Available online at www.sciencedirect.com
SCIENCE DIRECT®

ELSEVIER

J. Math. Anal. Appl. 308 (2005) 669–688

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Identities of the Rogers–Ramanujan–Bailey type

Andrew V. Sills

Department of Mathematics, Rutgers University, Hill Center-Busch Campus, Piscataway, NJ 08854, USA

Received 28 July 2004

Available online 25 February 2005

Submitted by B.C. Berndt

Abstract

A multiparameter generalization of the Bailey pair is defined in such a way as to include as special cases all Bailey pairs considered by W.N. Bailey in his paper [Identities of the Rogers–Ramanujan type, Proc. London Math. Soc. (2) 50 (1949) 421–435]. This leads to the derivation of a number of elegant new Rogers–Ramanujan type identities.

Keywords: Rogers–Ramanujan identities; Bailey pairs; q-series identities; Basic hypergeometric series

1. Introduction

1.1. Overview

Recall the famous Rogers–Ramanujan identities:

Theorem 1.1 (The Rogers–Ramanujan identities).

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{(q^2, q^3, q^5; q^5)_\infty}{(q;q)_\infty} \tag{1.1}$$

E-mail address: asills@math.rutgers.edu.
URL: http://www.math.rutgers.edu/~asills.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.11.061

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{(q,q^4,q^5;q^5)_\infty}{(q;q)_\infty}, \tag{1.2}$$

where

$$(a;q)_m = \prod_{j=0}^{m-1} (1-aq^j),$$
$$(a;q)_\infty = \prod_{j=0}^{\infty} (1-aq^j),$$

and

$$(a_1,a_2,\ldots,a_r;q)_s = (a_1;q)_s(a_2;q)_s\ldots(a_r;q)_s.$$

(Although the results in this paper may be considered purely from the point of view of formal power series, they also yield identities of analytic functions provided $|q| < 1$.)

The Rogers–Ramanujan identities are due to L.J. Rogers [18], and were rediscovered independently by S. Ramanujan [16] and I. Schur [20]. Rogers [18,19] (and later others) discovered many series-product identities similar in form to the Rogers–Ramanujan identities, and as such are referred to as "identities of the Rogers–Ramanujan type." A number of Rogers–Ramanujan type identities were recorded by Ramanujan in his Lost Notebook [17, Chapter 11]. During World War II, W.N. Bailey undertook a thorough study of Rogers' work connected with Rogers–Ramanujan type identities, and through the understanding he gained, was able to simplify and generalize Rogers' ideas in a pair of papers [8,9]. In the process, Bailey and Freeman Dyson (who served as referee for Bailey's two papers [12, p. 14]) discovered a number of new Rogers–Ramanujan type identities.

Bailey and his student L.J. Slater [23,24] only considered identities of single-fold series and infinite products. Bailey comments in passing [9, §4, p. 4] that "the most general formulae for basic series (apart from those already given) are too involved to be of any great interest" and as such, rejected multisums from consideration. However, G.E. Andrews reversed this prejudice against multisum Rogers–Ramanujan type identities by presenting very elegant examples of same in [1,3]. Since the appearance of those papers, many authors have presented multisum Rogers–Ramanujan type identities, with a particular emphasis on infinite families of results, as in Andrews' discovery of the "Bailey chain" [4, p. 28 ff], [3]. I will not be presenting such infinite families here as, in light of [3], it is a routine exercise to imbed any Rogers–Ramanujan type identity in such an infinite family.

There is much current interest in new Bailey pairs and innovations with Bailey chains (cf. [5,10,11,15,30,32]). We shall show in this paper that some very appealing Rogers–Ramanujan type identities are still to be found that are actually derivable from Bailey's original ideas (combined in some cases with q-hypergeometric transformations due to Verma and Jain [28,29]). For example,

$$\sum_{n,r\geqq 0} \frac{q^{2n^2+3r^2+4nr}}{(-q;q)_{n+r}(-q;q)_{n+2r}(q;q)_n(q;q)_r} = \frac{(q^4,q^5,q^9;q^9)_\infty}{(q^2;q^2)_\infty}, \tag{1.3}$$

$$\sum_{n,r\geqq 0} \frac{q^{n^2+3r^2+4nr}(-q;-q)_{2n+2r}}{(q^2;q^2)_{2n+2r}(q^2;q^2)_r(q^2;q^2)_n} = \frac{(q^6,q^8,q^{14};q^{14})_\infty (q^2;q^4)_\infty}{(q;q)_\infty}, \tag{1.4}$$

$$\sum_{n,r\geqq 0} \frac{q^{3n^2+6nr+6r^2}(q;q)_{3r}}{(q^3;q^3)_{2r}(q^3;q^3)_r(q^3;q^3)_n} = \frac{(q^7,q^8,q^{15};q^{15})_\infty}{(q^3;q^3)_\infty}, \tag{1.5}$$

$$\sum_{n,r\geqq 0} \frac{q^{2n^2+3r^2+4nr}}{(-q;q)_{2n+2r}(q;q^2)_r(q;q)_r(q^2;q^2)_n} = \frac{(q^{14},q^{16},q^{30};q^{30})_\infty}{(q^2;q^2)_\infty}, \tag{1.6}$$

$$\sum_{n,r\geqq 0} \frac{q^{n^2+2nr+3r^2}(-q;q^2)_{n+r}}{(q^2;q^2)_n(q^2;q^2)_r(q^2;q^4)_r} = \frac{(q^{16},q^{20},q^{36};q^{36})_\infty (q^2;q^4)_\infty}{(q;q)_\infty}, \tag{1.7}$$

$$\sum_{n,r\geqq 0} \frac{q^{n^2+2r^2}}{(q;q^2)_n(q^2;q^2)_r(q^2,q^4)_r(q;q)_{n-2r}} = \frac{(q^{28},q^{32},q^{60};q^{60})_\infty}{(q;q)_\infty}. \tag{1.8}$$

Furthermore, we note that the double sum identities I present here do not arise as merely “one level up” in the standard Bailey chain from well-known single-sum identities.

After reviewing the necessary background material in Section 1.2, I define the “standard multiparameter Bailey pair,” (SMPBP) in Section 2 and demonstrate that all of the Bailey pairs presented by Bailey in [8,9] may be viewed as special cases of the SMPBP.

In Section 3, I derive new Bailey pairs as special cases of the SMPBP, and finally in Section 4, I present a collection of new Rogers–Ramanujan type identities which are consequences of the Bailey pairs from Section 3. In Section 5, I conclude by relating this paper’s results to Slater’s list [24] and suggesting a possible direction of further research.

1.2. Background

In an effort to understand the mechanism which allowed Rogers to discover the Rogers–Ramanujan identities and other identities of similar type, Bailey discovered that the underlying engine was quite simple indeed. This engine was named the “Bailey transform” by Slater [25, §2.3].

Theorem 1.2 (The Bailey transform). *If*

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r}$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n},$$

then

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n.$$

Bailey remarks [9, p. 1] that "the proof is almost trivial" and indeed the proof merely involves reversing the order of summation in a double series.

Curiously, Bailey never uses the Bailey transform in this general form. He immediately specializes $u_n = 1/(q;q)_n$, $v_n = 1/(aq;q)_n$, and

$$\delta_n = \frac{(\rho_1;q)_n(\rho_2;q)_n(q^{-N};q)_n q^n}{(\rho_1\rho_2 q^{-N}a^{-1};q)_n}.$$

This, in turn, forces

$$\gamma_n = \frac{(aq/\rho_1;q)_N(aq/\rho_2;q)_N(-1)^n(\rho_1;q)_n(\rho_2;q)_n(q^{-N};q)_n(aq/\rho_1\rho_2)^n q^{nN-\binom{n}{2}}}{(aq;q)_N(aq/\rho_1\rho_2;q)_N(aq/\rho_1;q)_n(aq/\rho_2;q)_n(aq^{N+1};q)_n}.$$

Modern authors normally refer to the α and β appearing in the Bailey transform under the aforementioned specializations of u, v, and δ as a "Bailey pair."

