K is its reproducing kernel.*

Heo [9] introduced the concept of an $RKHC^*M$ and transferred some of the classical theorems to the setting of $RKHC^*M$s under an implicit assumption on self-duality. Note that self-duality is a rare property for Hilbert C^*-modules (e.g. the standard Hilbert C^*-module over a C^*-algebra A is self-dual if and only if A is finite-dimensional), and so it is too restrictive to be required; see [5] and references therein.

Here, we explain why $RKHC^*M$s are almost never self-dual; see [5]. For simplicity, we assume $S = \mathbb{N}$. The case of non-discrete S is technically more complicated, but is essentially similar. Then $\xi = \sum_s k_s a_s$ can be viewed as a sequence $\xi = (a_s)_{s\in\mathbb{N}}$, where $a_s \in \mathfrak{A}$, $s \in \mathbb{N}$. The inner product on $\mathcal{E}$ is given by

$$\langle \xi, \eta \rangle = \sum_{s,t\in\mathbb{N}} a_s^* K(s,t) b_t.$$

A sequence ξ lies in $\mathcal{E}$ if and only if the series $\sum_{s,t\in\mathbb{N}} a_s^* K(s,t) a_t$ is norm-convergent. Let $f = (f_s)$ be a sequence such that all partial sums $\sum_{s,t\in\mathbb{N}} f_s^* K(s,t) f_t$ are uniformly bounded. Then the formula $f(\xi) = \sum_{s,t\in\mathbb{N}} f_s^* K(s,t) a_t$ is well-defined (i.e., the series is norm-convergent) and gives a bounded $\mathfrak{A}$-linear map from $\mathcal{E}$ to $\mathfrak{A}$, that is, $f \in \mathcal{E}'$. It is easy to see that the two conditions, norm convergence and uniform boundedness, are different in most cases. For simplicity, assume that $\mathfrak{A}$ is commutative, and so is of the form $C_0(X)$ for a locally compact Hausdorff space X. Then $K(s,t)$s are functions on X. Suppose that $\cup_{s\in\mathbb{N}} \operatorname{supp} K(s,s)$ is infinite. Then one can find functions f_s, $s \in \mathbb{N}$, on X such that $\|K(s,s) f_s\| = 1$ (where $\|\cdot\|$ is the sup-norm on X) and $f_s f_t = 0$ when $s \neq t$. Then, obviously, $(f_s)_{s\in\mathbb{N}} \in \mathcal{E}' \setminus \mathcal{E}$. Note that since K is positive definite, the condition that $\cup_{s\in\mathbb{N}} \operatorname{supp} K(s,s)$ is finite, implies that $\operatorname{supp} K(s,t)$ is finite as well.

While every closed subspace of a Hilbert space is orthogonally complemented, it is well known that submodules in Hilbert C^*-modules are often not orthogonally complemented. Furthermore, it is shown in [12] that if there exists a full Hilbert $\mathfrak{A}$-module in which every closed submodule is orthogonally complemented, then $\mathfrak{A}$ is $*$-isomorphic to a C^*-algebra of (not necessarily all) compact operators on some Hilbert spaces; see [6] and references therein. The following theorem provides a class of orthogonally complemented submodules being $RKHC^*M$s. To achieve the next result, we need a lemma.

Lemma 2.5. *Let $\mathcal{E}$ be an $RKHC^*M$ on a set S with the kernel K. Then every orthogonally complemented submodule $\mathcal{E}_0$ of $\mathcal{E}$ can be endowed with an $RKHC^*M$ on S.*

Proof. Let $P : \mathcal{E} \to \mathcal{E}_0$ be the orthogonal projection onto $\mathcal{E}_0$. If k_s is the reproducing kernel at the point s in $\mathcal{E}$, then $P(k_s)$ evidently is the reproducing kernel for the point s in $\mathcal{E}_0$ satisfying the conditions of Definition 2.1. Note that

$$f(s) = \langle k_s, f \rangle = \langle k_s, P(f) \rangle = \langle P(k_s), f \rangle,$$

for all $f \in \mathcal{E}_0$. Thus $\mathcal{E}_0$ is an $RKHC^*M$ with the reproducing kernel $K_0(s,t) = \langle P(k_s), P(k_t) \rangle$. □

Theorem 2.6. *Suppose that $K(s_0, s_0)$ is invertible for some $s_0 \in S$. Then the following properties hold:*

(i) *The submodule $\mathcal{E}_0 = \{f \in \mathcal{E} : f(s_0) = 0\}$ is orthogonally complemented in $\mathcal{E}$.*
(ii) *$\mathcal{E}_0$ is the $RKHC^*M$ with the reproducing kernel*

$$K_0(s,t) = K(s,t) - K(s,s_0) K(s_0,s_0)^{-1} K(s_0,t).$$

Proof. (i) Note that k_{s_0} satisfying $\langle k_{s_0}, k_{s_0}\rangle = K(s_0, s_0)$ is invertible. Hence $P(f) = k_{s_0}\langle k_{s_0}, k_{s_0}\rangle^{-1}\langle k_{s_0}, f\rangle$ is the orthogonal projection onto the $\mathfrak{A}$-linear span of k_{s_0}. Then $\mathcal{E} = k_{s_0}\mathfrak{A} \oplus (k_{s_0}\mathfrak{A})^{\perp}$. The condition $f \perp k_{s_0}$ can be written as $0 = \langle k_{s_0}, f\rangle = f(s_0)$. Hence $(k_{s_0}\mathfrak{A})^{\perp} = \mathcal{E}_0$.

