

Nonlinear stability in reaction–diffusion systems via optimal Lyapunov functions

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Abstract

We define optimal Lyapunov functions to study nonlinear stability of constant solutions to reaction–diffusion systems. A computable and finite radius of attraction for the initial data is obtained. Applications are given to the well-known Brusselator model and a three-species model for the spatial spread of rabies among foxes.

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1. Introduction

Nonlinear reaction–diffusion systems are well suited to model a wide range of physical, chemical and biological pattern formation processes (see, for instance, [23,25,35]). The study of the stability–instability of a given basic solution is very important to understand the real world (Anderson [1], Straughan [34]).

Let us consider the perturbation equations of a given constant solution \tilde{U} to a reaction–diffusion system

$$U_t = D\Delta U + LU + N(U_1, \dots, U_n), \quad (1)$$

with initial condition

$$U(0) = U_0 \quad (2)$$

and suitable boundary conditions (usually zero Dirichlet or Neumann boundary conditions). The perturbation $U(x, t) = (U_1, U_2, \dots, U_n)^T$, with $x \in \Omega \subseteq \mathbb{R}^m$ and $t > 0$, is an element of a Hilbert space \mathcal{H} , and $U_0 \in \mathcal{H}$. Here we assume $\mathcal{H} = L^2(\Omega)$, where Ω is the space-domain of motion, $\Omega = (0, l_1) \times (0, l_2) \times \dots \times (0, l_m)$. $D = D_{ij}$ and $L = L_{ij}$, $i, j = 1, 2, \dots, n$, are constant matrices (depending on some *physical* parameters), Δ is the m -dimensional Laplacian and $N = (N_1, \dots, N_n)^T$ represents the nonlinearities (in some problems N_i , $i = 1, 2, \dots, n$, are polynomial in U_j and $D_{ij} = \delta_{ij} D_j$, with positive D_j , see [23,31]). Here we suppose that the initial value problem (1)–(2), with

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suitable boundary conditions, is well posed and the solutions exist globally at least for small initial data and do not discuss the question of global existence.

A fair amount of attention has been given to the application of Lyapunov methods to reaction–diffusion systems (see, for example, [3,6,13,14]). In [4,36] the stability–instability problem of a constant solution of (1) (with Neumann boundary conditions) is examined with the study of the stability properties of L and the stability of the principal submatrices of L and $L - D$, where D is a diagonal matrix. A nonlinear stability analysis for reaction–diffusion systems has also been given in [7,18,30], by introducing particular Lyapunov functions.

Here we shall consider the case of weakly coupled parabolic systems, i.e. systems with $D_{ij} = \delta_{ij} D_j$; this case usually happens in the applications. However, cross diffusion is important in certain biological situations (see [23]). A stability analysis of an epidemic model with cross diffusion is studied in [22] with an optimal Lyapunov function.

In order to study the stability–instability problem of \tilde{U} the *main classical methods* are:

- (a) *the method of the linearized instability*: this method provides a *critical parameter* R_c , for a given parameter R , above which \tilde{U} is unstable;
- (b) *the Lyapunov method* (with a Lyapunov function E): this method provides a *critical nonlinear Lyapunov parameter* R_E below which the basic solution \tilde{U} is nonlinearly stable (see [23,34]).

In general, we have $R_E \leq R_c$. If, in particular, $R_E = R_c$, one obtains *necessary and sufficient* conditions of linear and nonlinear (conditional or global) stability and we say that the *Lyapunov function* E is *optimal*. This, for instance, happens if the linear operator is symmetric or symmetrizable in the scalar product of the Hilbert space (see [5,10]).

We note that in the case of ordinary differential systems a linearization principle holds, the well-known Hartman–Grobman theorem, while for partial differential systems it has to prove case by case. It also holds for reaction–diffusion systems (see, for example, [14, pp. 98–100], [31] and references therein). It gives stability conditions up to the linear criticality but usually it does not give a computable radius of attractivity for initial data which is important in the applications.

In many physical problems, in PDEs case, the “classical energy” $E_0(t) = \|U\|^2/2$ is used as a good Lyapunov function to control the stability (see, for instance, [8,23,34]). In these cases one generally obtains $R_{E_0} \leq R_c$, in particular, $R_{E_0} = R_c$ if the linear operator A is *symmetric* [10,34]. If A has a skewsymmetric part, then $R_{E_0} < R_c$ and new Lyapunov functions, different from the classical energy E_0 , must be introduced, usually in a heuristically way, to improve the nonlinear critical stability thresholds (see [20,26,27,32,33]).

The aim of this paper is to obtain *necessary and sufficient nonlinear stability* conditions for basic solutions to reaction–diffusion systems with a *computable radius of attraction* for initial data and make some *applications*. To this end, we give a *general* analytical procedure to construct an optimal Lyapunov function, by means of a change of dependent variables, connected with a projection on eigenfunctions of the Laplacian, to control linear and nonlinear stability. We apply this method to study nonlinear stability of constant solutions of two interesting applications in chemical-physics and in epidemiology: the well-known Brusselator model of an autocatalytic chemical reaction studied by Prigogine and Lefever [28] and a three-species model for the spatial spread and control of rabies among foxes (see [1,23,24]). We remark that the present method gives sharp nonlinear stability conditions with a computable radius of attraction of initial data which is finite for all values of R less than R_c and it depends on the basic motion and the “physical” parameters of the system. This differs from the linearization principle where no estimation of the attracting radius for the initial data in terms of the “physical” parameters is given (see, for example [14, Chapter 5, Section 5.1, pp. 98–100]). This method obviously is also valid in ODEs. In this case, it is equivalent to other well-known classical methods to define an optimal Lyapunov function (see, for instance, [2,11,12]).

In Section 2 we give the general procedure to define optimal Lyapunov functions in reaction–diffusion systems. In Section 3 we apply the general procedure to the well-known Brusselator system and to a biological case of a three-species (SIR) model for the spatial spread and control of rabies among foxes. Our main conclusions and remarks are drawn in Section 4.

2. A general procedure

System (1) can be written as an evolution equation in a Hilbert space \mathcal{H} :

$$\dot{U} = AU + N(U), \quad (3)$$

where $A = D\Delta + L$ is an operator with suitable properties depending on the boundary conditions. We may assume that N is a nonlinear operator, sufficiently smooth, vanishing at 0 so that $\tilde{U} = 0$ is a solution of (1).

