



# Substochastic semigroups and densities of piecewise deterministic Markov processes

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## ABSTRACT

Necessary and sufficient conditions are given for a substochastic semigroup on  $L^1$  obtained through the Kato–Voigt perturbation theorem to be either stochastic or strongly stable. We show how such semigroups are related to piecewise deterministic Markov process, provide a probabilistic interpretation of our results, and apply them to fragmentation equations.

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## 1. Introduction

Piecewise deterministic Markov processes (PDMPs) are Markov processes involving deterministic motion punctuated by random jumps. A general theory for such processes was introduced in [13] within an abstract framework with numerous examples from queueing and control theory. The sample paths  $X(t)$  of the PDMP depend on three local characteristics: a flow  $\pi$ , a nonnegative jump rate function  $\varphi$ , and a stochastic transition kernel  $\mathcal{J}$ . Instead of a flow, here we consider semi-flows on a Borel state space  $E$  such that  $\pi_t(E) \subseteq E$ ,  $t \geq 0$ , which leads to PDMPs without active boundaries and allows us to use the more general formulation of stochastic models presented in [22]. Starting from  $x$  the process follows the trajectory  $\pi_t x$  until the first jump time  $t_1$  which occurs at a rate  $\varphi$ . The value of the process at the jump time  $t_1$  is selected from the distribution  $\mathcal{J}(\pi_{t_1} x, \cdot)$  and the process restarts afresh from this new point (see Section 5.1 for the construction). If the function  $\varphi$  is unbounded then it might happen that the process is only defined up to a finite random time, called an explosion time, so that we study the minimal PDMP with the given characteristics.

Let the state space be a  $\sigma$ -finite measure space  $(E, \mathcal{E}, m)$ . Suppose that the distribution of  $X(0)$  is absolutely continuous with respect to the measure  $m$ . One of our main objectives is to give sufficient conditions for the distribution of  $X(t)$  to be absolutely continuous with respect to  $m$  for all  $t > 0$ , and to derive rigorously an evolution equation for its density  $u(t, x)$ . This leads us to study equations of the form

$$\frac{\partial u(t, x)}{\partial t} = A_0 u(t, x) - \varphi(x)u(t, x) + P(\varphi u(t, \cdot))(x), \quad (1.1)$$

where  $P$  is a stochastic operator on  $L^1$  corresponding to the stochastic kernel  $\mathcal{J}$  (see Section 2) and  $A_0$  is the (infinitesimal) generator of a strongly continuous semigroup of stochastic operators (*stochastic semigroup*) corresponding to the determin-

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istic semi-flow  $\pi$ . Let us write

$$Au = A_0u - \varphi u \quad \text{and} \quad Cu = Au + P(\varphi u). \quad (1.2)$$

When  $\varphi$  is bounded, then the Cauchy problem associated with (1.1) is well posed, by the bounded perturbation theorem (see e.g. [16, Section III.1]), and  $C$  generates a stochastic semigroup. If  $\varphi$  is unbounded, then  $C$  is the sum of two unbounded operators and the existence and uniqueness of solutions to the Cauchy problem in  $L^1$  is problematic. The strategy which can be adapted to tackle such problems involves perturbation results for strongly continuous semigroups of positive contractions on  $L^1$  (*substochastic semigroups*). We refer the reader to the monograph [7] for an extensive overview on the subject. We make use of one such result (Theorem 3.1 in Section 3), which goes back to [24] in the case of a discrete state space and was subsequently developed in [33,2,4], from which it follows that the operator  $C$  has an extension  $\bar{C}$  generating a substochastic semigroup  $\{P(t)\}_{t \geq 0}$  provided that the operator  $A$  is the generator of a substochastic semigroup on  $L^1$  and  $C$  is defined on the domain  $\mathcal{D}(A)$  of  $A$ . In general, the semigroup  $\{P(t)\}_{t \geq 0}$  is stochastic if and only if the generator  $C$  is the minimal closed extension of  $(C, \mathcal{D}(A))$ . In that case, if  $u_0$  is nonnegative then the norm

$$\|P(t)u_0\| = \int_E P(t)u_0(x)m(dx), \quad t \geq 0,$$

is constant in time, meaning that there is conservation of mass. If  $C$  is not the minimal closed extension of  $(C, \mathcal{D}(A))$ , then we have

$$\|P(t)u_0\| < \|u_0\| \quad (1.3)$$

for some  $u_0$  and  $t > 0$ , meaning that there is a loss of mass. Our objective is to study the two extreme cases: either  $\{P(t)\}_{t \geq 0}$  is stochastic or it is *strongly stable*

$$\lim_{t \rightarrow \infty} \|P(t)u_0\| = 0 \quad \text{for all } u_0 \in L^1.$$

In Section 3 we provide general necessary and sufficient conditions for either to hold (Theorems 3.4–3.6). To the best of our knowledge all past investigations of the semigroup  $\{P(t)\}_{t \geq 0}$  concentrated on providing necessary and sufficient conditions for conservation of mass [7] and it was only shown in [9] that if there is a loss of mass for fragmentation models and explosive birth-death processes then (1.3) holds for every  $u_0$  and sufficiently large  $t$ . Thus, the study of strong stability seems to be new.

Formulating the problem in the context of piecewise deterministic Markov processes allows us to identify the corresponding semigroup  $\{P(t)\}_{t \geq 0}$  from a probabilistic point of view (Theorem 5.2). The combination of probabilistic and functional-analytic methods leads to rigorous results providing a derivation of an evolution equation for densities of such processes and necessary and sufficient conditions for the semigroup to be either stochastic or strongly stable. In the discrete state space, (1.1) with  $A_0 = 0$  is the forward Kolmogorov equation [17] and we recover the results of [24,30]. To illustrate our general approach, we use fragmentation models (Section 6) in our framework and provide a refined analysis of such models, previously studied extensively with either purely functional-analytic or probabilistic methods [29,18,27,15,28,21,19,8,3,1,5,34]. Our results can also be applied to stochastic differential equations with jumps [20,11].

The outline of this paper is as follows. In Section 2 we collect relevant definitions for stochastic operators and give necessary and sufficient conditions for strongly stable operators and semigroups. In Section 3 we recall the Kato–Voigt perturbation theorem and we prove necessary and sufficient conditions for the corresponding semigroup to be either stochastic or strongly stable. In Section 4 we describe the extension techniques introduced in [2], and further developed in [4], which provide the characterization of the generator and the evolution equation for densities. In Section 5 we study piecewise deterministic Markov processes. In Section 5.1 we describe a general construction of PDMPs and in Section 5.2 the relation of such stochastic models to the corresponding semigroups  $\{P(t)\}_{t \geq 0}$  on  $L^1$ . In Section 6 we let the operators  $P$  and  $A$  have definite forms and give a number of concrete examples of situations that fit directly into our framework.

## 2. Preliminaries

Let  $(E, \mathcal{E}, m)$  be a  $\sigma$ -finite measure space and  $L^p = L^p(E, \mathcal{E}, m)$  for all  $p \geq 1$ . A linear operator  $A : \mathcal{D} \rightarrow L^1$ , where  $\mathcal{D}$  is a linear subspace of  $L^1$ , is said to be *positive* if  $Au \geq 0$  for  $u \in \mathcal{D}_+ := \mathcal{D} \cap L^1_+$ . Then we write  $A \geq 0$ . Every positive operator  $A$  with  $\mathcal{D} = L^1$  is a bounded operator. In general, we will denote the domain of any operator  $A$  by  $\mathcal{D}(A)$ , its range by  $\text{Im}(A)$ ,  $\text{Im}(A) = \{Au : u \in \mathcal{D}(A)\}$ , and its null space by  $\text{Ker}(A)$ ,  $\text{Ker}(A) = \{u \in \mathcal{D}(A) : Au = 0\}$ . The *resolvent set*  $\rho(A)$  of  $A$  is the set of all complex numbers  $\lambda$  for which  $\lambda - A$  is invertible. The family  $R(\lambda, A) := (\lambda - A)^{-1}$ ,  $\lambda \in \rho(A)$ , of bounded linear operators is called the *resolvent* of  $A$ . Finally, if  $(A, \mathcal{D}(A))$  is the generator of a substochastic semigroup then  $R(\lambda, A)u \geq R(\mu, A)u \geq 0$  for  $\mu > \lambda > 0$  and  $u \in L^1_+$ .

Let  $D(m) \subset L^1$  be the set of all *densities* on  $E$ , i.e.

$$D(m) = \{u \in L^1 : u \geq 0, \|u\| = 1\},$$

where  $\|\cdot\|$  is the norm in  $L^1$ . A linear operator  $P : L^1 \rightarrow L^1$  such that  $P(D(m)) \subseteq D(m)$  is called *stochastic* or *Markov* [25].

Let  $\mathcal{J} : E \times \mathcal{E} \rightarrow [0, 1]$  be a *stochastic transition kernel*, i.e.  $\mathcal{J}(x, \cdot)$  is a probability measure for each  $x \in E$  and the function  $x \mapsto \mathcal{J}(x, B)$  is measurable for each  $B \in \mathcal{E}$ , and let  $P$  be a stochastic operator on  $L^1$ . If

$$\int_E \mathcal{J}(x, B) u(x) m(dx) = \int_B P u(x) m(dx) \quad \text{for all } B \in \mathcal{E}, u \in D(m),$$

then  $P$  is called the *transition operator* corresponding to  $\mathcal{J}$ . If  $p : E \times E \rightarrow [0, \infty)$  is a measurable function such that

$$\int_E p(x, y) m(dy) = 1, \quad x \in E,$$

then the operator  $P$  defined by

$$P u(x) = \int_E p(x, y) u(y) m(dy), \quad x \in E, u \in L^1,$$

is stochastic and it corresponds to the stochastic kernel

$$\mathcal{J}(x, B) = \int_B p(y, x) m(dy), \quad x \in E, B \in \mathcal{E}.$$

We simply say that  $P$  has *kernel*  $p$ .

A linear operator  $T$  on  $L^1$  is called *mean ergodic* if

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} T^n u \quad \text{exists for all } u \in L^1$$

and *strongly stable* if

$$\lim_{n \rightarrow \infty} \|T^n u\| = 0 \quad \text{for all } u \in L^1. \quad (2.1)$$

Note that a stochastic operator is never strongly stable. We have the following characterization of strongly stable positive contractions on  $L^1$ . The result seems to be known but we cannot find appropriate references. We include its very simple proof for the sake of completeness.

