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Traveling wave fronts of delayed non-local diffusion systems without quasimonotonicity

Shuxia Pan

Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou, Gansu 730050, People's Republic of China

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ABSTRACT

This paper is concerned with the existence of traveling wave fronts for delayed non-local diffusion systems without quasimonotonicity, which can not be answered by the known results. By using exponential order, upper–lower solutions and Schauder's fixed point theorem, we reduce the existence of monotone traveling wave fronts to the existence of upper–lower solutions without the requirement of monotonicity. To illustrate our results, we establish the existence of traveling wave fronts for two examples which are the delayed non-local diffusion version of the Nicholson's blowflies equation and the Belousov–Zhabotinskii model. These results imply that the traveling wave fronts of the delayed non-local diffusion systems without quasimonotonicity are persistent if the delay is small.

1. Introduction

Reaction diffusion system is the classical model in describing the spatial-temporal pattern, see, e.g., Britton [3], Pao [33], Smoller [36], Volpert et al. [39], Ye and Li [45]. However, as mentioned in Murray [28, pp. 244–246], the Laplacian operator in the reaction diffusion system is not sufficiently precise in modeling the spatial diffusion of the individuals in some cases, such as the embryological development process. One way of overcoming the imprecise is to introduce the following non-local diffusion model

$$\frac{\partial u_i(x,t)}{\partial t} = \int_{\mathbb{R}} J_i(x-y)\big[u_i(y,t)-u_i(x,t)\big]\,dy + f_i\big(u(x,t)\big), \quad x\in\mathbb{R}, \tag{1.1}$$

in which $i\in I=\{1,\dots,n\}$, $u=(u_1,\dots,u_n)\in\mathbb{R}^n$, $f_i:\mathbb{R}^n\to\mathbb{R}$ is a continuous mapping, and $J_i:\mathbb{R}\to\mathbb{R}$ is the kernel function describing the spatial migration of the individuals in population dynamics. In fact, the model similar to (1.1) was also proposed in other practical fields, for example, phase transition model [1], material science [2], Ising model [13,14,29], network model [15], thalamic model [6] and lattice dynamical systems [7,8]. In the past decades, the traveling wave fronts of (1.1) have been widely investigated due to the significant senses in several areas, and we refer to Bates et al. [1], Carr and Chamj [4], Chen [5], Chow et al. [8], Coville [9], Coville and Dupaigne [10–12].

It is well known that time delay seems to be inevitable in many evolutionary processes [42], and the traveling wave fronts of some delayed models similar to (1.1) have been studied by researchers, such as the delayed lattice dynamical system [19,20,27,43] and the delayed neural network [17]. In particular, Pan et al. [32] considered the traveling wave fronts of the following delayed non-local diffusion system

$$\frac{\partial u_i(x,t)}{\partial t} = \int_{\mathbb{R}} J_i(x-y)\big[u_i(y,t)-u_i(x,t)\big]\,dy + f_i\big(u_t(x)\big), \quad x\in\mathbb{R},\ i\in I, \tag{1.2}$$

E-mail address: sxpan03@lut.cn.

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where $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$, $u_t(x)$ is an element in $C([-\tau, 0], \mathbb{R}^n)$ parameterized by x, t and is defined by $u(x, t+s)$ for $s \in [-\tau, 0]$, herein $\tau > 0$ denotes the maximal time delay in the model [42], thus f_i maps $C([-\tau, 0], \mathbb{R}^n)$ to $\mathbb{R}$. Furthermore, there exists a vector $K = (k_1, \ldots, k_n) \in \mathbb{R}^n$ with $k_i > 0$, $i \in I$, such that $f_i(\widehat{0}) = f_i(\widehat{K}) = 0$, and $\widehat{\cdot}$ is the constant function in the space $C([-\tau, 0], \mathbb{R}^n)$. For the convenience of statement, the definition of a traveling wave front is given as follows.

Definition 1.1. A *traveling wave solution* of (1.2) is a special solution with form $u(x, t) = \Phi(x + ct)$, in which $c > 0$ is the speed parameter and $\Phi = (\phi_1, \ldots, \phi_n) \in C^1(\mathbb{R}, \mathbb{R}^n)$ is the wave profile function. Moreover, if $\Phi(t)$ is monotone in $t \in \mathbb{R}$, then it is called a *traveling wave front*.

Remark 1.2. In some models, $c \leqslant 0$ is admissible [1]. Our main interest in the paper is the case $c > 0$, so we give the above definition.

Substituting $u(x, t) = \Phi(x + ct)$ into (1.2) and replacing $x + ct$ by t, then

$$c\phi_i'(t) = \int_{\mathbb{R}} J_i(y - t)\big[\phi_i(y) - \phi_i(t)\big]\,dy + f_i^c(\Phi_t), \quad t \in \mathbb{R},\ i \in I, \tag{1.3}$$

where $f_i^c(\Phi_t)$ is defined by $f_i(\Phi(t + cs))$ for $s \in [-\tau, 0]$. Recalling the background of traveling wave fronts in several fields, e.g., material science [1,2], we also require that Φ satisfies the following asymptotic boundary conditions

$$\lim_{t \to -\infty} \phi_i(t) = 0, \qquad \lim_{t \to +\infty} \phi_i(t) = k_i, \quad i \in I. \tag{1.4}$$

Combining the upper–lower solutions with the Schauder's fixed point theorem, Pan et al. [32] established the existence of traveling wave fronts of (1.2), namely, the existence of monotone solutions of (1.3) and (1.4) when the reaction term f satisfies the following quasimonotone condition (in short, (QM))

(QM) For every $i \in I$, there exists a constant $\beta_i > 0$ such that

$$f_i(\Phi) - f_i(\Psi) + \beta_i\big[\phi_i(0) - \psi_i(0)\big] \geqslant \int_{\mathbb{R}} J_i(x)\,dx\big[\phi_i(0) - \psi_i(0)\big]$$

holds for any $\Phi(s) = (\phi_1, \ldots, \phi_n)$, $\Psi(s) = (\psi_1, \ldots, \psi_n) \in C([-c\tau, 0], \mathbb{R}^n)$ with $0 \leqslant \Psi(s) \leqslant \Phi(s) \leqslant K$, $s \in [-c\tau, 0]$.