We note that if U does not depend on the spatial variables, then $\Delta \equiv 0$, $\mathcal{H} = \mathbb{R}^m$, and we obtain the usual ODEs system. In this case, our approach is reduced, in a natural way, to the canonical reduction method based on the classical eigenvalues–eigenvectors problem. Indeed, by means of a change of dependent variables, we obtain *new canonical fields* $V = Q^{-1}U$ and a new (topologically equivalent) system

$$\dot{V} = BV + \bar{N}(V),$$

where $B = Q^{-1}AQ$ is a *similar matrix* to A (it is in a diagonal or a general Jordan form), $\bar{N}(V) = Q^{-1}N(QV)$. We recall that a *transformation matrix* Q is a non singular matrix of eigenvectors and generalized eigenvectors (in the case of a multiple eigenvalue with different geometric and algebraic multiplicity) of A , and Q^{-1} is its inverse. In the case of simple eigenvalues, Q is given by an n by n array such that the j th column is the j th eigenvector corresponding to the j th eigenvalue. If the j th eigenvalue is complex, the j th column and the $(j + 1)$ th column in the eigenvectors array are the real and imaginary parts corresponding to the j th eigenvalue. As it is well known, similar operators define ordinary differential equations that have the same dynamical properties (see [2,11,12]). We thus define the optimal Lyapunov function

$$E := \frac{1}{2}[V_1^2 + V_2^2 + \cdots + V_n^2],$$

which gives the critical linear and (at least local) nonlinear stability thresholds; we also obtain a computable radius of attraction of initial data. In particular cases we may obtain global stability.

In the PDEs case, we generalize the previous approach by introducing a change of dependent variables, connected with a projection on eigenfunctions of the Laplacian. Thus, we build as *optimal Lyapunov* function (at least for the linear problem) *the classical energy of the new canonical fields*:

$$E(t) = \frac{1}{2}\|V\|^2,$$

where (\cdot, \cdot) and $\|\cdot\|$ are the usual scalar product and norm in a Hilbert space \mathcal{H} (usually $\mathcal{H} = L^2(\Omega)$). For particular nonlinearities and dimensions of the space domain, sometimes, in order to control the nonlinearities, we have to add to $\frac{1}{2}\|V\|^2$ a complementary term. Now we define

$$E(t) = \frac{1}{2}\|V\|^2 + \tilde{b}E_2(t),$$

where $\tilde{b} \geq 0$ and $E_2(t)$ controls the nonlinearities (see [34]).

Let us consider the reaction–diffusion system

$$U_{i,t} = D_{ik}\Delta U_k + L_{ik}U_k + N_i(U_1, \dots, U_n), \quad (4)$$

with initial condition (2) and Dirichlet boundary conditions

$$U_i(x, t) = 0, \quad \forall (x, t) \in \partial\Omega \times (0, +\infty), \quad (5)$$

or Neumann boundary conditions, with zero average in Ω ,

$$\frac{\partial U_i(x, t)}{\partial n} = 0, \quad \forall (x, t) \in \partial\Omega \times (0, +\infty), \quad \langle U_i \rangle = \int_{\Omega} U_i(x, t) d\Omega = 0. \quad (6)$$

System (4) can be written in the form (3) as the initial value problem in \mathcal{H} ,

$$\dot{U} = AU + N(U), \quad U(0) = U_0, \quad (7)$$

$U_0 \in \mathcal{H}$, $A = D\Delta + L$, $D = D_{ij} = \delta_{ij}D_j$, $D_j > 0$, $L = L_{ij}$ where L_{ij} are real numbers.

We assume:

- (i) A is a densely defined closed operator with compact resolvent;

- (ii) the bilinear form associated with A is defined and bounded on a space \mathcal{H}_\star which is compactly embedded in \mathcal{H} (for instance, if A is the closure in $L_2(\Omega)$ of $D\Delta + L$ restricted to $C_0^2(\Omega)$, then $\mathcal{H}_\star = W_0^{1,2}(\Omega)$);
- (iii) N is a nonlinear operator, $N : D(N) \subseteq \mathcal{H} \rightarrow \mathcal{H}$, with $N(0) = 0$ and N satisfies a condition of type

$$|(NU, U)| \leq K_0 \|U\|^\alpha \|\nabla U\|^2 \quad (8)$$

(with K_0 and α positive numbers).

In these hypotheses, we have (see [9,16])

Theorem 2.1. *The spectrum of A consists entirely of an (at most) denumerable number of eigenvalues $\{\sigma_n\}_{n \in \mathbb{N}}$ with finite (algebraic and geometric) multiplicities and, moreover, such eigenvalues can cluster only at infinity.*

The eigenvalues can be ordered in the following way:

$$\operatorname{Re}(\sigma_1) \geq \operatorname{Re}(\sigma_2) \geq \dots \geq \operatorname{Re}(\sigma_n) \geq \dots$$

We recall some well-known stability definitions.

Definition 2.1. The zero solution of (7) is said to be linearly stable if $\operatorname{Re}(\sigma_1) < 0$.

Definition 2.2. The zero solution of (7) is said to be nonlinearly stable if

$$\forall \epsilon > 0 \exists \delta(\epsilon) > 0: \quad \|U_0\| < \delta(\epsilon) \quad \Rightarrow \quad \|U(t)\| < \epsilon, \quad \forall t \geq 0.$$

Definition 2.3. The zero solution of (7) is said to be asymptotically nonlinearly stable if it is nonlinearly stable and there exists $\gamma \in (0, \infty]$ such that

$$\|U_0\| < \gamma \quad \Rightarrow \quad \lim_{t \rightarrow \infty} \|U(t)\| = 0.$$

If $\gamma = \infty$ the zero solution is said to be unconditionally (or globally) nonlinearly stable.

The operator A is in general non-symmetric, although it allows a decomposition into two parts A_1 and A_2 such that

- (a) $A = A_1 + A_2$, $D(A_2) \supseteq D(A_1) = D(A)$,
- (b) A_1 is symmetric with compact resolvent,
- (c) A_2 is skewsymmetric in \mathcal{H} and bounded in \mathcal{H}^\star .

Thus the spectrum of A_1 satisfies the same type of property as that given above for the spectrum of A . But now the eigenvalues $\{\lambda_n\}$ are real and $\lambda_1 \geq \dots \geq \lambda_n \geq \dots$. The linear stability is reduced to studying the sign of $s = \operatorname{Re}(\sigma_1)$. In general s will depend on the basic motion through a dimensionless parameter R , such as reproduction number, Reynolds or Rayleigh numbers (and also on some other parameters). The value R_c of R at which linear instability sets in is the least value of R for which $\operatorname{Re}(\sigma_1) = 0$. It is called *critical value* (of linear instability).

