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NOTE

Meromorphic Functions That Share Four Small Functions[1]

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In this paper, we prove that if two nonconstant meromorphic functions f and g share two small functions CM* and share other two small functions IM*, then f must be a quasi-Möbius transformation of g. This result is a generalization of several results obtained by G. G. Gundersen and Li-Yang. © 2001 Academic Press

Key Words: meromorphic function; small function; sharing value.

1. INTRODUCTION

Let f be a nonconstant meromorphic function in the complex plane $\mathbb{C}$. We shall use the standard notations in Nevanlinna's value distribution theory of meromorphic functions such as $T(r, f)$, $N(r, f)$, and $m(r, f)$ (see, e.g., [3]). The notation $S(r, f)$ is defined to be any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ possibly outside a set of r of finite linear measure. A meromorphic function $a(\not\equiv \infty)$ is called a small function with respect to f provided that $T(r, a) = S(r, f)$.

Let f and g be two nonconstant meromorphic functions, and let a be a small function with respect to f and g. If $f - a$ and $g - a$ have the same zeros ignoring (counting) multiplicities, then we say that f and g share a IM (CM). We say f and g share ∞ IM (CM) if $1/f$ and $1/g$ share 0 IM (CM).

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Let $S(f=a=g)$ be the set of all common zeros of $f-a$ and $g-a$ ignoring multiplicities, and let $S_E(f=a=g)$ be the set of all common zeros of $f-a$ and $g-a$ with the same multiplicities. Furthermore, we denote by $S_{(k,l)}(f=a=g)$ the set of all points which are zeros of $f-a$ with multiplicity k as well as the zeros of $g-a$ with multiplicity l. Denote by $\overline{N}(r,f=a=g)$, $\overline{N}_E(r,f=a=g)$, and $\overline{N}_{(k,l)}(r,f=a=g)$ the reduced counting functions of f and g corresponding to the sets $S(f=a=g)$, $S_E(f=a=g)$, and $S_{(k,l)}(f=a=g)$, respectively. If

$$\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}\left(r,\frac{1}{g-a}\right)-2\overline{N}(r,f=a=g)=S(r,f)+S(r,g),$$

then we say that f and g share a IM*. If

$$\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}\left(r,\frac{1}{g-a}\right)-2\overline{N}_E(r,f=a=g)=S(r,f)+S(r,g),$$

then we say that f and g share a CM*. Obviously, any IM (CM) shared small function must be an IM* (CM*) shared small function. And we have

$$\overline{N}\left(r,\frac{1}{f-a}\right)=\sum_{k,l=1}\overline{N}_{(k,l)}(r,f=a=g)+S(r,f), \tag{1}$$

provided that f and g share a IM*.

In 1926, R. Nevanlinna [9] proved that if two meromorphic functions f and g share four values a_1, a_2, a_3, and a_4 CM, then f is a Möbius transformation of g. Since then many papers have been published on uniqueness theory and sharing values (see, e.g., [1, 8]). Most results on sharing values in the sense of IM or CM are still valid in the sense of IM* or CM*. For example, we have

THEOREM A [1]. *Let f and g be two nonconstant meromorphic functions, and let a_1, a_2, a_3, and a_4 be four distinct values in $\overline{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$. If f and g share a_1, a_2 CM* and share a_3, a_4 IM*, then f is a Möbius transformation of g.*

When the shared values are replaced by shared small functions, most problems will become difficult. In 1997, Li-Yang proved the following

THEOREM B [6]. *Let f and g be two nonconstant meromorphic functions, and let a_1, a_2, a_3, and a_4 be four distinct small functions with respect to f and*

g. If f and g share a_1, a_2, a_3 CM and share a_4 IM*, then f is a quasi-Möbius transformation of g, i.e.,*

$$f = \frac{\alpha_1 g + \beta_1}{\alpha_2 g + \beta_2},$$

where α_i, β_i $(i = 1, 2)$ are small functions with respect to f and g.

