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An iterative method for finding common solutions of equilibrium and fixed point problems

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Abstract

We introduce an iterative method for finding a common element of the set of solutions of an equilibrium problem and of the set of fixed points of a finite family of nonexpansive mappings in a Hilbert space. We prove the strong convergence of the proposed iterative algorithm to the unique solution of a variational inequality, which is the optimality condition for a minimization problem.

Keywords: Equilibrium problem; Fixed point; Nonexpansive mapping; Variational inequality; Iterative algorithm

1. Introduction

Let H be a real Hilbert space and A a bounded linear operator on H. Assume that A is strongly positive with coefficient $\bar{\gamma}$; that is, there is a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geqslant \bar{\gamma}\|x\|^2 \quad \forall x \in H.$$

Let T be a nonexpansive mapping on H (i.e. $\|Tx - Ty\| \leqslant \|x - y\|$ for $x, y \in H$). We denote by $\mathrm{Fix}(T)$ the set of fixed points of T. Namely, $\mathrm{Fix}(T) = \{x \in H\colon Tx = x\}$. It is well known that $\mathrm{Fix}(T)$ is always closed convex, and also nonempty provided T has a bounded trajectory (cf. [10]).

Finding an optimal point in the intersection F of the fixed points set of a finite family of nonexpansive mappings is a problem of interest in various branches of sciences; see [2,3,6,7,9,25] and also see [23] for solving the variational problems defined on the set of common fixed points of finitely many nonexpansive mappings.

Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings. Assume throughout the rest of this paper that

$$F := \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \neq \emptyset.$$

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For $n > N$, T_n is understood as $T_{n \bmod N}$ with the mod function taking values in $\{1, 2, \dots, N\}$.

Let u be a fixed element of H. In [21], Xu proved that the sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} = (I - \epsilon_{n+1} A) T_{n+1} x_n + \epsilon_{n+1} u \tag{1}$$

converges strongly to the solution of the quadratic minimization problem

$$\min_{x \in F} \frac{1}{2} \langle Ax, x \rangle - \langle x, u \rangle$$

under suitable hypotheses on $\{\epsilon_n\}$ and under the additional hypothesis,

$$F = \operatorname{Fix}(T_1 T_2 \cdots T_N) = \operatorname{Fix}(T_N T_1 \cdots T_{N-1}) = \cdots = \operatorname{Fix}(T_2 T_3 \cdots T_N T_1). \tag{2}$$

In [24], Yao modified the algorithm (1) without the assumption (2) by combining (1) with the viscosity approximation method of Moudafi [14] (see also Xu [22] and the recent work [12]). Also following Atsushiba and Takahashi [1], Yao defined the mappings

$$\begin{aligned}
U_{n,1} &:= \lambda_{n,1} T_1 + (1 - \lambda_{n,1}) I, \\
U_{n,2} &:= \lambda_{n,2} T_2 U_{n,1} + (1 - \lambda_{n,2}) I, \\
&\ \ \vdots \\
U_{n,N-1} &:= \lambda_{n,N-1} T_{N-1} U_{n,N-2} + (1 - \lambda_{n,N-1}) I, \\
W_n \equiv U_{n,N} &:= \lambda_{n,N} T_N U_{n,N-1} + (1 - \lambda_{n,N}) I
\end{aligned} \tag{3}$$

and introduced the iterative scheme

$$x_{n+1} := \epsilon_n \gamma f(x_n) + \beta x_n + \big((1 - \beta) I - \epsilon_n A\big) W_n x_n, \tag{4}$$

where $f : H \to H$ is an α-contraction (i.e. $\|f(x) - f(y)\| \leqslant \alpha \|x - y\|$ for all $x, y \in H$). Under suitable hypotheses on the sequences $\{\lambda_{n,i}\}_{i=1}^N$ and ϵ_n and under the further assumption

$$\|I - A\| \leqslant 1 - \alpha\gamma,$$

he tried to prove that the sequence generated by the explicit scheme (4) strongly converges to the unique solution x^* of the variational inequality

$$\big\langle (A - \gamma f) x^*, x - x^* \big\rangle \geqslant 0, \quad x \in F, \tag{5}$$

which is the optimality condition for the minimization problem

$$\min_{x \in F} \frac{1}{2} \langle Ax, x \rangle - h(x)$$

where h is a potential function for γf (i.e. $h'(x) = \gamma f(x)$ for $x \in H$).

Unfortunately, there is a gap in his proof due to the fact that for a double sequence $\{\nu_{n,k}\}_{n,k \in \mathbb{N}}$ of real numbers, the change of the order of the iterated limits

$$\lim_k \limsup_n \nu_{n,k} = \limsup_n \lim_k \nu_{n,k}$$

is not always true even if, for each k, the $\lim_n \nu_{n,k}$ exists and is independent of k (this is however true if the $\lim_n \nu_{n,k}$ exists and is attained uniformly in k). The same imperfect occurred in [1], where the technique used in [24] was initially introduced.

In our main result we shall use a different approach to get the convergence of a scheme which is more general than (4).

On the other hand, let C be a nonempty closed convex subset of H. Let $G : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for G is to determine its equilibrium points, i.e. the set

$$\operatorname{EP}(G) := \big\{ x \in C \colon\ G(x, y) \geqslant 0\ \forall y \in C \big\}. \tag{6}$$

Many problems in applied sciences reduce into finding some element of $\operatorname{EP}(G)$, see [4,8].

Given any $r > 0$. It is shown [8] that under suitable hypotheses on G (to be stated precisely in Section 2), the mapping $S_r : H \to C$ defined by

$$S_r(x) := \left\{ z \in C\colon\ G(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geqslant 0\ \forall y \in C \right\}$$

is single-valued and firmly nonexpansive and satisfies $\mathrm{Fix}(S_r) = \mathrm{EP}(G)$.

