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Three solutions for a mixed boundary value problem involving the one-dimensional p-Laplacian ☆

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Abstract

This paper deals with two mixed nonlinear boundary value problems depending on a parameter λ. For each of them we prove the existence of at least three generalized solutions when λ lies in an exactly determined open interval. Usefulness of this information on the interval is then emphasized by means of some consequences. Our main tool is a very recent three critical points theorem stated in [Topol. Methods Nonlinear Anal. 22 (2003) 93–104].

Keywords: Critical points; Three solutions; Mixed boundary value problem

1. Introduction

There seems to be increasing interest in multiple solutions to boundary value problems, because of their applications in many fields.

Results on this topic are usually achieved by multiple fixed-point theorems (see the book by Agarwal et al. [1] and references therein), or by variational methods. In particular, in the last years, a result of Ricceri (Theorem 1 of [8], see also Theorem 2.3 and Remark 2.2 of [6]) has been widely used.

Recently, the following three critical points theorem was established in [2].

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Theorem A (Theorem B of [2]). *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that*

(i) $\lim_{\|x\|\to+\infty}(\Phi(x) + \lambda\Psi(x)) = +\infty$ *for all* $\lambda \in [0, +\infty[$;
(ii) *there is $r \in \mathbb{R}$ such that*

$$\inf_X \Phi < r$$

and

$$\varphi_1(r) < \varphi_2(r),$$

where

$$\varphi_1(r) := \inf_{x\in\Phi^{-1}(]-\infty,r[)} \frac{\Psi(x) - \inf_{\overline{\Phi^{-1}(]-\infty,r[)}^w} \Psi}{r - \Phi(x)},$$

$$\varphi_2(r) := \inf_{x\in\Phi^{-1}(]-\infty,r[)} \sup_{y\in\Phi^{-1}([r,+\infty[)} \frac{\Psi(x) - \Psi(y)}{\Phi(y) - \Phi(x)},$$

and $\overline{\Phi^{-1}(]-\infty, r[)}^w$ is the closure of $\Phi^{-1}(]-\infty, r[)$ in the weak topology.

Then, for each $\lambda \in \left]\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)}\right[$ the functional $\Phi + \lambda\Psi$ has at least three critical points in X.

However, $\varphi_1(r)$ in Theorem A could be 0. In this and similar cases, we agree to read $1/0$ as $+\infty$.

The peculiarity of Theorem A, compared with Theorem 1 of [8] (see also Theorem 2.3 and Remark 2.2 of [6]), consists in the exact determination of the interval $\left]\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)}\right[$, which has several consequences.

Applications of Theorem A to multiplicity results for Dirichlet and Neumann boundary value problems have been given in [2–4].

The aim of this paper is to obtain further applications of Theorem A to the following two mixed problems:

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda f(t, u), \\ u(a) = u'(b) = 0, \end{cases} \tag{P$_\lambda$}$$

and

$$\begin{cases} -(|u'|^{p-2}u')' + |u|^{p-2}u = \lambda f(t, u), \\ u(a) = u'(b) = 0, \end{cases} \tag{P$'_\lambda$}$$

where $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, $p > 1$, and λ is a positive parameter.

In Section 3, under suitable hypotheses, we prove that for each of the problems (P_λ) and (P'_λ) there exist at least three solutions when λ lies in an exactly determined open interval (see Theorem 3.1 and Proposition 3.1).

Next, in Section 4 we point out some consequences, which emphasize the usefulness of this precise estimate of the interval. Here, as a way of example, we present two of them.

Theorem B (see Corollary 4.2). *Let $f : \mathbb{R} \to \mathbb{R}$ be a positive and bounded continuous function such that*

$$8\int_0^{1/2} f(\xi)\,d\xi < 1 < \frac{1}{3}\int_0^1 f(\xi)\,d\xi.$$

Then, the problem

$$\begin{cases} -u'' = f(u), \\ u(0) = u'(1) = 0, \end{cases} \tag{PA}$$

has at least three classical positive solutions.

Theorem C (see Corollary 4.9). *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with $f(0) = 0$, $f(x) \leqslant 0$ in a right-neighborhood of 0, and such that, for some $q \in]0, p-1[$, $\lim_{x\to+\infty} \frac{f(x)}{x^q} \in]0, +\infty[$. Then, there exists a positive real number $\bar{\lambda}$ such that, for each $\lambda > \bar{\lambda}$, the problem*

$$\begin{cases} -(|u'|^{p-2}u')' + |u|^{p-2}u = \lambda f(u), \\ u(a) = u'(b) = 0, \end{cases} \tag{PA$'_\lambda$}$$

has at least two nontrivial and nonnegative classical solutions.

Moreover, a comparison can be found in Remarks 4.1 and 4.3 which shows that the main result for the problem (P_λ) is essentially more general than the one for the problem (P'_λ).

Other recent results on multiple solutions to mixed boundary value problems can be found in [5,7,9].

2. Basic definitions

We recall that a function $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is said L^1-Carathéodory if

(a) $t \to f(t,x)$ is measurable for every $x \in \mathbb{R}$;
(b) $x \to f(t,x)$ is continuous for almost every $t \in [a,b]$;
(c) for every $\rho > 0$ there exists a function $l_\rho \in L^1([a,b])$ such that

$$\sup_{|x|\leqslant\rho} \left|f(t,x)\right| \leqslant l_\rho(t)$$

for almost every $t \in [a,b]$.