(ii) The reproducing kernel for $\mathcal{E}_0$ is given by

$$\begin{aligned}\langle (I-P)(k_s), (I-P)(k_t)\rangle &= \langle k_s, (I-P)(k_t)\rangle = \langle k_s, k_t - k_{s_0}\langle k_{s_0}, k_{s_0}\rangle^{-1}\langle k_{s_0}, k_t\rangle\rangle \\ &= K(s,t) - K(s,s_0)K(s_0,s_0)^{-1}K(s_0,t) \\ &= K_0(s,t). \quad \square\end{aligned}$$

The next result provides an interpolation theorem. We should notify that a finitely generated $\mathfrak{A}$-submodule is not necessarily closed. For example, when $\mathfrak{A} = C([0,1]) = \mathcal{E}$, the submodule singly generated by the function $f(x) = x$ is not closed, and its closure is not finitely generated.

Theorem 2.7. *Let $\mathcal{E}$ be an RKHC$^*M$ on S with the reproducing kernel K. Let $F = \{s_1, \dots, s_n\}$ be a subset of distinct elements of S such that the C^*-submodule generated by $k_{s_1}, \dots, k_{s_n}$ is closed in $\mathcal{E}$, and let $a_1, \dots, a_n \in \mathfrak{A}$. Then there is a function $f \in \mathcal{E}$ of minimal norm such that $f(s_i) = a_i$ for all $1 \le i \le n$ if and only if $(a_1, \dots, a_n)^t$ is in the range of the matrix $\big(K(s_j, s_i)\big) \in \mathbb{M}_n(\mathfrak{A})$.*

Proof. Assume that P_F is the orthogonal projection onto the orthogonally complemented submodule $\mathcal{E}_F$ finitely generated by $\{k_s : s \in F\}$. So that $\mathcal{E} = \mathcal{E}_F \oplus \mathcal{E}_F^{\perp}$, see [13, Lemma 2.3.7]. Let $f \in \mathcal{E}$ and let $P_F(f) = \sum_{j=1}^n k_{s_j} b_j \in \mathcal{E}_F$, where b_j's are in $\mathfrak{A}$. Then $P_F(f)(s) = f(s)$ for all $s \in F$, since $f(s) = \langle k_s, f\rangle = 0$ for all $s \in F$ if and only if $f \in \mathcal{E}_F^{\perp}$.

If there is a function $f \in \mathcal{E}$ such that $f(s_i) = a_i$ for all $1 \le i \le n$, then

$$a_i = f(s_i) = P_F(f)(s_i) = \langle k_{s_i}, P_F(f)\rangle = \left\langle k_{s_i}, \sum_{j=1}^n k_{s_j} b_j \right\rangle = \sum_{j=1}^n K(s_i, s_j) b_j.$$

Thus $(a_1, \dots, a_n)^t = \big(K(s_j, s_j)\big)(b_1, \dots, b_n)^t$ is in the range of the matrix $\big(K(s_j, s_i)\big)$.
In addition, if $g \in \mathcal{E}$ interpolates these points, then $(g - f)(s) = 0$ for all $s \in F$. Hence $g - f \in \mathcal{E}_F^{\perp}$, whence $g = f + h$ with $h \in \mathcal{E}_F^{\perp}$. Therefore,

$$\|P_F(f)\| = \|P_F(f + h)\| \le \|f + h\| = \|g\|.$$

Thus, $P_F(f)$ is the unique function of minimum norm that interpolates these values.

Conversely, if $(a_1, \dots, a_n)^t$ is in the range of $\big(K(s_j, s_i)\big)$ and $(a_1, \dots, a_n)^t = \big(K(s_i, s_j)\big)(b_1, \dots, b_n)^t$ for some $b_1, \dots, b_n \in \mathfrak{A}$, then

$$a_i = \sum_{j=1}^n K(s_i, s_j) b_j = \left\langle k_{s_i}, \sum_{j=1}^n k_{s_j} b_j \right\rangle.$$

Putting $f := \sum_{j=1}^n k_{s_j} b_j \in \mathcal{E}_F$, we get $f = P_F(f)$ and $a_i = \langle k_{s_i}, f\rangle = f(s_i)$ for $1 \le i \le n$. $\square$

Remark 2.8. If $\overline{a} = (a_1, \dots, a_n)^t$ and $\overline{b} = (b_1, \dots, b_n)^t$ are in the Hilbert C^*-module $\mathfrak{A}^n$ with its natural inner product (see [13]) and $(a_1, \dots, a_n)^t = \big(K(s_i, s_j)\big)(b_1, \dots, b_n)^t$, then we can choose f such that $\|f\| = \|\langle \overline{a}, \overline{b}\rangle\|^{1/2}$. In fact, if $f := \sum_{j=1}^n k_{s_j} b_j$, then

$$\|f\|^2 = \left\| \left\langle \sum_{i=1}^n k_{s_i} b_i, \sum_{j=1}^n k_{s_j} b_j \right\rangle \right\| = \left\| \sum_{1 \le i,j \le n} b_i^* K(s_i, s_j) b_j \right\|$$

$$= \left\| \left\langle (b_i)_i, \left(\sum_{j=1}^{n} K(s_i, s_j) b_j \right)_i \right\rangle \right\| = \|\langle \overline{b}, \overline{a} \rangle\|.$$

If K is strictly positive, then $\overline{b}$ is uniquely defined by $\overline{a}$. Thus, from the arguments at the first part of the proof of the above theorem, there is a unique $f \in \mathcal{E}_F$ satisfying the conditions of Theorem 2.7.

