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Sensitivity Analysis for Quasi-Variational Inclusions

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In this paper, we develop the sensitivity analysis for quasi variational inclusions by using the implicit resolvent equations technique without assuming the differentiability of the given data. © 1999 Academic Press

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1. INTRODUCTION

Variational inequalities arise in various models for a large number of mathematical, physical, regional, engineering, and other problems. Variational inequalities have been generalized and extended in different directions using novel and innovative techniques. For the recent state of the art, see [1–28] and the references therein. A useful and important generalization of the variational inequalities is called the quasi variational inclusion. Noor [12, 16–18], Uko [26], Ding [3], and Huang [6] have used the resolvent operator and resolvent equations techniques to study the existence of a solution of variational inclusions. Noor [12] has shown that the variational inclusions are equivalent to the resolvent equations. This equivalence has led to the suggestion and analysis of various iterative algorithms for solving quasi variational inclusions and related problems. Much attention has been given to developing a sensitivity analysis framework for classical variational inequalities and some results have been obtained with special structures. The techniques suggested so far vary with the problem being studied. Sensitivity analysis for variational inequalities (2.4) has been studied by many authors, including Tobin [25], Kyparisi [8, 9], Dafermos [2], Qiu and Magnanti [21], Yen [27], Noor [15, 19], and Liu

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[10] using quite different techniques. Moudafi and Noor [11] and Noor [19] have considered the sensitivity analysis for variational inclusions and mixed variational inequalities using the resolvent equations technique. In this paper, we extend this technique to study the sensitivity analysis of quasi variational inclusions. We first establish the equivalence between the mixed variational inequalities and resolvent equations. We use this equivalence to develop sensitivity analysis for the quasi variational inequalities (2.1) without assuming the differentiability of the given data. Using the ideas of Dafermos [2] as extended by Noor [14, 15, 19] and Moudafi and Noor [11], we develop a sensitivity analysis framework for quasi variational inclusions, which is based on a direct fixed point formulation of the mixed variational inequalities. This fixed point formulation is obtained by a suitable and appropriate rearrangement of the resolvent equations. We would like to point out that the resolvent equations technique is quite general and flexible. This technique has been used by Noor [12, 16, 17] to suggest some iterative methods for solving mixed variational inequalities.

2. PRELIMINARIES

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ respectively. Let $T\colon H \to H$ be a nonlinear operator and let $A(\cdot, \cdot)\colon H \times H \to R \cup \{+\infty\}$ be a maximal monotone operator with respect to the first argument.

We consider the problem of finding $u \in H$ such that

$$0 \in Tu + A(u, u). \tag{2.1}$$

Inclusion of type (2.1) is called the mixed quasi variational inclusion, which has many important and useful applications in pure and applied sciences; see, for example, [3, 6, 12, 16–18] and references therein.

We note that if $A(\cdot, \cdot) = \partial\varphi(\cdot, \cdot)$, where $\partial\varphi(\cdot, \cdot)$ is the subdifferential of a proper, convex and lower-semicontinuous function $\varphi(\cdot, \cdot)\colon H \times H \to R \cup \{+\infty\}$ with respect to the first argument, then problem (2.1) is equivalent to finding $u \in H$ such that

$$\langle Tu, v - u\rangle + \varphi(v, u) - \varphi(u, u) \geq 0, \qquad \text{for all } v \in H. \tag{2.2}$$

The problem of the type (2.2) is called the mixed quasi variational inequality problem, which has many important and significant applications in regional, physical, mathematical, pure, and applied sciences; see [1, 4, 7, 12–19].

If $\varphi(\cdot, u) \equiv \varphi(u)$ for all $u \in H$, then problem (2.2) is equivalent to finding $u \in H$ such that

$$\langle Tu, v - u\rangle + \varphi(v) - \varphi(u) \geq 0, \qquad \text{for all } v \in H, \tag{2.3}$$

which is known as the mixed variational inequality. For the applications, sensitivity analysis, and numerical methods, see [2, 3–7, 11–20].

We note that if φ is the indicator function of a closed convex set K in H, then problem (2.3) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u \rangle \geq 0, \qquad \text{for all } v \in K, \tag{2.4}$$

which is called the classical variational inequality, introduced and studied by Stampacchia [24] in 1964. For recent applications, numerical methods, sensitivity analysis, and physical formulations, see [1–28] and the references therein.