**Proposition 2.1.** *Let  $T$  be a positive contraction on  $L^1$  and  $T^* : L^\infty \rightarrow L^\infty$  be the adjoint of  $T$ . Then the following are equivalent:*

- (1)  $T$  is mean ergodic and  $\text{Ker}(I - T) = \{0\}$ .
- (2)  $T$  is strongly stable.
- (3) Condition (2.1) holds for some  $u \in L^1_+, u > 0$  a.e.
- (4) If for some  $f \in L^\infty_+$  we have  $T^* f = f$  then  $f = 0$ .
- (5)  $\lim_{n \rightarrow \infty} T^{*n} 1 = 0$  a.e.

**Proof.** First observe that (1) is equivalent to

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} T^n u = 0 \quad \text{for all } u \in L^1.$$

Since  $T$  is a positive contraction, the sequence  $(\|T^n u\|)$  is convergent for nonnegative  $u$ . Thus

$$\lim_{N \rightarrow \infty} \frac{1}{N} \left\| \sum_{n=0}^{N-1} T^n u \right\| = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \|T^n u\| = \lim_{n \rightarrow \infty} \|T^n u\|,$$

by additivity of the norm, which gives (1)  $\Leftrightarrow$  (2). The implications (5)  $\Rightarrow$  (2) and (2)  $\Rightarrow$  (3) are trivial. Now assume that (3) holds. Let  $f \in L^\infty_+$  be such that  $T^* f = f$ . We have

$$\int_E f u dm = \int_E T^{*n} f u dm = \int_E f T^n u dm \leq \|f\|_\infty \|T^n u\|,$$

which shows that  $f = 0$ . Finally, assume that (4) holds. Since  $T^* 1 \leq 1$ , the limit  $h := \lim_{n \rightarrow \infty} T^{*n} 1$  exists and  $T^* h = h$ . Thus  $h = 0$  by (4).  $\square$

**Remark 2.2.** Note that if  $T$  is a positive contraction with  $\text{Ker}(I - T) = \{0\}$  then  $T$  is mean ergodic if and only if  $\text{Ker}(I - T^*) = \{0\}$ , by Sine's theorem [31].

We now state for later use the inheritance of mean ergodicity under domination. This is a consequence of the Yosida–Kakutani ergodic theorem (see e.g. [35, Theorem VIII.3.2]).

**Proposition 2.3.** *Let  $T$  and  $K$  be positive contractions on  $L^1$  such that*

$$Tu \leq Ku \quad \text{for } u \in L^1_+.$$

*If  $K$  is mean ergodic then  $T$  is mean ergodic.*

A semigroup  $\{S(t)\}_{t \geq 0}$  is called *strongly stable* if

$$\lim_{t \rightarrow \infty} S(t)u = 0 \quad \text{for all } u \in L^1.$$

Note that a stochastic semigroup is never strongly stable. The mean ergodic theorem for semigroups [35, Chapter VIII.4] and additivity of the norm give the following characterization (see also [12, Theorems 2.1 and 7.7]).

**Proposition 2.4.** *Let  $\{S(t)\}_{t \geq 0}$  be a substochastic semigroup on  $L^1$  with generator  $A$ . Then the following are equivalent:*

- (1)  $\{S(t)\}_{t \geq 0}$  is strongly stable.
- (2) For every  $u \in L^1_+$

$$\lim_{\lambda \downarrow 0} \lambda R(\lambda, A)u = 0.$$

- (3)  $\text{Im}(A)$  is dense in  $L^1$ .

### 3. Perturbation of substochastic semigroups

In this section we consider two linear operators  $(A, \mathcal{D}(A))$  and  $(B, \mathcal{D}(B))$  in  $L^1$  which are assumed throughout to have the following properties:

- (G1)  $(A, \mathcal{D}(A))$  generates a substochastic semigroup  $\{S(t)\}_{t \geq 0}$ ;
- (G2)  $\mathcal{D}(B) \supseteq \mathcal{D}(A)$  and  $Bu \geq 0$  for  $u \in \mathcal{D}(A)_+$ ;
- (G3) for every  $u \in \mathcal{D}(A)_+$

$$\int_E (Au + Bu) dm = 0. \quad (3.1)$$

We refer to Sections 4 and 6 for examples of operators satisfying (G1)–(G3).

**Theorem 3.1.** (See [24, 33, 4].) *There exists a substochastic semigroup  $\{P(t)\}_{t \geq 0}$  on  $L^1$  generated by an extension  $C$  of the operator  $(A + B, \mathcal{D}(A))$ . The generator  $C$  is characterized by*

$$R(\lambda, C)u = \lim_{N \rightarrow \infty} R(\lambda, A) \sum_{n=0}^N (BR(\lambda, A))^n u, \quad u \in L^1, \lambda > 0, \quad (3.2)$$

*and  $\{P(t)\}_{t \geq 0}$  is the smallest substochastic semigroup whose generator is an extension of  $(A + B, \mathcal{D}(A))$ .*

*Moreover, the following are equivalent:*

- (1)  $\{P(t)\}_{t \geq 0}$  is a stochastic semigroup.
- (2) The generator  $C$  is the closure of  $(A + B, \mathcal{D}(A))$ .
- (3) For some  $\lambda > 0$

$$\lim_{n \rightarrow \infty} \|(BR(\lambda, A))^n u\| = 0 \quad \text{for all } u \in L^1. \quad (3.3)$$

The semigroup  $\{P(t)\}_{t \geq 0}$  from Theorem 3.1 can be obtained [4, 7] as the strong limit in  $L^1$  of semigroups  $\{P_r(t)\}_{t \geq 0}$  generated by  $(A + rB, \mathcal{D}(A))$  as  $r \uparrow 1$ . It satisfies the integral equation

$$P(t)u = S(t)u + \int_0^t P(t-s)BS(s)u ds \quad (3.4)$$

for any  $u \in \mathcal{D}(A)$  and  $t \geq 0$ , where  $\{S(t)\}_{t \geq 0}$  is the semigroup generated by  $(A, \mathcal{D}(A))$ , and it is also given by the Dyson–Phillips expansion

$$P(t)u = \sum_{n=0}^{\infty} S_n(t)u, \quad u \in \mathcal{D}(A), \quad t \geq 0, \quad (3.5)$$

where

$$S_0(t)u = S(t)u, \quad S_{n+1}(t)u = \int_0^t S_n(t-s)BS(s)u \, ds, \quad n \geq 0. \quad (3.6)$$

Let  $\lambda > 0$ . Since the generator  $C$  of the substochastic semigroup  $\{P(t)\}_{t \geq 0}$  is such that  $Cu = (A+B)u$  for  $u \in \mathcal{D}(A)$ , we have

$$(\lambda - C)R(\lambda, A)v = (\lambda - A - B)R(\lambda, A)v = (I - BR(\lambda, A))v$$

for  $v \in L^1$ . Thus  $\text{Ker}(I - BR(\lambda, A)) \subseteq \text{Ker}(R(\lambda, A))$  and

$$BR(\lambda, A)v + \lambda R(\lambda, A)v = v + (A+B)R(\lambda, A)v \quad \text{for } v \in L^1_+. \quad (3.7)$$

Combining this with (G2) and (G3), we obtain the following corollary.

**Corollary 3.2.** *Let  $\lambda > 0$ . Then*

$$\|BR(\lambda, A)u\| + \|\lambda R(\lambda, A)u\| = \|u\| \quad \text{for } u \in L^1_+ \quad (3.8)$$

and  $BR(\lambda, A)$  is a positive contraction with  $\text{Ker}(I - BR(\lambda, A)) = \{0\}$ .

**Remark 3.3.** Note that if  $u \in L^1_+$  then for each  $N \geq 0$

$$\left\| R(\lambda, A) \sum_{n=0}^N (BR(\lambda, A))^n u \right\| = \|u\| - \|(BR(\lambda, A))^{N+1} u\|. \quad (3.9)$$

In fact, since  $R(\lambda, A)v \in \mathcal{D}(A)_+$  for  $v \in L^1_+$ , we obtain, by (3.7) and (3.1),

$$\lambda \int_E R(\lambda, A)v \, dm = \int_E (v - BR(\lambda, A)v) \, dm,$$

which gives (3.9) for  $v = \sum_{n=0}^N (BR(\lambda, A))^n u$ .

We have the following result for stochastic semigroups.

**Theorem 3.4.** *Let  $\lambda > 0$ . The following are equivalent:*

- (1)  $\{P(t)\}_{t \geq 0}$  is a stochastic semigroup.
- (2) The operator  $BR(\lambda, A)$  is mean ergodic.
- (3)  $m\{x \in E: f_\lambda(x) > 0\} = 0$ , where

$$f_\lambda(x) = \lim_{n \rightarrow \infty} (BR(\lambda, A))^{*n} 1(x). \quad (3.10)$$

- (4) There is  $u \in L^1_+$ ,  $u > 0$  a.e. such that

$$\lim_{n \rightarrow \infty} \|(BR(\lambda, A))^n u\| = 0.$$

**Proof.** By Corollary 3.2, the operator  $BR(\lambda, A)$  is a positive contraction with  $\text{Ker}(I - BR(\lambda, A)) = \{0\}$ . First assume that (1) holds. Since the operator  $\lambda R(\lambda, C)$  is stochastic, we have  $\|\lambda R(\lambda, C)u\| = \|u\|$  for  $u \in L^1_+$ . Hence (4) follows from (3.2) and (3.9). The implications (4)  $\Rightarrow$  (3)  $\Rightarrow$  (2)  $\Rightarrow$  (1) follow from Proposition 2.1 and condition (3.3).  $\square$

Next, we consider strong stability.

**Theorem 3.5.** *The semigroup  $\{P(t)\}_{t \geq 0}$  is strongly stable if and only if*

$$m\left\{x \in E: \liminf_{\lambda \downarrow 0} f_\lambda(x) < 1\right\} = 0,$$

where  $f_\lambda$  is defined in (3.10).

**Proof.** It follows from (3.9) and the monotone convergence theorem that

$$\|\lambda R(\lambda, C)u\| = \|u\| - \|f_\lambda u\| \quad \text{for } u \in L_+^1.$$

Since  $f_\lambda \leq 1$  for all  $\lambda > 0$  and  $\|f_\mu u\| \leq \|f_\lambda u\|$  for  $\mu > \lambda$  and all  $u \in L_+^1$ , the claim follows from Proposition 2.4.  $\square$

We now prove the following general result which provides another sufficient condition for  $\{P(t)\}_{t \geq 0}$  to be stochastic.

**Theorem 3.6.** Define the operator  $K : L^1 \rightarrow L^1$  by

$$Ku = \lim_{\lambda \downarrow 0} BR(\lambda, A)u \quad \text{for } u \in L^1. \quad (3.11)$$

Then the following hold:

- (1)  $K$  is a positive contraction.
- (2)  $K$  is stochastic if and only if the semigroup  $\{S(t)\}_{t \geq 0}$  generated by  $A$  is strongly stable.
- (3) If  $K$  is mean ergodic then  $\{P(t)\}_{t \geq 0}$  is stochastic.