For the problem (7) an *energy equation* holds (remember that A_2 is skewsymmetric):

$$\frac{d}{dt} \frac{\|U\|^2}{2} = (AU, U) + (NU, U) = (A_1U, U) + (NU, U). \quad (9)$$

We assume that $(A_1U, U) = RI(U) - \|\nabla U\|^2$, where $I(U)$ is a quadratic form in U , in our hypotheses there exists a positive constant c (Poincaré's or Wirtinger's constant) such that $c\|U\|^2 \leq \|\nabla U\|^2$. From (9), we have

$$\frac{d}{dt} \frac{\|U\|^2}{2} \leq \left(\frac{RI(U)}{\|\nabla U\|^2} - 1 \right) \|\nabla U\|^2 + (NU, U) \leq \left(\frac{R}{R_E} - 1 \right) \|\nabla U\|^2 + (NU, U), \quad (10)$$

where

$$\frac{1}{R_E} = \max_S \frac{I(U)}{\|\nabla U\|^2} \quad (11)$$

and \mathcal{S} is the space of the *kinematically admissible fields*. (R_E is the *critical value of energy-nonlinear stability*.) From (10), because of the Poincaré's inequality and (8), we have

$$\dot{E} \leq 2c(-h + K_0 2^{\alpha/2} E^{\alpha/2})E,$$

where $h = 1 - \frac{R}{R_E}$. By assuming $R < R_E$ and $E(0) < (\frac{a}{b})^{2/\alpha}$, where $a = 2ch$ and $b = 2^{1+\alpha/2}cK_0$, integrating last differential inequality, we obtain the exponential decay

$$E(t) \leq \frac{E(0)e^{-at}}{(1 - \frac{bE(0)^{\alpha/2}}{a}(1 - e^{-\alpha at/2}))^{2/\alpha}}. \quad (12)$$

Thus, we have:

Theorem 2.2. *If $R < R_E$ and $E(0) < (\frac{a}{b})^{2/\alpha}$, with R_E given by (11), with $a = 2ch$ and $b = 2^{1+\alpha/2}cK_0$, then the zero solution of (7) is conditionally asymptotically stable according to (12).*

We note that, from the above arguments, it follows that while the linear stability problem reduced to studying the eigenvalue problem associated with all of A , nonlinear stability according to the *standard energy* method involves the eigenvalues of the *symmetric* part of A_1 only (see [34]). Moreover, whenever $A_2 = 0$, the two eigenvalue problems coincide and we have necessary and sufficient stability conditions ($R_E = R_C$). From a *physical* point of view, a skew-symmetric linear operator L_2 can represent a stabilizing effect (see [9,34]) that gets lost if we use the classical energy, in fact, now $(A_2 U, U) = 0$.

One reason to introduce a new optimal Lyapunov function (equivalent to the *energy norm* $\|\cdot\|$) is exactly to recover this stabilizing effect. However, we note that the present method contains the symmetrization case as a particular case.

Now we have to recall the well-known *linearization principle* [14,17,29]:

Theorem 2.3. *If $\operatorname{Re}(\sigma_1) < 0$, then there exist positive constants A' , B' and γ_0 such that $\|U\| \leq A'\|U_0\|e^{-B't}$ whenever $\|U_0\| < \gamma_0$.*

This result, though remarkable from a theoretical point of view, should not be considered completely satisfactory because, in particular, the constant γ_0 (which gives an attracting radius for the initial data) cannot be computed numerically and we do not know, in practice, how small the perturbations must initially be in order to have stability (see [9,15]).

The other reason to introduce a new optimal Lyapunov function is to obtain a computable and finite radius of attraction for the initial data.

Now we give the *main steps to construct an optimal* Lyapunov function for the reaction–diffusion system (4):

(1) First we linearize

$$U_{i,t} = D_{ik}\Delta U_k + L_{ik}U_k$$

and denote by ξ_p (p positive integer) the generic *eigenvalue* of the Laplacian with boundary conditions (5) or (6),

$\xi_p = \frac{p^2\pi^2}{l^2}$, with $l^{-2} = l_1^{-2} + l_2^{-2} + \dots + l_m^{-2}$. We define

$$A_\xi = -\xi D + L,$$

where ξ is the *principal eigenvalue* of the operator $D\Delta + L$ (i.e., the eigenvalue corresponding to the critical linearized instability parameter), and compute the eigenvalues of A_ξ .

(2) We introduce a *transformation matrix* Q of eigenvectors (and/or generalized eigenvectors) of the matrix A_ξ and its inverse Q^{-1} .

(3) We define the new field variables $V = Q^{-1}U$ and write the new (nonlinear) *reaction–diffusion system* equivalent to (4) (see also [23, p. 53])

$$V_{i,t} = F_{ik}\Delta V_k + G_{ik}V_k + \tilde{N}_i(V), \quad V(0) = V_0, \quad (13)$$

where

$$F = Q^{-1}DQ, \quad G = Q^{-1}LQ,$$

and

$$\tilde{N}(V) = Q^{-1}N(QV).$$

(4) We introduce the Lyapunov function

$$E_1(t) = \frac{1}{2} \|V(t)\|^2$$

for the new linearized system and study the (linear) stability of the zero solution.

(5) We write the balance equation for the Lyapunov function $E_1(t)$,

$$\dot{E}_1(t) = (GV, V) - (F\nabla V, \nabla V)$$

and we assume that the quadratic form $(F\nabla V, \nabla V)$ is positive definite. Then, we study the *maximum* problem

$$M = \max_{\mathcal{S}} \frac{(GV, V)}{(F\nabla V, \nabla V)}, \quad (14)$$

where \mathcal{S} is the space of the *admissible* functions (for example, in the case of zero Dirichlet boundary conditions, it is the Sobolev space $W_0^{1,2}(\Omega) - \{0\}$). M is obtained by solving the equation

$$\det(T_{ij}) = 0, \quad (15)$$

where

$$T_{ij} = \begin{cases} 2(G_{ij} - M(p)\xi_p F_{ij}) & \text{if } i = j, \\ G_{ij} + G_{ji} - M(p)\xi_p(F_{ij} + F_{ji}) & \text{if } i \neq j, \end{cases} \quad (16)$$

and maximizing the generic eigenvalue $M(p)$ with respect to the integer p .

(6) We consider the *nonlinear system* (13) and define the new Lyapunov function

$$E(t) = E_1(t) + \tilde{b}E_2(t), \quad \tilde{b} \geq 0,$$

with a *suitable* E_2 which controls the nonlinearities, and write the energy equation of $E(t)$ (in some problems, and for particular space dimensions, we can choose $\tilde{b} = 0$ and the optimal Lyapunov function $E(t)$ coincides with $E_1(t)$).

(7) Finally, we show that the condition $M < 1$ is the *nonlinear stability condition*, and $M = 1$ gives the critical Lyapunov number R_E which coincides with R_c . If the nonlinear term N satisfies a condition of type (8), we obtain the coincidence of the linear and nonlinear stability boundaries with a *computable* value for the radius of attraction of the initial data.

It is easy to see that:

Theorem 2.4. *The systems (7) and (13) are topologically equivalent.*

3. Applications

Here we apply the previous method to two well-known systems, the Brusselator and a three-species (SIR) model for the spatial spread and control of rabies among foxes. The linear stability–instability of this systems is well documented in the literature (see [23,25,28]). By using the present method, we obtain sharp nonlinear stability conditions with a computable finite radius of attraction for the initial data.

3.1. The Brusselator system

The Brusselator model, introduced by Prigogine and coauthors (see Nicolis and Prigogine [25], Prigogine and Lefever [28]), is a well-known example of an autocatalytic chemical reaction.