We recall that T is a maximal monotone operator, then the resolvent operator J_T associated with T is defined by

$$J_T(u) = (I + \rho T)^{-1}(u), \qquad \text{for all } u \in H,$$

where $\rho > 0$ is a constant and I is the identity operator. The resolvent operator J_T is a single-valued operator which is defined everywhere on the whole Hilbert space H and is nonexpansive.

Remark 2.1. Since the operator $A(\cdot, \cdot)$ is a maximal monotone operator with respect to the first argument, we define

$$J_{A(u)}u = (I + \rho A(u))^{-1}u, \qquad \text{for all } u \in H,$$

the implicit resolvent operator associated with $A(\cdot, u) \equiv A(u)$.

Related to the mixed variational inequality (2.1), we consider the problem of finding $z, u \in H$ such that

$$TJ_{A(u)}z + \rho^{-1}R_{A(u)}z = 0, \tag{2.5}$$

where $\rho > 0$ is a constant and $R_{A(u)} = I - J_{A(u)}$. Here I is the identity operator and $J_\varphi = (1 + \rho A(u))^{-1}$ is the resolvent operator. Equations of the type (2.5) are called the implicit resolvent equations. For the formulation and applications of the resolvent equations, see Noor [12, 16–19].

If $A(u, u) \equiv \partial\varphi(u, u)$, where $\partial\varphi(u, u)$ is the subdifferential of proper, convex and lower-semicontinuous function $\varphi(u, u)$, then problem (2.5) is equivalent to finding $z, u \in H$ such that

$$TJ_{\varphi(u)}z + \rho^{-1}R_{\varphi(u)}z = 0, \tag{2.6}$$

where $J_{\varphi(u)} = (I + \rho\, \partial\varphi(\cdot, u))^{-1}$ is the resolvent operator associated with $\partial\varphi(\cdot, u) \equiv \varphi(u)$. Noor [16] has established the equivalent between the problems (2.2) and (2.6). This equivalent was used to suggest some iterative algorithms for solving mixed quasi variational inequalities (2.2).

We note that if $\partial\varphi(\cdot, u) \equiv \varphi(u)$, where φ: $H \to R \cup \{+\infty\}$, then $T_{\varphi(u)} = T_{\varphi} = (I + \rho\,\partial\varphi)^{-1}$ and the problem (2.6) collapses to finding $z \in H$ such that

$$TJ_{\varphi} z + \rho^{-1} R_{\varphi} z = 0. \tag{2.7}$$

Equations of type (2.7) are called resolvent equations, which were introduced and studied by Noor [14]. For the applications of resolvent equations, see [12, 14, 16–18] and references therein.

We remark that if φ is the indicator function of a closed convex set K in H, then $J_{\varphi} = P_K$, the projection of H into K. Consequently, the resolvent equations (2.7) are equivalent to finding $z \in H$ such that

$$TP_K z + \rho^{-1} Q_K z = 0, \tag{2.8}$$

where $Q_K = I - P_K$. The equations (2.8) are called the Wiener–Hopf (normal maps) equations, which were introduced by Shi [23] and Robinson [22] independently. Using essentially the projection technique, Shi [23] and Robinson [22] have established the equivalence between the variational inequalities (2.4) and the Wiener–Hopf equations (2.8). For the generalization and the extensions of the Wiener–Hopf equations and their applications, see Noor [13–16, 19, 20] and references therein.

We now consider the parametric versions of the problems (2.1) and (2.5). To formulate the problem, let M be an open subset of H in which the parameter λ takes values. Let $T(u, \lambda)$ be given operator defined on $H \times M$ and taking value in H. from now on we denote $T_{\lambda}(\cdot) \equiv T(\cdot, \lambda)$ unless otherwise specified. The parametric quasi variational inclusion problem is to find $u \in H$ such that

$$0 \in T_{\lambda}(u) + A_{\lambda}(u, u). \tag{2.9}$$

We also assume that, for some $\overline{\lambda} \in M$, the problem (2.9) has a unique solution $\bar{u}$.

We remark that if $A(u, u) = \partial\varphi(u, u)$, the subdifferential of a proper, convex and lower-semicontinuous function $\varphi(u, u)$ with respect to the first argument, then problem (2.9) is equivalent to finding $u \in H$ such that

$$\langle T_{\lambda}(u), v - u\rangle + \varphi(v, u) - \varphi(u, u) \geq 0, \qquad \text{for all } v \in H, \tag{2.10}$$

which is called the parametric mixed quasi variational inequality.