Definition 1.3. A pair of sequences $(\alpha_n(a,q), \beta_n(a,q))$ is called a *Bailey pair* if for $n \geqq 0$,

$$\beta_n(a,q) = \sum_{r=0}^{n} \frac{\alpha_r(a,q)}{(q;q)_{n-r}(aq;q)_{n+r}}. \tag{1.9}$$

In [8,9], Bailey proved the fundamental result now known as "Bailey's lemma" (see also [4, Chapter 3]), which is actually just a consequence of the Bailey transform:

Theorem 1.4 (Bailey's lemma). *If $(\alpha_r(a,q), \beta_j(a,q))$ form a Bailey pair, then*

$$\begin{aligned}
&\frac{1}{(\frac{aq}{\rho_1};q)_n(\frac{aq}{\rho_2};q)_n} \sum_{j\geqq 0} \frac{(\rho_1;q)_j(\rho_2;q)_j(\frac{aq}{\rho_1\rho_2};q)_{n-j}}{(q;q)_{n-j}} \left(\frac{aq}{\rho_1\rho_2}\right)^j \beta_j(a,q) \\
&\quad = \sum_{r=0}^{n} \frac{(\rho_1;q)_r(\rho_2;q)_r}{(\frac{aq}{\rho_1};q)_r(\frac{aq}{\rho_2};q)_r(q;q)_{n-r}(aq;q)_{n+r}} \left(\frac{aq}{\rho_1\rho_2}\right)^r \alpha_r(a,q).
\end{aligned} \tag{1.10}$$

2. The standard multiparameter Bailey pair

Since the sequence β_n is completely determined for any α_n by (1.9), all we need to do is define the standard multiparameter Bailey pair via the α_n as

$$\begin{aligned}
&\alpha_n^{(d,e,k)}(a,b,q) \\
&\quad := \begin{cases} \frac{a^{(k-d+1)r/e}q^{(k-d+1)dr^2/e}(a^{1/e}q^{2d/e};q^{2d/e})_r(a^{1/e};q^{d/e})_r}{b^{r/e}(a^{1/e}b^{-1/e}q^{d/e};q^{d/e})_r(a^{1/e};q^{2d/e})_r(q^{d/e};q^{d/e})_r}, & \text{if } n = dr, \\ 0, & \text{otherwise,} \end{cases}
\end{aligned} \tag{2.1}$$

and the corresponding $\beta_n^{(d,e,k)}(a,b,q)$ will be determined by (1.9). Of course, the *form* in which $\beta_n^{(d,e,k)}$ is *presented* depends on which q-hypergeometric transformation or summation formula is employed. The mathematical interest lies in the fact that elegant Rogers–Ramanujan type identities will arise for many choices of d, e, and k, as we shall see in Section 4.

Remark 2.1. An easy calculation reveals that

$$\lim_{b\to 0} \alpha_n^{(d,e,k)}(a,b,q) = \lim_{b\to\infty} \alpha_n^{(d,e,k-1)}(a,b,q). \tag{2.2}$$

Remark 2.2. In all derivations of Rogers–Ramanujan type identities, Bailey lets $b \to 0$ or $b \to \infty$. In light of Remark 2.1, it will be sufficient to let $b \to 0$ from this point forward. Also, it will be convenient to replace a by a^e and q by q^e throughout. Thus, in practice, rather than (2.1), we will instead only need to consider the somewhat more manageable

$$\alpha_n^{(d,e,k)}(a^e,0,q^e) := \begin{cases} \frac{(-1)^r a^{(k-d)r} q^{(dk-d^2+\frac{d}{2})r^2-\frac{d}{2}r}(aq^{2d};q^{2d})_r(a;q^d)_r}{(a;q^{2d})_r(q^d;q^d)_r}, & \text{if } n = dr, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

We now proceed to find the corresponding β_n for the SMPBP:

$$\begin{aligned}
&\beta_n^{(d,e,k)}(a^e,0,q^e) \\
&\quad = \sum_{s=0}^{n} \frac{\alpha_s^{(d,e,k)}(a^e,0,q^e)}{(q^e;q^e)_{n-s}(a^eq^e;q^e)_{n+s}} \alpha_{d,k,m}(a^e,0,q^e) \\
&\quad = \frac{1}{(q^e;q^e)_n(a^eq^e;q^e)_n} \sum_{s=0}^{n} \frac{(-1)^s q^{ens-\frac{e}{2}s^2+\frac{e}{2}s}(q^{-en};q^e)_s}{(a^eq^{e(n+1)};q^e)_s} \alpha_s^{(d,e,k)}(a^e,0,q^e) \\
&\quad = \frac{1}{(q^e;q^e)_n(a^eq^e;q^e)_n} \sum_{r=0}^{\lfloor n/d\rfloor} \frac{(-1)^{dr} q^{endr-\frac{ed^2}{2}r^2+\frac{ed}{2}r}(q^{-en};q^e)_{dr}}{(a^eq^{e(n+1)};q^e)_{dr}} \\
&\qquad \times \alpha_{dr}^{(d,e,k)}(a^e,0,q^e) \\
&\quad = \frac{1}{(q^e;q^e)_n(a^eq^e;q^e)_n} \sum_{r=0}^{\lfloor n/d\rfloor} \frac{(a, q^d\sqrt{a}, -q^d\sqrt{a}; q^d)_r (q^{-en};q^e)_{dr}}{(q^d, \sqrt{a}, -\sqrt{a}; q^d)_r (a^eq^{e(n+1)};q^e)_{dr}} \\
&\qquad \times (-1)^{(d+1)r} a^{(k-d)r} q^{(2k-2d-ed+1)\frac{d}{2}r^2+(e-1)\frac{d}{2}r+endr}.
\end{aligned}$$

Note that

$$(q^{-en};q^e)_{dr} = \prod_{i=0}^{d-1} (q^{-en+ei};q^{de})_r, \tag{2.4}$$

and that each factor in the right-hand side of (2.4) can be factored into a product of e factors

$$(q^{-en+ei};q^{de})_r = \prod_{j=1}^{e} \left(\xi_e^j q^{-n+i};q^d\right)_r,$$

where ξ_e is a primitive eth root of unity, and that the complementary denominator factor $(a^eq^{e(n+1)};q^e)_{dr}$ can be split and factored analogously into a product of ed rising q-factorials.

Furthermore, $q^{(2k-2d-ed+1)\frac{d}{2}r^2}$ can be written as a limiting case of a product of $|2k-ed-2d+1|$ rising q-factorials. For example, supposing that $k=5$, $e=1$, $d=2$, one can write q^{5r^2} as a limit as $\tau \to 0$ of the product of $|2k-2d-ed+1|=5$ rising q factorials:

$$(-1)^r q^{5r^2} = \lim_{\tau\to 0} \tau^{5r} (q/\tau; q^2)_r^5.$$

Thus $\beta_n^{(d,e,k)}(a^e, 0, q^e)$ can be seen to be a product of $((q^e;q^e)_n(a^e q^e;q^e)_n)^{-1}$ and a limiting case of a very well poised ${}_{t+1}\phi_t$, where

$$t = ed + |2k-ed-2d+1| + 2,$$

and the basic hypergeometric series ${}_{p+1}\phi_p$ is defined by

$${}_{p+1}\phi_p\left[\begin{matrix} a_1, a_2, \ldots, a_{p+1} \\ b_1, b_2, \ldots, b_p \end{matrix}; q, z\right] = \sum_{r=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{p+1}; q)_r}{(q, b_1, b_2, \ldots, b_p; q)_r} z^r.$$

It will also be convenient to use the standard abbreviation

$$\begin{aligned} &{}_{p+1}W_p(a; a_3, a_4, \ldots, a_{p+1}; q; z) \\ &\quad := {}_{p+1}\phi_p\left[\begin{matrix} a, qa^{\frac12}, -qa^{\frac12}, a_3, a_4, \ldots, a_{p+1} \\ a^{\frac12}, -a^{\frac12}, \frac{aq}{a_3}, \frac{aq}{a_4}, \ldots, \frac{aq}{a_{p+1}} \end{matrix}; q, z\right]. \end{aligned}$$

3. Consequences of the SMPBP

3.1. Results of Rogers, Bailey, Dyson, and Slater

Bailey presented five Bailey pairs in total; he lists them as (i)–(v) in [9, pp. 5–6]. These Bailey pairs arise as the following specializations of the SMPBP:

- $(d,e,k) = (1,1,2)$ with $b \to 0$ is equivalent to Bailey's (i),
- $(d,e,k) = (1,2,2)$ is equivalent to (ii),
- $(d,e,k) = (1,3,2)$ with $b \to 0$ is equivalent to (iii),
- $(d,e,k) = (2,1,2)$ is equivalent to (iv), and
- $(d,e,k) = (3,1,4)$ with $b \to 0$ is equivalent to (v).

Bailey appears to have limited himself to these cases since these are the only ones where $\beta_n^{(d,e,k)}(a^e, 0, q^e)$ is representable as a finite product, and thus its insertion into the left-hand side of (1.10) will result in a *single*-fold sum.

The point I wish to emphasize here is that since $\beta_n^{(d,e,k)}(a^e, 0, q^e)$ is a finite product times a very well poised basic hypergeometric series, as long as one is willing to consider multisums, it is a priori plausible that elegant Rogers–Ramanujan type identities may be derivable for *any* triple (d,e,k) of positive integers, as long as one has in hand an appropriate q-hypergeometric transformation formula.