A specific possible form of the quasi-Möbius transformation in Theorem B can be found in [4]. In this paper, we generalize the above two theorems and prove the following

THEOREM 1. *Let f and g be two nonconstant meromorphic functions, and let a_1, a_2, a_3, and a_4 be four distinct small functions with respect to f and g. If f and g share a_1, a_2 CM* and share a_3, a_4 IM*, then f is a quasi-Möbius transformation of g.*

2. LEMMAS

LEMMA 1 [5]. *Let f and g be two nonconstant meromorphic functions, and let a $(\not\equiv 0, 1, \infty)$ be a small function with respect to f and g. Let*

$$\psi := (f-g)\left(\frac{f'g'}{g(f-1)} - \frac{f'g'}{f(g-1)} + \frac{f'(g'-a')}{f(g-a)} - \frac{g'(f'-a')}{g(f-a)} + \frac{g'(f'-a')}{(f-a)(g-1)} - \frac{f'(g'-a')}{(g-a)(f-1)}\right). \quad (2)$$

If f and g share 0, 1, ∞, a IM, then $T(r, \psi) = S(r, f) = S(r, g)$.*

LEMMA 2. *Let f and g be two nonconstant meromorphic functions satisfying $T(r, g) \le cT(r, f) + S(r, f)$, where c is a constant, and let a be a small function with respect to f and g. If f and g share a IM*, and*

$$\overline{N}_{(k,l)}(r, f = a = g) = S(r, f)$$

holds for all pairs (k, l) of positive integers, then $\overline{N}(r, 1/(f-a)) \le \varepsilon T(r, f) + S(r, f)$ holds for any positive number ε.

Proof. For any positive number ε, we select an integer n such that $c + 1 \le n\varepsilon$. It follows from the assumption and (1) that

$$\begin{aligned}
&\overline{N}\left(r, \frac{1}{f-a}\right)\\
&\quad = \sum_{k,l=1} \overline{N}_{(k,l)}(r, f = a = g) + S(r,f)\\
&\quad = \sum_{k+l>n} \overline{N}_{(k,l)}(r, f = a = g) + S(r,f)\\
&\quad \le \frac{1}{n}\left(\sum_{k+l>n} k\overline{N}_{(k,l)}\left(r, \frac{1}{f-a}\right) + \sum_{k+l>n} l\overline{N}_{(k,l)}\left(r, \frac{1}{g-a}\right)\right) + S(r,f)\\
&\quad \le \frac{1}{n}\left(N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{g-a}\right)\right) + S(r,f)\\
&\quad \le \frac{1}{n}(T(r,f) + T(r,g)) + S(r,f) \le \frac{1+c}{n} T(r,f) + S(r,f)\\
&\quad \le \varepsilon T(r,f) + S(r,f),
\end{aligned}$$

which completes the proof of the lemma. ∎

LEMMA 3. *Let f and g be two nonconstant meromorphic functions, and let a_1, a_2, a_3, and a_4 be four distinct small functions with respect to f and g. If f and g share a_1, a_2 CM* and share a_3, a_4 IM*, and if there exists a number $\varepsilon \in (0, 1/4)$, such that*

$$\overline{N}\left(r, \frac{1}{f-a_3}\right) - \overline{N}_E(r, f = a_3 = g) \le \varepsilon T(r,f) + S(r,f), \tag{3}$$

then f is a quasi-Möbius transformation of g.

Proof. Without loss of generality, we assume that $a_1 = 0$, $a_2 = \infty$, $a_3 = 1$, and $a_4 = a$, where a is a small function with respect to f and g, and $a \ne 0, 1, \infty$. Let

$$f_1 = \frac{f}{a}, \qquad g_1 = \frac{g}{a}, \qquad f_2 = \frac{f-1}{a-1}, \qquad g_2 = \frac{g-1}{a-1}.$$

Then f_1 and g_1 share $0, \infty$ CM* and share $1, 1/a$ IM*; furthermore, f_2 and g_2 share $-1/(a-1), \infty$ CM*, and share $0, 1$ IM*. Let

$$\beta = \frac{f'(f-a)}{f(f-1)} - \frac{g'(g-a)}{g(g-1)}, \qquad \beta_1 = \frac{f_1'(f_1 - 1/a)}{f_1(f_1-1)} - \frac{g_1'(g_1-1/a)}{g_1(g_1-1)}, \tag{4}$$

and

$$\beta_2 = \frac{f_2'(f_2 + 1/(a-1))}{f_2(f_2-1)} - \frac{g_2'(g_2 + 1/(a-1))}{g_2(g_2-1)}. \tag{5}$$

Note f and g share $0, \infty$ CM* and share $1, a$ IM*. By (3), we can easily get $T(r, \beta) \leq \varepsilon T(r, f) + S(r, f)$. If none of β, β_1, β_2 is identically zero, then

$$\overline{N}\left(r, \frac{1}{f-a}\right) \leq N(r, 1/\beta) + S(r, f) \leq \varepsilon T(r, f) + S(r, f). \tag{6}$$