**Proof.** We have  $\|BR(\lambda, A)u\| \leq \|u\|$  for  $u \in L_+^1$  and  $0 \leq BR(\mu, A)u \leq BR(\lambda, A)u$  for  $\mu > \lambda$ ,  $u \in L_+^1$ . Thus the limit  $\lim_{\lambda \downarrow 0} BR(\lambda, A)u$  exists and  $\|\lim_{\lambda \downarrow 0} BR(\lambda, A)u\| = \lim_{\lambda \downarrow 0} \|BR(\lambda, A)u\|$  for  $u \in L_+^1$ , by the monotone convergence theorem. Since the cone  $L_+^1$  is generating, i.e.  $L^1 = L_+^1 - L_+^1$ ,  $K$  is a well-defined positive contraction. From (3.8) it follows that

$$\|Ku\| = \|u\| - \lim_{\lambda \downarrow 0} \|R(\lambda, A)u\| \quad \text{for } u \in L_+^1,$$

which implies (2), by Proposition 2.4. Since  $BR(\lambda, A) \leq K$  for  $\lambda > 0$ , claim (3) is a consequence of Proposition 2.3 and Theorem 3.4.  $\square$

The semigroup  $\{P(t)\}_{t \geq 0}$  dominates  $\{S(t)\}_{t \geq 0}$ . By part (2) of Theorem 3.6, we obtain the following necessary condition for  $\{P(t)\}_{t \geq 0}$  to be strongly stable.

**Corollary 3.7.** If the semigroup  $\{P(t)\}_{t \geq 0}$  is strongly stable then the operator  $K$  defined by (3.11) is stochastic.

#### 4. Evolution equation

In this section we introduce an abstract setting in which the evolution equations for densities of PDMPs can be studied. Let  $P$  be a stochastic operator on  $L^1$ ,  $\varphi : E \rightarrow [0, \infty)$  be a measurable function, and let

$$L_\varphi^1 = \left\{ u \in L^1 : \int_E \varphi(x) |u(x)| m(dx) < \infty \right\}.$$

We assume that  $\{S(t)\}_{t \geq 0}$  is a substochastic semigroup on  $L^1$  with generator  $(A, \mathcal{D}(A))$  such that

$$\mathcal{D}(A) \subseteq L_\varphi^1 \quad \text{and} \quad \int_E Au \, dm = - \int_E \varphi u \, dm \quad \text{for } u \in \mathcal{D}(A)_+. \quad (4.1)$$

**Remark 4.1.** Note that (4.1) holds if

$$Au = A_0 u - \varphi u \quad \text{for } u \in \mathcal{D}(A) \subseteq \mathcal{D}(A_0) \cap L_\varphi^1,$$

where  $(A_0, \mathcal{D}(A_0))$  is the generator of a stochastic semigroup.

Define the operator  $B$  by  $Bu = P(\varphi u)$ ,  $u \in L_\varphi^1$ . Since  $P$  is positive and  $\|P(\varphi u)\| = \|\varphi u\|$  for  $u \in \mathcal{D}(A)_+$ , it follows from (4.1) that the operators  $(A, \mathcal{D}(A))$  and  $(B, L_\varphi^1)$  satisfy the assumptions (G1)–(G3) of Section 3 and, by Theorem 3.1, there exists a smallest substochastic semigroup  $\{P(t)\}_{t \geq 0}$  with generator  $(C, \mathcal{D}(C))$  which is an extension of the operator

$$Cu = Au + P(\varphi u) \quad \text{for } u \in \mathcal{D}(A). \quad (4.2)$$

Since  $(C, \mathcal{D}(C))$  is the generator of  $\{P(t)\}_{t \geq 0}$ , the Cauchy problem

$$u'(t) = Cu(t), \quad t \geq 0, \quad u(0) = u_0,$$

possesses a unique classical solution for all  $u_0 \in \mathcal{D}(C)$ , which is given by  $u(t) = P(t)u_0 \in \mathcal{D}(C)$ . However, as we do not know the operator  $C$ , we should rather work with the equation

$$u'(t) = Cu(t), \quad \text{where } Cu = \mathcal{A}u + P(\varphi u)$$

and  $\mathcal{A}$  and  $P$  are extensions of the operators  $A$  and  $P$  such that  $\mathcal{D}(C) \subseteq \mathcal{D}(\mathcal{C})$ . The existence of such extensions follows from the construction of [2, Section 2] (see [7, Section 6.3] for more details) which we now reformulate in terms of the operators that appear in (4.2).

We denote by  $L = L(E, \mathcal{E}, m)$  the space of equivalent classes of all measurable  $[-\infty, \infty]$ -valued function on  $E$  and by  $L^0$  the subspace of  $L$  consisting of all elements which are finite almost everywhere. If  $0 \leq u_n \leq u_{n+1}$ ,  $u_n \in L^1$ ,  $n \in \mathbb{N}$ , then the pointwise almost everywhere limit of  $u_n$  exists and will be denoted by  $\sup_n u_n$ , so that  $\sup_n u_n \in L$ . If  $T$  is a positive bounded linear operator, it may be extended pointwise and linearly beyond the space  $L^1$  in the following way: if  $u \in L_+$  then we define

$$Tu = \sup_n Tu_n \quad \text{for } u = \sup_n u_n, \quad u_n \in L_+^1$$

(note that  $Tu$  is independent of the particular approximating sequence  $u_n$ ), and if  $u \in L$  is such that  $T|u| \in L^0$  then we set  $Tu = Tu_+ - Tu_-$ . Since  $R(1, A)$  and  $P$  are positive contractions, they have pointwise extensions, which will be denoted in what follows by  $R(1, A)$  and  $P$ .

Let

$$\mathbb{F} = \{u \in L: R(1, A)|u| \in L^1\} \quad \text{and} \quad R_1 u = R(1, A)u \quad \text{for } u \in \mathbb{F}.$$

Then  $\mathbb{F} \subset L^0$  and the operator  $R_1: \mathbb{F} \rightarrow L^1$  is one-to-one [4, Lemma 3.1]. We can define the operator  $\mathcal{A}: \mathcal{D}(\mathcal{A}) \rightarrow L^0$  by

$$\mathcal{A}u = u - R_1^{-1}u \quad \text{for } u \in \mathcal{D}(\mathcal{A}) := \{R_1 v: v \in \mathbb{F}\} \quad (4.3)$$

and the operator  $\mathcal{B}: \mathcal{D}(\mathcal{B}) \rightarrow L^0$  by

$$\mathcal{B}u = P(\varphi u) \quad \text{for } u \in \mathcal{D}(\mathcal{B}) := \{u \in L^1: P(\varphi|u|) \in L^0\}.$$

Since  $L_\varphi^1 \subset \mathcal{D}(\mathcal{B})$  and  $\mathcal{A}$  is an extension of  $(A, \mathcal{D}(A))$ , the operator  $\mathcal{C}: \mathcal{D}(\mathcal{C}) \rightarrow L^1$  given by

$$\mathcal{C}u = \mathcal{A}u + \mathcal{B}u \quad \text{for } u \in \mathcal{D}(\mathcal{C}) = \{u \in \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\mathcal{B}): \mathcal{C}u \in L^1\}$$

is an extension of the operator  $(C, \mathcal{D}(A))$  defined by (4.2). Theorem 1 of [2] characterizes the generator  $(C, \mathcal{D}(C))$  of the semigroup  $\{P(t)\}_{t \geq 0}$  in the following way:

$$\mathcal{C}u = Cu \quad \text{for } u \in \mathcal{D}(C) = \left\{u \in \mathcal{D}(C): \lim_{n \rightarrow \infty} \|(R_1 \mathcal{B})^n u\| = 0\right\}.$$

Since  $(C, \mathcal{D}(C))$  is a closed extension of  $\mathcal{C}|_{\mathcal{D}(A)}$ , we obtain

$$\mathcal{D}(\overline{\mathcal{C}|_{\mathcal{D}(A)}}) \subseteq \mathcal{D}(C) \subseteq \mathcal{D}(\mathcal{C}).$$

Consequently, if  $u_0 \in \mathcal{D}(C) \cap D(m)$  then the equation

$$u'(t) = \mathcal{C}u(t), \quad t \geq 0, \quad u(0) = u_0, \quad (4.4)$$

has a nonnegative strongly differentiable solution  $u(t)$  which is given by  $u(t) = P(t)u_0$  for  $t \geq 0$  and if  $\{P(t)\}_{t \geq 0}$  is stochastic then this solution is unique in  $D(m)$ . Recall that  $D(m)$  is the set of densities.

**Remark 4.2.** Suppose that the operator  $P$  has kernel  $p$ . Then for every  $u \in L_+^1$  we obtain

$$P(\varphi u)(x) = \int_E p(x, y) \varphi(y) u(y) m(dy),$$

by the monotone convergence theorem.

If  $Au = -\varphi u$  for  $u \in L_\varphi^1$  then

$$\mathbb{F} = \left\{u \in L^0: \frac{u}{1+\varphi} \in L^1\right\} \quad \text{and} \quad \mathcal{A}u = -\varphi u \quad \text{for } u \in \mathcal{D}(\mathcal{A}) = L^1.$$

## 5. Piecewise deterministic Markov processes

### 5.1. Construction

Let  $E$  be a Borel subset of a Polish space (separable complete metric space) and let  $\mathcal{B}(E)$  be the Borel  $\sigma$ -algebra. We consider three local characteristics  $(\pi, \varphi, \mathcal{J})$ :

- (1) A *semidynamical system*  $\pi: \mathbb{R}_+ \times E \rightarrow E$  on  $E$ , i.e.  $\pi_0 x = x$ ,  $\pi_{t+s} x = \pi_t(\pi_s x)$  for  $x \in E$ ,  $s, t \in \mathbb{R}_+$ , and the mapping  $(t, x) \mapsto \pi_t x$  is continuous [25, Section 7.2].

(2) A *jump rate function*  $\varphi : E \rightarrow \mathbb{R}_+$  which is Borel measurable and such that for every  $x \in E$ ,  $t > 0$ , the function  $s \mapsto \varphi(\pi_s x)$  is integrable on  $[0, t]$ . We additionally assume that

$$\lim_{t \rightarrow \infty} \int_0^t \varphi(\pi_s x) ds = +\infty \quad \text{for all } x \in E. \quad (5.1)$$

(3) A *jump distribution*  $\mathcal{J} : E \times \mathcal{B}(E) \rightarrow [0, 1]$  which is a stochastic transition kernel such that  $\mathcal{J}(x, \{x\}) = 0$  for all  $x \in E$ .