Using this result, S. Takahashi and W. Takahashi [20] very recently introduced a viscosity approximation method for finding a common element of $\mathrm{EP}(G)$ and $\mathrm{Fix}(S)$, where S is a nonexpansive mapping.

Starting with an arbitrary element x_1 in H, they defined the sequences $\{u_n\}$ and $\{x_n\}$ recursively by

$$\begin{cases} G(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geqslant 0 \quad \forall y \in H, \\ x_{n+1} = \epsilon_n f(x_n) + (I - \epsilon_n)Su_n. \end{cases} \tag{7}$$

They proved that under certain appropriate conditions over ϵ_n and r_n, the sequences $\{x_n\}$ and $\{u_n\}$ both converge strongly to $z = P_{\mathrm{Fix}(S)\cap \mathrm{EP}(G)} f(z)$. (Here P_K denotes the nearest point projection from H onto a closed convex subset K of H.)

Moreover, it is shown in [13] that the sequence $\{x_n\}$ defined by the scheme

$$x_{n+1} = \epsilon_n \gamma f(x_n) + (I - \epsilon_n A)Sx_n \tag{8}$$

converges strongly to $z = P_{\mathrm{Fix}(S)}(I - A + \gamma f)(z)$.

By combining the schemes (7) and (8), Plubtieng and Punpaeng [15] proposed the following algorithm

$$\begin{cases} G(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geqslant 0, \quad \forall y \in H, \\ x_{n+1} = \epsilon_n \gamma f(x_n) + (I - \epsilon_n A)Su_n. \end{cases} \tag{9}$$

They proved that if the sequences $\{\epsilon_n\}$ and $\{r_n\}$ of parameters satisfy appropriate conditions, then the sequences $\{x_n\}$ and $\{u_n\}$ both converge to the unique solution z of the variational inequality

$$\big\langle (A - \gamma f)z, x - z\big\rangle \geqslant 0 \quad \forall x \in \mathrm{Fix}(S) \cap \mathrm{EP}(G), \tag{10}$$

which is the optimality condition for the minimization problem

$$\min_{x\in \mathrm{Fix}(S)\cap \mathrm{EP}(G)} \frac{1}{2}\langle Ax, x\rangle - h(x)$$

where h is a potential function for γf.

Note that the result in [13] is a particular case of this, corresponding to the choice $G(x, y) = 0$ (so that $u_n = x_n$).

In this paper we combine the scheme (4) for a finite family of nonexpansive mappings with the method (9) for the equilibrium problem and propose the following explicit scheme

$$\begin{cases} G(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geqslant 0 \quad \forall y \in H, \\ x_{n+1} = \epsilon_n \gamma f(x_n) + \beta x_n + \big((1 - \beta)I - \epsilon_n A\big)W_n u_n. \end{cases} \tag{11}$$

We prove under weaker hypotheses that both sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $x^* \in F$ which is an equilibrium point for G and is the unique solution of the variational inequality

$$\big\langle (A - \gamma f)x^*, x - x^*\big\rangle \geqslant 0 \quad \forall x \in F \cap \mathrm{EP}(G). \tag{12}$$

This result covers all previous schemes (1), (4), (7), (8) and (9).

2. Preliminaries

Let C be a closed convex subset of H. Recall that the (nearest point) projection P_C from H onto C assigns to each $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y\in C} \|x - y\|.$$

The following characterizes the projection P_C.

Lemma 2.1. *(See [19].) Given $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if there holds the inequality*

$$\langle x - y, y - z\rangle \geqslant 0 \quad \forall z \in C.$$

Lemma 2.2. *(See [16].) Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n$ and $\limsup_{n\to\infty} \beta_n < 1$. Suppose*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n$$

for all integers $n \geqslant 0$ and

$$\limsup_{n\to\infty}\bigl(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|\bigr) \leqslant 0.$$

Then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.

Lemma 2.3. *(See [21].) Assume $\{a_n\}$ is a sequence of nonnegative numbers such that*

$$a_{n+1} \leqslant (1 - \gamma_n)a_n + \delta_n, \quad n \geqslant 0,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and δ_n is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \leqslant 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4. *(See [13].) Let A be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma}$ and $0 < \rho \leqslant \|A\|^{-1}$. Then $\|I - \rho A\| \leqslant 1 - \rho\overline{\gamma}$.*

Lemma 2.5. *(See [8].) Let C be a nonempty closed convex subset of H and $G : C \times C \to \mathbb{R}$ satisfy*

(A1) $G(x, x) = 0$ *for all* $x \in C$;
(A2) *G is monotone, i.e.* $G(x, y) + G(y, x) \leqslant 0$ *for all* $x, y \in C$;
(A3) *for all* $x, y, z \in C$,

$$\liminf_{t\to 0} G\bigl(tz + (1 - t)x, y\bigr) \leqslant G(x, y);$$

(A4) *for all $x \in C$, $y \mapsto G(x, y)$ is convex and lower semicontinuous.*

For $x \in C$ and $r > 0$, set $S_r : H \to C$ to be

$$S_r(x) := \left\{z \in C\colon\ G(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geqslant 0,\ \forall y \in C\right\}.$$

Then S_r is well defined and the following hold:

(1) *S_r is single-valued*;
(2) *S_r is firmly nonexpansive, i.e.*

$$\|S_r x - S_r y\|^2 \leqslant \langle S_r x - S_r y, x - y\rangle,$$

for all $x, y \in H$;
(3) $\mathrm{Fix}(S_r) = \mathrm{EP}(G)$;
(4) *$\mathrm{EP}(G)$ is closed and convex.*