Now, we investigate the exterior tensor product of $RKHC^*M$s. Let $K_1 : X \times X \to \mathfrak{A}$ and $K_2 : S \times S \to \mathfrak{B}$ be positive definite kernels on sets X and S, respectively. It follows from Theorem 2.4 that there are $RKHC^*M$ $\mathcal{E}_1$ and $\mathcal{E}_2$ over $\mathfrak{A}$ and $\mathfrak{B}$ consisting of $\mathfrak{A}$-valued functions on X and $\mathfrak{B}$-valued functions on S, respectively. We define $K : (X \times S) \times (X \times S) \to \mathfrak{A} \otimes_* \mathfrak{B}$, where $\otimes_*$ denotes a fixed C^*-tensor product with the C^*-cross-norm $\|\cdot\|_*$ by

$$K((x, s), (y, t)) = K_1(x, y) \otimes K_2(s, t), \qquad (x, s), (y, t) \in X \times S.$$

Let $\xi_i = \sum_{k=1}^{n} a_i^k \otimes b_i^k \in \mathfrak{A} \otimes_{alg} \mathfrak{B}$, $i = 1, \ldots, m$. Then

$$\sum_{i,j=1}^{n} \xi_i^* K((x_i, s_i), (x_j, s_j)) \xi_j$$
$$= \sum_{k,l=1}^{n} \left(\sum_{i,j=1}^{n} (a_i^k)^* K_1(x_i, x_j) a_j^l \right) \otimes \left(\sum_{i,j=1}^{n} (b_i^k)^* K_2(s_i, s_j) b_j^l \right).$$

Set

$$\alpha_{kl} := \sum_{i,j=1}^{n} (a_i^k)^* K_1(x_i, x_j) a_j^l \in \mathfrak{A}, \quad \beta_{kl} := \sum_{i,j=1}^{n} (b_i^k)^* K_2(s_i, s_j) b_j^l \in \mathfrak{B}.$$

Then the matrices $(\alpha_{kl})_{k,l=1}^n$ and $(\beta_{kl})_{k,l=1}^n$ are positive elements of $M_n(\mathfrak{A})$ and of $M_n(\mathfrak{B})$, respectively. Then, by Lemma 4.3 of [11],

$$\sum_{i,j=1}^{n} \xi_i^* K((x_i, s_i), (x_j, s_j)) \xi_j = \sum_{k,l=1}^{n} \alpha_{kl} \otimes \beta_{kl} \geq 0. \tag{2.1}$$

Since the set of all positive elements in a C^*-algebra is closed, we conclude that (2.1) is also valid for every choice of elements ξ in $\mathfrak{A} \otimes_* \mathfrak{B}$. Thus K is a positive definite kernel. Again, in virtue of Theorem 2.4, there exists a Hilbert $\mathfrak{A} \otimes_* \mathfrak{B}$-module $\mathcal{E}$ of $\mathfrak{A} \otimes_* \mathfrak{B}$-valued functions on $X \times S$ such that K is its reproducing kernel. Now this question raises: What relations are there between $\mathcal{E}$, $\mathcal{E}_1$, and $\mathcal{E}_2$, where $\mathcal{E}_1$ and $\mathcal{E}_2$ are $RKHC^*M$s with kernels K_1 and K_2, respectively?

Recall that the exterior tensor product $\mathcal{E}_1 \otimes \mathcal{E}_2$ of Hilbert C^*-modules $\mathcal{E}_1$ over $\mathfrak{A}$ and $\mathcal{E}_2$ over $\mathfrak{B}$ is defined as the Hilbert C^*-module over $\mathfrak{A} \otimes_* \mathfrak{B}$ obtained by completion of $\mathcal{E}_1 \otimes_{alg} \mathcal{E}_2$ with respect to the norm

$$\|u\|^2 = \left\| \sum_{i,j=1}^{n} \langle f_i, f_j \rangle \otimes \langle g_i, g_j \rangle \right\|_*,$$

where $u = \sum_{i=1}^{n} f_i \otimes g_i \in \mathcal{E}_1 \otimes_{alg} \mathcal{E}_2$; see [11].

Set $k_y^1(x) := K_1(x, y)$, $k_t^2(s) := K_2(s, t)$, and $k_{(y,t)}(x, s) := K((x, s), (y, t))$. Clearly, $k_{(y,t)}(x, s) = k_y^1(x) \otimes k_t^2(s)$. By the assumption, the $\mathfrak{A}$-linear spans of $\{k_x^1 : x \in X\}$, $\{k_s^2 : s \in S\}$, and $\{k_{(x,s)} : (x, s) \in X \times S\}$ are dense in $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}$, respectively.