Moreover, Pan et al. [32] also established the existence of traveling wave fronts for the delayed non-local diffusion version of the Hutchinson equation and the Belousov–Zhabotinskii system. The effects of non-local diffusion and time delay were also discussed when the threshold of the wave speed was concerned. Recently, Pan [31] further investigated the following Nicholson's blowflies equation with non-local diffusion and delay

$$\frac{\partial u(x, t)}{\partial t} = \int_{-\infty}^{\infty} J(x - y)\big[u(y, t) - u(x, t)\big]\,dy - \delta u(x, t) + pu(x, t - \tau)e^{-au(x, t-\tau)}, \tag{1.5}$$

where all parameters are positive. More results on the Nicholson's model can be found in Li et al. [23]. By the results in Pan et al. [32], the existence of a nontrivial traveling wave front of (1.5) is proved if $1 < p/\delta \leqslant e$. But for the case of $p/\delta > e$ and $\tau > 0$, (1.5) does not satisfy (QM) such that the previous results are invalid in considering the existence of traveling wave fronts of (1.5). If $\tau = 0$ and $p > \delta$ hold in (1.5), then the existence of a traveling wave front is affirmative by applying the results in [10–12,32]. However, if $p/\delta > e$ is true, it remains an open problem whether such a traveling wave front persists [30] for $\tau > 0$. In order to answer this question, we shall further consider the existence of traveling wave fronts of (1.2) including (1.5) and this constitutes the purpose of the current paper.

To establish the existence of traveling wave fronts of (1.2) that at least includes (1.5) with $p > e\delta$, we shall introduce the following exponential quasimonotone condition (in short, (EQM)) which has less requirements than that of (QM) and is based on the exponential order [35].

(EQM) There exist constants $\beta_i > 0$, $i \in I$, such that

$$f_i(\Phi) - f_i(\Psi) + \beta_i\big[\phi_i(0) - \psi_i(0)\big] \geqslant \int_{\mathbb{R}} J_i(x)\,dx\big[\phi_i(0) - \psi_i(0)\big], \quad i \in I,$$

for any $\Phi(s) = (\phi_1, \ldots, \phi_n)$, $\Psi(s) = (\psi_1, \ldots, \psi_n) \in C([-c\tau, 0], \mathbb{R}^n)$ satisfying (i) $0 \leqslant \Psi(s) \leqslant \Phi(s) \leqslant K$, $s \in [-c\tau, 0]$, (ii) $e^{\frac{\beta_i s}{c}}(\phi_i(s) - \psi_i(s))$, $s \in [-c\tau, 0]$, is nondecreasing.

In what follows, under the assumption of (EQM), the existence of traveling wave fronts of (1.2) will be investigated by combining the Schauder's fixed point theorem with the upper–lower solutions. And we reduce the existence of monotone traveling wave fronts to the existence of upper–lower solutions without the requirement of monotonicity. To illustrate our results, we study the traveling wave fronts of (1.5) as well as the delayed non-local diffusion system with the Belousov–Zhabotinskii reaction. Moreover, we also show that (EQM) is very common in the mathematical literature.

Our current paper is partly motivated by the traveling wave fronts of the delayed reaction diffusion systems and lattice dynamical systems although we concern about a non-local diffusion model with delay, we refer to [18–22,24–26,34,37,40,41,43,44] and the references cited therein.

The rest of this paper is organized as follows. In Section 2, we give some preliminaries for the later sections. In Section 3, the traveling wave fronts of (1.2) will be studied by the Schauder's fixed point theorem and the upper–lower solutions. Finally, our abstract results will be applied to two models, and the existence of the traveling wave fronts will be established.

2. Preliminaries

Throughout the current paper, we shall use the usual notations for the standard partial ordering and order intervals in $\mathbb{R}^n$ or $\mathbb{R}$, and the monotonicity of vector function is in the sense of its components. Let $C_{[0,K]}(\mathbb{R},\mathbb{R}^n)$ be

$$C_{[0,K]}\big(\mathbb{R},\mathbb{R}^n\big)=\big\{u:\ u(t)\in C\big(\mathbb{R},\mathbb{R}^n\big),\ 0\leqslant u(t)\leqslant K \text{ for all } t\in\mathbb{R}\big\}.$$

For $\Phi=(\phi_1,\dots,\phi_n)\in C_{[0,K]}(\mathbb{R},\mathbb{R}^n)$, denote $H=(H_1,\dots,H_n):C_{[0,K]}(\mathbb{R},\mathbb{R}^n)\to C(\mathbb{R},\mathbb{R}^n)$ as follows

$$H_i(\Phi)(t)=\int\limits_{\mathbb{R}} J_i(y-t)\big[\phi_i(y)-\phi_i(t)\big]\,dy+\beta_i\phi_i(t)+f_i^c(\Phi_t),\quad t\in\mathbb{R},\ i\in I.$$

Then (1.3) is equivalent to

$$c\phi_i'(t)=-\beta_i\phi_i(t)+H_i(\Phi)(t),\quad i\in I. \tag{2.1}$$

Due to (2.1), we further define $F=(F_1,\dots,F_n):C_{[0,K]}(\mathbb{R},\mathbb{R}^n)\to C(\mathbb{R},\mathbb{R}^n)$ by

$$F_i(\Phi)(t)=\frac{1}{c}e^{-\frac{\beta_i}{c}t}\int\limits_{-\infty}^{t} e^{\frac{\beta_i}{c}s}H_i(\Phi)(s)\,ds,\quad t\in\mathbb{R},\ \Phi\in C_{[0,K]}\big(\mathbb{R},\mathbb{R}^n\big),\ i\in I.$$