Definition 2.6. Let C be a nonempty convex subset of a Banach space. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpansive mappings of C into itself and let $\lambda_1, \ldots, \lambda_N$ be real numbers such that $0 \leqslant \lambda_i \leqslant 1$ for every $i = 1, 2, \ldots, N$. We define a mapping W of C into itself as follows:

$$
\begin{aligned}
&U_1 := \lambda_1 T_1 + (1-\lambda_1)I,\\
&U_2 := \lambda_2 T_2 U_1 + (1-\lambda_2)I,\\
&\quad\vdots\\
&U_{N-1} := \lambda_{N-1} T_{N-1} U_{N-2} + (1-\lambda_{N-1})I,\\
&W := U_N = \lambda_N T_N U_{N-1} + (1-\lambda_N)I.
\end{aligned}
\tag{13}
$$

Such a mapping W is called the *W-mapping* generated by $T_1, \dots, T_N$ and $\lambda_1, \dots, \lambda_N$.

The concept of W-mappings was introduced in [17,18]. It is now one of the main tools in studying convergence of iterative methods for approaching a common fixed of nonlinear mappings; more recent progresses can be found in [1,5,11,24] and the references cited therein.

Lemma 2.7. *(See [1].) Let C be a nonempty closed convex set of a strictly convex Banach space. Let $T_1, \dots, T_N$ be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^N \mathrm{Fix}(T_i) \neq \emptyset$ and let $\lambda_1, \dots, \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, \dots, N-1$ and $0 < \lambda_N \leqslant 1$. Let W be the W-mapping of C into itself generated by $T_1, \dots, T_N$ and $\lambda_1, \dots, \lambda_N$. Then* $\mathrm{Fix}(W) = \bigcap_{i=1}^N \mathrm{Fix}(T_i)$.

Lemma 2.8. *Let C be a nonempty convex subset of a Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and $\{\lambda_{n,i}\}_{i=1}^N$ be sequences in* $[0,1]$ *such that $\lambda_{n,i} \to \lambda_i$ $(i = 1, \dots, N)$. Moreover for every $n \in \mathbb{N}$, let W and W_n be the W-mappings generated by $T_1, \dots, T_N$ and $\lambda_1, \dots, \lambda_N$ and $T_1, \dots, T_N$ and $\lambda_{n,1}, \dots, \lambda_{n,N}$ respectively. Then for every $x \in C$, it follows that*

$$
\lim_n \|W_n x - Wx\| = 0.
$$

Proof. Let $x \in C$ and U_k and $U_{n,k}$ be generated by $T_1, \dots, T_N$ and $\lambda_1, \dots, \lambda_N$ and $T_1, \dots, T_N$ and $\lambda_{n,1}, \dots, \lambda_{n,N}$ respectively, as in Definition 2.6. We have

$$
\begin{aligned}
\|U_{n,1}x - U_1 x\| &= \big\|\lambda_{n,1}T_1 x + (1-\lambda_{n,1})x - \lambda_1 T_1 x - (1-\lambda_1)x\big\| = \|\lambda_{n,1}T_1 x - \lambda_{n,1}x - \lambda_1 T_1 x + \lambda_1 x\|\\
&= |\lambda_{n,1} - \lambda_1| \|T_1 x - x\|.
\end{aligned}
$$

Let $k \in \{2, \dots, N\}$, then

$$
\begin{aligned}
\|U_{n,k}x - U_k x\| &= \big\|\lambda_{n,k}T_k U_{n,k-1}x + (1-\lambda_{n,k})x - \lambda_k T_k U_{k-1}x - (1-\lambda_k)x\big\|\\
&= \|\lambda_{n,k}T_k U_{n,k-1}x - \lambda_{n,k}x - \lambda_k T_k U_{k-1}x + \lambda_k x\|\\
&\leqslant \lambda_{n,k}\|T_k U_{n,k-1}x - T_k U_{k-1}x\| + |\lambda_{n,k} - \lambda_k| \|T_k U_{k-1}x\| + |\lambda_{n,k} - \lambda_k| \|x\|\\
&\leqslant \|U_{n,k-1}x - U_{k-1}x\| + |\lambda_{n,k} - \lambda_k| \big(\|T_k U_{k-1}x\| + \|x\|\big).
\end{aligned}
$$

Hence,

$$
\|W_n x - Wx\| = \|U_{n,N}x - U_N x\| \leqslant \sum_{k=2}^N |\lambda_{n,k} - \lambda_k| \big(\|T_k U_{k-1}x\| + \|x\|\big) + |\lambda_{n,1} - \lambda_1| \|T_1 x - x\|.
$$

Since for every $k \in \{1, \dots, N\}$, $\lim_n |\lambda_{n,k} - \lambda_k| = 0$, the result follows. $\square$

The following lemma is an immediate consequence of the inner product on H.

Lemma 2.9. *For all $x, y \in H$, there holds the inequality*

$$
\|x + y\|^2 \leqslant \|x\|^2 + 2\langle y, x + y\rangle.
$$

3. Main result

Theorem 3.1. *Let H be a Hilbert space, C a closed convex nonempty subset of H, $\{T_i\}_{i=1}^N$ a finite family of nonexpansive mappings from H into itself, $G : C \times C \to \mathbb{R}$ a bifunction, A a strongly positive bounded linear operator with coefficient $\overline{\gamma}$ and f an α-contraction on H for some $0 < \alpha < 1$. Moreover, let $\{\epsilon_n\}$ be a sequence in $(0, 1)$, $\{\lambda_{n,i}\}_{i=1}^N$ a sequence in $[a, b]$ with $0 < a \leqslant b < 1$, $\{r_n\}$ a sequence in $(0, \infty)$ and γ and β two real numbers such that $0 < \beta < 1$ and $0 < \gamma < \overline{\gamma}/\alpha$. Assume*