Let a, b be fixed concentrations of two chemical products and let

$$(X = a, Y = b/a)$$

be a constant solution of the Brusselator system

$$\begin{cases} X_t = a - (b + 1)X + X^2Y + D_1 \Delta X, \\ Y_t = bX - X^2Y + D_2 \Delta Y, \end{cases}$$

where D_1 and D_2 are the (positive) diffusion coefficients.

The perturbation equations to this solution are given by

$$\begin{cases} u_t = bu - u + a^2v + D_1 \Delta u + n_1(u, v), \\ v_t = -bu - a^2v + D_2 \Delta v + n_2(u, v), \end{cases} \quad (17)$$

in $\Omega \times (0, \infty)$, where $n_1(u, v) = -n_2(u, v) = \frac{b}{a}u^2 + 2auv + u^2v$ and $\Omega = (0, l_1)$ or $\Omega = (0, l_1) \times (0, l_2)$. Here we consider one-dimensional and bi-dimensional bounded domains. The system (17) is in the form (1) with $U = (u, v)^T$.

To the system (17) we add the initial condition $u(\mathbf{x}, 0) = u_0(\mathbf{x})$, $v(\mathbf{x}, 0) = v_0(\mathbf{x})$ (small enough to guarantee the global existence), the Neumann boundary conditions $\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0$, with the average conditions $\langle u \rangle = 0$, $\langle v \rangle = 0$, or the Dirichlet boundary conditions, $u = v = 0$ on the boundary.

If we consider the classical energy

$$E_0(t) = \frac{1}{2} \left[\|u\|^2 + \frac{a^2}{b} \|v\|^2 \right],$$

and use the Poincaré's inequality $\xi \|u\|^2 \leq \|\nabla u\|^2$, we have

$$\begin{aligned} \dot{E}_0 &= (b - 1) \|u\|^2 - \frac{a^4}{b} \|v\|^2 - D_1 \|\nabla u\|^2 - \frac{a^2}{b} D_2 \|\nabla v\|^2 + N(u, v) \\ &\leq (b - 1 - \xi D_1) \|u\|^2 - \frac{a^4}{b} \|v\|^2 - \frac{a^2}{b} D_2 \|\nabla v\|^2 + N(u, v), \end{aligned}$$

where $\xi = \xi_1 = \frac{\pi^2}{l^2}$ is the *first* eigenvalue of the Laplacian with the aforesaid boundary conditions, $l^{-2} = (l_1^{-2} + l_2^{-2})$ (or $l^{-2} = l_1^{-2}$ in the one-dimensional case) and $N(u, v) = (n_1, u) + (n_2, v)$. If

$$b < b_{E_0} := 1 + \xi D_1, \quad (18)$$

we have linear stability. It can be proved that we have also (conditional) nonlinear stability if we use a Lyapunov function $E_0 + \tilde{E}$ with suitable \tilde{E} .

Now we apply the general procedure. First we study the linear stability and compute the eigenvalues of the linearized problem. It is easy to see that the condition

$$b < b_c = \min_n \min \left(1 + \xi_n D_1 + \frac{a^2(1 + \xi_n D_1)}{\xi_n D_2}, 1 + a^2 + \xi_n (D_1 + D_2) \right)$$

is necessary and sufficient for (linear) stability (see [25]).

Here we consider separately the cases of complex and distinct real eigenvalues. The coincidence of real eigenvalues is very exceptional and will not consider here (nevertheless our method is still valid).

3.1.1. Complex eigenvalues

In the case of *complex eigenvalues* (Hopf instability) and for one-dimensional and bi-dimensional spaces, we have

$$b < 1 + a^2 + \xi_n (D_1 + D_2).$$

The minimum with respect to n of the right-hand side is achieved always for $n = \bar{n} = 1$, hence $\xi = \xi_1$. The critical linear stability number is given by

$$b_c = 1 + a^2 + \xi(D_1 + D_2). \quad (19)$$

From (19), we see that a^2 has always a *stabilizing effect*. This effect is *lost* if we use the energy E_0 (see (18)).

We introduce the matrix

$$A_\xi := \begin{pmatrix} b - 1 - \xi D_1 & a^2 \\ -b & -a^2 - \xi D_2 \end{pmatrix}.$$

The eigenvalues of the matrix A_ξ are given by $\bar{\lambda}_{1,2} = \nu \pm i\omega$, where

$$\nu = \frac{1}{2}[b - 1 - (D_1 + D_2)\xi - a^2], \quad \omega^2 = |A_\xi| - \nu^2.$$

Here $|A_\xi|$ denotes the determinant of the matrix A_ξ , and we have assumed that $\omega > 0$. A matrix of vectors Q and its inverse are given by

$$Q = \begin{pmatrix} (a^2 + \xi D_2 + \nu) & \omega \\ -b & 0 \end{pmatrix}, \quad Q^{-1} = \begin{pmatrix} 0 & \frac{-1}{b} \\ \frac{1}{\omega} & \frac{a^2 + \xi D_2 + \nu}{\omega b} \end{pmatrix}.$$

From the expression of the matrix Q^{-1} we have the new canonical fields $V = (\phi, \psi)^T$, given by

$$\begin{cases} \phi = -\frac{\nu}{b}, \\ \psi = \frac{1}{\omega} \left(u + \frac{a^2 + \xi D_2 + \nu}{b} v \right), \end{cases}$$

and, by means of some algebra, we obtain the new system (equivalent to (17))

$$\begin{cases} \phi_t = (\nu + \xi D_2)\phi + \omega\psi + D_2\Delta\phi + \bar{N}_1(\phi, \psi), \\ \psi_t = \frac{(\nu + D_1\xi)(\nu + D_2\xi) - a^2}{\omega}\phi + (\nu + \xi D_1)\psi + \frac{\alpha}{\omega}(D_1 - D_2)\Delta\phi + D_1\Delta\psi + \bar{N}_2(\phi, \psi), \end{cases}$$

where

$$\begin{aligned} \alpha &= a^2 + \xi D_2 + \nu, \\ \bar{N}_1(\phi, \psi) &= \left(\frac{\alpha^2}{a} - 2a\alpha \right) \phi^2 + \frac{\omega^2}{a} \psi^2 + 2\omega \left(\frac{\alpha}{a} - a \right) \phi\psi - \alpha^2 \phi^3 - 2\alpha\omega\phi^2\psi - \omega^2\phi\psi^2, \\ \bar{N}_2(\phi, \psi) &= \frac{b - \alpha}{\omega} \bar{N}_1(\phi, \psi). \end{aligned}$$

We write the energy equation for $E_1(t)$:

$$\dot{E}_1 = \mathcal{I}_1 - \mathcal{D}_1 + \mathcal{N}_1, \quad (20)$$

where

$$\begin{aligned} \mathcal{I}_1 &= (\nu + \xi D_2) \|\phi\|^2 + (\nu + \xi D_1) \|\psi\|^2 + \frac{\omega^2 + (\nu + D_1\xi)(\nu + D_2\xi) - a^2}{\omega} (\phi, \psi), \\ \mathcal{D}_1 &= \frac{\alpha(D_1 - D_2)}{\omega} (\nabla\phi, \nabla\psi) + D_2 \|\nabla\phi\|^2 + D_1 \|\nabla\psi\|^2, \\ \mathcal{N}_1 &= (\bar{N}_1(\phi, \psi), \phi) + (\bar{N}_2(\phi, \psi), \psi). \end{aligned} \quad (21)$$