A function $u : [a,b] \to \mathbb{R}$ is said a generalized solution to problem (P_λ) if $u \in C^1([a,b])$, $|u'|^{p-2}u' \in AC([a,b])$, $u(a) = u'(b) = 0$, and $-(|u'(t)|^{p-2}u'(t))' = \lambda f(t, u(t))$ for almost every $t \in [a,b]$.

We say that u is a weak solution to problem (P_λ) if $u \in W^{1,p}([a,b])$, $u(a) = 0$ and

$$\int_a^b \left|u'(t)\right|^{p-2} u'(t) v'(t)\,dt = \lambda \int_a^b f\bigl(t, u(t)\bigr) v(t)\,dt$$

for every $v \in W^{1,p}([a,b])$, with $v(a) = 0$.

Analogously, a function $u : [a,b] \to \mathbb{R}$ is a generalized solution to problem (P'_λ) if $u \in C^1([a,b])$, $|u'|^{p-2}u' \in AC([a,b])$, $u(a) = u'(b) = 0$, and $-(|u'(t)|^{p-2}u'(t))' + |u(t)|^{p-2}u(t) = \lambda f(t,u(t))$ for almost every $t \in [a,b]$, and it is a weak solution if $u \in W^{1,p}([a,b])$, $u(a) = 0$ and

$$\int_a^b \left|u'(t)\right|^{p-2} u'(t) v'(t)\,dt + \int_a^b \left|u(t)\right|^{p-2} u(t) v(t)\,dt = \lambda \int_a^b f\bigl(t, u(t)\bigr) v(t)\,dt$$

for every $v \in W^{1,p}([a,b])$, with $v(a) = 0$.

Standard methods show that generalized solutions to problem (P_λ) (respectively (P'_λ)) coincides with weak ones when f is an L^1-Carathéodory function.

For other basic notations and definitions we refer to [10].

3. Main results

Theorem 3.1. *Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ an L^1-Carathéodory function, and put $g(t,\xi) := \int_0^\xi f(t,x)\,dx$ for every $(t,\xi) \in [a,b] \times \mathbb{R}$. Assume that there exist three positive constants c, d, s, with $c < d$ and $s < p$, and a function $\mu \in L^1([a,b])$ such that*

(j) $$\frac{\int_a^b \max_{|\xi| \leqslant c} g(t,\xi)\,dt}{c^p} < \frac{1}{2^p}\left[\frac{(b-a)\left(\int_0^d g\left(\frac{b-a}{2d}x + a, x\right)dx + \int_d^{2d} g\left(\frac{b-a}{2d}x + a, d\right)dx\right)}{d^{p+1}} - \frac{2\int_a^b \max_{|\xi| \leqslant c} g(t,\xi)\,dt}{d^p}\right];$$

(jj) $g(t,\xi) \leqslant \mu(t)\bigl(1 + |\xi|^s\bigr)$ *for almost every $t \in [a,b]$ and for all $\xi \in \mathbb{R}$.*

Then, setting

$$\lambda' := \frac{\frac{1}{p}\left(\frac{2d}{b-a}\right)^p}{\frac{1}{d}\left[\int_0^d g\left(\frac{b-a}{2d}x + a, x\right)dx + \int_d^{2d} g\left(\frac{b-a}{2d}x + a, d\right)dx\right] - \frac{2}{b-a}\int_a^b \max_{|\xi| \leqslant c} g(t,\xi)\,dt}$$

and

$$\lambda'' := \frac{c^p}{p(b-a)^{p-1}\int_a^b \max_{|\xi| \leqslant c} g(t,\xi)\,dt},$$

for each $\lambda \in\,]\lambda', \lambda''[$ the problem (P_λ) admits at least three generalized solutions.

Proof. Let X be the reflexive Banach space $\{x \in W^{1,p}([a,b]):\ x(a)=0\}$ with the norm $\|x\| := (\int_a^b |x'(t)|^p\,dt)^{1/p}$ which is equivalent to the usual one. For each $x \in X$, put $\Phi(x) := \|x\|^p/p$ and $\Psi(x) := -\int_a^b g(t,x(t))\,dt$.

It is well known that Ψ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $x \in X$ is the functional $\Psi'(x) \in X^*$, given by

$$\Psi'(x)(v) = -\int_a^b f\big(t,x(t)\big)v(t)\,dt$$

for every $v \in X$, and that $\Psi' : X \to X^*$ is a continuous and compact operator.

Moreover, Φ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteuax derivative at the point $x \in X$ is the functional $\Phi'(x) \in X^*$, given by

$$\Phi'(x)(v) = \int_a^b \big|x'(t)\big|^{p-2} x'(t)v'(t)\,dt$$

for every $v \in X$, and that $\Phi' : X \to X^*$ admits a continuous inverse on X^*.

Since generalized solutions to problem (P_λ) coincides with weak ones, and these last are exactly the critical points of the functional $\Phi + \lambda\Psi$, our end is to apply Theorem A to Φ and Ψ.

Hypothesis (i) of Theorem A follows in a simple way, by (jj) and

$$\big|x(t)\big| \leqslant (b-a)^{(p-1)/p}\|x\|$$

for all $x \in X$ and for all $t \in [a,b]$.