(i) *the bifunction G satisfies*
  (A1) *$G(x, x) = 0$ for all $x \in C$;*
  (A2) *G is monotone, i.e. $G(x, y) + G(y, x) \leqslant 0$ for all $x, y \in C$;*
  (A3) *for all $x, y, z \in C$,*
$$\lim_{t\to 0} G\big(tz + (1 - t)x, y\big) \leqslant G(x, y);$$
  (A4) *for all $x \in C$, $y \mapsto G(x, y)$ is convex and lower semicontinuous*;
  (B1) *$F \cap \mathrm{EP}(G) \neq \emptyset$;*
(ii) *the sequence $\{\epsilon_n\}$ satisfies*
  (C1) $\lim_n \epsilon_n = 0$; *and*
  (C2) $\sum_{n=1}^{\infty} \epsilon_n = \infty$;
(iii) *the sequence $\{r_n\}$ satisfies*
  (D1) $\liminf_n r_n > 0$; *and*
  (D2) $\lim_n r_n/r_{n+1} = 1$ (*this is weaker than the condition* $\lim_n |r_{n+1} - r_n| = 0$);
(iv) *the finite family of sequences $\{\lambda_{n,i}\}_{i=1}^N$ satisfies*
  (E1) $\lim_n |\lambda_{n,i} - \lambda_{n-1,i}| = 0$ *for every $i \in \{1, \dots, N\}$.*

For every $n \in \mathbb{N}$, let W_n be the W-mapping generated by $T_1, \dots, T_N$ and $\lambda_{n,1}, \dots, \lambda_{n,N}$. If $\{x_n\}$ and $\{u_n\}$ are the sequences generated by $x_1 \in H$ and $\forall n \geqslant 1$

$$\begin{cases} G(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geqslant 0 \quad \forall y \in H, \\ x_{n+1} = \epsilon_n \gamma f(x_n) + \beta x_n + \big((1 - \beta)I - \epsilon_n A\big)W_n u_n \end{cases} \tag{14}$$

then both $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* \in F$, where x^* is an equilibrium point for G and is the unique solution of variational inequality (12)*, i.e.,*

$$x^* = P_{F\cap \mathrm{EP}(G)}\big(I - (A - \gamma f)\big)x^*.$$

Proof. By Lemma 2.5, it follows that for every $n \in \mathbb{N}$, there exists a nonexpansive mapping $S_{r_n} : H \to H$, such that $u_n = S_{r_n} x_n$ and $\mathrm{EP}(G) = \mathrm{Fix}(S_{r_n})$. Whenever needed, we shall equivalently write scheme (14) as

$$x_{n+1} = \epsilon_n \gamma f(x_n) + \beta x_n + \big((1 - \beta)I - \epsilon_n A\big)W_n S_{r_n} x_n. \tag{15}$$

Moreover, since $\epsilon_n \to 0$, we shall assume that $\epsilon_n \leqslant (1 - \beta)\|A\|^{-1}$ and $1 - \epsilon_n(\bar{\gamma} - \alpha\gamma) > 0$.

Observe that, if $\|u\| = 1$, then

$$\big\langle\big((1 - \beta)I - \epsilon_n A\big)u, u\big\rangle = (1 - \beta) - \epsilon_n\langle Au, u\rangle \geqslant \big(1 - \beta - \epsilon_n\|A\|\big) \geqslant 0.$$

By Lemma 2.4, we have

$$\big\|(1 - \beta)I - \epsilon_n A\big\| \leqslant 1 - \beta - \epsilon_n\overline{\gamma}. \tag{16}$$

We shall divide the proof into several steps.

Step 1. The sequence $\{x_n\}$ is bounded.
Proof of Step 1. Let $v \in \mathrm{EP}(G) \cap F$ and set

$$M = \max\big\{\|x_1 - v\|, \big\|\gamma f(v) - Av\big\|/(\overline{\gamma} - \alpha\gamma)\big\}.$$

We shall use induction to prove

$$\|x_n - v\| \leqslant M \tag{17}$$

for all $n \geqslant 1$. Clearly $\|x_1 - v\| \leqslant M$. Assume (17) holds for some $n > 1$. Then noting (15) and the fact that $v = W_n S_{r_n} v = S_{r_n} v$, we derive that

$$\begin{aligned}
\|x_{n+1} - v\| &= \big\|\big[(1-\beta)I - \epsilon_n A\big](W_n S_{r_n} x_n - W_n S_{r_n} v) + \epsilon_n \gamma\big[f(x_n) - f(v)\big] \\
&\quad + \epsilon_n\big(\gamma f(v) - Av\big) + \beta(x_n - v)\big\| \\
&\leqslant \big[1 - \epsilon_n(\overline{\gamma} - \alpha\gamma)\big]\|x_n - v\| + \epsilon_n(\overline{\gamma} - \alpha\gamma)\big\|\gamma f(v) - Av\big\|/(\overline{\gamma} - \alpha\gamma) \\
&\leqslant \max\big\{\|x_n - v\|, \big\|\gamma f(v) - Av\big\|/(\overline{\gamma} - \alpha\gamma)\big\} \leqslant M.
\end{aligned}$$

Step 2. Let $\{w_n\}$ be a bounded sequence in H. Then

$$\lim_{n\to\infty} \|W_{n+1} S_{r_{n+1}} w_n - W_{n+1} S_{r_n} w_n\| = 0. \tag{18}$$

Proof of Step 2. Since $\{w_n\}$ is bounded, we know that

$$L := \sup_n \big\{\|w_n\| + \|S_{r_{n+1}} w_n\|\big\} < \infty.$$

Now,

$$\|W_{n+1} S_{r_{n+1}} w_n - W_{n+1} S_{r_n} w_n\| \leqslant \|S_{r_{n+1}} w_n - S_{r_n} w_n\|.$$