Related to the parametric quasi variational inclusion (2.9), we consider the parametric resolvent equations. We consider the problem of finding

$z, u \in H$ such that

$$T_\lambda J_{A_\lambda(u)} z + \rho^{-1} R_{A_\lambda(u)} z = 0, \tag{2.11}$$

where $\rho > 0$ is a constant and $J_{A_\lambda} z$ and $R_{A_\lambda} z$ are defined on the set of (z, λ) with $\lambda \in M$ and take values in H. Equations of the type (2.11) are called the parametric implicit resolvent equations.

We now establish the equivalent between the problems (2.9) and (2.11), which is the main motivation of our next result. For a different proof, see Noor [16].

LEMMA 2.1. *The parametric quasi variational inclusion* (2.9) *has a solution $u \in H$ if and only if the parametric resolvent equations* (2.11) *have a solution $z, u \in H$, where*

$$u = J_{A_{\lambda(u)}} z \tag{2.12}$$

and

$$z = u - \rho T_\lambda(u). \tag{2.13}$$

Proof. Let $u \in H$ be a solution of (2.9). Then

$$\begin{aligned} &\rho T_\lambda(u) + \rho A(u, u) \ni 0 \qquad (\text{for a constant } \rho > 0) \\ &\Leftrightarrow \\ &-u + \rho T_\lambda(u) + u + \rho A(u, u) \ni 0 \\ &\Leftrightarrow \\ &u = J_{A_\lambda(u)}[u - \rho T_\lambda(u)]. \end{aligned}$$

Take $z = u - \rho T_\lambda(u)$. Then $z = J_{\lambda(u)} z$.

Thus

$$z = J_{A_\lambda(u)} z - \rho T_\lambda J_{A_\lambda(u)} z,$$

that is,

$$T_\lambda J_{A_\lambda(u)} z + \rho^{-1} R_{A_\lambda(u)} z = 0.$$

∎

From Lemma 2.1, we see that the parametric quasi variational inclusion (2.9) and the parametric resolvent equations (2.11) are equivalent. We use this equivalence to study the sensitivity analysis of the mixed variational inequalities. We assume that for some $\overline{\lambda} \in M$, problem (2.11) has a solution $\bar{z}$ and X is a close of a ball in H centered at $\bar{z}$. We want to investigate those conditions under which, for each λ in a neighborhood of

$\bar{\lambda}$, problem (2.11) has a unique solution $z(\lambda)$ near $\bar{z}$ and the function $z(\lambda)$ is continuous (Lipschitz continuous) and differentiable.

DEFINITION 2.2. Let T_λ be an operator on $X \times M$. Then, for all $\lambda \in M$, $u, v \in X$, the operator T_λ is said to be:

(a) *locally strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle T_\lambda(u) - T_\lambda(v), u - v \rangle \geq \alpha \|u - v\|^2;$$

(b) *locally Lipschitz continuous* if there exists a constant $\beta > 0$ such that

$$\|T_\lambda(u) - T_\lambda(v)\| \leq \beta \|u - v\|.$$

We also need the following condition for the operator $J_A(u)$.

Assumption 2.1. For all $u, v, w \in H$, the operator $J_A(u)$ satisfies the condition

$$\|J_{A(u)} w - J_{A(v)} w\| \leq \gamma \|u - v\|,$$

where $\gamma > 0$ is a constant.

3. MAIN RESULTS

We consider the case when the solutions of the parametric implicit resolvent equations (2.11) lie in the interior of X. Following the ideas of Dafermos [2] and Noor [14, 15, 19], we consider the map

$$\begin{aligned} F_\lambda(z) &= J_{A_\lambda(u)} z - \rho T_\lambda J_{A_\lambda(u)} z \qquad (\text{for all } (z, \lambda) \in X \times M) \\ &= u - \rho T_\lambda(u), \end{aligned} \tag{3.1}$$

where

$$u = J_{A_{\lambda(u)}} z. \tag{3.2}$$

We have to show that the map $F_\lambda(z)$ has a fixed point, which is a solution of the resolvent equations (2.11). First of all, we want to prove that the map $F_\lambda(z)$, defined by (3.1) is a contraction map with respect to z uniformly in $\lambda \in M$.