In order to prove (ii) of Theorem A, we claim that

$$\varphi_1(r) \leqslant \frac{\int_a^b \max_{|\xi| \leqslant [pr(b-a)^{p-1}]^{1/p}} g(t,\xi)\,dt}{r} \tag{C1}$$

for each $r > 0$, and

$$\varphi_2(r) \geqslant p\frac{\int_a^b g(t,y(t))\,dt - \int_a^b \max_{|\xi| \leqslant [pr(b-a)^{p-1}]^{1/p}} g(t,\xi)\,dt}{\|y\|^p} \tag{C2}$$

for each $r > 0$ and every $y \in X$ such that

$$\frac{1}{p}\|y\|^p \geqslant r \tag{3.1}$$

and

$$\int_a^b g\big(t,y(t)\big)\,dt \geqslant \int_a^b \max_{|\xi| \leqslant [pr(b-a)^{p-1}]^{1/p}} g(t,\xi)\,dt. \tag{3.2}$$

In fact, for $r > 0$, and taking into account that the function identically 0 obviously belongs to $\Phi^{-1}(]-\infty, r[)$, and that $\Psi(0) = 0$, we get

$$\varphi_1(r) \leqslant \frac{\sup_{\overline{\Phi^{-1}(]-\infty,r[)}^w} \int_a^b g(t,x(t))\,dt}{r},$$

and, since $\overline{\Phi^{-1}(]-\infty, r[)}^{w} = \Phi^{-1}(]-\infty, r])$, we have

$$\frac{\sup_{\overline{\Phi^{-1}(]-\infty,r[)}^{w}} \int_a^b g(t, x(t))\,dt}{r} = \frac{\sup_{\|x\|^p/p \leqslant r} \int_a^b g(t, x(t))\,dt}{r};$$

thus, from $|x(t)| \leqslant [pr(b-a)^{p-1}]^{1/p}$, for every $x \in X$ such that $\|x\|^p/p \leqslant r$ and for each $t \in [a, b]$, we obtain

$$\frac{\sup_{\|x\|^p/p \leqslant r} \int_a^b g(t, x(t))\,dt}{r} \leqslant \frac{\int_a^b \max_{|\xi| \leqslant [pr(b-a)^{p-1}]^{1/p}} g(t, \xi)\,dt}{r}.$$

So, (C1) is proved.

Moreover, for each $r > 0$ and each $y \in X$ such that $\|y\|^p/p \geqslant r$, we have

$$\varphi_2(r) \geqslant p \inf_{\|x\|^p/p<r} \frac{\int_a^b g(t, y(t))\,dt - \int_a^b g(t, x(t))\,dt}{\|y\|^p - \|x\|^p},$$

thus, from $|x(t)| \leqslant [pr(b-a)^{p-1}]^{1/p}$, for every $x \in X$ such that $\|x\|^p/p < r$ and for each $t \in [a, b]$, we obtain

$$\begin{aligned} &p \inf_{\|x\|^p/p<r} \frac{\int_a^b g(t, y(t))\,dt - \int_a^b g(t, x(t))\,dt}{\|y\|^p - \|x\|^p} \\ &\quad\geqslant p \inf_{\|x\|^p/p<r} \frac{\int_a^b g(t, y(t))\,dt - \int_a^b \max_{|\xi| \leqslant [pr(b-a)^{p-1}]^{1/p}} g(t, \xi)\,dt}{\|y\|^p - \|x\|^p}, \end{aligned}$$

from which, being $0 < \|y\|^p - \|x\|^p \leqslant \|y\|^p$ for every $x \in X$ such that $\|x\|^p/p < r$, and under further condition (3.2), we can write

$$\begin{aligned} &p \inf_{\|x\|^p/p<r} \frac{\int_a^b g(t, y(t))\,dt - \int_a^b \max_{|\xi| \leqslant [pr(b-a)^{p-1}]^{1/p}} g(t, \xi)\,dt}{\|y\|^p - \|x\|^p} \\ &\quad\geqslant p \frac{\int_a^b g(t, y(t))\,dt - \int_a^b \max_{|\xi| \leqslant [pr(b-a)^{p-1}]^{1/p}} g(t, \xi)\,dt}{\|y\|^p}. \end{aligned}$$

So, (C2) is also proved.

Now, in order to prove (ii) of Theorem A, taking into account (C1) and (C2), it suffices to find $r > 0$ and $y \in X$, which verifies (3.1), and

$$\begin{aligned} &\frac{\int_a^b \max_{|\xi| \leqslant [pr(b-a)^{p-1}]^{1/p}} g(t, \xi)\,dt}{r} \\ &\quad< p \frac{\int_a^b g(t, y(t))\,dt - \int_a^b \max_{|\xi| \leqslant [pr(b-a)^{p-1}]^{1/p}} g(t, \xi)\,dt}{\|y\|^p}. \end{aligned} \tag{3.3}$$

Notice that (3.2) is consequence of (3.3).

To this end, we define

$$y(t) := \begin{cases} \frac{2d}{b-a}(t-a) & \text{if } t \in \left[a, \frac{a+b}{2}\right], \\ d & \text{if } t \in \left[\frac{a+b}{2}, b\right], \end{cases} \tag{3.4}$$

and $r := \frac{c^p}{p(b-a)^{p-1}}$.

Clearly, $y \in X$ and $\|y\|^p = \frac{d^p 2^{p-1}}{(b-a)^{p-1}}$. Hence, since $c < d$, we have $\|y\|^p/p > r$.