Certain specializations of (d,e,k) yield classical Bailey pairs. The following table summarizes the best known classical results which follow from the SMPBP. The letter–number

(d, e, k)	Bailey pairs	RR type identities
$(1, 1, 1)$	H17	Euler's pentagonal number theorem (1)
$(1, 1, 2)$	B1, B3	Rogers–Ramanujan (18, 14); Göllnitz–Gordon (36, 34); Lebesgue (8, 12)
$(1, 2, 2)$	G1–G3	Rogers–Selberg (31–33); Rogers' mod 5 (19, 15)
$(1, 3, 2)$		Bailey's mod 9 identities (41–43)
$(2, 1, 2)$	C5, C7	Rogers' mod 10 identities (46, 44)
$(2, 1, 3)$	C1–C4	Rogers' mod 14 identities (59–61); Rogers' mod 20 (79)
$(3, 1, 4)$	J1–J6	Dyson's mod 27 identities (90–93); Slater (71–78, 107–116)

codes in the "Bailey pair" column refer to the codes used by Slater in her two papers [23, 24]. The reason that a single specialization of (d, e, k) may correspond to more than one of Slater's Bailey pairs is that she chose to specialize a before performing the required q-hypergeometric summation or transformation, and thus listed what corresponds to our $a = 1, q, q^2$, etc., and linear combinations thereof, as different Bailey pairs. By substituting the Bailey pairs into various limiting cases of Bailey's lemma, a variety of classical identities result. The parenthetical numbers in the third column refer to the numbers in Slater's list [24].

Remark 3.1. In addition to the classical Rogers–Ramanujan type identities mentioned above, certain identities discovered more recently can also be derived from the SMPBP. In particular, the Verma–Jain mod 17 identities [28, (3.1)–(3.8), pp. 247–248] arise from $(d, e, k) = (1, 6, 3)$, the Verma-Jain mod 19 identities [28, (3.9)–(3.17), pp. 248–250] from $(1, 6, 4)$, the Verma–Jain mod 22 identities [28, (3.18)–(3.22), pp. 250–251] from $(2, 1, 5)$, an identity of Ole Warnaar related to the modulus 11 [31, Theorem 1.3; $k = 4$, p. 246] from $(2, 2, 3)$, a mod 13 identity of George Andrews [3, (5.8); $k = 1$, p. 280] from $(2, 2, 4)$, and finally Dennis Stanton's mod 11 identity [27, (6.4), p. 65] from $(1, 2, 4)$.

3.2. New Bailey pairs

Remark 3.2. In [21], I presented a number of Rogers–Ramanujan type identities that arise from what may now be considered the $b = 0$, $e = 1$ case of the SMPBP. The interested reader is invited to consult [21] for additional examples.

We now consider specializations of (d, e, k) in

$$\beta_n^{(d,e,k)}(a^e, 0, q^e) = \sum_{s=0}^{n} \frac{\alpha_s^{(d,e,k)}(a^e, 0, q^e)}{(q^e; q^e)_{n-s}(a^e q^e; q^e)_{n+s}}, \tag{3.1}$$

which lead to new identities.

The most crucial step in each case will be the transformation of the very well poised series via one of the following known formulas:

Transformation 1 (Watson's q-analog of Whipple's theorem, [33], [13, Eq. (III.18), p. 360]).

$${}_8W_7\left(a; b, c, d, e, q^{-n}; q, \frac{a^2 q^{n+2}}{bcde}\right)$$

$$= \frac{(aq, aq/de; q)_n}{(aq/d, aq/e; q)_n} {}_4\phi_3 \left[\begin{matrix} aq/bc, d, e, q^{-n} \\ aq/b, aq/c, deq^{-n}/a \end{matrix} ; q, q \right] \tag{3.2}$$

is used to establish (3.8), (3.10), (3.11), and (3.15).

Transformation 2 (The first Verma–Jain ${}_{10}\phi_9$ transformation, [28, (1.3), p. 232], [13, (3.10.4), p. 97]).

$$\begin{aligned} &{}_{10}W_9\left(a; b, x, -x, y, -y, q^{-n}, -q^{-n}; q, -\frac{a^3q^{2n+3}}{bx^2y^2}\right) \\ &\quad = \frac{(a^2q^2, a^2q^2/x^2y^2; q^2)_n}{(a^2q^2/x^2, a^2q^2/y^2; q^2)_n} {}_5\phi_4 \left[\begin{matrix} q^{-2n}, x^2, y^2, -aq/b, -aq^2/b \\ x^2y^2q^{-2n}/a^2, a^2q^2/b^2, -aq, -aq^2 \end{matrix} ; q^2, q^2 \right] \end{aligned} \tag{3.3}$$

is used to establish (3.13), (3.15), (3.17), and (3.18).

Transformation 3 (The second Verma–Jain ${}_{10}\phi_9$ transformation, [28, (1.4), p. 232]).

$$\begin{aligned} &{}_{10}W_9\left(a; b, x, xq, y, yq, q^{1-n}, q^{-n}; q; \frac{a^3q^{2n+3}}{bx^2y^2}\right) \\ &\quad = \frac{(aq, aq/xy; q)_n}{(aq/x, aq/y; q)_n} {}_5\phi_4 \left[\begin{matrix} x, y, \sqrt{aq/b}, -\sqrt{aq/b}, q^{-n} \\ \sqrt{aq}, -\sqrt{aq}, aq/b, xyq^{-n}/a \end{matrix} ; q, q \right] \end{aligned} \tag{3.4}$$

is used to establish (3.19), (3.20), (3.23), and (3.26).

Transformation 4 (The first Verma–Jain ${}_{12}\phi_{11}$ transformation, [28, (1.4), p. 232]).

$$\begin{aligned} &{}_{12}W_{11}\left(a; x, \omega x, \omega^2 x, y, \omega y, \omega^2 y, q^{-n}, \omega q^{-n}, \omega^2 q^{-n}; q; -\frac{a^4q^{3n+4}}{x^3y^3}\right) \\ &\quad = \frac{(a^3q^3, \frac{a^3q^3}{x^3y^3}; q^3)_n}{(\frac{a^3q^3}{x^3}, \frac{a^3q^3}{y^3}; q^3)_n} {}_6\phi_5 \left[\begin{matrix} q^{-3n}, x^3, y^3, aq, aq^2, aq^3 \\ (aq)^{\frac{3}{2}}, -(aq)^{\frac{3}{2}}, a^{\frac{3}{2}}q^3, -a^{\frac{3}{2}}q^3, \frac{x^3y^3q^{-3n}}{a^3} \end{matrix} ; q^3, q^3 \right], \end{aligned} \tag{3.5}$$

where $\omega = \exp(2\pi i/3)$, is used to establish (3.12).

Transformation 5 (The second Verma–Jain ${}_{12}\phi_{11}$ transformation, [28, (1.5), p. 232]).

$$\begin{aligned} &{}_{12}W_{11}\left(a; x, xq, xq^2, y, yq, yq^2, q^{2-n}, q^{1-n}, q^{-n}; q, \frac{a^4q^{3n+3}}{x^3y^3}\right) \\ &\quad = \frac{(aq, aq/xy; q)_n}{(aq/x, aq/y; q)_n} {}_6\phi_5 \left[\begin{matrix} \sqrt[3]{a}, \omega\sqrt[3]{a}, \omega^2\sqrt[3]{a}, x, y, q^{-n} \\ \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, xyq^{-n}/a \end{matrix} ; q, q \right] \end{aligned} \tag{3.6}$$

is used to establish (3.24) and (3.25).

Finally,

Transformation 6 (Transformation of a very well poised ${}_8\phi_7$, [13, Eq. (3.4.7), reversed, p. 76]).

$$
\begin{aligned}
&{}_8W_7\left(a; y^{\frac{1}{2}}, -y^{\frac{1}{2}}, (yq)^{\frac{1}{2}}, -(yq)^{\frac{1}{2}}, x; q; \frac{a^2q}{y^2x}\right)\\
&\quad = \frac{(aq, \frac{a^2q}{y^2}; q)_\infty}{(\frac{aq}{y}, \frac{a^2q}{y}; q)_\infty}\, {}_2\phi_1\left[\begin{matrix} y, \frac{xy}{a} \\ \frac{aq}{x} \end{matrix}; q, \frac{a^2q}{y^2x}\right]
\end{aligned}
\tag{3.7}
$$

is used to establish (3.21) and (3.22).