We claim that $\mathcal{E}_1 \otimes \mathcal{E}_2$ is unitarily equivalent to $\mathcal{E}$. Let $f_i = \sum_{x\in X} k_x^1 a_x^i$ and $g_i = \sum_{s\in S} k_s^2 b_s^i$, where $a_x^i \in \mathfrak{A}$, $b_s^i \in \mathfrak{B}$, and both sums have a finite number of nonzero summands. Set

$$\Phi\left(\sum_{i=1}^{n}\sum_{x\in X,s\in S} k_x^1 a_x^i \otimes k_s^2 b_s^i\right) := \sum_{x\in X,s\in S} k_{(x,s)} \cdot \sum_{i=1}^{n} a_x^i \otimes b_s^i. \tag{2.2}$$

For $u \in \mathcal{E}_1 \otimes_{alg} \mathcal{E}_2$, define $\hat{u} \in \mathbb{F}(X \times S, \mathfrak{A} \otimes_* \mathfrak{B})$ by

$$\hat{u}(x,s) = \langle k_x^1 \otimes k_s^2, u\rangle, \qquad (x,s) \in X \times S.$$

Let $u = \sum_{i=1}^n f_i \otimes g_i$, where $f_i = \sum_{x\in X} k_x^1 a_x^i$, $g_i = \sum_{s\in S} k_s^2 b_s^i$, where $a_x^i \in \mathfrak{A}$, $b_s^i \in \mathfrak{B}$, and both sums have a finite number of nonzero summands. Then

$$\begin{aligned}\hat{u}(y,t) &= \sum_{i=1}^{n}\sum_{x\in X,s\in S} k_x^1(y) a_x^i \otimes k_s^2(t) b_s^i \\ &= \sum_{x\in X,s\in S} k_{(x,s)}(y,t) \sum_{i=1}^{n} a_x^i \otimes b_s^i = \Phi(u)(y,t),\end{aligned}$$

which shows that the map Φ, defined in (2.2), is well-defined and that $\hat{u} \in \mathcal{E}$.

It is clear that Φ is an isometry between dense subspaces of $\mathcal{E}_1 \otimes \mathcal{E}_2$ and of $\mathcal{E}$, hence it extends to a surjective isometry $\Phi : \mathcal{E}_1 \otimes \mathcal{E}_2 \to \mathcal{E}$.

Definition 2.9. Suppose that $K_1 : X \times X \to \mathfrak{A}$ and $K_2 : S \times S \to \mathfrak{B}$ are kernels. We call the map $K : (X \times S) \times (X \times S) \to \mathfrak{A} \otimes_* \mathfrak{B}$ defined by

$$K((x,s),(y,t)) = K_1(x,s) \otimes K_2(y,t), \qquad (x,s),(y,t) \in X \times S$$

the tensor product of the kernels K_1 and K_2, and denote it by $K_1 \otimes K_2$.

We summarize the above results in the following theorem.

Theorem 2.10. *Let K_1 and K_2 be positive definite kernels and let $\mathcal{E}_1$ and $\mathcal{E}_2$ be their associated Hilbert C^*-modules. Then $K_1 \otimes K_2$ is a positive definite kernel, and its associated Hilbert C^*-module is unitarily equivalent to the exterior tensor product of $\mathcal{E}_1$ and $\mathcal{E}_2$.*

3. Left multipliers of $RKHC^*M$s

If F_1 and F_2 are submodules of $\mathbb{F}(S,\mathfrak{A})$, then a function $f \in \mathbb{F}(S,\mathfrak{A})$ for which $fF_1 \subseteq F_2$ is called a left multiplier of F_1 into F_2. Note that

$$fF_1 = \{fh : h \in F_1\},$$

where fh is the multiplication of f and h. The set of all left multipliers of F_1 into F_2 is denoted by $\mathcal{M}(F_1,F_2)$. Clearly, $\mathcal{M}(F_1,F_2)$ is a linear space. Moreover, $\mathcal{M}(F)$ stands for $\mathcal{M}(F,F)$ being an algebra. For every $f \in \mathcal{M}(F_1,F_2)$, there is a linear map $M_f : F_1 \to F_2$ that is defined by $M_f(h) = fh$ for all $h \in F_1$.

The following lemma is a useful property of $RKHC^*M$s.

Lemma 3.1. *Suppose that $\mathcal{E}$ is an $RKHC^*M$ on a set S with the kernel K. If a sequence (h_n) in $\mathcal{E}$ converges to h, then $\lim_n h_n(s) = h(s)$ for each $s \in S$.*

Proof. It is easily concluded from

$$\|h_n(s) - h(s)\| = \|\langle k_s, h_n\rangle - \langle k_s, h\rangle\| \leq \|k_s\| \, \|h_n - h\|. \quad \square$$

Let $\mathcal{E}$ be an $RKHC^*M$ on S and let $g : S \to \mathfrak{A}$ be a function. Put

$$\mathcal{E}_g = \{gh : h \in \mathcal{E}\}.$$

In the next theorem, we endow the right $\mathfrak{A}$-submodule $\mathcal{E}_g$ of $\mathbb{F}(S, \mathfrak{A})$ with an $RKHC^*M$ structure.