From (j), taking into account the values of r and $\|y\|^p$ and that

$$\int_a^b g\big(t, y(t)\big)\,dt = \frac{b-a}{2d}\left[\int_0^d g\left(\frac{b-a}{2d}x + a, x\right)dx + \int_d^{2d} g\left(\frac{b-a}{2d}x + a,\, d\right)dx\right],$$

the inequality (3.3) follows easily.

Thus, the conclusion follows by Theorem A, by observing that

$$\frac{1}{\varphi_2(r)} \leqslant \frac{\frac{1}{p}(\frac{2d}{b-a})^p}{\frac{1}{d}[\int_0^d g(\frac{b-a}{2d}x+a,x)\,dx+\int_d^{2d} g(\frac{b-a}{2d}x+a,d)\,dx]-\frac{2}{b-a}\int_a^b \max_{|\xi|\leqslant c} g(t,\xi)\,dt}$$

and

$$\frac{1}{\varphi_1(r)} \geqslant \frac{c^p}{p(b-a)^{p-1}\int_a^b \max_{|\xi|\leqslant c} g(t,\xi)\,dt}. \qquad \Box$$

Remark 3.1. In Theorem 3.1, hypothesis (j) is related to the function y defined in (3.4). Different functions y would lead to several conditions, which are similar to (j); however, hypothesis (j) seems to be the simplest expression for these types of conditions.

Remark 3.2. In Theorem 3.1 instead of hypothesis (j) we can use the following less general, but a bit simpler:

$$\text{(j}'\text{)} \quad \frac{\int_a^b \max_{|\xi|\leqslant c} g(t,\xi)\,dt}{c^p} < \frac{b-a}{2^p+2}\,\frac{\int_0^d g\big(\frac{b-a}{2d}x + a, x\big)\,dx + \int_d^{2d} g\big(\frac{b-a}{2d}x + a, d\big)\,dx}{d^{p+1}}.$$

In fact, taking into account that $0 < c < d$, from (j′) we get

$$\begin{aligned}&\frac{b-a}{2^p+2}\,\frac{\int_0^d g\big(\frac{b-a}{2d}x + a, x\big)\,dx + \int_d^{2d} g\big(\frac{b-a}{2d}x + a, d\big)\,dx}{d^{p+1}}\\ &\quad< \frac{1}{2^p}\left[\frac{(b-a)\big(\int_0^d g\big(\frac{b-a}{2d}x + a, x\big)\,dx + \int_d^{2d} g\big(\frac{b-a}{2d}x + a, d\big)\,dx\big)}{d^{p+1}}\right.\\ &\qquad\left. - \frac{2\int_a^b \max_{|\xi|\leqslant c} g(t,\xi)\,dt}{d^p}\right],\end{aligned}$$

thus, using again (j′), hypothesis (j) of Theorem 3.1 follows.

Moreover, when f (and, consequently, g) does not depend on t, hypothesis (j′) becomes the following very simple condition:

$$\text{(j}''\text{)} \quad \frac{\max_{|\xi|\leqslant c} g(\xi)}{c^p} < \frac{1}{2^p+2}\,\frac{\frac{1}{d}\int_0^d g(x)\,dx + g(d)}{d^p}.$$

Remark 3.3. If we assume $f(t,0)=0$ for each $t \in [a,b]$, then putting

$$f^*(t,x) := \begin{cases} 0 & \text{if } t \in [a,b] \text{ and } x \leqslant 0, \\ f(t,x) & \text{if } t \in [a,b] \text{ and } x > 0, \end{cases}$$

and considering the problem

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda f^*(t,u), \\ u(a) = u'(b) = 0, \end{cases} \tag{P_λ^*}$$

we have that the generalized solutions to problem (P_λ^*) are nonnegative and, consequently, they are also solutions to problem (P_λ).

In fact, arguing by a contradiction, if we assume that a solution u to (P_λ^*) is negative at a point of $[a,b]$, then there exists an interval $]a',b'[\, \subset [a,b]$ such that $u_{|]a',b'[}$ is negative, hence $-(|u'(t)|^{p-2}u'(t))' = 0$ for every $t \in\,]a',b'[$, and, further, $u(a') = u(b') = 0$ or $u(a') = u'(b') = 0$ if $b' < b$ or $b' = b$, respectively. Thus $u(t) = 0$ for every $t \in\,]a',b'[$, which is a contradiction.

Concerning the problem (P'_λ), the following proposition can be proved in a very similar way to that used to prove Theorem 3.1, using the usual norm

$$\|x\| := \left(\int_a^b |x(t)|^p\, dt + \int_a^b |x'(t)|^p\, dt \right)^{1/p}$$

in X instead of that used in the proof of Theorem 3.1 (see also Remark 4.1 after Corollary 4.1).