By the definition of S_{r_n} (see Lemma 2.5) we have

$$G(S_{r_{n+1}} w_n, y) + \frac{1}{r_{n+1}}\langle y - S_{r_{n+1}} w_n, S_{r_{n+1}} w_n - w_n\rangle \geqslant 0 \quad \forall y \in C,$$

and

$$G(S_{r_n} w_n, y) + \frac{1}{r_n}\langle y - S_{r_n} w_n, S_{r_n} w_n - w_n\rangle \geqslant 0 \quad \forall y \in C.$$

In particular,

$$G(S_{r_{n+1}} w_n, S_{r_n} w_n) + \frac{1}{r_{n+1}}\langle S_{r_n} w_n - S_{r_{n+1}} w_n, S_{r_{n+1}} w_n - w_n\rangle \geqslant 0$$

and

$$G(S_{r_n} w_n, S_{r_{n+1}} w_n) + \frac{1}{r_n}\langle S_{r_{n+1}} w_n - S_{r_n} w_n, S_{r_n} w_n - w_n\rangle \geqslant 0.$$

Summing up the last two inequalities and using (A2), we obtain

$$\frac{1}{r_{n+1}}\langle S_{r_n} w_n - S_{r_{n+1}} w_n, S_{r_{n+1}} w_n - w_n\rangle + \frac{1}{r_n}\langle S_{r_{n+1}} w_n - S_{r_n} w_n, S_{r_n} w_n - w_n\rangle \geqslant 0.$$

It then follows that

$$\left\langle S_{r_n} w_n - S_{r_{n+1}} w_n, \frac{S_{r_{n+1}} w_n - w_n}{r_{n+1}} - \frac{S_{r_n} w_n - w_n}{r_n}\right\rangle \geqslant 0. \tag{19}$$

We derive from (19) that

$$\begin{aligned}
0 &\leqslant \left\langle S_{r_{n+1}} w_n - S_{r_n} w_n, S_{r_n} w_n - w_n - \frac{r_n}{r_{n+1}}(S_{r_{n+1}} w_n - w_n)\right\rangle \\
&= \left\langle S_{r_{n+1}} w_n - S_{r_n} w_n, S_{r_n} w_n - S_{r_{n+1}} w_n + S_{r_{n+1}} w_n - w_n - \frac{r_n}{r_{n+1}}(S_{r_{n+1}} w_n - w_n)\right\rangle \\
&= \left\langle S_{r_{n+1}} w_n - S_{r_n} w_n, (S_{r_n} w_n - S_{r_{n+1}} w_n) + \left(1 - \frac{r_n}{r_{n+1}}\right)(S_{r_{n+1}} w_n - w_n)\right\rangle
\end{aligned}$$

which implies that

$$\|S_{r_{n+1}}w_n - S_{r_n}w_n\|^2 \leqslant \left|1 - \frac{r_n}{r_{n+1}}\right| \|S_{r_{n+1}}w_n - S_{r_n}w_n\| \big(\|S_{r_{n+1}}w_n\| + \|w_n\|\big).$$

This implies that

$$\|S_{r_{n+1}}w_n - S_{r_n}w_n\| \leqslant L\left|1 - \frac{r_n}{r_{n+1}}\right|. \tag{20}$$

Therefore, (18) is a consequence of (20) and condition (D2).

Step 3. Let $\{w_n\}$ be a bounded sequence in H. Then

$$\lim_{n\to\infty} \|W_{n+1}w_n - W_n w_n\| = 0. \tag{21}$$

Proof of Step 3. Let $j \in \{0, \ldots, N-2\}$ and set

$$M := \sup_{n\in\mathbb{N}} \left\{ \|w_n\| + \|T_1 w_n\| + \sum_{j=2}^{N} \|T_j U_{n,j-1} w_n\| \right\} < \infty.$$

It follows from (3) that

$$\begin{aligned}
&\|U_{n+1,N-j}w_n - U_{n,N-j}w_n\| \\
&\quad = \big\|\lambda_{n+1,N-j}T_{N-j}U_{n+1,N-j-1}w_n + (1-\lambda_{n+1,N-j})w_n - \lambda_{n,N-j}T_{N-j}U_{n,N-j-1}w_n - (1-\lambda_{n,N-j})w_n\big\| \\
&\quad \leqslant \lambda_{n+1,N-j}\|T_{N-j}U_{n+1,N-j-1}w_n - T_{N-j}U_{n,N-j-1}w_n\| \\
&\qquad + |\lambda_{n+1,N-j} - \lambda_{n,N-j}|\|T_{N-j}U_{n,N-j-1}w_n\| + |\lambda_{n+1,N-j} - \lambda_{n,N-j}|\|w_n\| \\
&\quad \leqslant \|U_{n+1,N-j-1}w_n - U_{n,N-j-1}w_n\| + \big(\|w_n\| + \|T_{N-j}U_{n,N-j-1}w_n\|\big)|\lambda_{n+1,N-j} - \lambda_{n,N-j}| \\
&\quad \leqslant \|U_{n+1,N-j-1}w_n - U_{n,N-j-1}w_n\| + M|\lambda_{n+1,N-j} - \lambda_{n,N-j}|.
\end{aligned}$$

Thus, repeatedly using the above recursive inequalities, we deduce

$$\begin{aligned}
\|W_{n+1}w_n - W_n w_n\| &= \|U_{n+1,N}w_n - U_{n,N}w_n\| \\
&\leqslant M\sum_{j=2}^{N} |\lambda_{n+1,j} - \lambda_{n,j}| + |\lambda_{n+1,1} - \lambda_{n,1}|\big(\|w_n\| + \|T_1 w_n\|\big) \\
&\leqslant M\sum_{j=1}^{N} |\lambda_{n+1,j} - \lambda_{n,j}|.
\end{aligned} \tag{22}$$

Now by condition (E1) and using (22), we obtain (21) and Step 3 is proven.