The local characteristics  $(\pi, \varphi, \mathcal{J})$  determine a piecewise deterministic Markov process  $\{X(t)\}_{t \geq 0}$  (PDMP) on  $E$  (see e.g. [13,14,22]). Define the function

$$\Phi_x(t) = 1 - e^{-\phi_x(t)}, \quad t > 0, \quad x \in E, \quad \text{where } \phi_x(t) = \int_0^t \varphi(\pi_s x) ds.$$

From (2) it follows that for every  $x \in E$  the function  $\phi_x$  is non-decreasing and right-continuous, because  $\phi_x(\tau) \rightarrow 0$  as  $\tau \downarrow 0$  and  $\phi_x(t + \tau) = \phi_{\pi_t x}(\tau) + \phi_x(t)$  for all  $t, \tau > 0$ ,  $x \in E$ . This and (5.1) imply that  $\Phi_x$  is the distribution function of a positive finite random variable. Let  $\phi_x^{\leftarrow}$  be the *generalized inverse* of  $\phi_x$ , i.e.

$$\phi_x^{\leftarrow}(q) = \inf\{t : \phi_x(t) \geq q\}, \quad q \geq 0,$$

and let  $\kappa : [0, 1] \times E \rightarrow E$  be a measurable function such that

$$\mathcal{J}(x, B) = l_1\{q \in [0, 1] : \kappa(q, x) \in B\} \quad \text{for } x \in E, \quad B \in \mathcal{B}(E), \quad (5.2)$$

where  $l_1$  is the Lebesgue measure on  $([0, 1], \mathcal{B}([0, 1]))$ ; the existence of this function follows from (3) and the regularity of the space  $E$  [23, Lemma 3.22]. Observe that if  $\vartheta$  is a random variable uniformly distributed on  $(0, 1)$ , then  $\kappa(\vartheta, x)$  has distribution  $\mathcal{J}(x, \cdot)$  and if  $\varepsilon$  is exponentially distributed with mean 1, then  $\phi_x^{\leftarrow}(\varepsilon)$  has distribution  $\Phi_x$  (note that  $\varepsilon = -\log(1 - \vartheta)$ ).

Let  $\varepsilon_n, \vartheta_n$ ,  $n \in \mathbb{N}$ , be a sequence of independent random variables, where the  $\varepsilon_n$  are exponentially distributed with mean 1 and the  $\vartheta_n$  are uniformly distributed on  $(0, 1)$ . Let  $\Delta t_0 = \tau$ ,  $\tau \in \mathbb{R}_+$ , and let  $\xi_0 = x$ ,  $x \in E$ . Define recursively the sequence of *holding times* as

$$\Delta t_n := \phi_{\xi_{n-1}}^{\leftarrow}(\varepsilon_n),$$

and *post-jump positions* as

$$\xi_n := \kappa(\vartheta_n, \pi_{\Delta t_n}(\xi_{n-1})).$$

Then  $(\xi_n, \Delta t_n)$  is a discrete time-homogeneous Markov process on  $E \times \mathbb{R}_+$  with stochastic transition kernel given by

$$\mathcal{G}((x, \tau), B \times [0, t)) = \int_0^t \mathcal{J}(\pi_s x, B) \varphi(\pi_s x) e^{-\int_0^s \varphi(\pi_r x) dr} ds$$

for  $(x, \tau) \in E \times \mathbb{R}_+$ ,  $t \in \mathbb{R}_+$ , and  $B \in \mathcal{B}(E)$ . Let  $\mathbb{P}_{(x, \tau)}$  be the distribution of  $(\xi_n, \Delta t_n)$  starting at  $(\xi_0, \Delta t_0) = (x, \tau)$ . We write in an abbreviated fashion  $\mathbb{P}_x$  for  $\mathbb{P}_{(x, 0)}$  and  $\mathbb{E}_x$  for the integration with respect to  $\mathbb{P}_x$ ,  $x \in E$ .

Now let  $\Delta t_0 \equiv 0$  and define *jump times* as

$$t_n := \sum_{l=0}^n \Delta t_l \quad \text{for } n \geq 0.$$

Since  $\Delta t_n > 0$  for all  $n \geq 1$  with probability one, the sequence  $(t_n)$  is increasing and we can introduce the *explosion time*

$$t_\infty := \lim_{n \rightarrow \infty} t_n.$$

The sample path of the process  $\{X(t)\}_{t \geq 0}$  starting at  $X(0) = \xi_0 = x$  is now defined by

$$X(t) = \begin{cases} \pi_{t-t_n}(\xi_n), & \text{if } t_n \leq t < t_{n+1}, \quad n \geq 0, \\ \Delta, & \text{if } t \geq t_\infty, \end{cases}$$

where  $\Delta \notin E$  is some extra state representing a cemetery point for  $\{X(t)\}_{t \geq 0}$ . The process  $\{X(t)\}_{t \geq 0}$  is called the *minimal PDMP* corresponding to the characteristics  $(\pi, \varphi, \mathcal{J})$ . It has right continuous sample paths, by construction, and it is a strong Markov process, by [22, Theorem 8]. The process is called *non-explosive* if  $\mathbb{P}_x(t_\infty = \infty) = 1$  for all  $x \in E$ .

In particular, if  $\pi_t x = x$  for all  $t \geq 0$ ,  $x \in E$ , then  $\{X(t)\}_{t \geq 0}$  is the so-called *pure jump Markov process*. Observe that in this case condition (5.1) is equivalent to  $\varphi(x) > 0$  for every  $x \in E$  and  $\mathbb{P}_x(t_\infty = \infty) = 1$  is equivalent to

$$\sum_{n=1}^{\infty} \frac{\varepsilon_n}{\varphi(\xi_{n-1})} = \infty \quad \mathbb{P}_x\text{-a.e.}$$

We also have  $\mathbb{P}_x(t_\infty = \infty) = 1$  if and only if the series  $\sum_{n=1}^{\infty} \frac{1}{\varphi(\xi_{n-1})}$  diverges  $\mathbb{P}_x$ -a.e. (see e.g. [23, Proposition 12.19]). General sufficient conditions for the explosion of pure jump Markov processes are contained in [34, Section 2]. Note also that pure jump Markov processes on a countable set  $E$  are continuous-time Markov chains.

## 5.2. Existence of densities for PDMP

Let  $\{X(t)\}_{t \geq 0}$  be the minimal PDMP on  $E$  with characteristics  $(\pi, \varphi, \mathcal{J})$  as defined in Section 5.1 and let  $m$  be a  $\sigma$ -finite measure on  $\mathcal{E} = \mathcal{B}(E)$ . In this section we impose further restrictions on the characteristics  $(\pi, \varphi, \mathcal{J})$  which allow us to define a substochastic semigroup  $\{P(t)\}_{t \geq 0}$  on  $L^1$  corresponding to the Markov process  $\{X(t)\}_{t \geq 0}$  and to provide a probabilistic characterization of the analytic results from Section 3.

We assume that a stochastic operator  $P : L^1 \rightarrow L^1$  is the transition operator corresponding to  $\mathcal{J}$  and that a substochastic semigroup  $\{S(t)\}_{t \geq 0}$  on  $L^1$ , with generator  $(A, \mathcal{D}(A))$  satisfying (4.1), is such that

$$\int_E e^{-\int_0^t \varphi(\pi_r x) dr} 1_B(\pi_t x) u(x) m(dx) = \int_B S(t)u(x) m(dx) \quad (5.3)$$

for all  $t \geq 0$ ,  $u \in L^1_+$ ,  $B \in \mathcal{B}(E)$ . As shown in Section 4, there exists a smallest substochastic semigroup  $\{P(t)\}_{t \geq 0}$  on  $L^1$  whose generator is an extension of the operator  $(\mathcal{C}, \mathcal{D}(A))$  defined in (4.2). The semigroup  $\{P(t)\}_{t \geq 0}$  will be referred to as the *minimal semigroup on  $L^1$  corresponding to  $(\pi, \varphi, \mathcal{J})$* .

**Remark 5.1.** Observe that from (5.3) it follows that for every  $t > 0$  the transformation  $\pi_t : E \rightarrow E$  is nonsingular, i.e.  $m(\pi_t^{-1}(B)) = 0$  for all  $B \in \mathcal{B}(E)$  such that  $m(B) = 0$  [25, Section 3.2], and that there is a stochastic operator  $P_0(t)$  on  $L^1$  satisfying

$$\int_E 1_B(\pi_t x) v(x) m(dx) = \int_B P_0(t) v(x) m(dx), \quad B \in \mathcal{B}(E), \quad v \in L^1.$$

Hence,

$$S(t)u = P_0(t)v_t, \quad \text{where } v_t(x) = e^{-\int_0^t \varphi(\pi_r x) dr} u(x).$$

If  $\{P_0(t)\}_{t \geq 0}$  is a stochastic semigroup with generator  $A_0$  then one may expect that the minimal semigroup  $\{P(t)\}_{t \geq 0}$  solves (1.1) and that the operator  $A$  is as in (1.2) (see Section 6 for some examples).

Substituting  $B = E$  into (5.3) leads to

$$\|S(t)u\| = \int_E e^{-\int_0^t \varphi(\pi_r x) dr} u(x) m(dx) \quad \text{for all } u \in L^1_+,$$

which shows that  $\{S(t)\}_{t \geq 0}$  is strongly stable if and only if condition (5.1) holds. Thus, the operator  $K : L^1 \rightarrow L^1$  defined by

$$Ku = \lim_{\lambda \downarrow 0} P(\varphi R(\lambda, A)u) \quad \text{for } u \in L^1 \quad (5.4)$$

is stochastic, by Theorem 3.6.

The main result of this section is the following (we use the convention  $e^{-\infty} = 0$ ).