Proposition 3.1. *Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ an L^1-Carathéodory function, and put $g(t,\xi) := \int_0^\xi f(t,x)\,dx$ for every $(t,\xi) \in [a,b] \times \mathbb{R}$. Assume that there exist three positive constants c, d, s, with $c < d$ and $s < p$, and a function $\mu \in L^1([a,b])$ such that*

(k) $$\frac{\int_a^b \max_{|\xi| \leqslant c} g(t,\xi)\,dt}{c^p} < \frac{p+1}{(b-a)^p + (p+1)(b-a)^p + 2^p(p+1)} \times \left[\frac{(b-a)\left(\int_0^d g\left(\frac{b-a}{2d}x + a, x\right)dx + \int_d^{2d} g\left(\frac{b-a}{2d}x + a, d\right)dx\right)}{d^{p+1}} - \frac{2\int_a^b \max_{|\xi| \leqslant c} g(t,\xi)\,dt}{d^p} \right];$$

(jj) $g(t,\xi) \leqslant \mu(t)\left(1 + |\xi|^s\right)$ *for almost every $t \in [a,b]$ and for all $\xi \in \mathbb{R}$.*

Then, setting

$$\lambda' := \frac{\frac{(b-a)^p + (p+1)(b-a)^p + 2^p(p+1)}{p(p+1)(b-a)^p}\, d^p}{\frac{1}{d}\left[\int_0^d g\left(\frac{b-a}{2d}x + a, x\right)dx + \int_d^{2d} g\left(\frac{b-a}{2d}x + a, d\right)dx\right] - \frac{2}{b-a}\int_a^b \max_{|\xi| \leqslant c} g(t,\xi)\,dt}$$

and

$$\lambda'' := \frac{c^p}{p(b-a)^{p-1}\int_a^b \max_{|\xi|\leqslant c} g(t,\xi)\,dt},$$

for each $\lambda \in]\lambda', \lambda''[$ *the problem* (P'_λ) *admits at least three generalized solutions.*

Remark 3.4. In Proposition 3.1 instead of hypothesis (k) we can use the following less general, but a bit simpler:

$$(\mathrm{k}') \quad \frac{\int_a^b \max_{|\xi|\leqslant c} g(t,\xi)\,dt}{c^p} < \frac{(p+1)(b-a)}{(p+2)(b-a)^p + 2(p+1)(2^{p-1}+1)} \times \frac{\int_0^d g\left(\frac{b-a}{2d}x+a, x\right)dx + \int_d^{2d} g\left(\frac{b-a}{2d}x+a, d\right)dx}{d^{p+1}}.$$

In fact, taking into account that $0 < c < d$, from (k′) we get

$$\begin{aligned}
&\frac{(p+1)(b-a)}{(p+2)(b-a)^p + 2(p+1)(2^{p-1}+1)} \times \frac{\int_0^d g\left(\frac{b-a}{2d}x+a, x\right)dx + \int_d^{2d} g\left(\frac{b-a}{2d}x+a, d\right)dx}{d^{p+1}} \\
&< \frac{p+1}{(b-a)^p + (p+1)(b-a)^p + 2^p(p+1)} \\
&\quad \times \left[\frac{(b-a)\left(\int_0^d g\left(\frac{b-a}{2d}x+a, x\right)dx + \int_d^{2d} g\left(\frac{b-a}{2d}x+a, d\right)dx\right)}{d^{p+1}} - \frac{2\int_a^b \max_{|\xi|\leqslant c} g(t,\xi)\,dt}{d^p}\right],
\end{aligned}$$

thus, using again (k′), hypothesis (k) of Proposition 3.1 follows.

Moreover, when f (and, consequently, g) does not depend on t, hypotheses (k′) becomes the following very simple condition:

$$(\mathrm{k}'') \quad \frac{\max_{|\xi|\leqslant c} g(\xi)}{c^p} < \frac{p+1}{(p+2)(b-a)^p + 2(p+1)(2^{p-1}+1)} \frac{\frac{1}{d}\int_0^d g(x)\,dx + g(d)}{d^p},$$

which allows a direct comparison between our Proposition 3.1 and Theorem 2.1 of [9].

4. Consequences

Theorem 3.1 gives an estimate of the interval of the parameter λ for which the problem (P_λ) has at least three solutions. This information has several consequences. In this section we point out some of them.

First of all, we state the following straightforward corollary of Theorem 3.1.

Corollary 4.1. *Let* $\lambda > 0$ *be given, and let* f *and* g *be as in Theorem* 3.1. *Assume that there exist three positive constants* c, d, s, *with* $c < d$ *and* $s < p$, *and a function* $\mu \in L^1([a,b])$ *such that*

$$(\mathrm{j}^*)\quad \frac{\int_a^b \max_{|\xi|\leqslant c} g(t,\xi)\,dt}{c^p} < \frac{1}{\lambda p(b-a)^{p-1}} < \frac{1}{2^p}\left[\frac{(b-a)\left(\int_0^d g\left(\frac{b-a}{2d}x+a,x\right)dx + \int_d^{2d} g\left(\frac{b-a}{2d}x+a,d\right)dx\right)}{d^{p+1}} - \frac{2\int_a^b \max_{|\xi|\leqslant c} g(t,\xi)\,dt}{d^p}\right];$$

(jj) $g(t,\xi) \leqslant \mu(t)\left(1+|\xi|^s\right)$ *for almost every* $t \in [a,b]$ *and for all* $\xi \in \mathbb{R}$.

Then, the problem (P_λ) *admits at least three generalized solutions.*

Remark 4.1. Proposition 3.1 can be viewed also as a consequence of Corollary 4.1. In fact, under the hypotheses and given λ' and λ'' as in Proposition 3.1, simple calculations show that Corollary 4.1 can be used for each fixed $\lambda \in]\lambda', \lambda''[$ to obtain three generalized solutions of the problem

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda\left(f(t,u) - \frac{|u|^{p-2}u}{\lambda}\right), \\ u(a) = u'(b) = 0, \end{cases} \tag{PE_λ}$$

which, obviously for the fixed λ, is equivalent to problem (P'_λ).