Thus we can get $T(r, \beta_1) \leq \varepsilon T(r, f) + S(r, f)$. It follows that

$$\overline{N}\left(r, \frac{1}{f-1}\right) \leq N(r, 1/\beta_1) + S(r, f) \leq \varepsilon T(r, f) + S(r, f). \tag{7}$$

By (6), (7), and the definition of β_2, we get $T(r, \beta_2) \leq 2\varepsilon T(r, f) + S(r, f)$. Thus

$$\overline{N}\left(r, \frac{1}{f}\right) \leq N(r, 1/\beta_2) + S(r, f) \leq 2\varepsilon T(r, f) + S(r, f). \tag{8}$$

From (6), (7), (8), and the second fundamental theorem, we get $T(r, f) < 4\varepsilon T(r, f) + S(r, f)$, which is impossible for $\varepsilon \in (0, 1/4)$. Hence one of β, β_1, and β_2 must be identically zero. It follows that f and g share a CM*. Therefore, by Theorem B, f is a quasi-Möbius transformation of g. ∎

3. PROOF OF THE MAIN THEOREM

Without loss of generality, we assume that $a_1 = 0$, $a_2 = \infty$, $a_3 = 1$, $a_4 = a$, where a ($\not\equiv 0, 1, \infty$) is a small function with respect to f and g; otherwise, a quasi-Möbius transformation will do. Since f and g share three values IM*, by the second fundamental theorem, we can easily get $T(r, f) \leq 3T(r, g) + S(r, f)$ and $T(r, g) \leq 3T(r, f) + S(r, g)$. Hence

$S(r, f) = S(r, g) := S(r)$. If $\overline{N}(r, 1/(f-1)) = S(r)$ or $\overline{N}(r, 1/(f-a)) = S(r)$, then f and g share at least three of a_1, a_2, a_3, a_4 CM* and share another one IM*. By Theorem B, f is a quasi-Möbius transformation of g.

In the following, we assume that

$$\overline{N}\left(r, \frac{1}{f-1}\right) \neq S(r), \qquad \overline{N}\left(r, \frac{1}{f-a}\right) \neq S(r). \tag{9}$$

Furthermore, we assume that a is not constant; otherwise, by Theorem A, f is a quasi-Möbius transformation of g. Let

$$\varphi := \frac{f'}{f} - \frac{g'}{g} \tag{10}$$

and

$$\begin{aligned} \varphi_1 := & \left(\frac{f'(f-a)}{f(f-1)} - \frac{g'(g-a)}{g(g-1)}\right) \\ & \times \left(\frac{(af' - a'f)(f-1)}{af(f-a)} - \frac{(ag' - a'g)(g-1)}{ag(g-a)}\right) \qquad (11) \\ = & \left((1-a)\left(\frac{f'}{f-1} - \frac{g'}{g-1}\right) + a\varphi\right) \\ & \times \left(\left(1 - \frac{1}{a}\right)\left(\frac{f'-a'}{f-a} - \frac{g'-a'}{g-a}\right) + \frac{1}{a}\varphi\right). \qquad (12) \end{aligned}$$

By the lemma on the logarithmic derivative, we have $m(r, \varphi) = S(r)$ and $m(r, \varphi_1) = S(r)$. Since f and g share $0, \infty$ CM* and share $1, a$ IM*, it is easily seen from (10) and (11) that $N(r, \varphi) = S(r)$ and $N(r, \varphi_1) = S(r)$. Hence we have

$$T(r, \varphi) = S(r), \qquad T(r, \varphi_1) = S(r). \tag{13}$$

If $\varphi \equiv 0$, then f/g is a nonzero constant, and thus f is a Möbius transformation of g. If $\varphi_1 \equiv 0$, then it follows from (11) that f and g share $1, a$ CM*. Thus by Theorem B f is also a quasi-Möbius transformation of g.

If $\overline{N}_{(k,l)}(r, f = 1 = g)) = S(r)$ holds for all pairs (k, l) $(\max\{k, l\} > 1)$ of positive integers, then by the proof of Lemma 2, we have

$$\begin{aligned}\overline{N}\left(r, \frac{1}{f-1}\right) - \overline{N}_E(r, f = 1 = g) &\leq \sum_{\max\{k,l\}>1} \overline{N}_{(k,l)}(r, f = 1 = g) + S(r) \\ &\leq \frac{1}{5} T(r, f) + S(r).\end{aligned}$$

By Lemma 3, f is a quasi-Möbius transformation of g.