Theorem 3.2. *Suppose that $\mathcal{E}$ is an $RKHC^*M$ on a set S with the kernel K and that $g : S \to \mathfrak{A}$ is an arbitrary function. Then the following assertions hold:*

(i) $\mathcal{E}_0 := \{h \in \mathcal{E} : gh = 0\}$ *is closed.*
(ii) *If $\mathcal{E}_0$ is orthogonally complemented, then $\mathcal{E}_g$ is an $RKHC^*M$ with kernel $K'(s,t) = g(s)K(s,t)g(t)^*$.*

Proof. (i) It follows from Lemma 3.1 that $\mathcal{E}_0$ is closed.
(ii) It follows from the assumption that $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_0^\perp$. Therefore

$$\mathcal{E}_g = \{g\tilde{h} + gh^\# : \tilde{h} \in \mathcal{E}_0, h^\# \in \mathcal{E}_0^\perp\} = \{gh : h \in \mathcal{E}_0^\perp\}.$$

We define an $\mathfrak{A}$-valued inner product on $\mathcal{E}_g$ by $\langle gh_1, gh_2\rangle = \langle h_1, h_2\rangle$ for all $h_1, h_2 \in \mathcal{E}_0^\perp$. This is well-defined, since if $gh = gh'$ for $h, h' \in \mathcal{E}_0^\perp$, then $h - h' \in \mathcal{E}_0 \cap \mathcal{E}_0^\perp = \{0\}$. From the inner product on $\mathcal{E}_g$, it is clear that $\varphi_g : \mathcal{E}_0^\perp \to \mathcal{E}_g$ by $\varphi_g(h) = gh$ is a surjective linear isometry. Hence $\mathcal{E}_g$ is a Hilbert C^*-module isomorphic with $\mathcal{E}_0^\perp$. Thus the reproducing kernel structure of $\mathcal{E}_0^\perp$ constructed in Lemma 2.5 can be transferred onto $\mathcal{E}_g$. More precisely, for each $h \in \mathcal{E}_0^\perp$, we have

$$\begin{aligned}(gh)(s) = g(s)h(s) = g(s)\langle k_s, h\rangle = g(s)\langle k_s^\#, h\rangle = g(s)\langle gk_s^\#, gh\rangle \\ = g(s)\langle gk_s, gh\rangle = \langle gk_s g(s)^*, gh\rangle\end{aligned}$$

for some $k_s = \tilde{k_s} + k_s^\# \in \mathcal{E}_0 \oplus \mathcal{E}_0^\perp$. Hence the evaluation map δ_s can be represented by $\langle gk_s g(s)^*, \cdot\rangle$ with $k'_s = gk_s g(s)^* \in \mathcal{E}_g$. In addition, the corresponding reproducing kernel is

$$\begin{aligned}K'(s,t) = \langle k'_s, k'_t\rangle = \langle gk_s g(s)^*, gk_t g(t)^*\rangle = g(s)\langle gk_s, gk_t\rangle g(t)^* \\ = g(s)\langle gk_s^\#, gk_t^\#\rangle g(t)^* = g(s)\langle k_s^\#, k_t^\#\rangle g(t)^* + 0 \\ = g(s)\langle k_s^\#, k_t^\#\rangle g(t)^* + g(s)\tilde{k_t}(s)g(t)^* \qquad (\text{since } g\tilde{k_t} = 0) \\ = g(s)\langle k_s^\#, k_t^\#\rangle g(t)^* + g(s)\langle \tilde{k_s}, \tilde{k_t}\rangle = g(s)\langle k_s, k_t\rangle g(t)^* \\ = g(s)K(s,t)g(t)^*\end{aligned}$$

for every $s, t \in S$. $\square$

In the following theorem, k_s^1 and k_s^2 are the reproducing kernels at the point $s \in S$ for $RKHC^*M$s $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively.

Proposition 3.3. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be $RKHC^*M$s on a nonempty set S. If $f \in \mathcal{M}(\mathcal{E}_1, \mathcal{E}_2)$, then $M_f \in L(\mathcal{E}_1, \mathcal{E}_2)$ and $M_f^*(k_s^2) = k_s^1 f(s)^*$ for all $s \in S$.*

Proof. For every $h \in \mathcal{E}_1$, $s_1, \dots, s_n \in S$, and $a_1, \dots, a_n \in \mathfrak{A}$, we have

$$\left\langle \sum_{i=1}^{n} k_{s_i}^2 a_i, M_f(h) \right\rangle = \sum_{i=1}^{n} a_i^* \langle k_{s_i}^2, fh \rangle = \sum_{i=1}^{n} a_i^* f(s_i) h(s_i)$$
$$= \sum_{i=1}^{n} a_i^* f(s_i) \langle k_{s_i}^1, h \rangle = \left\langle \sum_{i=1}^{n} k_{s_i}^1 f(s_i)^* a_i, h \right\rangle.$$

Hence $M_f^* \left(\sum_{i=1}^n k_{s_i}^2 a_i \right) = \sum_{i=1}^n k_{s_i}^1 f(s_i)^* a_i$. In particular, $M_f^*(k_s^2) = k_s^1 f(s)^*$ for all $s \in S$. □

Thus, if $f \in \mathcal{M}(\mathcal{E})$ and $\mathfrak{A}$ is a unital C^*-algebra, then

$$f(s) = \langle k_s, fk_s \rangle K(s,s)^{-1} = \langle k_s, M_f(k_s) \rangle K(s,s)^{-1},$$

for every point $s \in S$ for which $K(s,s) \in \mathrm{Inv}(\mathfrak{A})$. Thus, we can present the following definition in the same manner as in the classical case [15] and transfer some known facts in the theory of $RKHS$s to context of $RKHC^*M$.

Definition 3.4. Let $\mathcal{E}$ be an $RKHC^*M$ on S over a unital C^*-algebra $\mathfrak{A}$. Let K be its associated kernel and let $T \in L(\mathcal{E})$ be arbitrary. Then the function

$$B_T : \{s \in S : K(s,s) \text{ is invertible}\} \longrightarrow \mathfrak{A}$$

defined by $B_T(s) = \langle k_s, T(k_s) \rangle K(s,s)^{-1}$ is called the Berezin transform of T associated by $\mathfrak{A}$.