Then it is clear that the fixed point of F also satisfies (1.3). Hence, it is sufficient to consider the existence of the fixed point of F in order to prove the existence of (1.3), and we shall complete the goal by applying the Schauder's fixed point theorem. For this purpose, we now introduce a Banach space. Let $\mu\in(0,\min_{1\leqslant i\leqslant n}\{\frac{\beta_i}{c}\})$, define

$$B_\mu\big(\mathbb{R},\mathbb{R}^n\big)=\Big\{u(t):\ u(t)\in C\big(\mathbb{R},\mathbb{R}^n\big) \text{ and } \sup_{t\in\mathbb{R}}\big|u(t)\big|e^{-\mu|t|}<\infty\Big\},$$

where $|\cdot|$ denotes the super norm in $\mathbb{R}^n$. Thus, $B_\mu(\mathbb{R},\mathbb{R}^n)$ is a Banach space when it is equipped with the norm $|\cdot|_\mu$ defined by

$$|u|_\mu=\sup_{t\in\mathbb{R}}\big|u(t)\big|e^{-\mu|t|}\quad \text{for } u\in B_\mu\big(\mathbb{R},\mathbb{R}^n\big).$$

For convenience, we list the necessary assumptions of (1.3) and all of them will be imposed throughout the next section.

(H1) $f(\widehat{0})=f(\widehat{K})=0$, where $\widehat{\cdot}$ means the constant value function in $C([-c\tau,0],\mathbb{R}^n)$.

(H2) For any $u,v\in C([-c\tau,0],\mathbb{R}^n)$ and $0\leqslant u,v\leqslant K$, there exists a constant $L>0$ such that

$$\big|f(u)-f(v)\big|\leqslant L\|u-v\|,$$

in which $\|\cdot\|$ denotes the upper norm in $C([-c\tau,0],\mathbb{R}^n)$.

(H3) $J_i(x)\geqslant 0$, $x\in\mathbb{R}$ and $\int_{\mathbb{R}} J_i(x)\,dx>0$ for all $i\in I$.

(H4) For $\mu\in(0,\min_{1\leqslant i\leqslant n}\{\frac{\beta_i}{c}\})$, $\int_{-\infty}^{\infty} J_i(x)e^{\mu|x|}\,dx<\infty$, $i\in I$.

Remark 2.1. An operator similar to F was earlier used in Wu and Zou [43] in proving the existence of traveling wave fronts for delayed lattice dynamical systems.

3. Main results

In order to apply the Schauder's fixed point theorem, it is very important to construct proper convex set, which will be defined by upper–lower solutions in this paper. So, we first give the following definition of upper–lower solutions.

Definition 3.1. A continuous vector function $\overline{\Phi}(t) = (\overline{\phi}_1(t), \dots, \overline{\phi}_n(t)) \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n)$ is called an *upper solution* of (1.3) if $\overline{\phi}_i'(t)$ is bounded for $t \in \mathbb{R} \setminus \mathbb{T}$ and satisfies

$$c\overline{\phi}_i'(t) \geqslant \int_{\mathbb{R}} J_i(y-t)\big[\overline{\phi}_i(y) - \overline{\phi}_i(t)\big]\,dy + f_i^c(\overline{\Phi}_t), \quad t \in \mathbb{R} \setminus \mathbb{T},\ i \in I, \tag{3.1}$$

where $\mathbb{T} = \{T_1, T_2, \dots, T_k\}$ with $T_1 < T_2 < \cdots < T_k$. The *lower solution* of (1.3) can be defined by reversing the inequality in (3.1).

In what follows, we assume that an upper solution $\overline{\Phi}(t) = (\overline{\phi}_1(t), \dots, \overline{\phi}_n(t))$ and a lower solution $\underline{\Phi}(t) = (\underline{\phi}_1(t), \dots, \underline{\phi}_n(t))$ of (1.3) are given such that

(A1) $0 \leqslant \underline{\Phi}(t) \leqslant \overline{\Phi}(t) \leqslant K$, $t \in \mathbb{R}$, with $\lim_{t\to-\infty} \underline{\Phi}(t) = 0$, $\lim_{t\to\infty} \overline{\Phi}(t) = K$;
(A2) let $A = (\inf_{t\in\mathbb{R}} \overline{\phi}_1(t), \dots, \inf_{t\in\mathbb{R}} \overline{\phi}_n(t))$ and $B = (\sup_{t\in\mathbb{R}} \underline{\phi}_1(t), \dots, \sup_{t\in\mathbb{R}} \underline{\phi}_n(t))$, then $B < K$ and $A > 0$;
(A3) $\sum_{i=1}^n f_i^2(\widehat{u}) \neq 0$ if $u \in (0, A] \cup [B, K)$;
(A4) the set $\Gamma(\underline{\Phi}, \overline{\Phi})$ is nonempty, where $\Gamma(\underline{\Phi}, \overline{\Phi})$ is defined by

$$\Gamma(\underline{\Phi}, \overline{\Phi}) = \left\{ \begin{array}{c|l} & \text{(i)}\ \ \underline{\Phi}(t) \leqslant \Phi(t) \leqslant \overline{\Phi}(t),\ t \in \mathbb{R}; \\ & \text{(ii)}\ \ \Phi(t) \text{ is nondecreasing in } t \in \mathbb{R}; \\ \Phi = (\phi_1, \dots, \phi_n) & \text{(iii)}\ \ e^{\frac{\beta_i}{c}t}[\overline{\phi}_i(t) - \phi_i(t)] \text{ and } e^{\frac{\beta_i}{c}t}[\phi_i(t) - \underline{\phi}_i(t)] \\ \in C(\mathbb{R}, \mathbb{R}^n) & \qquad \text{are nondecreasing in } t \in \mathbb{R},\ i \in I; \\ & \text{(iv)}\ \ e^{\frac{\beta_i}{c}t}[\phi_i(t+s) - \phi_i(t)] \text{ is nondecreasing} \\ & \qquad \text{in } t \in \mathbb{R} \text{ for every } s > 0,\ i \in I. \end{array} \right\}$$

Lemma 3.2. *Γ is a bounded and closed subset of $C(\mathbb{R}, \mathbb{R}^n)$ with respect to the decay norm $|\cdot|_\mu$. Moreover, it is also a convex subset of $C(\mathbb{R}, \mathbb{R}^n)$.*

The proof of Lemma 3.2 is similar to that of Huang et al. [20, Lemma 4.3] (also see Huang and Zou [21, Lemma 3.1]), so we omit it here. Moreover, by repeating the proof of Pan et al. [32, Lemma 3.5], the following result can be verified.