Let $(d, e, k) = (1, 2, 3)$. Then

$$
\begin{aligned}
&\beta_n^{(1,2,3)}(a^2, 0, q^2)\\
&\quad = \sum_{r=0}^{n} \frac{\alpha_r^{(1,2,3)}(a^2, 0, q^2)}{(q^2; q^2)_{n-r}(a^2q^2; q^2)_{n+r}}\\
&\quad = \frac{1}{(q^2; q^2)_n(a^2q^2; q^2)_n} \sum_{r=0}^{n} \frac{(-1)^r q^{2nr-r^2+r}(q^{-2n}; q^2)_r}{(a^2q^{2(n+1)}; q^2)_r}\alpha_r^{(1,2,3)}(a^2, 0, q^2)\\
&\quad = \frac{1}{(q^2; q^2)_n(a^2q^2; q^2)_n} \sum_{r=0}^{n} \frac{(a, q\sqrt{a}, -q\sqrt{a}, q^{-n}, -q^{-n}; q)_r a^{2r} q^{2nr+\frac{3}{2}r^2+\frac{r}{2}}}{(q, \sqrt{a}, -\sqrt{a}, aq^{n+1}, -aq^{n+1}; q)_r}\\
&\quad = \frac{1}{(q^2; q^2)_n(a^2q^2; q^2)_n}\\
&\qquad \times \lim_{\tau\to 0} {}_8\phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, q^{-n}, -q^{-n}, q/\tau, 1/\tau, q/\tau \\ \sqrt{a}, -\sqrt{a}, aq^{n+1}, -aq^{n+1}, \tau a, \tau aq, \tau a \end{matrix}; q, -a^2\tau^3q^{2n}\right]\\
&\quad = \lim_{\tau\to 0} \frac{(aq; q)_n(\tau^2 a; q)_n}{(\tau a; q)_n(\tau aq; q)_n(q^2; q^2)_n(a^2q^2; q^2)_n}\\
&\qquad \times {}_4\phi_3\left[\begin{matrix} -\tau aq^n, q/\tau, 1/\tau, q^{-n} \\ -aq^{n+1}, \tau a, q^{1-n}a\tau^2 \end{matrix}; q, q\right] \quad \big(\text{by (3.2)}\big)\\
&\quad = \frac{1}{(-q; q)_n} \sum_{r\geqq 0} \frac{a^r q^{r^2}}{(q; q)_r(q; q)_{n-r}(-aq; q)_{n+r}}.
\end{aligned}
$$

Thus, we have established

$$
\beta_n^{(1,2,3)}(a^2, 0, q^2) = \frac{1}{(-q; q)_n} \sum_{r\geqq 0} \frac{a^r q^{r^2}}{(q; q)_r(q; q)_{n-r}(-aq; q)_{n+r}}. \tag{3.8}
$$

Via analogous calculations, one can establish each of the following:

$$
\beta_n^{(1,2,4)}(a^2, 0, q^2) = \sum_{r\geqq 0} \frac{a^{2r} q^{2r^2}}{(-aq; q)_{2r}(q^2; q^2)_r(q^2; q^2)_{n-r}}, \tag{3.9}
$$

$$\beta_n^{(1,3,1)}(a^3,0,q^3) = \frac{(-1)^n a^{-n} q^{-\binom{n+1}{2}}(q;q)_n}{(q^3;q^3)_n (aq;q)_{2n}} \times \sum_{r \geqq 0} \frac{(-1)^r q^{\binom{r+1}{2}-nr}(aq;q)_{n+r}(aq;q)_{2n+r}}{(a^3q^3;q^3)_{n+r}(q;q)_r(q;q)_{n-r}}, \tag{3.10}$$

$$\beta_n^{(1,3,3)}(a^3,0,q^3) = \frac{(aq;q)_n}{(aq;q)_{2n}(q^3;q^3)_n} \sum_{r \geqq 0} \frac{a^r q^{r^2}(aq;q)_{2n+r}(aq;q)_{n+r}}{(a^3q^3;q^3)_{n+r}(q;q)_r(q;q)_{n-r}}, \tag{3.11}$$

$$\beta_n^{(1,3,5)}(a^3,0,q^3) = \sum_{r \geqq 0} \frac{a^{3r} q^{3r^2}(aq;q)_{3r}}{(a^3q^3;q^3)_{2r}(q^3;q^3)_r(q^3;q^3)_{n-r}}, \tag{3.12}$$

$$\beta_n^{(1,4,1)}(a^4,0,q^4) = \frac{(-1)^n q^{2n^2}}{(-a^2q^2;q^2)_{2n}} \sum_{r \geqq 0} \frac{q^{3r^2-4nr}}{(q^2;q^2)_r(-aq;q)_{2r},(q^4;q^4)_{n-r}}, \tag{3.13}$$

$$\beta_n^{(1,4,2)}(a^4,0,q^4) = \frac{i^n q^{n^2}(iq;q)_n(q;q)_n}{(q^4;q^4)_n(iaq;q)_{2n}^2} \times \sum_{r \geqq 0} \frac{(-i)^r q^{r^2-2nr}(iaq;q)_{2n+r}}{(q;q)_r(-iaq;q)_{n+r}(-aq;q)_{n+r}(q;q)_{n-r}(iq;q)_{n-r}} \tag{3.14}$$

(where here and throughout, $i = \sqrt{-1}$),

$$\beta_n^{(1,4,3)}(a^4,0,q^4) = \frac{(iq;q)_n(q;q)_n}{(q^4;q^4)_n(iaq;q)_{2n}^2} \times \sum_{r \geqq 0} \frac{a^r q^{r^2}(iaq;q)_{2n+r}}{(q;q)_r(-iaq;q)_{n+r}(-aq;q)_{n+r}(q;q)_{n-r}(iq;q)_{n-r}}, \tag{3.15}$$

$$\beta_n^{(1,4,4)}(a^4,0,q^4) = \frac{1}{(-a^2q^2;q^2)_{2n}} \sum_{r \geqq 0} \frac{a^{2r} q^{2r^2}}{(q^2;q^2)_r(-aq;q)_{2r},(q^4;q^4)_{n-r}}, \tag{3.16}$$

$$\beta_n^{(1,6,3)}(a^6,0,q^6) = \frac{1}{(a^6q^6;q^6)_{2n}} \sum_{r \geqq 0} \frac{(-1)^r a^{2r} q^{3r^2}(a^2q^2;q^2)_{3n-r}}{(q^2;q^2)_r(-aq;q)_{2r}(q^6;q^6)_{n-r}}, \tag{3.17}$$

$$\beta_n^{(1,6,4)}(a^6,0,q^6) = \frac{1}{(a^6q^6;q^6)_{2n}} \sum_{r \geqq 0} \frac{a^{2r} q^{2r^2}(a^2q^2;q^2)_{3n-r}}{(q^2;q^2)_r(-aq;q)_{2r}(q^6;q^6)_{n-r}}, \tag{3.18}$$

$$\beta_n^{(2,1,5)}(a,0,q) = \sum_{r \geqq 0} \frac{a^r q^{r^2}}{(q;q)_r(aq;q^2)_r(q;q)_{n-r}}, \tag{3.19}$$

$$\beta_n^{(2,2,2)}(a^2,0,q^2)=\frac{(-1)^n q^{n^2}}{(-aq;q)_{2n}}\sum_{r\geqq 0}\frac{(-1)^r q^{\frac{3}{2}r^2-\frac{1}{2}r-2nr}}{(aq;q^2)_r(q;q)_r(q^2;q^2)_{n-r}}, \tag{3.20}$$

$$\beta_n^{(2,2,3)}(a^2,0,q^2)=\frac{(aq^2;q^2)_n}{(a^2q^2;q^2)_{2n}}\sum_{r\geqq 0}\frac{a^r q^{2nr}}{(q^2;q^2)_r(q^2;q^2)_{n-r}}, \tag{3.21}$$

$$\beta_n^{(2,2,4)}(a^2,0,q^2)=\frac{(aq^2;q^2)_n}{(a^2q^2;q^2)_{2n}}\sum_{r\geqq 0}\frac{a^r q^{2r^2}}{(q^2;q^2)_r(q^2;q^2)_{n-r}}, \tag{3.22}$$

$$\beta_n^{(2,2,5)}(a^2,0,q^2)=\frac{1}{(-aq;q)_{2n}}\sum_{r\geqq 0}\frac{a^r q^{r^2}}{(q;q)_r(aq;q^2)_r(q^2;q^2)_{n-r}}, \tag{3.23}$$

$$\beta_n^{(3,2,7)}(a^2,0,q^2)=\frac{1}{(-aq;q)_{2n}}\sum_{r\geqq 0}\frac{a^r q^{r^2}(a;q^3)_r}{(a;q)_{2r}(q;q)_r(q^2;q^2)_{n-r}}, \tag{3.24}$$

$$\beta_n^{(3,3,7)}(a^3,0,q^3)=\frac{1}{(a^3q^3;q^3)_{2n}}\sum_{r\geqq 0}\frac{a^r q^{3n^2+r}(a;q^3)_r(aq;q)_{3n-r}}{(q;q)_r(a;q)_{2r}(q^3;q^3)_{n-r}}, \tag{3.25}$$

$$\beta_n^{(4,1,7)}(a,0,q)=\frac{1}{(aq;q^2)_n}\sum_{r\geqq 0}\frac{a^r q^{2r^2}}{(q^2;q^2)_r(aq^2;q^4)_r(q;q)_{n-2r}}. \tag{3.26}$$

4. A list of Rogers–Ramanujan–Bailey type identities

For easy reference, we restate the Bailey lemma with the SMPBP inserted:

Theorem 4.1 (Bailey's lemma).

$$\begin{aligned}
&\frac{1}{(\frac{a^eq^e}{\rho_1^e};q^e)_N(\frac{a^eq^e}{\rho_2^e};q^e)_N}\\
&\quad\times\sum_{j\geqq 0}\frac{(\rho_1^e;q^e)_j(\rho_2^e;q^e)_j((\frac{aq}{\rho_1\rho_2})^e;q^e)_{N-j}}{(q^e;q^e)_{N-j}}\left(\frac{aq}{\rho_1\rho_2}\right)^{ej}\beta_j^{(d,e,k)}(a^e,b^e,q^e)\\
&\quad=\sum_{r=0}^{\lfloor N/d\rfloor}\frac{(\rho_1^e;q^e)_{dr}(\rho_2^e;q^e)_{dr}}{((\frac{aq}{\rho_1})^e;q^e)_{dr}((\frac{aq}{\rho_2})^e;q^e)_{dr}(q^e;q^e)_{N-dr}(a^eq^e;q^e)_{N+dr}}\left(\frac{aq}{\rho_1\rho_2}\right)^{der}\\
&\quad\times\alpha_{dr}^{(d,e,k)}(a^e,b^e,q^e).
\end{aligned}\tag{4.1}$$

Note that the rather general identities presented by Bailey as Eqs. (6.1)–(6.4) in [9, p. 6], from which all of the other identities in [8,9] can be derived, are simply cases $(d,e,k)=(1,2,2)$, $(1,3,2)$, $(2,1,2)$, and $(3,1,4)$, respectively, of (4.1).