**Theorem 5.2.** Let  $(t_n)$  be the sequence of jump times and  $t_\infty = \lim_{n \rightarrow \infty} t_n$  be the explosion time for  $\{X(t)\}_{t \geq 0}$ . Then the following hold:

(1) For any  $\lambda > 0$

$$\lim_{n \rightarrow \infty} (P(\varphi R(\lambda, A)))^{*n} 1(x) = \mathbb{E}_x(e^{-\lambda t_\infty}) \quad m\text{-a.e. } x.$$

(2) For any  $B \in \mathcal{B}(E)$ ,  $u \in \mathcal{D}(A)_+$ , and  $t > 0$

$$\int_B P(t)u(x) m(dx) = \int_E \mathbb{P}_x(X(t) \in B, t < t_\infty) u(x) m(dx).$$

(3) The operator  $K$  as defined in (5.4) is the transition operator corresponding to the discrete-time Markov process  $(X(t_n))_{n \geq 0}$  with stochastic kernel

$$\mathcal{K}(x, B) = \int_0^\infty \mathcal{J}(\pi_s x, B) \varphi(\pi_s x) e^{-\int_0^s \varphi(\pi_r x) dr} ds, \quad x \in E, \quad B \in \mathcal{B}(E).$$

**Proof.** Let  $M(E)_+$  (respectively  $BM(E)_+$ ) be the space of all (bounded) Borel measurable nonnegative functions on  $E$ . From (5.3) we obtain, by approximation,

$$\int_E e^{-\int_0^t \varphi(\pi_r x) dr} f(\pi_t x) u(x) m(dx) = \int_E f(x) S(t) u(x) m(dx) \quad (5.5)$$

for all  $t \geq 0$ ,  $u \in L_+^1$ ,  $f \in M(E)_+$ . Let  $\lambda > 0$  and

$$G^\lambda f(x) = \int_0^\infty e^{-\lambda s} T_0(s)(\varphi \mathcal{J} f)(x) ds, \quad x \in E, \quad f \in BM(E)_+,$$

where the operators  $\mathcal{J}$  and  $T_0(s)$  are defined by

$$\mathcal{J} f(x) = \int_E f(y) \mathcal{J}(x, dy), \quad x \in E, \quad f \in BM(E)_+,$$

and

$$T_0(s) f(x) = e^{-\int_0^s \varphi(\pi_r x) dr} f(\pi_s x), \quad x \in E, \quad f \in M(E)_+, \quad s \geq 0.$$

From (5.5) and Fubini's theorem it follows that

$$\int_E G^\lambda f(x) u(x) m(dx) = \int_E f(x) P(\varphi R(\lambda, A) u)(x) m(dx) \quad (5.6)$$

for  $f \in BM(E)_+$ ,  $u \in L_+^1$ , which gives  $(P(\varphi R(\lambda, A)))^* f = G^\lambda f$ . On the other hand, the construction of the sequence  $(t_n, X(t_n))$  yields

$$(G^\lambda)^n f(x) = \mathbb{E}_x(f(X(t_n)) e^{-\lambda t_n}), \quad x \in E, \quad n \in \mathbb{N}, \quad f \in BM(E)_+, \quad (5.7)$$

which, by the monotone convergence theorem, leads to

$$\lim_{n \rightarrow \infty} (G^\lambda)^n 1(x) = \mathbb{E}_x(e^{-\lambda t_\infty})$$

and proves (1).

In order to show (2), for each  $n \geq 0$  we define

$$T_n(t) f(x) = \mathbb{E}_x(f(X(t)) 1_{\{t < t_{n+1}\}}), \quad x \in E, \quad t \in \mathbb{R}_+, \quad f \in BM(E)_+.$$

Let  $B \in \mathcal{B}(E)$  and  $u \in \mathcal{D}(A)_+$ . From the construction of the process and the strong Markov property it follows that [22, Theorem 9]

$$T_n(t) 1_B(x) = T_0(t) 1_B(x) + \int_0^t T_0(s)(\varphi \mathcal{J}(T_{n-1}(t-s) 1_B))(x) ds$$

for all  $x \in E$ ,  $t \geq 0$ , and  $n \geq 1$ . Hence, by induction,

$$\int_E T_n(t) 1_B(x) u(x) m(dx) = \int_B \sum_{j=0}^n S_j(t) u(x) m(dx), \quad n \geq 0, \quad t > 0,$$

where the  $S_j$  are defined in (3.6). From (3.5) we obtain

$$\lim_{n \rightarrow \infty} \int_B \sum_{j=0}^n S_j(t) u(x) m(dx) = \int_B P(t) u(x) m(dx).$$

On the other hand,

$$T_n(t) 1_B(x) = \mathbb{P}_x(X(t) \in B, \quad t < t_n) \uparrow \mathbb{P}_x(X(t) \in B, \quad t < t_\infty)$$

for all  $x \in E$ , which proves (2).

Finally, from (5.7) we conclude that

$$\lim_{\lambda \downarrow 0} G^\lambda 1_B(x) = \mathbb{E}_x(1_B(X(t_1))) = \mathcal{K}(x, B),$$

which completes the proof of (3), by (5.6).  $\square$

As a direct consequence of Theorem 5.2 and Theorem 3.4 we obtain the following corollary.

**Corollary 5.3.** The semigroup  $\{P(t)\}_{t \geq 0}$  is stochastic if and only if

$$m\{x \in E: \mathbb{P}_x(t_\infty < \infty) > 0\} = 0.$$

In that case, if the distribution of  $X(0)$  has a density  $u_0 \in \mathcal{D}(A)$  then  $X(t)$  has the density  $P(t)u_0$  for all  $t > 0$ .

Furthermore, as a consequence of Theorem 5.2 and Theorem 3.5 we obtain the following result.

**Corollary 5.4.** The semigroup  $\{P(t)\}_{t \geq 0}$  is strongly stable if and only if

$$m\{x \in E: \mathbb{P}_x(t_\infty = \infty) > 0\} = 0.$$

**Remark 5.5.** Note that for every density  $u \in \mathcal{D}(A)_+$  we obtain

$$\int_E P(t)u(x)m(dx) = \int_E \mathbb{P}_x(t_\infty > t)u(x)m(dx) \quad \text{for all } t > 0, \quad (5.8)$$

by part (2) of Theorem 5.2. In particular, if  $\mathcal{D}(A)$  is such that for every  $u \in L_+^1$  we can find a non-decreasing sequence  $u_n \in \mathcal{D}(A)_+$  such that  $u_n \uparrow u$  then (5.8) holds for all  $u \in L_+^1$ .

**Corollary 5.6.** Let  $E$  be a countable set,  $m$  be the counting measure on  $E$ ,  $\varphi > 0$ , and  $\{X(t)\}_{t \geq 0}$  be a pure jump Markov process on  $E$ . Then the semigroup  $\{P(t)\}_{t \geq 0}$  is stochastic if and only if the process  $\{X(t)\}_{t \geq 0}$  is non-explosive.

## 6. Fragmentation models revisited

In this section we illustrate the applicability of our results to fragmentation models described by linear rate equations [27,15,28,8,1,5]. For a recent survey of analytic methods for such models we refer the reader to [6]. See also [10] for a different probabilistic treatment of so-called random fragmentation processes.

Let  $E = (0, \infty)$ ,  $\mathcal{E} = \mathcal{B}(E)$ , and  $m(dx) = x dx$ . Let  $b: E \times E \rightarrow \mathbb{R}_+$  be a Borel measurable function such that for every  $y > 0$

$$\int_0^y b(x, y)x dx = y \quad \text{and} \quad b(x, y) = 0 \quad \text{for } x \geq y. \quad (6.1)$$

The stochastic kernel defined by

$$\mathcal{J}(x, B) = \frac{1}{x} \int_0^x 1_B(y)b(y, x)y dy \quad \text{for } x \in E, \quad B \in \mathcal{B}(E), \quad (6.2)$$

will be referred to as the *fragmentation kernel*. According to (5.2), we have  $\mathcal{J}(x, B) = l_1\{q \in [0, 1]: \kappa(q, x) \in B\}$ , where

$$\kappa(q, x) = H_x^{\leftarrow}(q)x \quad \text{for } q \in [0, 1], \quad x > 0,$$

and  $H_x^{\leftarrow}(q) = \inf\{r \in [0, 1]: H_x(r) \geq q\}$ ,  $q \in [0, 1]$ , is the generalized inverse of the distribution function

$$H_x(r) = \int_0^r b(xz, x)xz dz \quad \text{for } r \in [0, 1].$$

Note that  $0 < H_x^{\leftarrow}(q) \leq 1$  for all  $x$  and  $q \in (0, 1)$ .

The kernel  $\mathcal{J}$  is called *homogenous* if  $b$  is of the form

$$b(x, y) = \frac{1}{y} h\left(\frac{x}{y}\right) \quad \text{for } 0 < x < y, \quad (6.3)$$

where  $h: (0, 1) \rightarrow \mathbb{R}_+$  is a Borel measurable function with  $\int_0^1 h(z)z dz = 1$ . Since  $H_x(r)$  does not depend on  $x$ , we obtain

$$\kappa(q, x) = H^{\leftarrow}(q)x, \quad \text{where } H(r) = \int_0^r h(z)z dz.$$

The kernel  $\mathcal{J}$  is called *separable* if  $b$  is of the form

$$b(x, y) = \frac{\beta(x)y}{\Lambda(y)} \quad \text{for } x < y, \quad \text{where } \Lambda(y) = \int_0^y \beta(z)z dz$$

and  $\beta$  is a nonnegative Borel measurable function on  $E$  such that  $\Lambda(y)$  is finite and positive for all  $y > 0$ . We have  $H_x(r) = \Lambda(xr)/\Lambda(x)$  for  $r \in [0, 1]$ . Hence  $H_x^\leftarrow(q) = \Lambda^\leftarrow(q\Lambda(x))/x$  and in this case

$$\kappa(q, x) = \Lambda^\leftarrow(q\Lambda(x)).$$

Since  $\Lambda(\Lambda^\leftarrow x) = x$  for all  $x > 0$ , the mapping  $x \mapsto \Lambda(x)$  transforms this case into the homogenous fragmentation with  $H(r) = r$  for  $r \in (0, 1)$ .

In what follows we assume that  $\varepsilon_n, \vartheta_n, n \in \mathbb{N}$ , and  $\xi_0$  are independent random variables, where the  $\varepsilon_n$  are exponentially distributed with mean 1, the  $\vartheta_n$  are uniformly distributed on  $(0, 1)$ , and  $\xi_0$  is an  $E$ -valued random variable.

### 6.1. Pure fragmentation

In this section we consider the pure fragmentation equation [27–29]

$$\frac{\partial u(t, x)}{\partial t} = \int_x^\infty b(x, y)\varphi(y)u(t, y)dy - \varphi(x)u(t, x), \quad t > 0, \quad x > 0, \quad (6.4)$$

where  $b$  satisfies (6.1) and  $\varphi$  is a positive Borel measurable function. If we let  $\psi(y, x) = b(x, y)\varphi(y)$  then (6.4) has the same form as in [29] in the absence of coagulation. For a discussion of the model we refer the reader to [7, Chapter 8].