Lemma 3.3. *$F : C_{[0,K]}(\mathbb{R}, \mathbb{R}^n) \to C(\mathbb{R}, \mathbb{R}^n)$ is continuous with respect to the norm $|\cdot|_\mu$.*

Lemma 3.4. *Assume that* (EQM) *holds. Then*

(i) *$H(\Phi)(t)$ and $F(\Phi)(t)$ are nondecreasing in $t \in \mathbb{R}$ if $\Phi \in \Gamma$;*
(ii) *for $t \in \mathbb{R}$, $H(\Psi)(t) \leqslant H(\Phi)(t)$, $F(\Psi)(t) \leqslant F(\Phi)(t)$, if $\Psi(t) = (\psi_1, \dots, \psi_n)$, $\Phi(t) = (\phi_1, \dots, \phi_n) \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n)$ satisfy that* (a) *$0 \leqslant \Psi(t) \leqslant \Phi(t) \leqslant K$, $t \in \mathbb{R}$,* (b) *for every $i \in I$, $e^{\frac{\beta_i t}{c}}(\phi_i(t) - \psi_i(t))$ is nondecreasing in $t \in \mathbb{R}$.*

Lemma 3.4 is clear by the condition (EQM) and the definitions of H and F, so the proof is omitted here.

Lemma 3.5. *Assume that* (EQM) *holds. Then $F : \Gamma \to \Gamma$.*

Proof. For any $\Phi \in \Gamma$, we shall show that $F(\Phi)(t)$ satisfies the items (i)–(iv) in the definition of Γ. For the item (i), it suffices to prove that

$$F(\underline{\Phi})(t) \geqslant \underline{\Phi}(t) \quad \text{and} \quad \overline{\Phi}(t) \geqslant F(\overline{\Phi})(t), \quad t \in \mathbb{R}, \tag{3.2}$$

by Lemma 3.4. Let $T_0 = -\infty$, $T_{k+1} = +\infty$, then

$$\begin{aligned} F_i(\overline{\Phi})(t) &= \frac{1}{c}\int_{-\infty}^{t} e^{-\frac{\beta_i}{c}(t-s)} H_i(\overline{\Phi})(s)\,ds \\ &\leqslant \frac{1}{c}\left\{\left(\sum_{j=1}^{l-1}\int_{T_{j-1}}^{T_j} + \int_{T_{l-1}}^{t}\right) e^{-\frac{\beta_i}{c}(t-s)}\big[c\overline{\phi}_i'(s) + \beta_i\overline{\phi}_i(s)\big]\,ds\right\} \\ &= \overline{\phi}_i(t), \quad i \in I, \end{aligned}$$

for $t \in (T_{l-1}, T_l)$ with $l = 1, 2, \ldots, k+1$. Thus the continuity of $F_i(\overline{\Phi})(t)$ and $\overline{\phi}_i(t)$ implies that $\overline{\phi}_i(t) \geqslant F_i(\overline{\Phi})(t)$, $t \in \mathbb{R}$, $i \in I$. Similarly, we can prove that $F(\underline{\Phi})(t) \geqslant \underline{\Phi}(t)$, $t \in \mathbb{R}$. This completes the proof of the item (i).

The proof of the item (ii) is clear by Lemma 3.4, so we omit it here.

For $t \in (T_{l-1}, T_l)$ with $l = 1, 2, \ldots, k+1$, direct calculations imply that

$$\begin{aligned} e^{\frac{\beta_i t}{c}}\big[\overline{\phi}_i(t) - F_i(\Phi)(t)\big] &= \frac{e^{\frac{\beta_i t}{c}}}{c}\left\{\left(\sum_{j=1}^{l-1}\int_{T_{j-1}}^{T_j} + \int_{T_{l-1}}^{t}\right)e^{-\frac{\beta_i}{c}(t-s)}\big[c\overline{\phi}_i'(s) + \beta_i\overline{\phi}_i(s)\big]\,ds\right\} \\ &\quad - \frac{e^{\frac{\beta_i t}{c}}}{c}\left\{\left(\sum_{j=1}^{l-1}\int_{T_{j-1}}^{T_j} + \int_{T_{l-1}}^{t}\right)e^{-\frac{\beta_i}{c}(t-s)}H_i(\Phi)(s)\,ds\right\} \\ &= \frac{1}{c}\left(\sum_{j=1}^{l-1}\int_{T_{j-1}}^{T_j} + \int_{T_{l-1}}^{t}\right)e^{\frac{\beta_i s}{c}}\big(c\overline{\phi}_i'(s) + \beta_i\overline{\phi}_i(s) - H_i(\Phi)(s)\big)\,ds, \quad i \in I. \end{aligned}$$

Moreover, Lemma 3.4 and Definition 3.1 indicate that

$$c\overline{\phi}_i'(s) + \beta_i\overline{\phi}_i(s) - H_i(\Phi)(s) \geqslant c\overline{\phi}_i'(s) + \beta_i\overline{\phi}_i(s) - H_i(\overline{\Phi})(s) \geqslant 0, \quad s \in \mathbb{R}\setminus\mathbb{T},\ i \in I.$$

Then the nondecreasing of $e^{\frac{\beta_i t}{c}}[\overline{\phi}_i(t) - F_i(\Phi)(t)]$ is clear. In a similar way, we can prove that for every $i \in I$, $e^{\frac{\beta_i t}{c}}[F_i(\Phi)(t) - \underline{\phi}_i(t)]$ is nondecreasing in $t \in \mathbb{R}$. Therefore, the item (iii) is true.