I have compiled a list of Rogers–Ramanujan type identities which I believe to be new. Each is a direct consequence of (4.1), with parameters specialized as indicated in brackets. Note that the final form of the sum side for many of the identities was obtained only after reversing the order of summation and replacing n by $n+r$. Of course, in order to obtain each infinite product representation, Jacobi's triple product identity [13, Eq. (1.6.1), p. 15] is applied to the right-hand side after a is specialized.

This list is by no means exhaustive; I have merely chosen some examples to illustrate the power of the SMPBP.

Remark 4.2. The identity which arises from inserting a given $(\alpha_n^{(d,e,k)}(a,q), \beta_n^{(d,e,k}(a,q))$ into (4.1) in the case where $\rho_1, \rho_2, N \to \infty$, and $a=1$ is just one of a set of $d(e-1)+k$ identities. The other identities can be found via a system of q-difference equations. This phenomenon is explored in [22].

$$\sum_{n,r\geqq 0} \frac{q^{n^2+2nr+2r^2}(-q;q^2)_{n+r}}{(-q;q)_{n+r}(-q;q)_{n+2r}(q;q)_n(q;q)_r} = \frac{(q^3,q^4,q^7;q^7)_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty} \tag{4.2}$$

$(N, \rho_1 \to \infty, \rho_2 = -\sqrt{q}, a=1, b\to 0, (d,e,k)=(1,2,3))$,

$$\sum_{n,r\geqq 0} \frac{(-1)^n q^{\frac{5}{2}n^2-\frac{1}{2}n+4nr+2r^2}(q;q)_{n+r}(q;q)_{n+2r}(q;q)_{2n+3r}}{(q^3;q^3)_{n+r}(q^3;q^3)_{n+2r}(q;q)_n(q;q)_r(q;q)_{2n+2r}}$$
$$= \frac{(q^3,q^4,q^7;q^7)_\infty}{(q^3;q^3)_\infty} \tag{4.3}$$

$(N, \rho_1, \rho_2 \to \infty, a=1, b\to 0, (d,e,k)=(1,3,1))$,

$$\sum_{n,r\geqq 0} \frac{(-1)^n q^{2n^2-n+2nr+r^2}(-q^3;q^6)_{n+r}(q^2;q^2)_{n+r}(q^2;q^2)_{n+2r}(q^2;q^2)_{2n+3r}}{(q^6;q^6)_{n+r}(q^6;q^6)_{n+2r}(q^2;q^2)_n(q^2;q^2)_r(q^2;q^2)_{2n+2r}}$$
$$= \frac{(q^3,q^5,q^8;q^8)_\infty(-q^3;q^6)_\infty}{(q^6;q^6)_\infty} \tag{4.4}$$

$(N, \rho_1 \to \infty, \rho_2 = -\sqrt{q}, a=1, b\to 0, (d,e,k)=(1,3,1))$,

$$\sum_{n,r\geqq 0} \frac{q^{2n^2+2n+4nr+3r^2+3r}}{(-q;q)_{n+r}(-q;q)_{n+2r+1}(q;q)_n(q;q)_r} = \frac{(q,q^8,q^9;q^9)_\infty}{(q^2;q^2)_\infty} \tag{4.5}$$

$(N, \rho_1, \rho_2 \to \infty, a=q, b\to 0, (d,e,k)=(1,2,3))$,

$$\sum_{n,r\geqq 0} \frac{q^{2n^2+4nr+3r^2}}{(-q;q)_{n+r}(-q;q)_{n+2r}(q;q)_n(q;q)_r} = \frac{(q^4,q^5,q^9;q^9)_\infty}{(q^2;q^2)_\infty} \tag{4.6}$$

$(N, \rho_1, \rho_2 \to \infty, a=1, b\to 0, (d,e,k)=(1,2,3))$,

$$\sum_{n,r\geqq 0} \frac{q^{n^2+2nr+3r^2}(-q;q^2)_{n+r}}{(-q;q)_{2r}(q^2;q^2)_n(q^2;q^2)_r} = \frac{(q^4,q^5,q^9;q^9)_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty} \tag{4.7}$$

$(N, \rho_1 \to \infty,\ \rho_2 = -\sqrt{q},\ a = 1,\ b \to 0,\ (d, e, k) = (1, 2, 4))$,

$$\sum_{n,r \geqq 0} \frac{(-1)^{n+r} q^{6n^2+8nr+5r^2+4n+4r}}{(-q^2;q^2)_{2n+2r+1}(q^2;q^2)_r(-q;q)_{2r+1}(q^4;q^4)_n} = \frac{(q, q^8, q^9; q^9)_\infty}{(q^4;q^4)_\infty} \tag{4.8}$$

$(N, \rho_1, \rho_2 \to \infty,\ a = q,\ b \to 0,\ (d, e, k) = (1, 4, 1))$,

$$\sum_{n,r \geqq 0} \frac{(-1)^{n+r} q^{6n^2+8nr+5r^2}}{(-q^2;q^2)_{2n+2r}(q^2;q^2)_r(-q;q)_{2r}(q^4;q^4)_n} = \frac{(q^4, q^5, q^9; q^9)_\infty}{(q^4;q^4)_\infty} \tag{4.9}$$

$(N, \rho_1, \rho_2 \to \infty,\ a = 1,\ b \to 0,\ (d, e, k) = (1, 4, 1))$,

$$\sum_{n,r \geqq 0} \frac{(-1)^n q^{2n^2+2nr+\frac{3}{2}r^2-\frac{r}{2}}(-q;q^2)_{n+r}}{(-q;q)_{2n+2r}(q;q)_r(q;q^2)_r(q^2;q^2)_n}$$
$$= \frac{(q^4, q^6, q^{10}; q^{10})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty} \tag{4.10}$$

$(N, \rho_1 \to \infty,\ \rho_2 = -\sqrt{q},\ a = 1,\ b \to 0,\ (d, e, k) = (2, 2, 2))$,

$$\sum_{n,r \geqq 0} \frac{(-1)^{n+r} q^{8n^2+8nr+6r^2}(-q^4;q^8)_{n+r}}{(-q^4;q^4)_{2n+2r}(q^4;q^4)_r(-q^2;q^2)_{2r}(q^8;q^8)_n}$$
$$= \frac{(q^4, q^6, q^{10}; q^{10})_\infty(-q^4;q^8)_\infty}{(q^8;q^8)_\infty} \tag{4.11}$$

$(N, \rho_1 \to \infty,\ \rho_2 = -\sqrt{q},\ a = 1,\ b \to 0,\ (d, e, k) = (1, 4, 1))$,

$$\sum_{n,r \geqq 0} \frac{q^{n^2+2r^2+3nr+2n+3r}(q;q)_{n+r+1}}{(q;q)_{2n+2r+2}(q;q)_r(q;q)_n} = \frac{(q, q^{10}, q^{11}; q^{11})_\infty}{(q;q)_\infty} \tag{4.12}$$

$(N, \rho_1, \rho_2 \to \infty,\ a = q,\ b \to 0,\ (d, e, k) = (2, 2, 3))$,

$$\sum_{n,r \geqq 0} \frac{q^{n^2+2r^2+3nr}(q;q)_{n+r}}{(q;q)_{2n+2r}(q;q)_r(q;q)_n} = \frac{(q^5, q^6, q^{11}; q^{11})_\infty}{(q;q)_\infty} \tag{4.13}$$

$(N, \rho_1, \rho_2 \to \infty,\ a = 1,\ b \to 0,\ (d, e, k) = (2, 2, 3)$, due to S.O. Warnaar [31, Theorem 1.3 with $k = 4$, p. 247]),

$$\sum_{n,r \geqq 0} \frac{q^{2n^2+2n+4nr+4r^2+4r}}{(-q;q)_{2r+1}(q^2;q^2)_n(q^2;q^2)_r} = \frac{(q, q^{10}, q^{11}; q^{11})_\infty}{(q^2;q^2)_\infty} \tag{4.14}$$