We rewrite Eq. (6.4) in the form (4.4) with the stochastic operator  $P$  on  $L^1$  given by

$$Pu(x) = \int_x^\infty b(x, y)u(y)dy, \quad u \in L^1, \quad (6.5)$$

and

$$Au = -\varphi u, \quad u \in L_\varphi^1 = \left\{ u \in L^1 : \int_0^\infty \varphi(x)|u(x)|x dx < \infty \right\}.$$

Observe that  $P$  is the transition operator corresponding to  $\mathcal{J}$  as defined in (6.2). The operator  $(A, L_\varphi^1)$  generates a sub-stochastic semigroup  $\{S(t)\}_{t \geq 0}$  on  $L^1$  where  $S(t)u(x) = e^{-\varphi(x)t}u(x)$ ,  $t \geq 0$ ,  $x \in E$ ,  $u \in L^1$ . Hence, (5.3) holds with  $\pi_t x = x$ ,  $t \geq 0$ ,  $x \in E$ .

Let  $\{X(t)\}_{t \geq 0}$  be the minimal pure jump Markov process with characteristics  $(\pi, \varphi, \mathcal{J})$  and let  $\{P(t)\}_{t \geq 0}$  be the minimal semigroup on  $L^1$  corresponding to  $(\pi, \varphi, \mathcal{J})$  as defined in Section 5.2. The sequences of jump times  $t_n$  and post-jump positions  $\xi_n = X(t_n)$  satisfy

$$t_n = \sum_{k=1}^n \frac{\varepsilon_k}{\varphi(\xi_{k-1})}, \quad \xi_n = H_{\xi_{n-1}}^\leftarrow(\vartheta_n)\xi_{n-1}, \quad n \geq 1.$$

Since the sequence  $(\xi_n)$  is non-increasing, we can take  $\Delta = 0$  and write for the explosion time

$$t_\infty = \inf\{t > 0 : X(t) = 0\}.$$

As a consequence of Corollary 5.3 we obtain the following result of [28].

**Corollary 6.1.** *If  $\varphi$  is bounded on bounded subsets of  $(0, \infty)$  then  $\{P(t)\}_{t \geq 0}$  is stochastic.*

**Proof.** Let  $N > 0$  and let  $M_N < \infty$  be such that  $\varphi(x) \leq M_N$  for all  $x \leq N$ . Thus, if  $\xi_0 \leq N$  then  $\xi_k \leq N$  for all  $k$ , and  $t_n \geq \sum_{k=1}^n \varepsilon_k / M_N$  for all  $n$ . As a result  $\mathbb{P}_x(t_\infty < \infty) = 0$  for all  $x \leq N$ , and the claim follows from Corollary 5.3.  $\square$

From Corollary 5.4 we obtain the following result.

**Corollary 6.2.** *Let  $V$  be a nonnegative Borel measurable function such that  $V(x)\varphi(x) \geq 1$  for all  $x > 0$ . If*

$$m \left\{ x \in E : \mathbb{P}_x \left( \sum_{n=1}^\infty \varepsilon_n V(\xi_n) = \infty \right) > 0 \right\} = 0$$

*then  $\{P(t)\}_{t \geq 0}$  is strongly stable.*

**Example 6.1.** Consider a homogenous kernel as in (6.3) and let  $V(x) = x^\gamma/a$ , where  $\gamma, a > 0$ . The random variable

$$\tau = \sum_{k=1}^\infty \varepsilon_k \prod_{l=1}^{k-1} H^\leftarrow(\vartheta_l)^\gamma$$

is finite with probability 1, by [32, Theorem 1.6]. Thus, if  $\varphi(x) \geq a/x^\gamma$  for  $x > 0$ , then  $\{P(t)\}_{t \geq 0}$  is strongly stable, by Corollary 6.2. Since  $t_\infty \leq V(\xi_0)\tau$ , we have for every  $u \in L^1_+$ , by Remark 5.5,

$$\int_0^\infty P(t)u(x)x dx \leq \int_0^\infty (1 - F_\tau(atx^{-\gamma}))u(x)x dx \quad \text{for all } t > 0, \quad (6.6)$$

with equality when  $\varphi(x) = a/x^\gamma$ , where  $F_\tau$  is the distribution function of  $\tau$ .

In particular, if  $h(z) = (\nu + 2)z^\nu$  with  $\nu + 2 > 0$ , then  $H^\leftarrow(\vartheta_1) = \vartheta_1^{1/(\nu+2)}$  and  $\tau$  has the gamma distribution [32, Example 3.8] such that

$$1 - F_\tau(q) = \frac{1}{\Gamma(1 + (\nu + 2)/\gamma)} \int_q^\infty s^{(\nu+2)/\gamma} e^{-s} ds, \quad q \geq 0,$$

where  $\Gamma$  is the Gamma function. When  $\varphi(x) = 1/x^\gamma$  the equality in (6.6) coincides with the heuristic results of [27]; for values of  $\nu$  and  $\gamma$  such that  $(\nu + 2)/\gamma \in \mathbb{N} \cup \{0\}$  we obtain

$$\int_0^\infty P(t)u(x)x dx = \int_0^\infty e^{-tx^{-\gamma}} \sum_{k=0}^{(\nu+2)/\gamma} \frac{(tx^{-\gamma})^k}{k!} u(x)x dx$$

for all  $t > 0$  and  $u \in L^1_+$ . See [6, Example 6.5] for quite involved calculations for the specific choice of  $\nu = 0$  and  $\gamma = 1$ .

**Remark 6.3.** Since the sequence  $(\xi_n)$  is non-increasing, it converges with probability one to some random variable. In particular, when the kernel is either homogenous or separable the limiting random variable is zero. Then it is sufficient to look only at a neighborhood of zero to decide whether the semigroup is stochastic or not.

## 6.2. Fragmentation with growth

Pure fragmentation, described by (6.4), may occur together with other phenomena. In this section we study fragmentation processes with continuous growth, where the growth process is described by a semidynamical system  $\pi$  satisfying the equation

$$\frac{\partial}{\partial t} \pi_t x = g(\pi_t x) \quad \text{for } x, t > 0, \quad (6.7)$$

where  $g$  is a strictly positive continuous function. We refer the reader to [1,5,26] for related examples. We denote by  $L^1_{\text{loc}}$  the space of all Borel measurable functions on  $E$  which are integrable on compact subsets of  $E$  and by  $AC$  the space of absolutely continuous functions on  $E$ .

Our first task is to construct the minimal PDMP  $\{X(t)\}_{t \geq 0}$  on  $E$  with characteristics  $(\pi, \varphi, \mathcal{J})$ , where  $\pi$  satisfies (6.7),  $\varphi \in L^1_{\text{loc}}$  is nonnegative, and  $\mathcal{J}$  is the fragmentation kernel (6.2). We assume throughout this section that there is  $\bar{x} > 0$  such that

$$\int_{\bar{x}}^\infty \frac{1}{g(z)} dz = \infty \quad \text{and} \quad \int_{\bar{x}}^\infty \frac{\varphi(z)}{g(z)} dz = \infty. \quad (6.8)$$

Since  $1/g \in L^1_{\text{loc}}$  and  $\varphi/g \in L^1_{\text{loc}}$ , we can define

$$G(x) = \int_{x_0}^x \frac{1}{g(z)} dz \quad \text{and} \quad Q(x) = \int_{x_1}^x \frac{\varphi(z)}{g(z)} dz, \quad (6.9)$$

where  $x_0 = 0$  and  $x_1 = 0$  when the integrals exist for all  $x$  and, otherwise,  $x_0, x_1$  are any points in  $E$ .

The function  $G$  is increasing, invertible, continuously differentiable on  $E$ , and the formula  $r(t, x) = G^{-1}(G(x) + t)$  defines a monotone continuous function in each variable. Since  $G(\infty) = +\infty$ , the function  $r(t, x)$  is well defined for all  $t \geq 0, x \in E$  and determines a semidynamical system on  $E$

$$\pi_t x = G^{-1}(G(x) + t). \quad (6.10)$$

In the case when  $G(0) = -\infty$  the function  $r(t, x)$  is well defined for all  $t \in \mathbb{R}$  and  $x \in E$ , so that we have, in fact, a flow  $\pi_t$  on  $E$  such that  $\pi_t(E) = E$ . In any case, for any given  $x > 0$  we have  $\pi_{-t}x = r(-t, x) \in E$  for all  $t > 0$  such that  $t < G(x) - G(0)$ .

The function  $Q$  is non-decreasing. Let  $Q^\leftarrow$  be the generalized inverse of  $Q$ , which is defined and finite for all  $q \in \mathbb{R}$ , by (6.8). We have

$$\phi_x(t) = \int_0^t \varphi(\pi_r x) dr = \int_x^{\pi_t x} \frac{\varphi(z)}{g(z)} dz = Q(\pi_t x) - Q(x) \quad \text{for } x > 0, t \geq 0,$$

so that (5.1) holds if and only if  $Q(\infty) = \infty$ , which is our assumption (6.8). From (6.10) it follows that

$$\phi_x^\leftarrow(q) = G(Q^\leftarrow(Q(x) + q)) - G(x) \quad (6.11)$$

and

$$\pi_{\phi_x^\leftarrow(q)} x = Q^\leftarrow(Q(x) + q) \quad \text{for } x > 0, q \geq 0. \quad (6.12)$$

Consequently, the random variables  $t_n$  and  $\xi_n = X(t_n)$ ,  $n \geq 1$ , now satisfy

$$t_n = \sum_{k=1}^n \phi_{\xi_{k-1}}^\leftarrow(\varepsilon_k), \quad \xi_n = H_{Q^\leftarrow(Q(\xi_{n-1}) + \varepsilon_n)}^\leftarrow(\vartheta_n) Q^\leftarrow(Q(\xi_{n-1}) + \varepsilon_n). \quad (6.13)$$

**Remark 6.4.** If  $\varphi$  is bounded above by a constant  $a$  then

$$\phi_x^\leftarrow(q) = G(Q^\leftarrow(Q(x) + q)) - G(x) \geq \frac{1}{a}(Q(Q^\leftarrow(Q(x) + q)) - Q(x)) \geq \frac{q}{a}.$$

Thus  $t_n \geq \frac{1}{a} \sum_{k=1}^n \varepsilon_k$  for every  $n$ , so that  $\mathbb{P}_x(t_\infty < \infty) = 0$  for all  $x > 0$ .

**Remark 6.5.** Observe that if

$$m\left\{x \in E: \mathbb{P}_x\left(\limsup_{n \rightarrow \infty} \xi_n < \infty\right) > 0\right\} = 0,$$

then  $m\{x \in E: \mathbb{P}_x(t_\infty < \infty) > 0\} = 0$ . This is a consequence of  $G(\infty) = \infty$  and  $t_n \geq G(Q^\leftarrow(Q(\xi_{n-1}) + \varepsilon_n)) - G(\xi_0)$  for  $n \geq 1$ .