Next, accordingly with Remark 3.2, Theorem 3.1 leads to very easy propositions for autonomous problems, like the following

Corollary 4.2 (see Theorem B in the Introduction). *Let* $f:\mathbb{R}\to\mathbb{R}$ *be a nonnegative and bounded continuous function such that*

$$8\int_0^{1/2} f(\xi)\,d\xi < 1 < \frac{1}{3}\int_0^1 f(\xi)\,d\xi.$$

Then, the problem

$$\begin{cases} -u'' = f(u), \\ u(0) = u'(1) = 0, \end{cases} \tag{PA}$$

has at least three classical solutions.

Remark 4.2. Theorem B is an immediate consequence of Corollary 4.2.

Corollary 4.3. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be a continuous function with* $f(0) = 0$ $f(x) \leqslant 0$ *in a right-neighborhood of* 0, *and such that, for some* $q \in]0, p-1[$, $\lim_{x\to+\infty} \frac{f(x)}{x^q} \in]0, +\infty[$. *Then,* *there exists a positive real number $\bar{\lambda}$ such that, for each $\lambda > \bar{\lambda}$, the problem*

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda f(u), \\ u(a) = u'(b) = 0, \end{cases} \tag{PA_λ}$$

has at least two nontrivial and nonnegative classical solutions.

Proof. In virtue of Remark 3.3, we put $f(x) = 0$ for $x < 0$. Clearly, there exists $c > 0$ such that $\max_{|\xi| \leqslant c} g(\xi) = 0$. Moreover, since $\lim_{x\to+\infty} \frac{f(x)}{x^q} \in]0, +\infty[$, there exists $d > c$ such that $\int_0^d g(\xi)\,d\xi > 0$ and $g(d) > 0$. Finally, there exists $\mu > 0$ such that $g(\xi) \leqslant \mu(1 + |\xi|^{1+q})$ for all $\xi \in \mathbb{R}$. Therefore, we can use Theorem 3.1 to reach the conclusion.

However, we obtain only two nontrivial and nonnegative solutions because $f(0) = 0$; obviously they are classical solutions in virtue of the continuity of f. □

Corollary 4.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with $f(0) = 0$, $f(x) \geqslant 0$ in a left-neighborhood of* 0*, and such that, for some $q \in]0, p-1[$, $\lim_{x\to-\infty} \frac{f(x)}{|x|^q} \in]-\infty, 0[$. Then, there exists a positive real number $\bar{\lambda}$ such that, for each $\lambda > \bar{\lambda}$, the problem* (PA_λ) *has at least two nontrivial and nonpositive classical solutions.*

Proof. It is enough to apply Corollary 4.3 to the function $f^*(x) := -f(-x)$. □

Next we prove another application of Theorem 3.1, which shows that (under simple conditions) for sufficiently large intervals the mixed problem has two nontrivial and nonnegative generalized solutions.

Corollary 4.5. *Let $\alpha \in L^1([0, +\infty[)$ be a function such that* $\inf \alpha > 0$ *and let $\beta : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function. Let us suppose that*

$$\beta(d) > 0$$

for some $d > 0$,

$$\lim_{x\to 0^+} \frac{\beta(x)}{x^{p-1}} = 0,$$

and

$$\lim_{x\to+\infty} \frac{\beta(x)}{x^q} \in \mathbb{R}$$

for some $q \in]0, p-1[$. Then, for every $\lambda > 0$ and every

$$b > \bar{b} := \frac{2d}{(\lambda p \inf \alpha \int_0^d \beta(x)\,dx)^{1/p}},$$

the problem

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda\alpha(t)\beta(u), \\ u(0) = u'(b) = 0, \end{cases} \tag{PS_λ}$$

admits at least two nontrivial and nonnegative generalized solutions.

Proof. In virtue of Remark 3.3, we put $\beta(x)=0$ for $x<0$. Fix $\lambda>0$ and $b>\bar{b}$. Since

$$\lim_{x\to 0^+}\frac{\beta(x)}{x^{p-1}}=0,$$

we have

$$\lim_{x\to 0^+}\frac{\int_0^x\beta(\xi)\,d\xi}{x^p}=0,$$

thus, taking into account the hypotheses on the sign of α and β and that $\beta(d)>0$, we can choose $c>0$ such that $c<d$,

$$\begin{aligned}&\frac{\int_0^c\beta(\xi)\,d\xi}{c^p}\\&\quad<\min\left\{\frac{b}{2^p+2}\frac{\int_0^d g\left(\frac{b}{2d}x,x\right)dx+\int_d^{2d} g\left(\frac{b}{2d}x,d\right)dx}{d^{p+1}\int_0^b\alpha(t)\,dt},\ \frac{1}{\lambda p b^{p-1}\int_0^b\alpha(t)\,dt}\right\}\end{aligned}$$

and

$$\frac{\frac{1}{p}\left(\frac{2d}{b-a}\right)^p}{\frac{1}{d}\left[\int_0^d g\left(\frac{b}{2d}x,x\right)dx+\int_d^{2d} g\left(\frac{b}{2d}x,d\right)dx\right]-\frac{2}{b}\int_0^b\alpha(t)\,dt\,\int_0^c\beta(\xi)\,d\xi}<\lambda.$$

Moreover, the existence of a function $\mu\in L^1([0,b])$ such that $g(t,\xi)\leqslant\mu(1+|\xi|^{q+1})$, for every $t\in[0,b]$ and for all $\xi\in\mathbb{R}$, follows easily from $\lim_{x\to+\infty}\frac{\beta(x)}{x^q}\in\mathbb{R}$.