$(N, \rho_1, \rho_2 \to \infty,\ a = q,\ b \to 0,\ (d, e, k) = (1, 2, 4))$,

$$\sum_{n,r \geqq 0} \frac{q^{3n^2+6nr+4r^2+3n+4r}(q;q)_{n+r}(q;q)_{2n+3r+1}(q;q)_{n+2r+1}}{(q^3;q^3)_{n+r}(q^3;q^3)_{n+2r+1}(q;q)_n(q;q)_r(q;q)_{2n+2r+1}}$$
$$= \frac{(q, q^{10}, q^{11}; q^{11})_\infty}{(q^3;q^3)_\infty} \tag{4.15}$$

$(N, \rho_1, \rho_2 \to \infty, a = q, b \to 0, (d, e, k) = (1, 3, 3))$,

$$\sum_{n,r \geqq 0} \frac{q^{3n^2+6nr+4r^2}(q;q)_{n+r}(q;q)_{2n+3r}(q;q)_{n+2r}}{(q^3;q^3)_{n+r}(q^3;q^3)_{n+2r}(q;q)_n(q;q)_r(q;q)_{2n+2r}} = \frac{(q^5,q^6,q^{11};q^{11})_\infty}{(q^3;q^3)_\infty} \tag{4.16}$$

$(N, \rho_1, \rho_2 \to \infty, a = 1, b \to 0, (d, e, k) = (1, 3, 3))$,

$$\sum_{n,r \geqq 0} \frac{i^n q^{5n^2+8nr+4r^2}(iq;q)_{n+r}(q;q)_{n+r}(iq;q)_{2n+3r}}{(q^4;q^4)_{n+r}(iq;q)^2_{2n+2r}(q;q)_r(-iq;q)_{n+2r}(-q;q)_{n+2r}(q;q)_n(iq;q)_n} = \frac{(q^5,q^6,q^{11};q^{11})_\infty}{(q^4;q^4)_\infty} \tag{4.17}$$

$(N, \rho_1, \rho_2 \to \infty, a = 1, b \to 0, (d, e, k) = (1, 4, 2))$. Note: This is a different series expansion of the infinite product considered by Andrews [2, (1.10), p. 332].

$$\sum_{n,r \geqq 0} \frac{q^{2n^2+4nr+4r^2}(-q^2;q^4)_{n+r}}{(-q^2;q^2)_{2n+2r}(q^2;q^2)_r(-q;q)_{2r}(q^4;q^4)_n} = \frac{(q^5,q^6,q^{11};q^{11})_\infty(-q^2;q^4)_\infty}{(q^4;q^4)_\infty} \tag{4.18}$$

$(N, \rho_1 \to \infty, \rho_2 = -\sqrt{q}, a = 1, b \to 0, (d, e, k) = (1, 4, 4))$,

$$\sum_{n,r \geqq 0} \frac{(-1)^r q^{3n^2+6nr+6r^2}(-q^3;q^6)_{n+r}(q^2;q^2)_{3n+2r}}{(q^6;q^6)_{2n+2r}(q^2;q^2)_r(-q;q)_{2r}(q^6;q^6)_n} = \frac{(q^5,q^6,q^{11};q^{11})_\infty(-q^3;q^6)_\infty}{(q^6;q^6)_\infty} \tag{4.19}$$

$(N, \rho_1 \to \infty, \rho_2 = -\sqrt{q}, a = 1, b \to 0, (d, e, k) = (1, 6, 3))$,

$$\sum_{n,r \geqq 0} \frac{q^{n^2+2r^2+2nr+2n+3r}(q;q)_{n+r+1}}{(q;q)_{2n+2r+2}(q;q)_r(q;q)_n} = \frac{(q,q^{12},q^{13};q^{13})_\infty}{(q;q)_\infty} \tag{4.20}$$

$(N, \rho_1, \rho_2 \to \infty, a = q, b \to 0, (d, e, k) = (2, 2, 4))$,

$$\sum_{n,r \geqq 0} \frac{q^{n^2+2r^2+2nr}(q;q)_{n+r}}{(q;q)_{2n+2r}(q;q)_r(q;q)_n} = \frac{(q^6,q^7,q^{13};q^{13})_\infty}{(q;q)_\infty} \tag{4.21}$$

$(N, \rho_1, \rho_2 \to \infty, a = 1, b \to 0, (d, e, k) = (2, 2, 4)$, due to G.E. Andrews [3, Eq. (5.8) with $k = 1$, p. 280]),

$$\sum_{n,r\geqq 0} \frac{q^{4n^2+8nr+5r^2}(iq;q)_{n+r}(q;q)_{n+r}(iq;q)_{2n+3r}}{(q^4;q^4)_{n+r}(iq;q)^2_{2n+2r}(q;q)_r(-iq;q)_{n+2r}(-q;q)_{n+2r}(q;q)_n(iq;q)_n}$$
$$= \frac{(q^6,q^7,q^{13};q^{13})_\infty}{(q^4;q^4)_\infty} \tag{4.22}$$

$(N, \rho_1, \rho_2 \to \infty,\ a = 1,\ b \to 0,\ (d,e,k) = (1,4,3))$,

$$\sum_{n,r\geqq 0} \frac{q^{3n^2+6nr+5r^2}(-q^3;q^6)_{n+r}(q^2;q^2)_{3n+2r}}{(q^6;q^6)_{2n+2r}(q^2;q^2)_r(-q;q)_{2r}(q^6;q^6)_n}$$
$$= \frac{(q^6,q^7,q^{13};q^{13})_\infty(-q^3;q^6)_\infty}{(q^6;q^6)_\infty} \tag{4.23}$$

$(N, \rho_1 \to \infty,\ \rho_2 = -\sqrt{q},\ a = 1,\ b \to 0,\ (d,e,k) = (1,6,4))$,

$$\sum_{n,r\geqq 0} \frac{q^{n^2+3r^2+4nr}(-q;q^2)_{n+r}(q^2;q^2)_{n+r}}{(q^2;q^2)_{2n+2r}(q^2;q^2)_r(q^2;q^2)_n} = \frac{(q^6,q^8,q^{14};q^{14})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty} \tag{4.24}$$

$(N, \rho_1 \to \infty,\ a = 1,\ \rho_2 = -\sqrt{q},\ b \to 0,\ (d,e,k) = (2,2,3))$,

$$\sum_{n,r\geqq 0} \frac{i^n q^{6n^2+8nr+4r^2}(-q^4;q^8)_{n+r}(iq^2;q^2)_{n+r}(q^2;q^2)_{n+r}(iq^2;q^2)_{2n+3r}}{(q^8;q^8)_{n+r}(iq^2;q^2)^2_{2n+2r}(q^2;q^2)_r(-iq^2;q^2)_{n+2r}(-q^2;q^2)_{n+2r}(q^2;q^2)_n(iq^2;q^2)_n}$$
$$= \frac{(q^6,q^8,q^{14};q^{14})_\infty(-q^4;q^8)_\infty}{(q^8;q^8)_\infty} \tag{4.25}$$

$(N, \rho_1, \to \infty,\ \rho_2 = -\sqrt{q},\ a = 1,\ b \to 0,\ (d,e,k) = (1,4,2))$,

$$\sum_{n,r\geqq 0} \frac{q^{3n^2+6nr+6r^2+3n+6r}(q;q)_{3r+1}}{(q^3;q^3)_{2r+1}(q^3;q^3)_r(q^3;q^3)_n} = \frac{(q,q^{14},q^{15};q^{15})_\infty}{(q^3;q^3)_\infty} \tag{4.26}$$

$(N, \rho_1, \rho_2 \to \infty,\ a = q,\ b \to 0,\ (d,e,k) = (1,3,5))$,

$$\sum_{n,r\geqq 0} \frac{q^{3n^2+6nr+6r^2}(q;q)_{3r}}{(q^3;q^3)_{2r}(q^3;q^3)_r(q^3;q^3)_n} = \frac{(q^7,q^8,q^{15};q^{15})_\infty}{(q^3;q^3)_\infty} \tag{4.27}$$

$(N, \rho_1, \rho_2 \to \infty,\ a = 1,\ b \to 0,\ (d,e,k) = (1,3,5))$,

$$\sum_{n,r\geqq 0} \frac{q^{4n^2+8nr+6r^2+4n+6r}}{(-q^2;q^2)_{2n+2r+1}(q^2;q^2)_r(-q;q)_{2r+1}(q^4;q^4)_n}$$
$$= \frac{(q,q^{14},q^{15};q^{15})_\infty}{(q^4;q^4)_\infty} \tag{4.28}$$

$(N, \rho_1, \rho_2 \to \infty,\ a = q,\ b \to 0,\ (d,e,k) = (1,4,4))$,

$$\sum_{n,r\geqq 0} \frac{q^{4n^2+8nr+6r^2}}{(-q^2;q^2)_{2n+2r}(q^2;q^2)_r(-q;q)_{2r}(q^4;q^4)_n} = \frac{(q^7,q^8,q^{15};q^{15})_\infty}{(q^4;q^4)_\infty} \tag{4.29}$$