For simplicity, we consider the case of Dirichlet boundary conditions (all the results hold also for Neumann b.c. with zero average in Ω). Following the general procedure, step (5) in Section 2, we obtain the equation for the maximum

$$M = \max_{W_0^{1,2}(\Omega) - 0} \frac{\mathcal{I}_1}{\mathcal{D}_1}.$$

It is obtained by solving (15) and (16), where now

$$\begin{aligned} G_{11} &= \nu + \xi D_2, & G_{12} &= \omega, & F_{11} &= D_2, & F_{12} &= 0, \\ G_{21} &= \frac{(\nu + D_1 \xi)(\nu + D_2 \xi) - a^2}{\omega}, & G_{22} &= \nu + \xi D_1, \\ F_{21} &= \frac{\alpha}{\omega}(D_1 - D_2), & F_{22} &= D_1. \end{aligned}$$

We also assume that the positive-definite condition of \mathcal{D}_1 holds

$$\frac{\alpha^2}{\omega^2}(D_1 - D_2)^2 - 4D_1 D_2 < 0.$$

We easily find

$$M = 1 + \frac{2\nu}{\xi[D_1 + D_2 + \frac{|D_1 - D_2|}{\omega}\sqrt{\omega^2 + \alpha^2}]}.$$

From the energy equation (20), we obtain the estimate

$$\dot{E}_1 \leq (M - 1)\mathcal{D}_1 + \mathcal{N}_1. \quad (22)$$

In the particular cases (in one-dimensional space): $D_1 = D_2$; $D_1 = 8 \times 10^{-3}$, $D_2 = 4 \times 10^{-3}$, $a = 2$, $l = 1$; $D_1 = 8 \times 10^{-3}$, $D_2 = 1.6 \times 10^{-3}$, $a = 2$, $l = 0.11$, we can easily verify that $M < 1 \Leftrightarrow \nu < 0$. This is equivalent to $b < b_c$.

Now we estimate the nonlinear terms.

Case (a₁). In the one-dimensional case, we choose $E(t) = E_1(t)$.

From the definitions of E and \mathcal{D}_1 , we easily have the inequalities

$$\begin{aligned} \|\phi\| &\leq \sqrt{2}E^{1/2}, & \|\phi_x\| &\leq \left[\frac{2}{D_1 + D_2 - \frac{|D_1 - D_2|}{\omega}\sqrt{\alpha^2 + \omega^2}} \right]^{1/2} \mathcal{D}_1^{1/2}, \\ \mathcal{D}_1 &\geq \left[\frac{D_1 + D_2 - \frac{|D_1 - D_2|}{\omega}\sqrt{\alpha^2 + \omega^2}}{2} \right] (\|\phi_x\|^2 + \|\psi_x\|^2) \end{aligned}$$

(also ψ satisfies the first two inequalities). Moreover, by the Poincaré inequality, we obtain

$$E \leq \frac{l_1^2}{2\pi^2} (\|\phi_x\|^2 + \|\psi_x\|^2) \leq \frac{2l_1^2 \mathcal{D}_1}{\pi^2 (D_1 + D_2 - \frac{|D_1 - D_2|}{\omega}\sqrt{\alpha^2 + \omega^2})}.$$

From (21) and (22), we have

$$\dot{E}_1 \leq (M - 1)\mathcal{D}_1 + (\bar{N}_1(\phi, \psi), \phi) + (\bar{N}_2(\phi, \psi), \psi),$$

where

$$\begin{aligned} (\bar{N}_1(\phi, \psi), \phi) &= \left(\frac{\alpha^2}{a} - 2a\alpha \right) (\phi^2, \phi) + \frac{\omega^2}{a} (\psi^2, \phi) + 2\omega \left(\frac{\alpha}{a} - a \right) (\phi^2, \psi) \\ &\quad - \alpha^2 (\phi^2, \phi^2) - 2\alpha\omega (\phi^3, \psi) - \omega^2 (\phi^2, \psi^2) \\ &\leq \left| \frac{\alpha^2}{a} - 2a\alpha \right| |(\phi^2, \phi)| + \frac{\omega^2}{a} |(\psi^2, \phi)| + 2\omega \left| \frac{\alpha}{a} - a \right| |(\phi^2, \psi)| + 2|\alpha|\omega |(\phi^3, \psi)| \end{aligned}$$

and

$$\begin{aligned} (\bar{N}_2(\phi, \psi), \psi) &\leq \left| \frac{b - \alpha}{\omega} \right| \left[\left| \frac{\alpha^2}{a} - 2a\alpha \right| |(\phi^2, \phi)| + \frac{\omega^2}{a} |(\psi^2, \phi)| + 2\omega \left| \frac{\alpha}{a} - a \right| |(\phi^2, \psi)| + \alpha^2 |(\phi^2, \phi^2)| \right. \\ &\quad \left. + 2|\alpha|\omega |(\phi^3, \psi)| + \omega^2 |(\phi^2, \psi^2)| \right]. \end{aligned}$$

Since in 1 dimension $W_0^{1,2}(0, l_1) \subset C^0(0, l_1)$, we have

$$\max_{[0, l_1]} |f| \leq \sqrt{l_1} \|f_x\|. \quad (23)$$

From this inequality and from the Poincaré inequality, we easily obtain

$$\begin{aligned} |(\phi, \phi^2)| &\leq \frac{l_1 \sqrt{l_1}}{\pi} \|\phi_x\|^2 \|\phi\|, & |(\psi, \phi^2)| &\leq \frac{l_1 \sqrt{l_1}}{\pi} \|\phi_x\|^2 \|\psi\|, \\ |(\phi, \psi^2)| &\leq \frac{l_1 \sqrt{l_1}}{\pi} \|\psi_x\|^2 \|\phi\|, & |(\psi, \phi^3)| &\leq l_1 \|\phi_x\| \|\psi_x\| \|\phi\|^2, \\ |(\phi^2, \psi^2)| &\leq l_1 \|\phi_x\|^2 \|\psi\|^2, & |(\phi, \psi^3)| &\leq l_1 \|\phi_x\| \|\psi_x\| \|\psi\|^2. \end{aligned}$$

Collecting all these inequalities, we have

$$\dot{E} \leq \mathcal{D}_1 [M - 1 + C_1 E^{1/2} + B_1 E], \quad (24)$$

where

$$\begin{aligned} C_1 &= \frac{l_1 \sqrt{2l_1}}{\pi} k_0 \left(\frac{|b - \alpha|}{\omega} + 1 \right) \left(\left| \frac{\alpha^2}{a} - 2a\alpha^2 \right| + \frac{\omega^2}{a} + 2\omega \left| \frac{\alpha}{a} - a \right| \right), \\ B_1 &= 2k_0 l_1 \left(2|\alpha|\omega \left(1 + \frac{|b - \alpha|}{\omega} \right) + |b - \alpha| \left(\frac{\alpha^2}{\omega} + 1 \right) \right), \end{aligned}$$

and

$$k_0 = \frac{2}{D_1 + D_2 - \frac{|D_1 - D_2|}{\omega} \sqrt{\alpha^2 + \omega^2}}.$$