LEMMA 3.1. *Let T be locally strongly monotone with constant $\alpha \geq 0$ and locally Lipschitz continuous with constant $\beta \geq 0$. If Assumption* 2.1 *holds,* *then, for all* $z_1, z_2 \in X$ *and* $\lambda \in M$, *we have*

$$\|F_\lambda(z_1) - F_\lambda(z_2)\| \leq \theta \|z_1 - z_2\|,$$

where

$$\theta = \left(\sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\right)\Big/(1 - \gamma) < 1$$

for

$$|\rho - \alpha/\beta^2| < \sqrt{\alpha^2 - \beta^2\gamma(2 - \gamma)}\Big/\beta^2 \tag{3.3}$$

$$\alpha > \beta\sqrt{\gamma(2 - \gamma)}, \qquad \gamma < 1. \tag{3.4}$$

Proof. For all $z_1, z_2 \in X$, $\lambda \in M$, we have, from (3.1),

$$\begin{aligned}
\|F_\lambda(z_1) - F_\lambda(z_2)\|^2 &= \|u_1 - u_2 - \rho(T_\lambda(u_1) - T_\lambda(u_2))\|^2 \\
&= \|u_1 - u_2\|^2 - 2\rho\langle T_\lambda(u_1) - T_\lambda(u_2), u_1 - u_2\rangle \\
&\quad + \rho^2\|T_\lambda(u_1) - T_\lambda(u_2)\|^2 \\
&\leq (1 - 2\rho\alpha + \rho^2\beta^2)\|u_1 - u_2\|^2,
\end{aligned}$$

which implies that

$$\|F_\lambda(z_1) - F_\lambda(z_2)\| \leq \left(\sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\right)\|u_1 - u_2\|. \tag{3.5}$$

Now from (3.2) and Assumption 2.1, we have

$$\begin{aligned}
\|u_1 - u_2\| &= \|J_{A_\lambda(u_1)}z_1 - J_{A_\lambda(u_2)}z_2\| \\
&\leq \|J_{A_\lambda(u_1)}z_1 - J_{A_\lambda(u_2)}z_1\| + \|J_{A_\lambda(u_2)}z_1 - J_{A_\lambda(u_2)}z_2\| \\
&\leq \gamma\|u_1 - u_2\| + \|z_1 - z_2\|,
\end{aligned}$$

which implies that

$$\|u_1 - u_2\| \leq (1/(1 - \gamma))\|z_1 - z_2\|. \tag{3.6}$$

Combining (3.5) and (3.6), we have

$$\begin{aligned}
\|F_\lambda(z_1) - F_\lambda(z_2)\| &\leq \left[\left(\sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\right)\Big/(1 - \gamma)\right]\|z_1 - z_2\| \\
&= \theta\|z_1 - z_2\|,
\end{aligned}$$

where $\theta = (\sqrt{1 - 2\rho\alpha + \rho^2\beta^2})/(1 - \gamma) < 1$.

From (3.3) and (3.4), it follows that $\theta < 1$ and consequently the map $F_\lambda(z)$ defined by (3.1) is a contraction map and has a fixed point $z(\lambda)$, which is the solution of the resolvent equation (2.11). ∎

Remark 3.1. From Lemma 3.1, we see that the map $F_\lambda(z)$ defined by (3.1) has a unique fixed point $z(\lambda)$, that is,

$$z(\lambda) = F_\lambda(z).$$

Also, by assumption, the function $\bar{z}$, for $\lambda = \bar{\lambda}$ is a solution of the parametric resolvent equations (2.11). Again using Lemma 3.1, we see that $\bar{z}$, for $\lambda = \bar{\lambda}$, is a fixed point of $F_\lambda(z)$ and it is a fixed point of $F_{\bar{\lambda}}(z)$. Consequently, we conclude that

$$z(\bar{\lambda}) = \bar{z} = F_{\bar{\lambda}}\big(z(\bar{\lambda})\big). \tag{3.7}$$

Using Lemma 3.1, we prove the continuity of the solution $z(\lambda)$ of the parametric resolvent equations (2.7), which is the main motivation of our next result.