Theorem 3.5. *Let $\mathcal{E}$ be an $RKHC^*M$ on S with the reproducing kernel K over a unital C^*-algebra $\mathfrak{A}$. Let*

$$L = \{M_f : f \in \mathcal{M}(\mathcal{E}) \text{ and } f(s) = 0 \text{ whenever } K(s,s) \text{ is not invertible}\}.$$

Then L is a unital subalgebra of $L(\mathcal{E})$.

Furthermore, if $\{M_{f_\alpha}\}_{\alpha \in I}$ is a net in L such that $\langle M_{f_\alpha} h_1, h_2 \rangle \to \langle Th_1, h_2 \rangle$ $(h_1, h_2 \in \mathcal{E})$ for some $T \in L(\mathcal{E})$, then $T = M_f$ for some $f \in \mathbb{F}(S, \mathfrak{A})$.

Proof. Since

$$\lambda M_f + M_g = M_{\lambda f + g}, \ M_f \circ M_g = M_{fg} \quad (f, g \in \mathcal{M}(\mathcal{E}), \lambda \in \mathbb{C}),$$

L is an algebra. Moreover, M_1 is the unit of L, where $1 \in \mathcal{M}(\mathcal{E})$ is the constant function onto the unit of $\mathfrak{A}$.

Next, we show that $T = M_f$ for some $f \in \mathcal{M}(\mathcal{E})$. We have

$$\lim_\alpha f_\alpha(s) = \lim_\alpha \langle k_s, M_{f_\alpha}(k_s) \rangle K(s,s)^{-1} = \langle k_s, T(k_s) \rangle K(s,s)^{-1} = B_T(s),$$

for every $s \in S$ for which $K(s,s)$ is invertible. Set $f(s) := B_T(s)$ whenever $K(s,s)$ is invertible and $f(s) := 0$ otherwise. To complete the proof, we shall show that $T = M_f$. We have

$$\left\langle \sum_{i=1}^{n} k_{s_i} a_i, Th \right\rangle = \lim_\alpha \left\langle \sum_{i=1}^{n} k_{s_i} a_i, M_{f_\alpha} h \right\rangle = \lim_\alpha \left\langle \sum_{i=1}^{n} k_{s_i} a_i, f_\alpha h \right\rangle$$
$$= \lim_\alpha \sum_{i=1}^{n} a_i^* f_\alpha(s_i) h(s_i) = \sum_{i=1}^{n} a_i^* f(s_i) h(s_i)$$

$$= \sum_{i=1}^{n} a_i^* \langle k_{s_i}, fh \rangle = \left\langle \sum_{i=1}^{n} k_{s_i} a_i, M_f h \right\rangle,$$

for every $h \in \mathcal{E}$, $s_1, \ldots, s_n \in S$, and $a_1, \ldots, a_n \in \mathfrak{A}$. Thus $T = M_f$. □

4. Papadakis theorem for $RKHC^*M$s

We recall the following definitions from [7].

Definition 4.1. Let J be an arbitrary subset of $\mathbb{N}$ and let $\mathcal{E}$ be a Hilbert C^*-module over a unital C^*-algebra $\mathfrak{A}$. A sequence $(x_j)_{j\in J}$ in $\mathcal{E}$ is said to be a (standard) frame if there are real numbers $C, D > 0$ such that

$$C\langle x, x\rangle \leq \sum_{j\in J} \langle x, x_j\rangle\langle x_j, x\rangle \leq D\langle x, x\rangle \tag{4.1}$$

for every $x \in \mathcal{E}$ in which the sum in the middle of inequality (4.1) converges in norm.

The sharp numbers (i.e., maximal for C and minimal for D) are called frame bounds. A frame $\{x_j : j \in J\}$ is said to be a tight frame if $C = D$, and normalized if $C = D = 1$. Therefore, a set $\{x_j : j \in J\}$ is a normalized tight frame whenever the equality

$$\langle x, x\rangle = \sum_{j\in J} \langle x, x_j\rangle\langle x_j, x\rangle \tag{4.2}$$

is valid for every $x \in \mathcal{E}$.

Now, we extend the Papadakis theorem to $RKHC^*M$s.

Theorem 4.2. *Let $\mathcal{E}$ be an $RKHC^*M$ on a set S over a unital C^*-algebra $\mathfrak{A}$ and let K be its corresponding reproducing kernel. Then $\{f_j : j \in J\} \subseteq \mathcal{E}$ is a normalized tight frame for $\mathcal{E}$ if and only if*

$$K(s,t) = \sum_{j\in J} f_j(s)^* f_j(t) \quad (s,t \in S), \tag{4.3}$$

where the sum is convergent in norm.