If

$$M < 1$$

and

$$E(0) < \left(\frac{\sqrt{C_1^2 - 4B_1(M - 1)} - C_1}{2B_1} \right)^2$$

we have

$$\dot{E}_1(0) \leq \mathcal{D}_1 [M - 1 + C_1 E(0)^{1/2} + B_1 E(0)],$$

and by a recursive argument (see [34]) we obtain the exponential decay

$$E(t) \leq E(0) \exp \left\{ \frac{\pi^2}{l_1^2 k_0} [M - 1 + C_1 E(0)^{1/2} + B_1 E(0)] t \right\}.$$

Case (a₂). In the bi-dimensional case, the estimates leading to (24) are based on inequality (23) which does not hold in two dimensions. Now we use the energy

$$E = E_1 + \tilde{b} E_2,$$

with the complementary energy E_2 introduced by Straughan [34],

$$E_2 = \frac{1}{4} (\lambda_1 \|\zeta\|^2 + \lambda_2 \|\eta\|^2),$$

where $\zeta = \phi^2$, $\eta = \psi^2$ and λ_1, λ_2 are positive parameters to be chosen. Proceeding as in [34], we can prove that the energy equation for E_2 is given by

$$\dot{E}_2 = \mathcal{I}_2 - \mathcal{D}_2 + \mathcal{N}_2,$$

where

$$\begin{aligned}\mathcal{I}_2 &= \lambda_1(v + D_2\xi)\|\zeta\|^2 + \lambda_2(v + D_1\xi)\|\eta\|^2 + \lambda_1\omega(\zeta\phi, \psi) - \alpha^2\|\zeta\|^2 \\ &\quad + \lambda_2\frac{(v + D_1\xi)(v + D_2\xi) - a^2}{\omega}(\eta\phi, \psi) + \lambda_2\frac{\alpha(D_1 - D_2)}{\omega}(\eta\psi, \Delta\phi), \\ \mathcal{D}_2 &= \frac{3}{4}[\lambda_1 D_2\|\nabla\zeta\|^2 + \lambda_2 D_1\|\nabla\eta\|^2], \quad \mathcal{N}_2 = \lambda_1(\zeta\phi, \bar{n}_1) + \lambda_2\left(\eta\psi, \frac{b - \alpha}{\omega}\bar{n}_1\right).\end{aligned}$$

By using classical embedding theorems in the Sobolev spaces, and the Poincaré's inequality, with some calculations (details of such calculations, in another context, may be found in [34, p. 36]), we can obtain the inequality

$$\mathcal{N}_1 + \mathcal{I}_2 + \mathcal{N}_2 \leq H_0 \mathcal{D}_3 E^{1/2}, \quad (25)$$

where H_0 is a positive *computable* constant depending on the parameters $a, b, D_1, D_2, 1 - M, l_1, l_2$, and λ_1, λ_2 (the last ones can be chosen to obtaining the best radius of attractivity for the initial data),

$$\mathcal{D}_3 = (1 - M)\mathcal{D}_1 + \mathcal{D}_2. \quad (26)$$

Because of the Poincaré's inequality, we have $\mathcal{D}_3 \geq 2\pi^2(1 - M)\bar{\delta}E$, where $\bar{\delta} = \min(D_1, D_2)$. From (25) and (26) we obtain

$$\dot{E} \leq -2\pi^2\bar{\delta}(1 - M)E[1 - H(E(t))^{1/2}].$$

By assuming

$$M < 1,$$

i.e., $b < b_c$ and

$$E(0) < H^{-2},$$

we easily find nonlinear stability according to the exponential decay

$$E(t) \leq E(0) \exp\{-2\pi^2\bar{\delta}(1 - M)(1 - HE(0)^{1/2})t\}.$$

3.1.2. Real eigenvalues

Now we consider the case of *distinct real eigenvalues*.

It is proved (see [25,28]) that the critical linear instability value b_c is given by

$$b_c = \min_n \left(1 + \xi_n D_1 + \frac{a^2(1 + \xi_n D_1)}{\xi_n D_2} \right).$$

By increasing b , the Turing instability sets in for $b = b_c$ which corresponds to the integer n_c nearest to the minimum (μ, b_μ) of the critical curve. In one-dimensional case, one obtains

$$\mu = \frac{l\sqrt{a}}{\pi(D_1 D_2)^{1/4}}, \quad b_\mu = \left(1 + \sqrt{\frac{D_1}{D_2}a} \right)^2.$$

For example, for the values $D_1 = 1.6 \times 10^{-3}$, $D_2 = 8 \times 10^{-3}$, $a = 2$, $l = 1$, considered in [25, §7.4], one finds: $n_c = 8$, $b_c = 3.6022$, $\mu = 7.5260$, and $b_\mu = 3.5888$.

If we use the classical energy E_0 , also here the stabilizing effect of a^2 is lost. As before, the eigenvalues of the matrix

$$A_\xi := \begin{pmatrix} b - 1 - \xi D_1 & a^2 \\ -b & -a^2 - \xi D_2 \end{pmatrix},$$

where now $\xi = \xi_{n_c}$ are

$$\lambda^\pm = v \pm \sqrt{v^2 - |A_\xi|}.$$

A matrix of eigenvectors Q (depending on two parameters c_1 and c_2) and its inverse are obtained as before and we have the new system (13). The nonlinear terms are obtained (and controlled) as before. By solving the maximum equation, we find that $M < 1$ is equivalent to $b < b_c$. For example, by choosing $D_1 = 1.6 \times 10^{-3}$, $D_2 = 8 \times 10^{-3}$, $a = 2$, $l = 1$ (values given in [25]) it can be proved that the constant c_1 and c_2 can be chosen in such a way that the quadratic form $(F \nabla V, \nabla V)$ is positive definite. We have $M = 1$ if and only if $b = b_c$.

3.2. Three-species (SIR) model for the spatial spread and control of rabies among foxes

Let us consider the three-component reaction–diffusion system (system (13.73) of Murray [23])

$$\begin{cases} S_t = (a - b) \left(1 - \frac{N}{K}\right) S - \beta RS + D_1 \Delta S, \\ I_t = \beta RS - \sigma I - \left[b + (a - b) \frac{N}{K}\right] I + D_1 \Delta I, \\ R_t = \sigma I - \left[b + (a - b) \frac{N}{K}\right] R - \alpha R + D \Delta R, \\ N = S + I + R, \end{cases}$$

in $\Omega \times (0, +\infty)$, with $\Omega = (0, L) \times (0, L)$, $L > 0$. The populations densities (fox/km²) S , I , R , represent susceptible foxes, infected, but noninfectious, foxes and rabid (infectious) foxes, respectively. N is the total population. The positive parameters a , b , α , β , σ , K , D_1 , D are: the average birth rate, the average intrinsic death rate, the average rabies death rate ($1/\alpha$ is the average duration of clinical disease), the disease transmission coefficient, the average infected \rightarrow infectious per capita rate ($1/\sigma$ is the average incubation time), the carrying capacity, the diffusion coefficient of susceptibles and infected, the diffusion coefficient of infectious individuals. (For the basic model assumptions and biological meanings, see [1,23,24,37].)