We now turn our attention to the minimal semigroup  $\{P(t)\}_{t \geq 0}$  on  $L^1$  corresponding to  $(\pi, \varphi, \mathcal{J})$ . For  $t > 0$  we define the operators  $S(t)$  on  $L^1$  by

$$S(t)u(x) = \mathbf{1}_E(\pi_{-t}x)u(\pi_{-t}x) \frac{\pi_{-t}xg(\pi_{-t}x)}{xg(x)} e^{Q(\pi_{-t}x) - Q(x)}, \quad x \in E, \quad (6.14)$$

for  $u \in L^1$ . Then  $\{S(t)\}_{t \geq 0}$  is a substochastic semigroup on  $L^1$  satisfying (5.3), whose generator is of the form

$$Au(x) = -\frac{1}{x} \frac{d}{dx} (xg(x)u(x)) - \varphi(x)u(x), \quad u \in \mathcal{D}(A) = \mathcal{D}_0 \cap L_\varphi^1,$$

where  $u \in \mathcal{D}_0$  if and only if the function  $\tilde{u}(x) = xg(x)u(x)$  is such that  $\tilde{u} \in AC$ ,  $\tilde{u}'(x)/x$  belongs to  $L^1$ , and, additionally  $\lim_{x \rightarrow 0} \tilde{u}(x) = 0$  when  $G(0) = 0$ . This can be derived from [26, Theorem 5] by an isomorphic transformation of the space  $L^1$ . The resolvent operator  $R(1, A)$  is given by

$$R(1, A)u(x) = \frac{1}{xg(x)} e^{-G(x) - Q(x)} \int_0^x e^{G(y) + Q(y)} u(y) y dy, \quad u \in L^1,$$

and its extension  $R_1$ , as described in Section 4, is defined by the same integral expression. It can be proved as in [3, Lemma 4.1] that the extension  $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$  defined in (4.3) is of the form

$$\mathcal{A}u(x) = -\frac{1}{x} \frac{d}{dx} (xg(x)u(x)) - \varphi(x)u(x)$$

for  $u \in \mathcal{D}(\mathcal{A}) \subseteq \{u \in L^1: \tilde{g}u \in AC\}$ , where  $\tilde{g}(x) = xg(x)$ ,  $x > 0$ . Consequently, the corresponding evolution equation on  $L^1$  for  $u(t, x) = P(t)u_0(x)$  is of the form

$$\frac{\partial u(t, x)}{\partial t} = -\frac{1}{x} \frac{\partial}{\partial x} (xg(x)u(t, x)) - \varphi(x)u(t, x) + \int_x^\infty b(x, y)\varphi(y)u(t, y) dy$$

with  $u(0, x) = u_0(x)$ .

Finally, we apply our results from Sections 3 and 5.2 to the minimal semigroup  $\{P(t)\}_{t \geq 0}$ . It is easily seen, by Theorem 5.2, that the operator  $K$  as defined in (5.4) is a stochastic operator with kernel

$$k(x, y) = \int_{\max\{x, y\}}^{\infty} b(x, z) \frac{\varphi(z)}{zg(z)} e^{Q(y) - Q(z)} dz, \quad x, y \in (0, \infty). \quad (6.15)$$

By Theorem 3.6,  $\{P(t)\}_{t \geq 0}$  is stochastic, if  $K$  is mean ergodic. In particular,  $K$  is mean ergodic, if  $K$  is *asymptotically stable*, i.e. there is  $u_* \in D(m)$  such that  $Ku_* = u_*$  and

$$\lim_{n \rightarrow \infty} \|K^n u - u_*\| = 0 \quad \text{for all } u \in D(m).$$

General sufficient conditions for the latter to hold are contained in [25, Chapter 5] and we have the following result.

**Theorem 6.6.** (See [25, Theorem 5.7.1].) *If the kernel  $k$  satisfies*

$$\int_0^{\infty} \inf_{0 < y < r} k(x, y) m(dx) > 0 \quad \text{for every } r > 0$$

*and has a Lyapunov function  $V : (0, \infty) \rightarrow [0, \infty)$ , i.e.  $\lim_{x \rightarrow \infty} V(x) = \infty$  and for some constants  $0 \leq c < 1$ ,  $d \geq 0$*

$$\int_0^{\infty} V(x) Ku(x) m(dx) \leq c \int_0^{\infty} V(x) u(x) m(dx) + d \quad \text{for } u \in D(m),$$

*then the operator  $K$  is asymptotically stable.*

In the reminder of this section, we will study the semigroup  $\{P(t)\}_{t \geq 0}$  under the assumption that the kernel  $\mathcal{J}$  is homogeneous as in (6.3).

**Corollary 6.7.** *Assume that there are  $r, \gamma > 0$  such that*

$$\int_0^r \frac{\varphi(z)}{g(z)} dz < \infty \quad \text{and} \quad \liminf_{x \rightarrow \infty} \frac{\varphi(x)}{x^{\gamma-1} g(x)} > 0. \quad (6.16)$$

*Then the operator  $K$  is asymptotically stable and the semigroup  $\{P(t)\}_{t \geq 0}$  is stochastic.*

**Proof.** Since  $Q(0) = 0$ , we have

$$k(x, y) \geq \int_x^{\infty} h\left(\frac{x}{z}\right) \frac{\varphi(z)}{z^2 g(z)} e^{-Q(z)} dz \quad \text{for } 0 < y < r < x,$$

which shows that the first condition in Theorem 6.6 holds, by (6.16). We also have  $\lim_{x \rightarrow \infty} x^{\gamma} e^{-Q(x)} = 0$  and

$$\int_0^{\infty} x^{\gamma} k(x, y) x dx = \int_0^1 z^{\gamma} zh(z) dz \left( y^{\gamma} + \gamma e^{Q(y)} \int_y^{\infty} z^{\gamma-1} e^{-Q(z)} dz \right) \leq cy^{\gamma} + d,$$

where  $c := \int_0^1 z^{\gamma} zh(z) dz < 1 = \int_0^1 zh(z) dz$ , which shows that  $V(x) = x^{\gamma}$  is a Lyapunov function.  $\square$

The assumptions in Corollary 6.7 cannot be essentially weakened.

**Example 6.2.** Suppose that  $\varphi(x) = g(x)/x$  for all  $x > 0$ . Then  $Q(x) = \log x$  for  $x > 0$ , so that the sequence  $\xi_n$  is of the form

$$\xi_n = \xi_0 \prod_{k=1}^n H^{\leftarrow}(\vartheta_k) e^{\varepsilon_k} \quad \text{for } n \geq 1.$$

Let  $\mu_0 = \int_0^1 \log zh(z) dz$ . Observe that  $\mu_0$  is always negative and might be equal to  $-\infty$ . If  $\mu_0 \geq -1$  then for any  $x$  we have  $\mathbb{P}_x(\limsup_{n \rightarrow \infty} \xi_n = \infty) = 1$ . If  $\mu_0 < -1$  then, by the strong law of large numbers,  $\mathbb{P}_x(\lim_{n \rightarrow \infty} \xi_n = 0) = 1$  for all  $x$ . Thus  $K$  is not asymptotically stable in both cases and  $\{P(t)\}_{t \geq 0}$  is stochastic when  $\mu_0 \geq -1$ , by Remark 6.5. In any case, if  $g(x) \leq \bar{a}x$  then  $\{P(t)\}_{t \geq 0}$  is stochastic, by Remark 6.4.

**Example 6.3.** Suppose that  $g(x) = x^{1-\beta}$  and  $\varphi(x) = ax^\alpha$  for  $x > 0$ , where  $a > 0$ . Then condition (6.8) holds if and only if  $\beta \geq 0$  and  $\alpha + \beta \geq 0$ . If  $\alpha + \beta > 0$  then (6.16) holds, thus  $\{P(t)\}_{t \geq 0}$  is stochastic. Now suppose that  $\alpha + \beta = 0$ . If either  $\beta = 0$  or  $\mu_0 = \int_0^1 \log zh(z)z dz \geq -1/a$  then  $\{P(t)\}_{t \geq 0}$  is also stochastic, as in the preceding example. If  $\beta > 0$  and  $\mu_0 < -1/a$  then  $\{P(t)\}_{t \geq 0}$  is strongly stable. This follows from the representation

$$t_n = \frac{1}{\beta} \sum_{k=1}^n (e^{\beta \varepsilon_k/a} - 1) \xi_{k-1}^\beta = \frac{\xi_0^\beta}{\beta} \sum_{k=1}^n (e^{\beta \varepsilon_k} - 1) \prod_{l=1}^{k-1} H^{\leftarrow}(\vartheta_l)^\beta e^{\beta \varepsilon_l/a}$$

and the fact that the random variable

$$\tau = \sum_{k=1}^{\infty} (e^{\beta \varepsilon_k/a} - 1) \prod_{l=1}^{k-1} H^{\leftarrow}(\vartheta_l)^\beta e^{\beta \varepsilon_l/a}$$

is finite with probability 1 precisely when  $\mu_0 < -1/a$  [32, Theorem 1.6].

### 6.3. Fragmentation with decay

In this section we consider fragmentation processes with continuous degradation, where the degradation process is described by a semidynamical system  $\pi$  satisfying the equation

$$\frac{\partial}{\partial t} \pi_t x = -g(\pi_t x) \quad \text{for } x, t > 0, \quad (6.17)$$

where  $g$  is a strictly positive continuous function. Now the corresponding linear evolution equation will be of the form

$$\frac{\partial u(t, x)}{\partial t} = \frac{1}{x} \frac{\partial}{\partial x} (xg(x)u(t, x)) - \varphi(x)u(t, x) + \int_x^\infty b(x, y)\varphi(y)u(t, y) dy.$$

We refer the reader to [15,8,3] for related examples.

To construct the minimal PDMP  $\{X(t)\}_{t \geq 0}$  with characteristics  $(\pi, \varphi, \mathcal{J})$ , where  $\pi$  satisfies (6.17),  $\varphi \in L^1_{\text{loc}}$  is nonnegative, and  $\mathcal{J}$  is given by (6.2), we redefine the functions  $G$  and  $Q$  from (6.9) in such a way that formulas (6.10)–(6.13) remain valid. We assume that there is  $\bar{x} > 0$  such that

$$\int_0^{\bar{x}} \frac{1}{g(z)} dz = \infty \quad \text{and} \quad \int_0^{\bar{x}} \frac{\varphi(z)}{g(z)} dz = \infty, \quad (6.18)$$

and define

$$G(x) = \int_x^{x_0} \frac{1}{g(z)} dz \quad \text{and} \quad Q(x) = \int_x^{x_1} \frac{\varphi(z)}{g(z)} dz, \quad (6.19)$$

where  $x_0 = +\infty$  and  $x_1 = +\infty$  when the integrals exist for all  $x$ , and, otherwise,  $x_0, x_1$  are any points from  $E$ . The semidynamical system  $\pi$  defined by (6.10) satisfies (6.17). The function  $G$  is now decreasing and  $Q$  is non-increasing. As the generalized inverse of  $Q$  we take

$$Q^{\leftarrow}(q) = \begin{cases} \sup\{x: Q(x) \geq q\}, & q > Q(\infty), \\ 0, & q \leq Q(\infty) \text{ and } Q(\infty) > -\infty. \end{cases}$$

With these alterations Eqs. (6.11) and (6.12) remain valid. The random variables  $t_n$  and  $\xi_n = X(t_n)$ ,  $n \geq 1$ , again satisfy (6.13).