Proof. Suppose that $\{f_j : j \in J\}$ is a normalized tight frame for $\mathcal{E}$. It follows from (4.2) that

$$\langle f, f\rangle = \sum_{j\in J} \langle f, f_j\rangle\langle f_j, f\rangle,$$

for every $f \in \mathcal{E}$. Therefore, by the polarization identity, we can write

$$\begin{aligned} K(s,t) = \langle k_s, k_t\rangle &= \frac{1}{4}\sum_{k=0}^{3} i^k \langle k_t + i^k k_s, k_t + i^k k_s\rangle \\ &= \frac{1}{4}\sum_{k=0}^{3} i^k \sum_{j\in J} \langle k_t + i^k k_s, f_j\rangle\langle f_j, k_t + i^k k_s\rangle \\ &= \frac{1}{4}\sum_{k=0}^{3} i^k \sum_{j\in J} (f_j(t) + i^k f_j(s))^* (f_j(t) + i^k f_j(s)) \end{aligned}$$

$$= \sum_{j\in J} \frac{1}{4} \sum_{k=0}^{3} i^k \langle f_j(t) + i^k f_j(s), f_j(t) + i^k f_j(s)\rangle$$
$$= \sum_{j\in J} \langle f_j(s), f_j(t)\rangle = \sum_{j\in J} f_j(s)^* f_j(t)$$

for all $s, t \in S$.

Conversely, let (4.3) hold for some family $\{f_j : j \in J\}$ and let the sum in (4.3) converge in the norm topology. Then

$$\langle k_s, k_s\rangle = K(s,s) = \sum_{j\in J} f_j(s)^* f_j(s) = \sum_{j\in J} \langle k_s, f_j\rangle\langle f_j, k_s\rangle$$

for every $s \in S$. Hence, by the density of $\mathfrak{A}$-linear span of $\{k_s : s \in S\}$ in $\mathcal{E}$ and the joint continuity of inner product, we derive

$$\langle f, f\rangle = \sum_{j\in J} \langle f, f_j\rangle\langle f_j, f\rangle$$

for all $f \in \mathcal{E}$. It follows from (4.2) that $\{f_j : j \in J\}$ is a normalized tight frame. □

As we already mentioned in the introduction, $RKHC^*M$s are rarely self-dual. Recall that $\mathcal{E}'$ denotes the dual module of $\mathcal{E}$.

Lemma 4.3. *Elements of $\mathcal{E}'$ can be thought of as functions on S, that is, there is an inclusion $\mathcal{E}' \subset \mathbb{F}(S, \mathfrak{A})$ that extends the inclusion $\mathcal{E} \subset \mathbb{F}(S, \mathfrak{A})$.*

Proof. Suppose that $F \in \mathcal{E}'$. Set $F(s) := F(k_s)$. This gives us a map $\mathcal{E}' \to \mathbb{F}(S, \mathfrak{A})$. To show that this map is faithful, suppose that $F(s) = 0$ for every $s \in S$. Then F vanishes on a dense subset of $\mathcal{E}$, and hence is zero. □

It is clear that if K is a kernel of the form (4.3), then it is a positive definite kernel; see [8] for the Kolmogorov decomposition at the setting of Hilbert C^*-modules. Hence, employing Theorem 2.4, there exists a Hilbert $\mathfrak{A}$-module consisting of $\mathfrak{A}$-valued functions on S such that K is its reproducing kernel. This is a motivation for the following definition.

Definition 4.4. Let $K : S \times S \to \mathfrak{A}$ be the positive definite kernel defined by

$$K(s,t) = \sum_{\alpha\in I} e_\alpha(s)^* e_\alpha(t) \quad (s, t \in S),$$

where $\{e_\alpha\}_{\alpha\in I}$ is a family in $\mathbb{F}(S, \mathfrak{A})$ with the property that $\sum_{\alpha\in I} e_\alpha(s)^* e_\alpha(s)$ converges in $\mathfrak{A}$. Then K is called the Papadakis kernel, and the Hilbert $\mathfrak{A}$-module consisting of $\mathfrak{A}$-valued functions on S, given by Theorem 2.4, is called the Papadakis Hilbert $\mathfrak{A}$-module.

Let K be a Papadakis kernel on S for some family $\{e_\alpha\}_{\alpha\in I} \subseteq \mathbb{F}(S, \mathfrak{A})$ and let $\mathcal{E}$ be the associated Papadakis Hilbert C^*-module. The following theorem shows that the multiplication of an element of $\mathbb{F}(S, \mathfrak{A})$ satisfying suitable conditions and that e_α ($\alpha \in I$) is an element of $\mathcal{E}$. Note that $e_\alpha^* : S \to \mathfrak{A}$ is defined by $e_\alpha^*(s) = e_\alpha(s)^*$ for all $\alpha \in I$ and $s \in S$. To achieve our next result, we mimic some ideas of [17].

Definition 4.5. A subset P of S is said to be a set of uniqueness of $\mathcal{E} \subseteq \mathbb{F}(S, \mathfrak{A})$ if the $\mathfrak{A}$-linear span of k_p, $p \in P$, is dense in $\mathcal{E}$. In this case, we write $P \in \mathbb{U}(\mathcal{E})$.

Note that if $f, g \in \mathcal{E}$ with $f(p) = g(p)$ for any $p \in P$, then $f = \sum_{p\in P} k_p a_p$ and $g = \sum_{p\in P} k_p b_p$, where $a_p, b_p \in \mathfrak{A}$. Then $f = g$.