Hence, taking into account the Remarks 3.2 and 3.3 and applying Theorem 3.1 in the interval $[0,b]$, the conclusion follows since $\lambda'<\lambda<\lambda''$.

However, we obtain only two nontrivial and nonnegative solutions because $\beta(0)=0$. □

Finally, we give the following other application, in which the dependence on the variable t is investigated in order to obtain two nontrivial and nonnegative generalized solutions for mixed problems.

Corollary 4.6. *Let $\beta:\mathbb{R}\to\mathbb{R}$ be a nonnegative bounded continuous function. Let us suppose that*

$$\beta(\bar{x})>0$$

for some $\bar{x}>0$, and

$$\lim_{x\to 0^+}\frac{\beta(x)}{x^{p-1}}=0.$$

Then, for every $\lambda>0$ and every nonnegative $\alpha\in L^1([a,b])$, with

$$\|\alpha_{|[a+(b-a)/4,b]}\|_1>\frac{2^{2p+1}\bar{x}^{p+1}}{\lambda p(b-a)^p\int_0^{\bar{x}}\beta(\xi)\,d\xi},$$

the problem (PS_λ) admits at least two nontrivial and nonnegative generalized solutions.

Proof. In virtue of Remark 3.3, we put $\beta(x) = 0$ for $x < 0$. Fix $\lambda > 0$ and put $d := 2\bar{x}$. Clearly $\int_0^d \beta(\xi)\,d\xi > \int_0^{d/2} \beta(\xi)\,d\xi > 0$. Let $\alpha \in L^1([a,b])$, with

$$\|\alpha_{|[a+(b-a)/4,b]}\|_1 > \frac{2^p d^{p+1}}{\lambda p(b-a)^p \int_0^{d/2} \beta(\xi)\,d\xi}.$$

Clearly $\|\alpha\|_1 \geqslant \|\alpha_{|[a+(b-a)/4,b]}\|_1$. Moreover, a simple calculation shows that

$$\int_0^d g\left(\frac{b-a}{2d}x + a, x\right) dx + \int_d^{2d} g\left(\frac{b-a}{2d}x + a, d\right) dx$$
$$\geqslant \|\alpha_{|[a+(b-a)/4,b]}\|_1 \int_0^{d/2} \beta(\xi)\,d\xi.$$

Now, since

$$\lim_{x\to 0^+} \frac{\beta(x)}{x^{p-1}} = 0,$$

we get

$$\lim_{x\to 0^+} \frac{\max_{|\xi|\leqslant x} \int_0^x \beta(\xi)\,d\xi}{x^p} = 0.$$

From this, and taking into account that

$$\lim_{c\to 0^+} \frac{\frac{1}{p}\left(\frac{2d}{b-a}\right)^p}{\frac{1}{d}\left[\int_0^d g\left(\frac{b-a}{2d}x+a,x\right)dx + \int_d^{2d} g\left(\frac{b-a}{2d}x+a,d\right)dx\right] - \frac{2}{b-a}\int_a^b \max_{|\xi|\leqslant c} g(t,\xi)\,dt}$$
$$= \frac{\frac{1}{p}\left(\frac{2d}{b-a}\right)^p}{\frac{1}{d}\left[\int_0^d g\left(\frac{b-a}{2d}x+a,x\right)dx + \int_d^{2d} g\left(\frac{b-a}{2d}x+a,d\right)dx\right]}$$
$$\leqslant \frac{2^p d^{p+1}}{\|\alpha_{|[a+(b-a)/4,b]}\|_1 p(b-a)^p \int_0^{d/2} \beta(\xi)\,d\xi} < \lambda,$$

there exists $c > 0$, with $c < d$, verifying (j′) of Remark 3.2,

$$\lambda' := \frac{\frac{1}{p}\left(\frac{2d}{b-a}\right)^p}{\frac{1}{d}\left[\int_0^d g\left(\frac{b-a}{2d}x+a,x\right)dx + \int_d^{2d} g\left(\frac{b-a}{2d}x+a,d\right)dx\right] - \frac{2}{b-a}\int_a^b \max_{|\xi|\leqslant c} g(t,\xi)\,dt} < \lambda$$

and

$$\lambda'' := \frac{c^p}{p(b-a)^{p-1}\int_a^b \max_{|\xi|\leqslant c} g(t,\xi)\,dt} > \lambda.$$

Moreover, (jj) of Theorem 3.1 follows easily from the boundedness of β.

Hence, taking into account the Remarks 3.2 and 3.3 and applying Theorem 3.1, the conclusion follows.

However, we obtain only two nontrivial and nonnegative solutions because $\beta(0) = 0$. $\square$

Although Proposition 3.1 can be deduced from Corollary 4.1 (see Remark 4.1), its consequences (corresponding to Corollaries 4.1–4.6) are more transparent if related directly to Proposition 3.1 rather than to Theorem 3.1. We state them without proof.