The semigroup  $\{S(t)\}_{t \geq 0}$ , defined by (6.14), is a substochastic semigroup on  $L^1$  satisfying (5.3), whose generator is of the form

$$Au(x) = \frac{1}{x} \frac{d}{dx} (xg(x)u(x)) - \varphi(x)u(x), \quad u \in \mathcal{D}(A) = \mathcal{D}_0 \cap L^1_\varphi,$$

where  $u \in \mathcal{D}_0$  if and only if the function  $\tilde{u}(x) = xg(x)u(x)$  is such that  $\tilde{u} \in AC$ ,  $\tilde{u}'(x)/x$  belongs to  $L^1$ , and, additionally  $\lim_{x \rightarrow \infty} \tilde{u}(x) = 0$  when  $G(\infty) = 0$ . This can be derived from [26, Theorem 7].

We conclude this section with the following characterization of the minimal semigroup  $\{P(t)\}_{t \geq 0}$  on  $L^1$  corresponding to  $(\pi, \varphi, \mathcal{J})$ .

**Corollary 6.8.** If  $\varphi$  is bounded on bounded subsets of  $(0, \infty)$  then  $\{P(t)\}_{t \geq 0}$  is stochastic.

If  $V$  is a non-decreasing function such that  $V(x)\varphi(x) \geq 1$  for all  $x > 0$  and

$$m \left\{ x \in E: \mathbb{P}_x \left( \sum_{n=1}^{\infty} \varepsilon_n V(\xi_n) = \infty \right) > 0 \right\} = 0,$$

then  $\{P(t)\}_{t \geq 0}$  is strongly stable.

**Proof.** Observe that for all  $x, q > 0$  we have  $Q^{\leftarrow}(Q(x) + q) < x$ . Since  $H_x^{\leftarrow}(q) \leq 1$  for all  $x > 0, q \in [0, 1]$ , we obtain

$$\xi_n \leq Q^{\leftarrow}(Q(\xi_{n-1}) + \varepsilon_n) < \xi_{n-1}, \quad n \geq 1,$$

which shows that the sequence  $\xi_n$  is decreasing. We also have

$$\frac{q}{\sup_{z \in I_{x,q}} \varphi(z)} \leq G(Q^{\leftarrow}(Q(x) + q)) - G(x) \leq \frac{q}{\inf_{z \in I_{x,q}} \varphi(z)}, \quad (6.20)$$

where  $I_{x,q} = [Q^{\leftarrow}(Q(x) + q), x]$ . The first inequality in (6.20) implies that

$$\phi_{\xi_{k-1}}^{\leftarrow}(\varepsilon_k) \geq \frac{\varepsilon_k}{\sup_{z \leq \xi_{k-1}} \varphi(z)} \geq \frac{\varepsilon_k}{\sup_{z \leq \xi_0} \varphi(z)}, \quad k \geq 1,$$

which proves the first assertion, as in the proof of Corollary 6.8. The second statement is a direct consequence of the second inequality in (6.20) and Corollary 5.4.  $\square$

**Example 6.4.** Consider a homogenous kernel as in (6.3). First suppose that  $\varphi(x) \geq a/x^\gamma$  for  $x > 0$ , where  $a, \gamma > 0$ . Then  $\{P(t)\}_{t \geq 0}$  is strongly stable and condition (6.6) holds for every density  $u \in \mathcal{D}(A)$ .

For the particular choice of  $g(x) = x^{1-\beta}$  and  $\varphi(x) = ax^\alpha$  for  $x > 0$ , condition (6.18) holds if and only if  $\beta \leq 0$  and  $\alpha + \beta \leq 0$ . Hence, if  $\beta \leq 0$  then  $\{P(t)\}_{t \geq 0}$  is stochastic when  $0 \leq \alpha \leq -\beta$  and it is strongly stable when  $\alpha < 0$ .

Observe also that the case when  $G(0) < \infty$ , which in this example holds when  $\beta > 0$ , corresponds to the situation when for every  $x > 0$  there is  $t \in (0, \infty)$  such that  $\pi_t x = 0$ , so that 0 is reached in a finite time from every point.

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## References

- [1] O. Arino, R. Rudnicki, Stability of phytoplankton dynamics, C. R. Biologies 327 (2004) 961–969.
- [2] L. Arlotti, A perturbation theorem for positive contraction semigroups on  $L^1$ -spaces with applications to transport equations and Kolmogorov's differential equations, Acta Appl. Math. 23 (1991) 129–144.
- [3] L. Arlotti, J. Banasiak, Strictly substochastic semigroups with application to conservative and shattering solutions to fragmentation equations with mass loss, J. Math. Anal. Appl. 293 (2004) 693–720.
- [4] J. Banasiak, On an extension of the Kato–Voigt perturbation theorem for substochastic semigroups and its application, Taiwanese J. Math. 5 (2001) 169–191.
- [5] J. Banasiak, On conservativity and shattering for an equation of phytoplankton dynamics, C. R. Biologies 327 (2004) 1025–1036.
- [6] J. Banasiak, Shattering and non-uniqueness in fragmentation models—an analytic approach, Phys. D 222 (2006) 63–72.
- [7] J. Banasiak, L. Arlotti, Perturbations of Positive Semigroups with Applications, Springer Monogr. Math., Springer-Verlag London Ltd., London, 2006.
- [8] J. Banasiak, W. Lamb, On the application of substochastic semigroup theory to fragmentation models with mass loss, J. Math. Anal. Appl. 284 (2003) 9–30.
- [9] J. Banasiak, M. Mokhtar-Kharroubi, Universality of dishonesty of substochastic semigroups: Shattering fragmentation and explosive birth-and-death processes, Discrete Contin. Dyn. Syst. Ser. B 5 (2005) 529–542.
- [10] J. Bertoin, Random Fragmentation and Coagulation Processes, Cambridge Stud. Adv. Math., vol. 102, Cambridge University Press, Cambridge, 2006.
- [11] A. Bobrowski, T. Lipniacki, K. Pichór, R. Rudnicki, Asymptotic behavior of distributions of mRNA and protein levels in a model of stochastic gene expression, J. Math. Anal. Appl. 333 (2007) 753–769.
- [12] R. Chill, Y. Tomilov, Stability of operator semigroups: Ideas and results, in: Perspectives in Operator Theory, in: Banach Center Publ., vol. 75, Polish Acad. Sci., Warsaw, 2007, pp. 71–109.
- [13] M.H.A. Davis, Piecewise-deterministic Markov processes: A general class of nondiffusion stochastic models, J. Roy. Statist. Soc. Ser. B 46 (1984) 353–388.
- [14] M.H.A. Davis, Markov Models and Optimization, Monogr. Statist. Appl. Probab., vol. 49, Chapman & Hall, London, 1993.
- [15] B.F. Edwards, M. Cai, H. Han, Rate equation and scaling for fragmentation with mass loss, Phys. Rev. A 41 (1990) 5755–5757.
- [16] K.-J. Engel, R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Grad. Texts in Math., vol. 194, Springer-Verlag, New York, 2000.
- [17] W. Feller, On the integro-differential equations of purely discontinuous Markoff processes, Trans. Amer. Math. Soc. 48 (1940) 488–515.
- [18] A. Filippov, On the distribution of the sizes of particles which undergo splitting, Theory Probab. Appl. 6 (1961) 275–294.
- [19] N. Fournier, J.-S. Giet, On small particles in coagulation–fragmentation equations, J. Stat. Phys. 111 (2003) 1299–1329.
- [20] N. Fournier, J.-S. Giet, Existence of densities for jumping stochastic differential equations, Stochastic Process. Appl. 116 (2006) 643–661.
- [21] B. Haas, Loss of mass in deterministic and random fragmentations, Stochastic Process. Appl. 106 (2003) 245–277.

- [22] J. Jacod, A.V. Skorokhod, Jumping Markov processes, *Ann. Inst. H. Poincaré Probab. Statist.* 32 (1996) 11–67.
- [23] O. Kallenberg, *Foundations of Modern Probability, Probability and Its Applications*, second ed., Springer-Verlag, New York, 2002.
- [24] T. Kato, On the semi-groups generated by Kolmogoroff's differential equations, *J. Math. Soc. Japan* 6 (1954) 1–15.
- [25] A. Lasota, M.C. Mackey, *Chaos, Fractals, and Noise*, Appl. Math. Sci., vol. 97, Springer-Verlag, New York, 1994.
- [26] M.C. Mackey, M. Tyran-Kamińska, Dynamics and density evolution in piecewise deterministic growth processes, *Ann. Polon. Math.* 94 (2008) 111–129.
- [27] E.D. McGrady, R.M. Ziff, “Shattering” transition in fragmentation, *Phys. Rev. Lett.* 58 (1987) 892–895.
- [28] D.J. McLaughlin, W. Lamb, A.C. McBride, A semigroup approach to fragmentation models, *SIAM J. Math. Anal.* 28 (1997) 1158–1172.
- [29] Z. Melzak, A scalar transport equation, *Trans. Amer. Math. Soc.* 85 (1957) 547–560.
- [30] G.E.H. Reuter, Denumerable Markov processes and the associated contraction semigroups on  $l$ , *Acta Math.* 97 (1957) 1–46.
- [31] R. Sine, A mean ergodic theorem, *Proc. Amer. Math. Soc.* 24 (1970) 438–439.
- [32] W. Vervaat, On a stochastic difference equation and a representation of nonnegative infinitely divisible random variables, *Adv. in Appl. Probab.* 11 (1979) 750–783.
- [33] J. Voigt, On substochastic  $C_0$ -semigroups and their generators, *Transport Theory Statist. Phys.* 16 (1987) 453–466.
- [34] W. Wagner, Explosion phenomena in stochastic coagulation–fragmentation models, *Ann. Appl. Probab.* 15 (2005) 2081–2112.
- [35] K. Yosida, *Functional Analysis*, sixth ed., Springer-Verlag, Berlin, 1980.