$(N, \rho_1, \rho_2 \to \infty,\ a = 1,\ b \to 0,\ (d, e, k) = (1, 4, 4))$,

$$\sum_{n,r \geqq 0} \frac{q^{3n^2+6nr+5r^2}(-q^3; q^6)_{n+r}(q^2; q^2)_{n+r}(q^2; q^2)_{2n+3r}(q^2; q^2)_{n+2r}}{(q^6; q^6)_{n+r}(q^6; q^6)_{n+2r}(q^2; q^2)_n(q^2; q^2)_r(q^2; q^2)_{2n+2r}} = \frac{(q^7, q^9, q^{16}; q^{16})_\infty(-q^3; q^6)_\infty}{(q^6; q^6)_\infty} \tag{4.30}$$

$(N, \rho_1 \to \infty,\ \rho_2 = -\sqrt{q},\ a = 1,\ b \to 0,\ (d, e, k) = (1, 3, 3))$,

$$\sum_{n,r \geqq 0} \frac{(-1)^n q^{3n^2+4n+4nr+\frac{5}{2}r^2+\frac{7}{2}r}}{(-q; q)_{2n+2r+2}(q; q)_r(q; q^2)_{r+1}(q^2; q^2)_n} = \frac{(q^2, q^{16}, q^{18}; q^{18})_\infty}{(q^2; q^2)_\infty} \tag{4.31}$$

$(N, \rho_1, \rho_2 \to \infty,\ a = q^2,\ b \to 0,\ (d, e, k) = (2, 2, 2))$,

$$\sum_{n,r \geqq 0} \frac{(-1)^n q^{3n^2+4nr+\frac{5}{2}r^2-\frac{r}{2}}}{(-q; q)_{2n+2r}(q; q)_r(q; q^2)_r(q^2; q^2)_n} = \frac{(q^8, q^{10}, q^{18}; q^{18})_\infty}{(q^2; q^2)_\infty} \tag{4.32}$$

$(N, \rho_1, \rho_2 \to \infty,\ a = 1,\ b \to 0,\ (d, e, k) = (2, 2, 2))$,

$$\sum_{n,r \geqq 0} \frac{q^{4n^2+8nr+6r^2}(-q^4;q^8)_{n+r}(iq^2;q^2)_{n+r}(q^2;q^2)_{n+r}(iq^2;q^2)_{2n+3r}}{(q^8;q^8)_{n+r}(iq^2;q^2)^2_{2n+2r}(q^2;q^2)_r(-iq^2;q^2)_{n+2r}(-q^2;q^2)_{n+2r}(q^2;q^2)_n(iq^2;q^2)_n} = \frac{(q^8, q^{10}, q^{18}; q^{18})_\infty(-q^4; q^8)_\infty}{(q^8; q^8)_\infty} \tag{4.33}$$

$(N, \rho_1 \to \infty,\ \rho_2 = -\sqrt{q},\ a = 1,\ b \to 0,\ (d, e, k) = (1, 4, 3))$,

$$\sum_{n,r \geqq 0} \frac{q^{n^2+2nr+3r^2}(-q; q^2)_{n+r}(q^2; q^2)_{n+r}}{(q^2; q^2)_{2n+2r}(q^2; q^2)_r(q^2; q^2)_n} = \frac{(q^8, q^{10}, q^{18}; q^{18})_\infty(-q; q^2)_\infty}{(q^2; q^2)_\infty} \tag{4.34}$$

$(N, \rho_1, \to \infty,\ \rho_2 = -\sqrt{q},\ a = 1,\ b \to 0,\ (d, e, k) = (2, 2, 4))$,

$$\sum_{n,r \geqq 0} \frac{q^{\frac{1}{2}n^2+\frac{1}{2}n+nr+\frac{3}{2}r^2+\frac{1}{2}r}(-1; q)_{n+r}}{(q; q)_n(q; q)_r(q; q^2)_r} = \frac{(q^9, q^9, q^{18}; q^{18})_\infty(-q; q)_\infty}{(q; q)_\infty} \tag{4.35}$$

$(N, \rho_1 \to \infty,\ \rho_2 = -1,\ a = 1,\ b \to 0,\ (d, e, k) = (2, 1, 5))$,

$$\sum_{n,r \geqq 0} \frac{q^{n^2+2r^2+2nr}(-q; q^2)_{n+r}}{(-q; q)_{2n+2r}(q; q^2)_r(q; q)_r(q; q)_n} = \frac{(q^{10}, q^{12}, q^{22}; q^{22})_\infty(-q; q^2)_\infty}{(q^2; q^2)_\infty} \tag{4.36}$$

$(N, \rho_1 \to \infty,\ \rho_2 = -\sqrt{q},\ a = 1,\ b \to 0,\ (d, e, k) = (2, 2, 5))$,

$$\sum_{n,r\geqq 0} \frac{q^{3n^2+6nr+9r^2}(-q^3;q^6)_{n+r}(q^2;q^2)_{3r}}{(q^6;q^6)_{2r}(q^6;q^6)_r(q^6;q^6)_n} = \frac{(q^{11},q^{13},q^{24};q^{24})_\infty(-q^3;q^6)_\infty}{(q^6;q^6)_\infty} \tag{4.37}$$

$(N, \rho_1, \to \infty,\ \rho_2 = -\sqrt{q},\ a = 1,\ b \to 0,\ (d,e,k) = (1,3,5))$,

$$\sum_{n,r\geqq 0} \frac{q^{2n^2+3r^2+4nr+4n+6r}}{(-q;q)_{2n+2r+2}(q;q^2)_{r+1}(q;q)_r(q^2;q^2)_n} = \frac{(q^2,q^{28},q^{30};q^{30})_\infty}{(q^2;q^2)_\infty} \tag{4.38}$$

$(N, \rho_1, \rho_2 \to \infty,\ a = q^2,\ b \to 0,\ (d,e,k) = (2,2,5))$,

$$\sum_{n,r\geqq 0} \frac{q^{2n^2+3r^2+4nr}}{(-q;q)_{2n+2r}(q;q^2)_r(q;q)_r(q^2;q^2)_n} = \frac{(q^{14},q^{16},q^{30};q^{30})_\infty}{(q^2;q^2)_\infty} \tag{4.39}$$

$(N, \rho_1, \rho_2 \to \infty,\ a = 1,\ b \to 0,\ (d,e,k) = (2,2,5))$,

$$\sum_{n,r\geqq 0} \frac{q^{n^2+2nr+3r^2}(-q;q^2)_{n+r}}{(q^2;q^2)_n(q^2;q^2)_r(q^2;q^4)_r} = \frac{(q^{16},q^{20},q^{36};q^{36})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty} \tag{4.40}$$

$(N, \rho_1 \to \infty,\ \rho_2 = -\sqrt{q},\ a = 1,\ b \to 0,\ (d,e,k) = (2,1,5))$,

$$\sum_{n,r\geqq 0} \frac{q^{n(n+1)/2+2r^2}(-1;q)_n}{(q;q^2)_n(q^2;q^2)_r(q^2,q^4)_r(q;q)_{n-2r}} = \frac{(q^{22},q^{22},q^{44};q^{44})_\infty(-q;q)_\infty}{(q;q)_\infty} \tag{4.41}$$

$(N, \rho_1 \to \infty,\ \rho_2 = -1,\ a = 1,\ b \to 0,\ (d,e,k) = (4,1,7))$,

$$\sum_{n,r\geqq 0} \frac{q^{n^2+2r^2+4n+4r}}{(q;q^2)_{n+2}(q^2;q^2)_r(q^2,q^4)_{r+1}(q;q)_{n-2r}} = \frac{(q^4,q^{56},q^{60};q^{60})_\infty}{(q;q)_\infty} \tag{4.42}$$

$(N, \rho_1, \rho_2 \to \infty,\ a = q^4,\ b \to 0,\ (d,e,k) = (4,1,7))$,

$$\sum_{n,r\geqq 0} \frac{q^{n^2+2r^2}}{(q;q^2)_n(q^2;q^2)_r(q^2,q^4)_r(q;q)_{n-2r}} = \frac{(q^{28},q^{32},q^{60};q^{60})_\infty}{(q;q)_\infty} \tag{4.43}$$

$(N, \rho_1, \rho_2 \to \infty,\ a = 1,\ b \to 0,\ (d,e,k) = (4,1,7))$,

$$\sum_{n,r\geqq 0} \frac{q^{2n^2+4nr+3r^2+6n+9r}(q^3;q^3)_r}{(-q;q)_{2n+2r+3}(q;q)_{2r+2}(q;q)_r(q^2;q^2)_n} = \frac{(q^3,q^{60},q^{63};q^{63})_\infty}{(q^2;q^2)_\infty} \tag{4.44}$$

$(N, \rho_1, \rho_2 \to \infty,\ a = q^3,\ b \to 0,\ (d,e,k) = (3,2,7))$,

$$\sum_{n,r\geqq 0} \frac{q^{3n^2+4r^2+6nr+9n+12r}(q^3;q^3)_r(q;q)_{3n+2r+3}}{(q^3;q^3)_{2n+2r+3}(q;q)_r(q;q)_{2r+2}(q^3;q^3)_n} = \frac{(q^3,q^{78},q^{81};q^{81})_\infty}{(q^3;q^3)_\infty} \tag{4.45}$$

$(N, \rho_1, \rho_2 \to \infty, a = q^3, b \to 0, (d, e, k) = (3, 3, 7)),$

$$\sum_{n,r \geqq 0} \frac{q^{n^2+4r^2}(-q;q^2)_n}{(q^2;q^4)_n(q^4;q^4)_r(q^4;q^8)_r(q^2;q^2)_{n-2r}} = \frac{(q^{40}, q^{48}, q^{88}; q^{88})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty} \tag{4.46}$$

$(N, \rho_1 \to \infty, \rho_2 = -\sqrt{q}, a = 1, b \to 0, (d, e, k) = (4, 1, 7)).$

5. Conclusion

As remarked earlier, after discovering his lemma and transform, Bailey considered a total of five Bailey pairs, and together with Dyson, derived quite a few identities from them [8,9]. Soon after Bailey completed his work on Rogers–Ramanujan type identities, Slater, by her own count, found 94 Bailey pairs, leading to 130 identities (although both of these totals are admittedly somewhat inflated as redundancies exist in both).