Corollary 4.7. *Let $\lambda > 0$ be given, and let f and g be as in Proposition* 3.1. *Assume that there exist three positive constants c, d, s, with $c < d$ and $s < p$, and a function $\mu \in L^1([a,b])$ such that*

$$\text{(k}^*\text{)}\quad \frac{\int_a^b \max_{|\xi|\leqslant c} g(t,\xi)\,dt}{c^p} < \frac{1}{\lambda p(b-a)^{p-1}} < \frac{p+1}{(b-a)^p + (p+1)(b-a)^p + 2^p(p+1)} \times \left[\frac{(b-a)\left(\int_0^d g\left(\frac{b-a}{2d}x + a, x\right)dx + \int_d^{2d} g\left(\frac{b-a}{2d}x + a, d\right)dx\right)}{d^{p+1}} - \frac{2\int_a^b \max_{|\xi|\leqslant c} g(t,\xi)\,dt}{d^p}\right];$$

(jj) $g(t,\xi) \leqslant \mu(t)\left(1 + |\xi|^s\right)$ *for almost every $t \in [a,b]$ and for all $\xi \in \mathbb{R}$.*

Then, the problem (P'_λ) *admits at least three generalized solutions.*

Remark 4.3. Unlike Remark 4.1, Corollary 4.1 cannot be proved making use of Corollary 4.7 on the problem

$$\begin{cases} -(|u'|^{p-2}u')' + |u|^{p-2}u = \lambda\left(f(t,u) + \frac{|u|^{p-2}u}{\lambda}\right), \\ u(a) = u'(b) = 0. \end{cases} \tag{PE'_λ}$$

In fact, for $f \geqslant 0$, $b - a \geqslant 1$, and $\lambda > 0$, the first inequality in hypothesis (k*) of Corollary 4.7 can never be satisfied.

Corollary 4.8. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative and bounded continuous function such that*

$$8\int_0^{1/2} f(\xi)\,d\xi < 1 < \frac{3}{11}\int_0^1 f(\xi)\,d\xi.$$

Then, the problem

$$\begin{cases} -u'' + u = f(u), \\ u(0) = u'(1) = 0, \end{cases} \tag{PA'}$$

has at least three classical solutions.

Corollary 4.9 (see Theorem C in the Introduction). *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with $f(0) = 0$, $f(x) \leqslant 0$ in a right-neighborhood of* 0, *and such that, for some*

$q \in\,]0, p-1[$, $\lim_{x\to+\infty} \frac{f(x)}{x^q} \in\,]0, +\infty[$. *Then, there exists a positive real number* $\bar{\lambda}$ *such that, for each* $\lambda > \bar{\lambda}$, *the problem*

$$\begin{cases} -(|u'|^{p-2}u')' + |u|^{p-2}u = \lambda f(u), \\ u(a) = u'(b) = 0, \end{cases} \tag{PA$'_\lambda$}$$

has at least two nontrivial and nonnegative classical solutions.

Corollary 4.10. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be a continuous function with* $f(0)=0$, $f(x)\geqslant 0$ *in a left-neighborhood of* 0, *and such that, for some* $q \in\,]0, p-1[$, $\lim_{x\to-\infty} \frac{f(x)}{|x|^q} \in\,]-\infty, 0[$. *Then, there exists a positive real number* $\bar{\lambda}$ *such that, for each* $\lambda > \bar{\lambda}$, *the problem* (PA$'_\lambda$) *has at least two nontrivial and nonpositive classical solutions.*

Corollary 4.11. *Let* $\alpha \in L^1([0, +\infty[)$ *be a function such that* $\inf \alpha > 0$ *and let* $\beta:\mathbb{R}\to\mathbb{R}$ *be a nonnegative continuous function. Let us suppose that*

$$\beta(d) > 0$$

for some $d > 0$,

$$\lim_{x\to 0^+} \frac{\beta(x)}{x^{p-1}} = 0,$$

and

$$\lim_{x\to+\infty} \frac{\beta(x)}{x^q} \in \mathbb{R}$$

for some $q \in\,]0, p-1[$. *Then, for every*

$$\lambda > \bar{\lambda} := \frac{(p+2)d^p}{p(p+1)\inf\alpha \int_0^d \beta(\xi)\,d\xi}$$

and every

$$b > \bar{b} := \frac{2d}{((\lambda-\bar{\lambda})p \inf\alpha \int_0^d \beta(x)\,dx)^{1/p}},$$

the problem

$$\begin{cases} -(|u'|^{p-2}u')' + |u|^{p-2}u = \lambda\alpha(t)\beta(u), \\ u(0) = u'(b) = 0, \end{cases} \tag{PS$'_\lambda$}$$

admits at least two nontrivial and nonnegative generalized solutions.

Corollary 4.12. *Let* $\beta:\mathbb{R}\to\mathbb{R}$ *be a nonnegative bounded continuous function. Let us suppose that*

$$\beta(\bar{x}) > 0$$

for some $\bar{x} > 0$, *and*

$$\lim_{x\to 0^+} \frac{\beta(x)}{x^{p-1}} = 0.$$

Then, for every $\lambda > 0$ *and every nonnegative* $\alpha \in L^1([a, b])$, *with*

$$\|\alpha_{|[a+(b-a)/4,b]}\|_1 > \frac{[(b-a)^p + (p+1)(b-a)^p + 2^p(p+1)](2\bar{x})^{p+1}}{\lambda p(p+1)(b-a)^p \int_0^{\bar{x}} \beta(\xi)\, d\xi},$$

the problem (PS'_λ) *admits at least two nontrivial and nonnegative generalized solutions.*

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