Suppose that f is not any quasi-Möbius transformation of g. Then $\varphi \not\equiv 0$, $\varphi_1 \not\equiv 0$. And there exists a pair (k, l) $(\max\{k, l\} > 1)$ of positive integers such that

$$\overline{N}_{(k,l)}(r, f = 1 = g) \neq S(r). \tag{14}$$

Similarly, there exists a pair (k_1, l_1) $(\max\{k_1, l_1\} > 1)$ of positive integers such that

$$\overline{N}_{(k_1,l_1)}(r, f = a = g) \neq S(r). \tag{15}$$

If $k > 1$ and $l > 1$, then $\varphi(z) = 0$ holds for all $z \in S_{(k,l)}(f = 1 = g)$. Since $\varphi \not\equiv 0$, we have $\overline{N}_{(k,l)}(r, f = 1 = g) \leq \overline{N}(r, 1/\varphi) = S(r)$, which contradicts (14). Hence $\min\{k, l\} = 1$. Similarly, we have $\min\{k_1, l_1\} = 1$.

Let S_0 be the set of all zeros, 1-points, and poles of $a(z)$ or $\varphi(z)$ or $\varphi_1(z)$. Let ψ be the function defined in (2) and let $z_1 \in S_{(k,l)}(f = 1 = g) \setminus S_0$. By a simple computation, we have

$$\psi(z_1) = \frac{(k-l)a'(z_1) + \max\{k, l\} a(z_1) \varphi(z_1)}{1 - a(z_1)} \varphi(z_1).$$

From this and (14), we deduce that

$$\psi \equiv \frac{(k-l)a' + \max\{k, l\} a \varphi}{1 - a} \varphi. \tag{16}$$

Similarly, by considering the value of ψ at the point $z_a \in S_{(k_1,l_1)}(f = a = g) \setminus S_0$, we can get

$$\psi \equiv \frac{(l_1 - k_1)a' + \max\{k_1, l_1\} a \varphi}{1 - a} \varphi. \tag{17}$$

The above two equations yield $(\max\{k,l\}-\max\{k_1,l_1\})\varphi \equiv (l-k+l_1-k_1)(a'/a)$, which gives

$$(f/g)^{\max\{k,l\}-\max\{k_1,l_1\}} \equiv c a^{l-k+l_1-k_1}, \tag{18}$$

where c is a nonzero constant. Since f is not any quasi-Möbius transformation of g, and a is not constant, (18) implies $\max\{k,l\}-\max\{k_1,l_1\}=0$ and $l-k+l_1-k_1=0$. Note that $\max\{k,l\}>1$, $\min\{k,l\}=1$, $\max\{k_1,l_1\}>1$, and $\min\{k_1,l_1\}=1$. We get $l_1=k>1$, $k_1=l=1$ or $k_1=l>1$, $l_1=k=1$.

Without loss of generality, we assume that $l_1=k>1$, $k_1=l=1$. Then we have

$$\overline{N}_{(k,1)}(r, f=1=g) \neq S(r), \qquad \overline{N}_{(1,k)}(r, f=a=g) \neq S(r). \tag{19}$$

Equations (16) and (17) become

$$\psi \equiv \frac{(k-1)a' + ka\varphi}{1-a}\varphi, \tag{20}$$

where $k>1$ is a integer. Moreover, we have $\overline{N}_{(k_2,l_2)}(r, f=1=g)=S(r)$ and $\overline{N}_{(k_3,l_3)}(r, f=a=g)=S(r)$, where $(k_2,l_2)\neq(k,1)$ and $(k_3,l_3)\neq(1,k)$ are pairs of positive integers.

Let $z_1 \in S_{(k,1)}(f=1=g)\setminus S_0$. Then $f'(z_1)=0$ and $\varphi(z_1)=-g'(z_1)$. From (11) and by a simple computation, we get

$$\varphi_1(z_1) = (1-k)\varphi(z_1)\left(\varphi(z_1) + \frac{a'(z_1)}{a(z_1)}\right).$$

Note that $\overline{N}_{(k,1)}(r, f=1=g)\neq S(r)$. We obtain

$$\varphi_1 \equiv (1-k)\varphi\left(\varphi + \frac{a'}{a}\right). \tag{21}$$

Let

$$h_1 := (1-a)\left(\frac{f'}{f-1} - \frac{g'}{g-1}\right) + a\varphi$$

and

$$h_2 := \left(1-\frac{1}{a}\right)\left(\frac{f'-a'}{f-a} - \frac{g'-a'}{g-a}\right) + \frac{1}{a}\varphi.$$