In order to verify the item (iv), let $s > 0$ be any given constant, then

$$\begin{aligned} e^{\frac{\beta_i}{c}t}\big[F_i(\Phi)(t+s) - F_i(\Phi)(t)\big] &= e^{\frac{\beta_i}{c}t}\left[\int_{-\infty}^{t+s} e^{-\frac{\beta_i}{c}(t+s-z)}H_i(\Phi)(z)\,dz - \int_{-\infty}^{t} e^{-\frac{\beta_i}{c}(t-z)}H_i(\Phi)(z)\,dz\right] \\ &= \int_{-\infty}^{t+s} e^{\frac{\beta_i}{c}(z-s)}H_i(\Phi)(z)\,dz - \int_{-\infty}^{t} e^{\frac{\beta_i}{c}z}H_i(\Phi)(z)\,dz \\ &= \int_{-\infty}^{t} e^{\frac{\beta_i}{c}z}\big[H_i(\Phi)(z+s) - H_i(\Phi)(z)\big]\,dz, \quad i \in I. \end{aligned}$$

Since $H_i(\Phi)(z+s) - H_i(\Phi)(z) \geqslant 0$, $z \in \mathbb{R}$ by Lemma 3.4, then $e^{\frac{\beta_i}{c}t}[F_i(\Phi)(t+s) - F_i(\Phi)(t)]$ is nondecreasing in $t \in \mathbb{R}$, $i \in I$. Thus, the item (iv) holds. The proof is complete. □

Lemma 3.6. *$F : \Gamma \to \Gamma$ is compact with respect to the decay norm $|\cdot|_\mu$.*

The proof of Lemma 3.6 is similar to that of Huang et al. [20, Lemma 3.5], Huang and Zou [21, Lemma 3.4], Pan et al. [32, Lemma 3.7], so it is omitted here.

Theorem 3.7. *Assume that* (EQM) *holds and* (1.3) *has a pair of upper–lower solutions such that* (A1)–(A4) *are satisfied. Then* (1.3)–(1.4) *has a monotone solution which is a traveling wave front of* (1.2)*.*

Proof. By Lemmas 3.2–3.6 and the Schauder's fixed point theorem, F has a fixed point $\Phi^* \in \Gamma$. Then $\lim_{t\to\pm\infty}\Phi^*(t)$ exist by the items (i)–(ii), and we denote them by $\Phi_\pm$. The assumption (H2) further implies that $f(\widehat{\Phi_\pm}) = 0$. Combining these with the assumptions (A2) and (A3), then $\Phi_- = 0$ and $\Phi_+ = K$ hold. The proof is complete. □

4. Applications

In this section, we first apply Theorem 3.7 to two examples and prove the existence of traveling wave fronts by constructing upper–lower solutions. Then we will show that (EQM) is very common in mathematical models.

Example 4.1. Let us consider the following delayed non-local diffusion model

$$\frac{\partial u(x,t)}{\partial t} = \int_{-\infty}^{\infty} J(x-y)\big[u(y,t) - u(x,t)\big]\,dy - \delta u(x,t) + pu(x,t-\tau)e^{-au(x,t-\tau)}, \tag{4.1}$$

in which $x, u \in \mathbb{R}$ and all parameters are positive. This model is the non-local diffusion version of the Nicholson's blowflies equation with delay, herein δ denotes the per capita daily adult death rate, p measures the maximal per capita daily egg production rate, a^{-1} describes the size at which the blowfly population reproduces at its maximum rate and time delay τ depends on the generation time. We refer to Gurney et al. [16] for the delayed Nicholson's model and Li et al. [23] for the non-local delay version of the Nicholson's model.

It is clear that (4.1) has a trivial equilibrium 0 and a positive spatial homogeneous equilibrium $k = \frac{1}{a}\ln\frac{p}{\delta}$ provided that $p > \delta$ holds. Moreover, the following assumptions will be imposed throughout the two examples in this section.

(J1) $J : \mathbb{R} \to \mathbb{R}$, $J(x) = J(-x) \geqslant 0$, $x \in \mathbb{R}$ and $\int_{\mathbb{R}} J(x)\,dx > 0$.
(J2) For any $\lambda > 0$, $\int_{\mathbb{R}} J(x)e^{\lambda x}\,dx < \infty$.

Let $u(x,t) = \phi(x+ct)$ be a traveling wave front of (4.1), then ϕ must satisfy

$$c\phi'(t) = \int_{-\infty}^{\infty} J(t-y)\big[\phi(y)-\phi(t)\big]\,dy - \delta\phi(t) + p\phi(t-c\tau)e^{-a\phi(t-c\tau)}, \quad t \in \mathbb{R}, \tag{4.2}$$

and we are interested in the following asymptotic boundary conditions

$$\lim_{t\to-\infty}\phi(t) = 0, \qquad \lim_{t\to\infty}\phi(t) = k. \tag{4.3}$$

For any $\phi(s) \in C([-c\tau,0],\mathbb{R})$ with $0 \leqslant \phi(s) \leqslant k$, $s \in [-c\tau,0]$, define

$$f(\phi) = -\delta\phi(0) + p\phi(-c\tau)e^{-a\phi(-c\tau)}.$$

Lemma 4.2. *Assume that τ is small enough and $p > e\delta$ holds. Then f satisfies* (EQM).