For about a quarter century following the work of Bailey and Slater, a handful of mathematicians did work related to the Rogers–Ramanujan identities, notably Henry Alder, George Andrews, David Bressoud, Leonard Carlitz, and Basil Gordon. Then around 1980, an explosion of interest in Rogers–Ramanujan occurred in the mathematics and physics communities as connections were found with Lie algebras (thanks to Jim Lepowsky, Steve Milne, and Robert Wilson) and statistical mechanics (through the efforts of Rodney Baxter, Alex Berkovich, Barry McCoy, Anne Schilling, Ole Warnaar and others). A bit later, as the computer revolution in mathematics began, important contributions related to Rogers–Ramanujan were made by Peter Paule, Axel Riese, Herb Wilf, Doron Zeilberger and others. Some of these practitioners have improved and extended Bailey's lemma. For example, Andrews [3] showed how the Bailey lemma is self-replicating, leading to the so-called "Bailey chain." Andrews et al. [7] found an "A_2 Bailey lemma." Andrews and Berkovich [5,6] extend the Bailey chain to a "Bailey tree." Further innovations and extensions of the Bailey chain are given by Berkovich and Warnaar [10]; Bressoud et al. [11]; Jeremy Lovejoy [15]; and Warnaar [32]. Ismail and Stanton obtained Rogers–Ramanujan type identities via tribasic integration [14]. In [26], V. Spiridonov found an elliptic analog of the Bailey chain. And the list goes on and on. For additional references, see the end notes of Chapter 2 in the new edition of Gasper and Rahman [13].

The inspiration for this current paper comes from a desire to "return to the basics" and to gain an understanding of Bailey's contributions via a unification of his work. Accordingly, I refer to (3.1) as the *standard* multiparameter Bailey pair because so many of the classical Rogers–Ramanujan type identities are direct consequences of it. As noted earlier, all identities in Bailey's papers [8,9] may be derived from the SMPBP. A rough count indicates that at least 40 percent of Slater's list [24] may be derivable from the SMPBP. However, it seems plausible that other multiparameter Bailey pairs, analogous to the SMPBP in some way, could be defined, perhaps accounting for the rest of (or at least large portions of the rest of) Slater's list, and incidentally revealing many new identities of the Rogers–Ramanujan type. This clearly warrants further investigation.

Acknowledgments

I thank Christian Krattenthaler for writing the HYPQ Mathematica package and making it freely available. The use of the HYPQ package was invaluable for this project. Also, I thank Bruce Berndt, Ole Warnaar, and the anonymous referee for helpful comments, and George Andrews for his continued encouragement.

References

[1] G.E. Andrews, An analytic generalization of the Rogers–Ramanujan identities for odd moduli, Proc. Nat. Acad. Sci. USA 71 (1974) 4082–4085.
[2] G.E. Andrews, On Rogers–Ramanujan type identities related to the modulus 11, Proc. London Math. Soc. (3) 30 (1975) 330–346.
[3] G.E. Andrews, Multiple series Rogers–Ramanujan type identities, Pacific J. Math. 114 (1984) 267–283.
[4] G.E. Andrews, q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra, CBMS Regional Conf. Ser. in Math., vol. 66, American Mathematical Society, Providence, RI, 1986.
[5] G.E. Andrews, Bailey's transform, lemma, chains and tree, in: J. Bustoz, et al. (Eds.), Special Functions 2000: Current Perspective and Future Directions, Kluwer Academic, Dordrecht, 2001, pp. 1–22.
[6] G.E. Andrews, A. Berkovich, The WP-Bailey tree, J. London Math. Soc. (2) 66 (2002) 529–549.
[7] G.E. Andrews, A. Schilling, S.O. Warnaar, An A_2 Bailey lemma and Rogers–Ramanujan type identities, J. Amer. Math. Soc. 12 (1999) 677–702.
[8] W.N. Bailey, Some identities in combinatory analysis, Proc. London Math. Soc. (2) 49 (1947) 421–435.
[9] W.N. Bailey, Identities of the Rogers–Ramanujan type, Proc. London Math. Soc. (2) 50 (1949) 1–10.
[10] A. Berkovich, S.O. Warnaar, Positivity preserving transformations for q-binomial coefficients, Trans. Amer. Math. Soc., in press.
[11] D.M. Bressoud, M.E.H. Ismail, D. Stanton, Change of base in Bailey pairs, Ramanujan J. 4 (2000) 435–453.
[12] F.J. Dyson, A walk through Ramanujan's garden, in: Lecture given at the Ramanujan Centenary Conference, University of Illinois, June 2, 1987, G.E. Andrews, et al. (Eds.), Ramanujan Revisited, Academic Press, San Diego, 1988, pp. 7–28.
[13] G. Gasper, M. Rahman, Basic Hypergeometric Series, second ed., Cambridge Univ. Press, 2004.
[14] M.E.H. Ismail, D. Stanton, Tribasic integrals and identities of the Rogers–Ramanujan type, Trans. Amer. Math. Soc. 355 (2003) 4061–4091.
[15] J. Lovejoy, A Bailey lattice, Proc. Amer. Math. Soc. 132 (2004) 1507–1516.
[16] P.A. MacMahon, Combinatory Analysis, vol. 2, Cambridge Univ. Press, London, 1918.
[17] S. Ramanujan, in: G. Andrews, B. Berndt (Eds.), Ramanujan's Lost Notebook, vol. 1, Springer, 2005.
[18] L.J. Rogers, Second memoir on the expansion of certain infinite products, Proc. London Math. Soc. 25 (1894) 318–343.
[19] L.J. Rogers, On two theorems of combinatory analysis and some allied identities, Proc. London Math. Soc. 16 (1917) 315–336.
[20] I. Schur, Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüche, in: Sitzungsberichte der Berliner Akademie, 1917, pp. 302–321.
[21] A.V. Sills, On identities of the Rogers–Ramanujan type, Ramanujan J., in press.
[22] A.V. Sills, q-difference equations and identities of the Rogers–Ramanujan–Bailey type, J. Differ. Equations Appl. 10 (2004) 1069–1084.
[23] L.J. Slater, A new proof of Rogers transformation of infinite series, Proc. London Math. Soc. (2) 53 (1951) 460–475.
[24] L.J. Slater, Further identities of the Rogers–Ramanujan type, Proc. London Math. Soc. (2) 54 (1952) 147–167.
[25] L.J. Slater, Generalized Hypergeometric Functions, Cambridge Univ. Press, 1966.
[26] V.P. Spiridonov, An elliptic incarnation of the Bailey chain, Internat. Math. Res. Notices 37 (2002) 1945–1977.

[27] D. Stanton, The Bailey–Rogers–Ramanujan group, in: q-Series with Applications to Combinatorics, Number Theory, and Physics (Urbana, IL, 2000), in: Contemp. Math., vol. 291, American Mathematical Society, Providence, RI, 2001, pp. 55–70.
[28] A. Verma, V.K. Jain, Transformations between basic hypergeometric series on different bases and identities of the Rogers–Ramanujan type, J. Math. Anal. Appl. 76 (1980) 230–269.
[29] A. Verma, V.K. Jain, Transformations of non-terminating basic hypergeometric series, their contour integrals and applications to Rogers–Ramanujan identities, J. Math. Anal. Appl. 87 (1982) 9–44.
[30] S.O. Warnaar, 50 years of Bailey's lemma, in: Algebraic Combinatorics and Applications, Springer, Berlin, 2001, pp. 333–347.
[31] S.O. Warnaar, The generalized Borwein conjecture, in: q-Series with Applications to Combinatorics, Number Theory, and Physics (Urbana, IL, 2000), in: Contemp. Math., vol. 291, American Mathematical Society, Providence, RI, 2001, pp. 243–267.
[32] S.O. Warnnar, Extensions of the well-poised and elliptic well-poised Bailey lemma, Indag. Math. (N.S.) 14 (2003) 571–588.
[33] G.N. Watson, A new proof of the Rogers–Ramanujan identities, J. London Math. Soc. 4 (1929) 4–9.