We adimensionalize the system by means of the transformation

$$\begin{aligned} S &= sK, & I &= qK, & R &= rK, & N &= nK, & t &= \frac{t^*}{K\beta}, & \mathbf{x} &= \mathbf{x}^* \sqrt{\frac{D_1}{K\beta}}, \\ \epsilon &= \frac{a-b}{K\beta}, & \mu &= \frac{\sigma}{K\beta}, & \delta &= \frac{b}{K\beta}, & d &= \frac{\alpha+b}{K\beta}, & \vartheta &= \frac{D}{D_1}. \end{aligned}$$

On dropping the asterisks for notational simplicity, we have

$$\begin{cases} s_t = \epsilon(1-n)s - rs + \Delta s, \\ q_t = rs - (\mu + \delta + \epsilon n)q + \Delta q, \\ r_t = \mu q - (d + \epsilon n)r + \vartheta \Delta r, \\ n = s + q + r. \end{cases}$$

The perturbation $\mathbf{u} = (u(\mathbf{x}, t), v(\mathbf{x}, t), w(\mathbf{x}, t))^T$ to the constant solution $(s, q, r)^T = (1, 0, 0)^T$, is governed by the system

$$\begin{cases} u_t = -\epsilon u - \epsilon v - (\epsilon + 1)w + \Delta u - [\epsilon u^2 + \epsilon uv + (\epsilon + 1)uw], \\ v_t = -(\mu + \delta + \epsilon)v + w + \Delta v - [\epsilon(uv + v^2 + vw) - uw], \\ w_t = \mu v - (d + \epsilon)w + \vartheta \Delta w - [\epsilon(u + v + w)w], \end{cases}$$

where now we have $\Omega = (0, \tilde{L}) \times (0, \tilde{L})$, with $\tilde{L} = L\sqrt{\frac{K\beta}{D_1}}$. To this system we add the Dirichlet or Neumann boundary conditions, respectively,

$$\begin{aligned} \mathbf{u}(0, y) &= \mathbf{u}(\tilde{L}, y) = \mathbf{u}(x, 0) = \mathbf{u}(x, \tilde{L}) = 0, \\ \mathbf{u}_y(0, y) &= \mathbf{u}_y(\tilde{L}, y) = \mathbf{u}_x(x, 0) = \mathbf{u}_x(x, \tilde{L}) = 0. \end{aligned} \tag{27}$$

In order to study the linear and nonlinear stability of the aforesaid solution, we first consider the linear system

$$\begin{cases} u_t = -\epsilon u - \epsilon v - (\epsilon + 1)w + \Delta u, \\ v_t = -(\mu + \delta + \epsilon)v + w + \Delta v, \\ w_t = \mu v - (d + \epsilon)w + \vartheta \Delta w. \end{cases} \quad (28)$$

Because of the boundary conditions, we may introduce an exponential time dependence in \mathbf{u} so that $\mathbf{u} = \mathbf{u}_0 e^{\lambda t} \sin \frac{n\pi}{L} x \sin \frac{n\pi}{L} y$, $n \in \mathbb{N}^+$, in the case (27) and $\mathbf{u} = \mathbf{u}_0 e^{\lambda t} \cos \frac{n\pi}{L} x \cos \frac{n\pi}{L} y$, $n \in \mathbb{N}$, for Neumann boundary conditions. We obtain the real eigenvalues

$$\lambda_{1,2} = -\frac{\nu + (1 + \vartheta)\xi_n}{2} \pm \sqrt{\left(\frac{\nu + (1 + \vartheta)\xi_n}{2}\right)^2 - \omega_n}, \quad \lambda_3 = -\epsilon - \xi_n,$$

where $\nu = 2\epsilon + \mu + \delta + d$, $\omega_n = (\epsilon + \mu + \delta + \xi_n)(\epsilon + d + \vartheta \xi_n) - \mu$, and $\xi_n = \frac{2n^2\pi^2}{L^2}$ is an eigenvalue of the spatial eigenvalue problem $\Delta \mathbf{u} + \xi_n \mathbf{u} = 0$.

We have instability whenever $\lambda_+ = \lambda_1 > 0$, that is

$$(\epsilon + \mu + \delta + \xi_n)(\epsilon + d + \vartheta \xi_n) < \mu.$$

From this inequality, we can obtain a necessary condition for the existence of unstable modes for Neumann boundary conditions (onset of epidemic wave, see [23]),

$$0 < d < \frac{\mu}{\epsilon + \mu + \delta} - \epsilon,$$

and, for Dirichlet boundary conditions,

$$0 < d < \frac{\mu}{\epsilon + \mu + \delta + \xi_1} - \epsilon - \vartheta \xi_1.$$

If $1 - (\epsilon + d + \vartheta \xi) \leq 0$, we have linear stability for any μ and δ . If $1 - (\epsilon + d + \vartheta \xi) > 0$, we obtain linear stability whenever $\mu < \mu_c$, with the critical value given by

$$\mu_c = \frac{(\epsilon + \delta + \xi)(\epsilon + d + \vartheta \xi)}{1 - (\epsilon + d + \vartheta \xi)},$$

where $\xi = \xi_1$ for Dirichlet b.c. and $\xi = 0$ for Neumann b.c.

We note that the linearized system (28) has two equations uncoupled from the other. Moreover, the sub-system of Eqs. (28)_{2–3} is symmetrizable (see [34]), and we can find optimal Lyapunov parameters in the classical energy to obtaining the coincidence of linear and nonlinear stability regions. In fact, if we used the energy

$$E_0 = \frac{1}{2}(a\|u\|^2 + \mu\|v\|^2 + \|w\|^2),$$

where a is a positive parameter to be chosen with a variational and optimization method, we could obtain the critical nonlinear energy stability parameter μ_{E_0} which coincides with μ_c . Instead here we use our procedure and observe that our method is more general than the best coupling parameters method (i.e. the classical energy with optimal Lyapunov parameters) and contains the results of the symmetrization method as a particular case. For this, in the present example, we apply the canonical reduction method in the case of Dirichlet boundary conditions. The more realistic case of Neumann boundary conditions will be studied in a forthcoming paper.