Proof. For any $\phi, \psi \in C([-c\tau,0],\mathbb{R})$ with (i) $0 \leqslant \phi \leqslant \psi \leqslant k$; (ii) $e^{\frac{\beta s}{c}}[\psi(s)-\phi(s)]$ is nondecreasing in $s \in [-c\tau,0]$, where $\beta > 0$ will be clarified later. Then we have

$$\begin{aligned} f(\psi) - f(\phi) &= -\delta\psi(0) + p\psi(-c\tau)e^{-a\psi(-c\tau)} + \delta\phi(0) - p\phi(-c\tau)e^{-a\phi(-c\tau)} \\ &\geqslant -\delta\big[\psi(0)-\phi(0)\big] - p\ln\frac{p}{\delta}\big[\psi(-c\tau)-\phi(-c\tau)\big] \\ &= -\delta\big[\psi(0)-\phi(0)\big] - p\ln\frac{p}{\delta}e^{\beta\tau}e^{-\beta\tau}\big[\psi(-c\tau)-\phi(-c\tau)\big] \\ &\geqslant -\left(\delta + p\ln\frac{p}{\delta}e^{\beta\tau}\right)\big[\psi(0)-\phi(0)\big]. \end{aligned}$$

If τ is small enough, then we can choose $\beta > 0$ such that $\beta > \int_{\mathbb{R}} J(x)\,dx + \delta + p\ln\frac{p}{\delta}e^{\beta\tau}$, which implies that f satisfies (EQM) for small $\tau > 0$. The proof is complete. $\square$

Remark 4.3. If $1 < p/\delta \leqslant e$ holds, then f satisfies (QM) (see [31]).

Now, we are in a position to construct proper upper–lower solutions for (4.2). For $\lambda \geqslant 0$ and $c \geqslant 0$, define

$$\Delta(\lambda,c) = \int_{-\infty}^{\infty} J(y)\big[e^{\lambda y}-1\big]\,dy - c\lambda - \delta + pe^{-\lambda c\tau}. \tag{4.4}$$

Then $\Delta(\lambda,c)$ is well defined by (J1)–(J2). We further show the properties of $\Delta(\lambda,c)$ by the following lemma.

Lemma 4.4. *There exists a constant $c^* > 0$ such that* (4.4) *has two distinct positive roots if $c > c^*$ while* (4.4) *has no real zero if $c < c^*$. More precisely, $c > c^*$ implies that there exist $\lambda_2(c) > \lambda_1(c) > 0$ such that*

$$\Delta(\lambda,c)\begin{cases} >0 & \text{for } 0<\lambda<\lambda_1(c), \\ =0 & \text{for } \lambda=\lambda_1(c),\lambda_2(c), \\ <0 & \text{for } \lambda_1(c)<\lambda<\lambda_2(c), \\ >0 & \text{for } \lambda>\lambda_2(c). \end{cases}$$

By the constants in Lemma 4.4, define the continuous functions as follows

$$\overline{\phi}(t) = \min\big\{k, ke^{\lambda_1(c)t}\big\}, \qquad \underline{\phi}(t) = \max\big\{0, k\big[e^{\lambda_1(c)t} - qe^{\eta\lambda_1(c)t}\big]\big\},$$

where $q > 1$ is large enough and η is fixed such that

$$\eta \in \left(1, \min\left\{2, \frac{\lambda_2}{\lambda_1}\right\}\right). \tag{4.5}$$

Lemma 4.5. *If τ is small enough, then $\overline{\phi}(t)$ is an upper solution of* (4.2).

Proof. It is sufficient to prove that $\overline{\phi}(t)$ satisfies the definition of an upper solution. If $t < 0$, then $\overline{\phi}(t) = ke^{\lambda_1(c)t}$ and

$$\begin{aligned}
&\int_{-\infty}^{\infty} J(t-y)\big[\overline{\phi}(y) - \overline{\phi}(t)\big]\,dy - \delta\overline{\phi}(t) + p\overline{\phi}(t-c\tau)e^{-a\overline{\phi}(t-c\tau)} - c\overline{\phi}'(t)\\
&\quad \leqslant k\left\{\int_{-\infty}^{\infty} J(t-y)\big[e^{\lambda_1 y} - e^{\lambda_1 t}\big]\,dy - \delta e^{\lambda_1 t} + pe^{\lambda_1(t-c\tau)} - c\lambda_1 e^{\lambda_1 t}\right\}\\
&\quad = ke^{\lambda_1 t}\left\{\int_{-\infty}^{\infty} J(y)\big[e^{\lambda_1 y} - 1\big]\,dy - \delta + pe^{-\lambda_1 c\tau} - c\lambda_1\right\}\\
&\quad = 0.
\end{aligned}$$

When $t > c\tau$ holds, then $\overline{\phi}(t) = \overline{\phi}(t-c\tau) = k$ and the result is clear.

Note that $\tau > 0$ is small enough, then there exists $\mu < 0$ such that

$$\int_{-\infty}^{\infty} J(t-y)\big[\overline{\phi}(y) - \overline{\phi}(t)\big]\,dy \leqslant k\int_{-\infty}^{-c\tau} J(s)\big[e^{\lambda_1(c\tau+s)} - 1\big]\,ds \triangleq \mu, \quad t \in [0, c\tau]$$

since (J1) indicates that $\int_{-\infty}^{-c\tau} J(y)\,dy > 0$. Moreover, if $\tau > 0$ is small enough, then

$$\sup_{\theta\in[-c\tau,0]}\big[-\delta k + p\overline{\phi}(\theta)e^{-a\overline{\phi}(\theta)}\big] \leqslant -\mu.$$

Thus, $t \in [0, c\tau]$ with $\tau > 0$ small enough implies that

$$\begin{aligned}
&\int_{-\infty}^{\infty} J(t-y)\big[\overline{\phi}(y) - \overline{\phi}(t)\big]\,dy - \delta\overline{\phi}(t) + p\overline{\phi}(t-c\tau)e^{-a\overline{\phi}(t-c\tau)} - c\overline{\phi}'(t)\\
&\quad = \int_{-\infty}^{0} J(t-y)\big[\overline{\phi}(y) - \overline{\phi}(t)\big]\,dy - \delta k + p\overline{\phi}(t-c\tau)e^{-a\overline{\phi}(t-c\tau)}\\
&\quad \leqslant -\mu + \mu = 0.
\end{aligned}$$

The proof is complete. □

Lemma 4.6. *If $q > 1$ is large enough, then $\underline{\phi}(t)$ is a lower solution of* (4.2).