Step 4. $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Proof of Step 4. Define a sequence $\{z_n\}$ by $z_n = (x_{n+1} - \beta x_n)/(1-\beta)$ so that

$$x_{n+1} = \beta x_n + (1-\beta)z_n.$$

We now compute

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \frac{1}{1-\beta}\big\|(x_{n+2} - \beta x_{n+1}) - (x_{n+1} - \beta x_n)\big\| \\
&= \frac{1}{1-\beta}\big\|\gamma\big[\epsilon_{n+1} f(x_{n+1}) - \epsilon_n f(x_n)\big] + \big[(1-\beta)I - \epsilon_{n+1}A\big]W_{n+1}S_{r_{n+1}}x_{n+1} \\
&\quad - \big[(1-\beta)I - \epsilon_n A\big]W_n S_{r_n}x_n\big\| \\
&= \bigg\|\frac{\gamma}{1-\beta}\big[\epsilon_{n+1} f(x_{n+1}) - \epsilon_n f(x_n)\big] - \frac{1}{1-\beta}(\epsilon_{n+1}AW_{n+1}S_{r_{n+1}}x_{n+1} - \epsilon_n AW_n S_{r_n}x_n) \\
&\quad + W_{n+1}S_{r_{n+1}}x_{n+1} - W_n S_{r_n}x_n\bigg\|.
\end{aligned} \tag{23}$$

Since $\{x_n\}$ is bounded and by (23), we have, for some big enough constant $K > 0$,

$$\begin{aligned}\|z_{n+1} - z_n\| &\leqslant \|W_{n+1}S_{r_{n+1}}x_{n+1} - W_nS_{r_n}x_n\| + K(\epsilon_{n+1} + \epsilon_n)\\ &\leqslant \|W_{n+1}S_{r_{n+1}}x_{n+1} - W_{n+1}S_{r_{n+1}}x_n\| + \|W_{n+1}S_{r_{n+1}}x_n - W_nS_{r_n}x_n\| + K(\epsilon_{n+1} + \epsilon_n)\\ &\leqslant \|x_{n+1} - x_n\| + \|W_{n+1}S_{r_{n+1}}x_n - W_{n+1}S_{r_n}x_n\| + \|W_{n+1}u_n - W_nu_n\| + K(\epsilon_{n+1} + \epsilon_n)\end{aligned} \tag{24}$$

where $u_n = S_{r_n}x_n$. Now since $\epsilon_n \to 0$ and by Steps 2 and 3, we immediately conclude from (24) that

$$\begin{aligned}&\limsup_{n\to\infty}\big(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|\big)\\ &\quad\leqslant \limsup_{n\to\infty}\big(\|W_{n+1}S_{r_{n+1}}x_n - W_{n+1}S_{r_n}x_n\| + \|W_{n+1}u_n - W_nu_n\| + K(\epsilon_{n+1} + \epsilon_n)\big) \leqslant 0.\end{aligned}$$

Apply Lemma 2.2 to get $\lim_n \|x_{n+1} - x_n\| = (1-\beta)\lim_n \|x_n - z_n\| = 0$.

Step 5. $\lim_n \|x_n - W_nu_n\| = 0$, where $u_n = S_{r_n}x_n$.
Proof of Step 5. Indeed we have

$$\begin{aligned}\|x_n - W_nu_n\| &\leqslant \|x_n - x_{n+1}\| + \|x_{n+1} - W_nu_n\|\\ &= \|x_n - x_{n+1}\| + \big|\epsilon_n\big[\gamma f(x_n) - AW_nu_n\big] + \beta(x_n - W_nu_n)\big\|\\ &\leqslant \|x_n - x_{n+1}\| + \epsilon_n\big(\gamma\big\|f(x_n)\big\| + \|AW_nu_n\|\big) + \beta\|x_n - W_nu_n\|.\end{aligned}$$

It follows from Step 4 that

$$\|x_n - W_nu_n\| \leqslant \frac{1}{1-\beta}\big(\|x_n - x_{n+1}\| + \epsilon_n\big(\gamma\big\|f(x_n)\big\| + \|AW_nu_n\|\big)\big) \to 0.$$

Step 6. $\lim_n \|x_n - S_{r_n}x_n\| = 0$.
Proof of Step 6. Let $v \in F \cap \mathrm{EP}(G)$. Since S_{r_n} is firmly nonexpansive, we obtain

$$\begin{aligned}\|v - S_{r_n}x_n\|^2 &= \|S_{r_n}v - S_{r_n}x_n\|^2\\ &\leqslant \langle S_{r_n}x_n - S_{r_n}v, x_n - v\rangle = \langle S_{r_n}x_n - v, x_n - v\rangle\\ &= \frac{1}{2}\big(\|S_{r_n}x_n - v\|^2 + \|x_n - v\|^2 - \|x_n - S_{r_n}x_n\|^2\big).\end{aligned}$$

It follows that

$$\|S_{r_n}x_n - v\|^2 \leqslant \|x_n - v\|^2 - \|x_n - S_{r_n}x_n\|^2. \tag{25}$$