Theorem 4.6. *Let K be the Papadakis kernel for some family $\{e_\alpha\}_{\alpha\in I} \subseteq \mathbb{F}(S, \mathfrak{A})$ and let $\mathcal{E}$ be the associated Papadakis Hilbert C^*-module. Let $e_\alpha \in \mathcal{M}(\mathcal{E}', \mathcal{E})$ and let $X \in \mathbb{U}(\mathcal{E})$. Assume that $\psi : X \to \mathcal{Z}(\mathfrak{A})$ is a function and that $c > 0$ is such that*

$$\sum_{i,j=1}^{n} a_i^* K(x_i, x_j)(c^2 - \psi(x_i)^*\psi(x_j))a_j \geq 0, \tag{4.4}$$

for all $n \in \mathbb{N}, x_1, x_2, \ldots, x_n \in X$, and $a_1, a_2, \ldots, a_n \in \mathfrak{A}$. Then for every $\alpha \in I$, there is a unique function $\varphi_\alpha \in \mathcal{E}$ such that

$$\varphi_\alpha(x) = e_\alpha(x)\psi(x), \qquad x \in X$$

or, equivalently,

$$\varphi_\alpha = e_\alpha\psi$$

and

$$e_\alpha\varphi_\beta = \varphi_\alpha e_\beta,$$

for all $\alpha, \beta \in I$. Furthermore, if $\operatorname{ran}(e_\alpha) \subseteq \mathcal{Z}(\mathfrak{A})$ and $K(s, s)$ is invertible for every $s \in S$, then

$$|\varphi_\alpha(s)| \leq c|e_\alpha(s)|, (s \in S).$$

Proof. Inequality (4.4) can be restated as follows:

$$\begin{aligned} c^2 \left\langle \sum_{i=1}^{n} k_{x_i} a_i, \sum_{j=1}^{n} k_{x_j} a_j \right\rangle &\geq \sum_{\alpha\in I} \left(\sum_{i=1}^{n} e_\alpha(x_i) a_i \psi(x_i) \right)^* \left(\sum_{j=1}^{n} e_\alpha(x_j) a_j \psi(x_j) \right) \\ &= \sum_{\alpha\in I} \left| \sum_{i=1}^{n} e_\alpha(x_i) a_i \psi(x_i) \right|^2 \end{aligned} \tag{4.5}$$

for every $n \in \mathbb{N}, x_1, x_2, \ldots, x_n \in X$ and $a_1, a_2, \ldots, a_n \in \mathfrak{A}$. Put

$$D = \left\{ \sum_{i=1}^{n} k_{x_i} a_i : n \in \mathbb{N}, x_1, x_2, \ldots, x_n \in X, a_1, a_2, \ldots, a_n \in \mathfrak{A} \right\}.$$

For every $\alpha \in I$, we define $\varphi_\alpha : D \to \mathfrak{A}$ by

$$\varphi_\alpha \left(\sum_{i=1}^{n} k_{x_i} a_i \right) = \sum_{i=1}^{n} e_\alpha(x_i) a_i \psi(x_i), \tag{4.6}$$

where $n \in \mathbb{N}, x_1, x_2, \ldots, x_n \in X$, and $a_1, a_2, \ldots, a_n \in \mathfrak{A}$. Set $b := \sum_{i=1}^{n} k_{x_i} a_i$. From (4.5), we conclude that

$$c^2\langle b, b\rangle \geq \varphi_\alpha(b)^*\varphi_\alpha(b)$$

for every $\alpha \in I$. Hence φ_α is a well-defined bounded $\mathfrak{A}$-linear map. Since $X \in \mathbb{U}(\mathcal{E})$, the set D is dense in $\mathcal{E}$. Hence, we can extend φ_α to $\mathcal{E}$. For simplicity, we denote it by the same φ_α, so that $\varphi_\alpha \in \mathcal{E}'$. From (4.5), we reach

$$c^2|g|^2 \geqslant \sum_{\alpha\in I} |\varphi_\alpha(g)|^2 \quad (g \in \mathcal{E}). \tag{4.7}$$

Utilizing Lemma 4.3 and (4.6), we arrive at

$$\varphi_\alpha(x) = \varphi_\alpha(k_x) = e_\alpha(x)\psi(x) \quad (x \in X, \alpha \in I).$$

Then

$$e_\alpha(x)\varphi_\beta(x) = \varphi_\alpha(x)e_\beta(x) \quad (x \in X,\ \alpha, \beta \in I). \tag{4.8}$$

Since $e_\alpha \in \mathcal{M}(\mathcal{E})$ and $X \in \mathbb{U}(\mathcal{E})$, from (4.8), we infer that

$$e_\alpha(s)\varphi_\beta(s) = \varphi_\alpha(s)e_\beta(s), \quad (s \in S).$$

Now, fix $\alpha \in I$ and $s \in S$. Putting $g = k_s$ in (4.7), we arrive at

$$c^2K(s,s) \geq \sum_{\beta\in I} \varphi_\beta(s)^*\varphi_\beta(s).$$

Since $\mathrm{ran}(e_\alpha) \subseteq \mathcal{Z}(\mathfrak{A})$, we have

$$\begin{aligned} c^2K(s,s)e_\alpha(s)^*e_\alpha(s) &\geq \sum_{\beta\in I}(e_\alpha(s)\varphi_\beta(s))^*e_\alpha(s)\varphi_\beta(s) \\ &= \sum_{\beta\in I} e_\beta(s)^*\varphi_\alpha(s)^*e_\beta(s)\varphi_\alpha(s) \\ &= |\varphi_\alpha(s)|^2K(s,s). \end{aligned}$$

Now, the invertibility of $K(s,s)$ entails that $|\varphi_\alpha(s)| \leq c|e_\alpha(s)|$. □

Declaration of competing interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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