Proof. We only need to verify that $\underline{\phi}(t)$ satisfies the definition of a lower solution. In particular, let M be a positive constant such that

$$\big|pue^{-au} - pu\big| \leqslant Mu^2, \quad u \in [0, k].$$

If $\underline{\phi}(t) = k[e^{\lambda_1(c)t} - qe^{\eta\lambda_1(c)t}]$, then $\underline{\phi}(t-c\tau) = k[e^{\lambda_1(c)(t-c\tau)} - qe^{\eta\lambda_1(c)(t-c\tau)}]$ and

$$\begin{aligned}
&\int_{-\infty}^{\infty} J(t-y)\big[\underline{\phi}(y) - \underline{\phi}(t)\big]\,dy - \delta\underline{\phi}(t) + p\underline{\phi}(t-c\tau)e^{-a\underline{\phi}(t-c\tau)} - c\underline{\phi}'(t)\\
&\quad \geqslant k\left\{\int_{-\infty}^{\infty} J(t-y)\big[e^{\lambda_1 y} - qe^{\eta\lambda_1 y} - \big(e^{\lambda_1 t} - qe^{\eta\lambda_1 t}\big)\big]\,dy - \delta\big(e^{\lambda_1 t} - qe^{\eta\lambda_1 t}\big)\right.\\
&\qquad \left. + pe^{\lambda_1(t-c\tau)} - pe^{\eta\lambda_1(t-c\tau)} - Mke^{2\lambda_1(t-c\tau)} - c\big(\lambda_1 e^{\lambda_1 t} - q\eta\lambda_1 e^{\eta\lambda_1 t}\big)\right\}
\end{aligned}$$

$$= ke^{\lambda_1 t}\left\{\int_{-\infty}^{\infty} J(y)\left[e^{\lambda_1 y}-1\right]dy-\delta+pe^{-\lambda_1 c\tau}-c\lambda_1\right\}-Mk^2e^{2\lambda_1(t-c\tau)}$$
$$-qke^{\eta\lambda_1 t}\left\{\int_{-\infty}^{\infty} J(y)\left[e^{\eta\lambda_1 y}-1\right]dy-\delta+pe^{-\eta\lambda_1 c\tau}-c\eta\lambda_1\right\}$$
$$-qk\Delta(\eta\lambda_1,c)e^{\eta\lambda_1 t}-Mk^2e^{2\lambda_1(t-c\tau)}.$$

Choose $q > \frac{Mk}{-\Delta(\eta\lambda_1,c)}+1$, then $-qk\Delta(\eta\lambda_1,c)e^{\eta\lambda_1 t}-Mk^2e^{2\lambda_1(t-c\tau)}>0$ and this implies that $\underline{\phi}(t)$ is the lower solution of (4.2) if $\underline{\phi}(t)=k[e^{\lambda_1(c)t}-qe^{\eta\lambda_1(c)t}]$.

When $\underline{\phi}(t)=0$, then $\underline{\phi}(t-c\tau)\geqslant 0$ and the result is clear. The proof is complete. □

Theorem 4.7. *Let $p>e\delta$ be true, c^* and $\lambda_1(c)$ be defined by Lemma 4.4. Assume that $\tau>0$ is small enough. Then for every $c>c^*$, (4.1) has a traveling wave front $\phi(x+ct)$ such that $\lim_{t\to-\infty}\phi(t)e^{-\lambda_1(c)t}>0$ holds.*

Proof. In order to apply Theorem 3.7, it suffices to prove that the set $\Gamma(\underline{\phi},\overline{\phi})$ is nonempty by Lemmas 4.2 and 4.5–4.6. Define the continuous function $\phi^*(t)=\frac{k}{1+e^{-\lambda_1 t}}$, then it is clear that $\phi^*(t)\in\Gamma(\underline{\phi},\overline{\phi})$ for $\beta>c\lambda_1$. The asymptotic behavior $\lim_{t\to-\infty}\phi(t)e^{-\lambda_1(c)t}>0$ holds by those of upper–lower solutions. The proof is complete. □

Remark 4.8. Let $\tau=0$, then the result of Theorem 4.7 remains true. Thus, the traveling wave front of (4.1) is persistent with respect to small delay.

Example 4.9. Let us consider the following delayed system

$$\begin{cases}\dfrac{\partial u(x,t)}{\partial t}=\displaystyle\int_{\mathbb{R}} J(x-y)\left[u(y,t)-u(x,t)\right]dy+u(x,t)\left[1-u(x,t-\tau_1)-rv(x,t-\tau_2)\right],\\ \dfrac{\partial v(x,t)}{\partial t}=\displaystyle\int_{\mathbb{R}} J(x-y)\left[v(y,t)-v(x,t)\right]dy-bu(x,t)v(x,t),\end{cases}\tag{4.6}$$

where $x,u,v\in\mathbb{R}$ and all parameters are positive. System (4.6) can be regarded as the delayed non-local diffusion version of the Belousov–Zhabotinskii model which was proposed to describe the concentration of Bromic acid and bromide ion in chemical reactions. Recently, the traveling wave fronts in different versions of Belousov–Zhabotinskii model have been widely studied by many authors, see, e.g., [21,25,28,38].

Similar to that in [21,25], letting $v^*=1-v$ in (4.6), and omitting the asterisks for notational simplicity, then (4.6) becomes

$$\begin{cases}\dfrac{\partial u(x,t)}{\partial t}=\displaystyle\int_{\mathbb{R}} J(x-y)\left[u(y,t)-u(x,t)\right]dy+u(x,t)\left[1-r-u(x,t-\tau_1)+rv(x,t-\tau_2)\right],\\ \dfrac{\partial v(x,t)}{\partial t}=\displaystyle\int_{\mathbb{R}} J(x-y)\left[v(y,t)-v(x,t)\right]dy+bu(x,t)\left[1-v(x,t)\right].\end{cases}\tag{4.7}$$