Set $y_n = \gamma f(x_n) - AW_nS_{r_n}x_n$ and let $\lambda > 0$ be a constant such that

$$\lambda > \sup_{n,k}\big\{\|y_n\|, \|x_k - v\|\big\}.$$

Using Lemma 2.9 and noting that $\|\cdot\|^2$ is convex, we derive, using (25)

$$\begin{aligned}\|x_{n+1} - v\|^2 &= \big\|\big[(1-\beta)(W_nS_{r_n}x_n - v) + \beta(x_n - v)\big] + \epsilon_n\big[\gamma f(x_n) - AW_nS_{r_n}x_n\big]\big\|^2\\ &\leqslant \big\|(1-\beta)(W_nS_{r_n}x_n - v) + \beta(x_n - v)\big\|^2 + 2\epsilon_n\langle y_n, x_{n+1} - v\rangle\\ &\leqslant (1-\beta)\|W_nS_{r_n}x_n - v\|^2 + \beta\|x_n - v\|^2 + 2\lambda^2\epsilon_n\\ &\leqslant (1-\beta)\|S_{r_n}x_n - v\|^2 + \beta\|x_n - v\|^2 + 2\lambda^2\epsilon_n\\ &= (1-\beta)\big(\|x_n - v\|^2 - \|x_n - S_{r_n}x_n\|^2\big) + \beta\|x_n - v\|^2 + 2\lambda^2\epsilon_n\\ &= \|x_n - v\|^2 - (1-\beta)\|x_n - S_{r_n}x_n\|^2 + 2\lambda^2\epsilon_n.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_n - S_{r_n}x_n\|^2 &\leqslant \frac{1}{1-\beta}\big(\|x_n - v\|^2 - \|x_{n+1} - v\|^2 + 2\lambda^2\epsilon_n\big)\\ &= \frac{1}{1-\beta}\big(\|x_n - x_{n+1}\|^2 + 2\langle x_n - x_{n+1}, x_{n+1} - v\rangle + 2\lambda^2\epsilon_n\big)\\ &\leqslant \frac{1}{1-\beta}\big(\|x_n - x_{n+1}\|^2 + 2\lambda\|x_n - x_{n+1}\| + 2\lambda^2\epsilon_n\big)\\ &\to 0\end{aligned}$$

by Step 4 and condition (C1).

Step 7. The weak ω-limit set of (x_n), $\omega_w(x_n)$, is a subset of $F \cap \mathrm{EP}(G)$.

Proof of Step 7. Let $z \in \omega_w(x_n)$ and let $\{x_{n_m}\}$ be a subsequence of $\{x_n\}$ weakly converging to z. We need to show that $z \in F \cap \mathrm{EP}(G)$.

At first, note that by (A2) and given $y \in C$ we have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geqslant G(y, u_n).$$

In particular,

$$\left\langle y - u_{n_m}, \frac{u_{n_m} - x_{n_m}}{r_{n_m}}\right\rangle \geqslant G(y, u_{n_m}). \tag{26}$$

By condition (A4), $G(y,\cdot)$ is lower semicontinuous and convex, and thus weakly semicontinuous. Step 6 and condition (D1) imply that $(u_{n_m} - x_{n_m})/r_{n_m} \to 0$ in norm. Therefore, letting $m \to \infty$ in (26) yields

$$G(y,z) \leqslant \lim_{m\to\infty} G(y, u_m) \leqslant 0, \quad y \in H.$$

Replacing y with $y_t := ty + (1-t)z$ with $t \in [0,1]$ and using (A1) and (A4), we obtain

$$0 = G(y_t, y_t) \leqslant tG(y_t, y) + (1-t)G(y_t, z) \leqslant tG(y_t, y).$$

Hence

$$G\big(ty + (1-t)z, y\big) \geqslant 0, \quad t \in (0,1],\ y \in H.$$

Letting $t \to 0^+$ and using assumption (A3), we conclude

$$G(z,y) \geqslant 0, \quad y \in H.$$

Therefore, $z \in \mathrm{EP}(G)$.

It remains to prove that $z \in F$. To see this, we observe that we may assume (by passing to a further subsequence if necessary)

$$\lambda_{n_m,k} \to \lambda_k \in (0,1) \quad (k = 1, 2, \dots N).$$

Let W be the W-mapping generated by $T_1, \dots, T_N$ and $\lambda_1, \dots, \lambda_N$. Then by Lemma 2.8, we have, for every $x \in H$,

$$W_{n_m}x \to Wx. \tag{27}$$

Moreover, from Lemma 2.7 it follows that $F = \mathrm{Fix}(W)$. Assume that $z \notin F$; then $z \neq Wz$. Since $z \in \mathrm{EP}(G) = \mathrm{Fix}(S_{r_n})$, by Step 5, (27) and using Opial's property of a Hilbert space, we have

$$\begin{aligned}\liminf_m \|x_{n_m} - z\| &< \liminf_m \|x_{n_m} - Wz\|\\ &\leqslant \liminf_m\big(\|x_{n_m} - W_{n_m}S_{r_{n_m}}x_{n_m}\| + \|W_{n_m}S_{r_{n_m}}x_{n_m} - W_{n_m}S_{r_{n_m}}z\| + \|W_{n_m}z - Wz\|\big)\\ &\leqslant \liminf_m \|x_{n_m} - z\|.\end{aligned}$$

This is a contradiction. Therefore, z must belong to F.

Step 8. Let x^* be the unique solution of the variational inequality (12). That is,

$$\langle (A-\gamma f)x^*, x-x^*\rangle \geqslant 0, \quad x\in F\cap \mathrm{EP}(G). \tag{28}$$

Then

$$\limsup_n \langle (\gamma f - A)x^*, x_n - x^*\rangle \leqslant 0. \tag{29}$$

Proof of Step 8. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_k \langle (\gamma f - A)x^*, x_{n_k} - x^*\rangle = \limsup_n \langle (\gamma f - A)x^*, x_n - x^*\rangle. \tag{30}$$

Without loss of generality, we can assume that (x_{n_k}) weakly converges to some z in C. By Step 7, $z\in F\cap \mathrm{EP}(G)$. Thus combining (30) and (28), we get

$$\limsup_n \langle (\gamma f - A)x^*, x_n - x^*\rangle = \langle (\gamma f - A)x^*, z - x^*\rangle \leqslant 0$$

as required.