We fix $\xi_n = \xi_1$ in the expression of the eigenvalues and compute the transformation matrix Q and its inverse Q^{-1} , we obtain the transformed system

$$\begin{cases} V_{1t} = G_{11}V_1 + G_{12}V_2 + F_{11}\Delta V_1 + F_{12}\Delta V_2 + N_1, \\ V_{2t} = G_{21}V_1 + G_{22}V_2 + F_{21}\Delta V_1 + F_{22}\Delta V_2 + N_2, \\ V_{3t} = G_{31}V_1 + G_{32}V_2 - \epsilon V_3 + F_{31}\Delta V_1 + F_{32}\Delta V_2 + \Delta V_3 + N_3, \end{cases}$$

where G_{ij} , F_{ij} and N_i are easily computed.

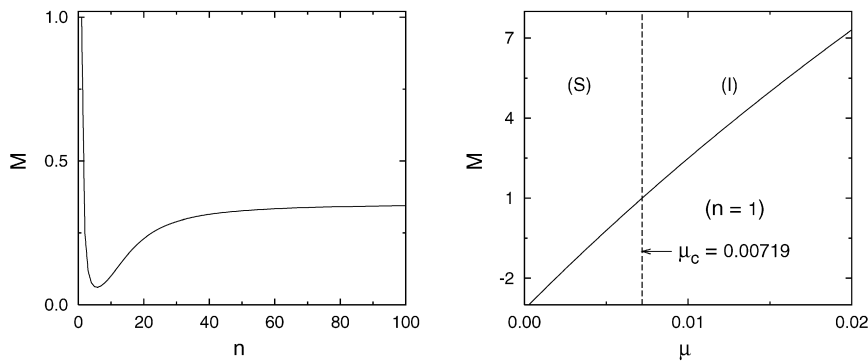


Fig. 1. On the left we report the *maximum* M of the solutions of Eq. (15), near the critical instability parameters μ_c , as a function of the integer n . On the right we report the *maximum* M , evaluated for $n = 1$, as a function of μ and we observe two different, stability (S) and instability (I), regions.

For the new Lyapunov function $E(t) = \frac{1}{2} \|V\|^2$, we have the identity

$$\begin{aligned} \dot{E} = & G_{11} \|V_1\|^2 + (G_{12} + G_{21})(V_1, V_2) + G_{22} \|V_2\|^2 - F_{11} \|\nabla V_1\|^2 \\ & - (F_{12} + F_{21})(\nabla V_1, \nabla V_2) - F_{22} \|\nabla V_2\|^2 + G_{31}(V_1, V_3) + G_{32}(V_2, V_3) \\ & - \epsilon \|V_3\|^2 - F_{31}(\nabla V_1, \nabla V_3) - F_{32}(\nabla V_2, \nabla V_3) - \|\nabla V_3\|^2 + (\tilde{N}(V), V). \end{aligned}$$

By solving the maximum problem as in (14)–(16), it can be proved that if $\mu < \mu_c$, we have nonlinear stability. The nonlinear terms can be controlled as in the Brusselator model, and the constants c_i can be chosen in an optimal way to obtaining a computable radius of attraction for the initial data.

Following the general theory, here we consider as set of input physical parameters (see [23]), the values: $a = 1 \text{ year}^{-1}$, $b = 0.5 \text{ year}^{-1}$, $\alpha = 0.2 \text{ day}^{-1}$, $\beta = 80 \text{ km}^2 \text{ year}^{-1} \text{ fox}^{-1}$, $\sigma = 0.036 \text{ day}^{-1}$, $D = 200 \text{ km}^2 \text{ year}^{-1}$, $D_1 = 0.5D$, $K = 2 \text{ fox km}^{-2}$, $L = 79.06 \text{ km}$ and we obtain as stability–instability critical parameter the value $\mu_c \approx 0.00719$.

In Fig. 1 we report the solution of the *maximum* problem given by (14)–(16). In particular, on the left of Fig. 1 we report the maximum M of the solutions of equation $\det(T_{ij}) = 0$, evaluated near the critical instability parameter μ_c , as a function of the integer n . The numerical results show that for all $n \geq 1$ the maximum M is less than or equal to 1 and in particular, only in correspondence of the critical value μ_c , $M = 1$ for $n = 1$.

On the right of Fig. 1 we report the numerical results of stability–instability regimes for the physical system as function of the parameter μ . In fact, Fig. 1 shows the maximum M , evaluated for $n = 1$, as a function of the μ parameter, and we can observe two different regions in which we have stability regime (S) for $\mu < \mu_c$ ($M < 1$) and instability regime (I) for $\mu > \mu_c$ ($M > 1$).

We now observe that the nonlinear term $(\tilde{N}(V), V)$ is the sum of cubic terms of the type $\int_{\Omega} fgh \, d\Omega$. By using the Hölder and Poincaré inequalities, and the well-known inequalities $\|f\|_6^2 \leq \frac{\sqrt[3]{4}}{\sqrt{3}\sqrt{\pi^2}} \|\nabla f\|^2$, $\|f\|_{3/2} \leq |\Omega|^{1/6} \|f\|$, we easily find a known positive constant C_1 such that the inequality

$$\dot{E}_1(t) \leq 2\xi[(M-1) + C_1 E_1^{1/2}] E_1$$

holds. From this, by assuming that $E_1(0) < \frac{(1-M)^2}{C_1^2}$, we obtain the exponential decay (12) where now $a = 2(1-M)\xi$, $\alpha = 1$ and $b = 2\xi C_1$.

4. Concluding remarks

We have studied the nonlinear stability of constant solutions to some reaction–diffusion systems by defining an optimal Lyapunov function which gives sharp stability thresholds, and have applied the general method to two well-known and interesting problems. These examples of the procedure are helpful in understanding its implementation to

other cases. The method is valid also in the ODEs case and it is equivalent to classical methods (see, for example, [2,11,12]).

We stress that it is important—both for theoretical and applicative problems—to have a procedure to construct a Lyapunov function to yield an optimal stability threshold in PDEs systems. Sometimes the classical energy E_0 gives optimal results for instance in the symmetric or symmetrizable case (see e.g. [10,32,33]). However, there are instances where *ad hoc* functions have been devised to yield sharp results, see e.g. [20,26,27,34]. The present method contains the symmetrizable case a particular case. We also note that the procedure is independent of the dimension n of the systems. Moreover, the critical stability value R_E and the maximum M given by Eqs. (14)–(16) can be easily found by a computer algebra system or numerical programs especially for large dimension. We obtain also a known (computable and finite) radius of attraction of initial data which depends explicitly on the parameters of the system (this differs from the linearization principle). The method is sufficiently general to be applied also to other partial differential systems (for example in fluid dynamics and flows in porous media), and some preliminary results have been obtained in [19,21].

We note that there are still many *open problems*, for instance:

- (i) to find optimal Lyapunov functions in the reaction–diffusion systems with the linear operator A depending explicitly on the spatial variables \mathbf{x} and possibly depending on ∇U ,
- (ii) to find optimal Lyapunov functions in the reaction–diffusion systems with general boundary conditions (in the linear and nonlinear case).

Some of these issues will be the main topics of further research.

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