Assume that $(u(x,t),v(x,t))=(\phi_1(x+ct),\phi_2(x+ct))$ is a traveling wave front of (4.7), then $(\phi_1(t),\phi_2(t))$, $t\in\mathbb{R}$, must satisfy

$$\begin{cases}c\phi_1'(t)=\displaystyle\int_{-\infty}^{\infty} J(t-y)\left[\phi_1(y)-\phi_1(t)\right]dy+\phi_1(t)\left[1-r-\phi_1(t-c\tau_1)+r\phi_2(t-c\tau_2)\right],\\ c\phi_2'(t)=\displaystyle\int_{-\infty}^{\infty} J(t-y)\left[\phi_2(y)-\phi_2(t)\right]dy+b\phi_1(t)\left[1-\phi_2(t)\right].\end{cases}\tag{4.8}$$

Recalling the background of Belousov–Zhabotinskii model [21,25,28,38], we will prove the existence of (4.8) with the following asymptotic boundary conditions

$$\lim_{t\to-\infty}\phi_1(t)=\lim_{t\to-\infty}\phi_2(t)=0,\qquad \lim_{t\to\infty}\phi_1(t)=\lim_{t\to\infty}\phi_2(t)=1.\tag{4.9}$$

Let $\tau=\max\{\tau_1,\tau_2\}$. For $(\phi_1,\phi_2)\in C([-c\tau,0],\mathbb{R}^2)$, define $f=(f_1,f_2)$ by

$$\begin{cases}f_1=\phi_1(0)\left[1-r-\phi_1(-c\tau_1)+r\phi_2(-c\tau_2)\right],\\ f_2=b\phi_1(0)\left[1-\phi_2(0)\right].\end{cases}$$

Lemma 4.10. *Assume that τ_1 is small enough. Then f satisfies* (EQM)*.*

The proof is similar to that of [21, Lemma 4.1], so we omit it here.

For $\lambda \geqslant 0$, $c > 0$, define

$$\Delta(\gamma, c) = \int_{-\infty}^{\infty} J(y)\left[e^{\gamma y} - 1\right] dy - c\gamma + 1 - r. \tag{4.10}$$

Then (J1)–(J2) imply that $\Delta(\gamma, c)$ is well defined and the following result holds.

Lemma 4.11. *There exists a constant $c_* > 0$ such that* (4.10) *has two distinct positive zeros if $c > c_*$ while* (4.10) *has no real root if $c < c_*$. More precisely, if $c > c^*$ holds, then there exist $\gamma_2(c) > \gamma_1(c) > 0$ such that*

$$\Delta(\gamma, c) \begin{cases} > 0 & \text{for } 0 < \gamma < \gamma_1(c), \\ = 0 & \text{for } \gamma = \gamma_1(c), \gamma_2(c), \\ < 0 & \text{for } \gamma_1(c) < \gamma < \gamma_2(c), \\ > 0 & \text{for } \gamma > \gamma_2(c). \end{cases}$$

By the constants in Lemma 4.11, define continuous functions as follows

$$\overline{\phi}_1(t) = \min\left\{e^{\gamma_1 t}, 1\right\}, \qquad \overline{\phi}_2(t) = \min\left\{e^{\gamma_1 (t - c\tau_1)}, 1\right\},$$
$$\underline{\phi}_1(t) = \max\left\{(1-r)\left(e^{\gamma_1 t} - q e^{\eta \gamma_1 t}\right), 0\right\}, \qquad \underline{\phi}_2(t) = 0,$$

where $\eta \in (1, \min\{2, \frac{\gamma_2(c)}{\gamma_1(c)}\})$ and $q > 1$ is a constant. Moreover, similar to the proof of Lemmas 4.5–4.6, we can verify the following results.

Lemma 4.12. *Assume that τ_1 is small enough and $b < 1 - r$. Then $(\overline{\phi}_1(t), \overline{\phi}_2(t))$ is an upper solution and $(\underline{\phi}_1(t), \underline{\phi}_2(t))$ is a lower solution of* (4.8) *if $q > 1$ is large enough.*

Theorem 4.13. *Assume that τ_1 is small enough and $b < 1 - r$. Then* (4.6) *has a traveling wave front with wave speed $c > c^*$.*

Proof. In order to apply Theorem 3.7, it is sufficient to prove that $\Gamma([\underline{\phi}_1, \underline{\phi}_2], [\overline{\phi}_1, \overline{\phi}_2])$ is nonempty by Lemma 4.12. In fact,

$$\left(\frac{1}{1 + e^{-\gamma_1 t}}, \frac{1}{1 + e^{-\gamma_1 (t - c\tau_1)}}\right) \in \Gamma$$

for some $\beta > 0$ large enough. The proof is complete. □

Examples 4.1 and 4.9 imply that the construction of upper and lower solutions is possible when we establish the existence of traveling wave fronts of (1.2) by Theorem 3.7. Furthermore, (EQM) is very common in mathematical literature besides that the (QM) system must be the (EQM) system. For example, let $n = 1$ in (1.2)–(1.3), and denote f_1, J_1 by f, J, then we have the following result.

Proposition 4.14. *Assume that* (H1) *and* (H2) *hold. Then f in* (1.3) *satisfies* (EQM) *if $\tau > 0$ is small enough.*

Proof. Assume that $\phi(s), \psi(s) \in C([-c\tau, 0], \mathbb{R})$ satisfy (i) $0 \leqslant \psi(s) \leqslant \phi(s) \leqslant K$, $s \in [-c\tau, 0]$, (ii) $e^{\frac{\beta s}{c}}(\phi(s) - \psi(s))$, $s \in [-c\tau, 0]$ is nondecreasing for $\beta > 0$ clarified later. Then

$$f(\phi) - f(\psi) \geqslant -L\|\phi - \psi\| \geqslant -Le^{\beta\tau}\left[\phi(0) - \psi(0)\right]$$

by (H2). Since τ is small enough, then there exists $\beta > 0$ such that

$$\beta \geqslant Le^{\beta\tau} + \int_{\mathbb{R}} J(x)\, dx$$

holds. The proof is complete. □

Remark 4.15. It should be noted that β in Lemma 4.2 and Proposition 4.14 are independent of the wave speed c.

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