Step 9. The sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to x^*.
Proof of Step 9. By the definition (14) (or equivalently, (15)) of $\{x_n\}$ and using Lemmas 2.4 and 2.9, we have (note $u_n = S_{r_n}x_n$)

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \big\|\big[\big((1-\beta)I-\epsilon_n A\big)\big(W_n u_n - x^*\big) + \beta\big(x_n - x^*\big)\big] + \epsilon_n\big(\gamma f(x_n) - Ax^*\big)\big\|^2 \\
&\leqslant \big\|\big((1-\beta)I-\epsilon_n A\big)\big(W_n u_n - x^*\big) + \beta\big(x_n - x^*\big)\big\|^2 + 2\epsilon_n\langle \gamma f(x_n) - Ax^*, x_{n+1}-x^*\rangle \\
&= \left\|(1-\beta)\frac{((1-\beta)I-\epsilon_n A)}{(1-\beta)}\big(W_n u_n - x^*\big) + \beta\big(x_n - x^*\big)\right\|^2 \\
&\quad + 2\epsilon_n\gamma\langle f(x_n) - f\big(x^*\big), x_{n+1}-x^*\rangle + 2\epsilon_n\langle \gamma f\big(x^*\big) - Ax^*, x_{n+1}-x^*\rangle \\
&\leqslant (1-\beta)\left\|\frac{((1-\beta)I-\epsilon_n A)}{(1-\beta)}\big(W_n u_n - x^*\big)\right\|^2 + \beta\|x_n - x^*\|^2 \\
&\quad + 2\epsilon_n\gamma\alpha\|x_n - x^*\|\cdot\|x_{n+1}-x^*\| + 2\epsilon_n\langle \gamma f\big(x^*\big) - Ax^*, x_{n+1}-x^*\rangle \\
&\leqslant \frac{\|(1-\beta)I-\epsilon_n A\|^2}{1-\beta}\|W_n u_n - x^*\|^2 + \beta\|x_n - x^*\|^2 \\
&\quad + \epsilon_n\gamma\alpha\big(\|x_n - x^*\|^2 + \|x_{n+1}-x^*\|^2\big) + 2\epsilon_n\langle \gamma f\big(x^*\big) - Ax^*, x_{n+1}-x^*\rangle \\
&\leqslant \left(\frac{((1-\beta)-\overline{\gamma}\epsilon_n)^2}{1-\beta} + \beta + \epsilon_n\gamma\alpha\right)\|x_n - x^*\|^2 + \epsilon_n\gamma\alpha\|x_{n+1}-x^*\|^2 \\
&\quad + 2\epsilon_n\langle \gamma f\big(x^*\big) - Ax^*, x_{n+1}-x^*\rangle \\
&= \left(1-(2\overline{\gamma}-\alpha\gamma)\epsilon_n + \frac{\overline{\gamma}^2\epsilon_n^2}{1-\beta}\right)\|x_n - x^*\|^2 + \alpha\gamma\epsilon_n\|x_{n+1}-x^*\|^2 \\
&\quad + 2\epsilon_n\langle \gamma f\big(x^*\big) - Ax^*, x_{n+1}-x^*\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\leqslant \left(1-\frac{2(\overline{\gamma}-\alpha\gamma)\epsilon_n}{1-\alpha\gamma\epsilon_n}\right)\|x_n - x^*\|^2 \\
&\quad + \frac{\epsilon_n}{1-\alpha\gamma\epsilon_n}\left[2\langle \gamma f\big(x^*\big) - Ax^*, x_{n+1}-x^*\rangle + \frac{\overline{\gamma}^2\epsilon_n}{1-\beta}\|x_n - x^*\|^2\right].
\end{aligned} \tag{31}$$

Set

$$a_n = \|x_n - x^*\|^2, \qquad \gamma_n = \frac{2(\overline{\gamma} - \alpha\gamma)\epsilon_n}{1 - \alpha\gamma\epsilon_n},$$
$$\delta_n = \frac{\epsilon_n}{1 - \alpha\gamma\epsilon_n}\left[2\langle\gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle + \frac{\overline{\gamma}^2\epsilon_n}{1-\beta}\|x_n - x^*\|^2\right].$$

Then we can rewrite (31) as

$$a_{n+1} \leqslant (1 - \gamma_n)a_n + \delta_n. \tag{32}$$

It is easily verified from conditions (C1) and (C2), and Step 8 that

$$\gamma_n \to 0, \quad \sum_{n=1}^{\infty} \gamma_n = \infty, \qquad \limsup_{n\to\infty} \delta_n/\gamma_n \leqslant 0.$$

Therefore we can apply Lemma 2.3 to (32) to conclude that $a_n \to 0$. Namely, $x_n \to x^*$ in norm.

Finally, noticing

$$\|u_n - x^*\| = \|S_{r_n}x_n - S_{r_n}x^*\| \leqslant \|x_n - x^*\|$$

we also conclude that $u_n \to x^*$ in norm. □

Remark. If we take $N = 1$, $T_1 = S$ and $\beta = 0$, then we obtain the result of Theorem 3.3 in [15], without the hypothesis

$$\sum_{n=1}^{\infty} |\epsilon_{n+1} - \epsilon_n| < \infty.$$

Moreover, if we set $G = 0$ in Theorem 3.1, we arrive at [24, Theorem 1] without the assumption

$$\|I - A\| \leqslant 1 - \alpha\gamma.$$

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