Then from (12) we have $\varphi_1 = h_1 h_2$. Note that f and g share ∞ CM*. We can see that the zeros and poles of h_1 which is not in S_0 are simple and comes from the a-points and 1-points of f, respectively. Therefore, we have

$$\overline{N}\left(r, \frac{1}{f-1}\right) \le N\left(r, \frac{1}{h_2}\right) + S(r,f) \le \overline{N}\left(r, \frac{1}{f-a}\right) + S(r,f),$$

and

$$\overline{N}\left(r, \frac{1}{f-a}\right) \le N\left(r, \frac{1}{h_1}\right) + S(r,f) \le \overline{N}\left(r, \frac{1}{f-1}\right) + S(r,f).$$

Hence

$$\overline{N}\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-1}\right) + S(r,f). \tag{22}$$

Let $z_0 \in S(f = 0 = g) \setminus S_0$. A simple computation shows that $h_1^2(z_0) - 2a(z_0)\varphi(z_0)h_1(z_0) + a^2(z_0)\varphi_1(z_0) = 0$. If $h_1^2 - 2a\varphi h_1 + a^2\varphi_1 \equiv 0$, then we get $T(r, h_1) = S(r, f)$, and thus $\overline{N}(r, 1/(f-1)) = S(r, f)$. This contradicts (9). Hence $h_1^2 - 2a\varphi h_1 + a^2\varphi_1 \not\equiv 0$. Therefore,

$$\overline{N}\left(r, \frac{1}{f}\right) \le 2T(r, h_1) + S(r,f) \le 2\overline{N}\left(r, \frac{1}{f-1}\right) + S(r,f).$$

From this, (22), and the second fundamental theorem, we get

$$T(r,f) < 4\overline{N}\left(r, \frac{1}{f-1}\right) + S(r,f). \tag{23}$$

We define three auxiliary functions,

$$f_1 := \frac{h_1(g-1)}{(1-k)(f-a)\varphi}, \qquad f_2 := \frac{(1-k)(g-1)\varphi}{h_2(f-a)}, \qquad f_3 := \frac{f}{g}.$$

It is easily seen that $\overline{N}(r, f_i) + \overline{N}(r, 1/f_i) = S(r)$ for $i = 1, 2, 3$. By simple computation, we get

$$h_1(z_1)(g(z_1) - 1) = (1-k)(1 - a(z_1))\varphi(z_1),$$
$$h_2(z_a)(f(z_a) - a(z_a)) = (1-k)(a(z_a) - 1)\varphi(z_a),$$

where $z_1 \in S_{(k,1)}(f = 1 = g) \setminus S_0$ and $z_a \in S_{(1,k)}(f = a = g) \setminus S_0$. Hence we have $f_1(z_1) = f_3(z_1) = 1$ and $f_2(z_a) = f_3(z_a) = 1$. Therefore, by (22) and (23), we get

$$T(r,f) < 4\overline{N}\left(r, \frac{1}{f_i - 1}\right) + S(r,f), \qquad i = 1,2,3. \tag{24}$$

In terms of Lemma 7 in [7], there exist two pairs of integers (m_1, n_1) and (m_2, n_2) such that $f_1^{m_1} = f_3^{n_1}$ and $f_2^{m_2} = f_3^{n_2}$. It follows that

$$\left(\frac{(1-k)^2 \varphi^2}{\varphi_1}\right)^{m_1 m_2} = \left(\frac{f_2}{f_1}\right)^{m_1 m_2} = (f_3)^{m_1 n_2 - m_2 n_1}. \tag{25}$$

Since $T(r, f_3) \neq S(r)$, Eq. (25) implies $m_1 n_2 - m_2 n_1 = 0$, and thus

$$c(1-k)^2 \varphi^2 = \varphi_1, \tag{26}$$

where c is constant and $c^{m_1 m_2} = 1$. From this and (21), we get

$$(c(k-1) + 1)\varphi + \frac{a'}{a} = 0. \tag{27}$$

If f_3 has no zeros and poles, then the above equation leads to $a(f_3)^{c(k-1)+1} = c_1$, where c_1 is a constant. Therefore, $a(z_1) = c_1$. Note that $\overline{N}(r, 1/(f-1)) \neq S(r)$. We get $a = c_1$. This is impossible. Suppose that f_3 has some zeros or poles. By considering the residues of $\varphi = f_3'/f_3$ and a'/a, we see that $c(k-1) + 1$ must be a rational number. Let $c(k-1) + 1 = p/q$, where p and q are integers. From the above equation, we get $a^q(f_3)^p = c_2$, where c_2 is a constant. It follows that $T(r, f_3) = S(r)$. This contradicts (24) and completes the proof of the theorem.

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