LEMMA 3.2. *If the operator T_λ is locally strongly monotone Lipschitz continuous and the map $\lambda \rightarrow J_{A_\lambda(u)}z$ is continuous* (*or Lipschitz continuous*), *then the function $z(\lambda)$ satisfying* (2.11) *is continuous* (*or Lipschitz continuous*) *at $\lambda = \bar{\lambda}$.*

Proof. For all $\lambda \in M$, invoking Lemma 3.1 and the triangle inequality, we have

$$\begin{aligned}\|z(\lambda) - z(\bar{\lambda})\| &= \|F_\lambda(z(\lambda)) - F_{\bar{\lambda}}\big(z(\bar{\lambda})\big)\| \\ &\le \|F_\lambda(z(\lambda)) - F_\lambda\big(z(\bar{\lambda})\big)\| + \|F_\lambda\big(z(\bar{\lambda})\big) - F_{\bar{\lambda}}\big(z(\bar{\lambda})\big)\| \\ &\le \theta\|z(\lambda) - z(\bar{\lambda})\| + \|F_\lambda\big(z(\bar{\lambda})\big) - F_{\bar{\lambda}}\big(z(\bar{\lambda})\big)\|. \end{aligned} \tag{3.8}$$

From (3.1) and the fact that the operator T_λ is Lipschitz continuous, we have

$$\begin{aligned}\|F_\lambda\big(z(\bar{\lambda})\big) - F_{\bar{\lambda}}\big(z(\bar{\lambda})\big)\| &= \|u(\bar{\lambda}) - u(\bar{\lambda}) - \rho\big(T_\lambda\big(u(\bar{\lambda})\big) - T_{\bar{\lambda}}\big(u(\bar{\lambda})\big)\| \\ &\le \rho\|T_\lambda\big(u(\bar{\lambda})\big) - T_{\bar{\lambda}}\big(u(\bar{\lambda})\big)\| \\ &\le \rho\mu\|\lambda - \bar{\lambda}\|, \end{aligned} \tag{3.9}$$

where $\mu > 0$ is a Lipschitz continuity constant of T_λ,

Combining (3.8) and (3.9), we obtain

$$\|z(\lambda) - z(\bar{\lambda})\| \le \frac{\rho\mu}{1-\theta}\|\lambda - \bar{\lambda}\|, \qquad \text{for all } \lambda, \bar{\lambda} \in X,$$

from which the required result follows. ∎

Now using the technique of Dafermos [2], we can prove the following result.

LEMMA 3.3. *If the assumptions of Lemma* 3.2 *hold, then there exists a neighborhood* $N \subset M$ *of* $\bar{\lambda}$ *such that, for* $\lambda \in N$, $z(\lambda)$ *is the unique solution of the resolvent equations* (2.11) *in the interior of* X.

THEOREM 3.1. *Let* $\bar{u}$ *be the solution of the parametric quasi variational inclusion* (2.9) *and let* $\bar{z}$ *be the solution of the parametric resolvent equations* (2.11) *for* $\lambda = \bar{\lambda}$. *Let* $T_\lambda(u)$ *be the locally strongly monotone Lipschitz continuous operator for all* $u, v \in X$. *If the map* $\lambda \to J_{A_\lambda(u)}(z)$ *is continuous* (*or Lipschitz continuous*) *at* $\lambda = \bar{\lambda}$, *then there exists a neighborhood* $N \subset M$ *of* $\bar{\lambda}$ *such that, for* $\lambda \in N$, *the parametric resolvent equations* (2.11) *have a unique solution* $z(\lambda)$ *in the interior of* X, $z(\bar{\lambda}) = \bar{z}$, *and* $z(\lambda)$ *is continuous* (*or Lipschitz continuous*) *at* $\lambda = \bar{\lambda}$.

Proof. The proof follows from Lemmas 3.1, 3.2, and 3.3 and Remark 3.1.

Remark 3.2. Since the quasi variational inclusions include the classical variational inequalities, mixed (quasi) variational inequalities, and complementarity problems as special cases, the technique developed in this paper can be used to study the sensitivity analysis of these problems. In particular, if $A(u,u) \equiv \partial\varphi(u,u)$, we can obtain the sensitivity analysis of mixed quasi variational inequalities (2.2), which appears to be a new result. The fixed formulation of the quasi variational inclusions allows us to study the Hölder and Lipschitz continuity of the solution of the parametric problems essentially using the technique of Yen [27]. In fact, our results represent a refinement and significant improvement of previous known results of Dafermos [2], Moudafi and Noor [11], and Noor [19], and others in this field. It is worth mentioning that the resolvent equations technique does not require the differentiability of the given data.

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