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On the periodic Schrödinger–Boussinesq system

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ABSTRACT

We study the local and global well-posedness of the periodic boundary value problem for the nonlinear Schrödinger–Boussinesq system. The existence of periodic traveling-wave solutions as well as the orbital stability of such solutions are also considered.

1. Introduction

In this paper, we consider the periodic Schrödinger–Boussinesq system (hereafter referred to as the SB-system)

$$\begin{cases} iu_t + u_{xx} = \alpha v u, \\ v_{tt} - v_{xx} + v_{xxxx} = \beta\left(|u|^2\right)_{xx}, \end{cases} \tag{1}$$

where $t > 0$, $x \in [0, L]$, for some $L > 0$, and α, β are real constants.

Here, u and v are, respectively, a complex-valued and a real-valued function defined in space-time $[0, L] \times \mathbb{R}$. The SB-system may be considered as a model of interactions between short and intermediate long waves, which is derived in describing the dynamics of Langmuir soliton formation and interaction in a plasma [28] and diatomic lattice system [32]. The short wave term $u(x,t) : [0, L] \times \mathbb{R} \to \mathbb{C}$ is described by a Schrödinger-type equation with a potential $v(x,t) : [0, L] \times \mathbb{R} \to \mathbb{R}$ satisfying some sort of Boussinesq equation, and representing the intermediate long wave.

The nonlinear Schrödinger (NLS) equation models a wide range of physical phenomena including self-focusing of optical beams in nonlinear media, propagation of Langmuir waves in plasmas, etc. For an introduction in this topic, we refer the reader to [26]. The Boussinesq equation, as a model of long waves, was originally derived by Boussinesq [8] in his study of nonlinear, dispersive wave propagation. It should be remarked that it was the first equation proposed in the literature to describe this kind of physical phenomena. This equation was also used by Zakharov [34], as a model of nonlinear string, and by Falk et al. [13] in their study of shape-memory alloys.

Our first aim in the current paper, is to study the well-posedness of the periodic boundary value problem (BVP) for the SB-system (1), that is, we are interested in the solvability of system (1) subject to the initial conditions

$$u(x,0) = u_0(x); \qquad v(x,0) = v_0(x); \qquad v_t(x,0) = (v_1)_x(x). \tag{2}$$

Concerning the local well-posedness question, some results has been obtained for the SB-system (1) in the continuous case. Indeed, Linares and Navas [25] proved that (1) is locally well-posed for initial data $u_0 \in L^2(\mathbb{R})$, $v_0 \in L^2(\mathbb{R})$, $v_1 = h_x$

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with $h \in H^{-1}(\mathbb{R})$, and $u_0 \in H^1(\mathbb{R})$, $v_0 \in H^1(\mathbb{R})$, $v_1 = h_x$ with $h \in L^2(\mathbb{R})$. Moreover, by using some conservations laws, in the latter case, the solutions can be extended globally. Yongqian [33] established a similar result when $u_0 \in H^s(\mathbb{R})$, $v_0 \in H^s(\mathbb{R})$, $v_1 = h_{xx}$ with $h \in H^s(\mathbb{R})$ for $s \geqslant 0$. Assuming $s \geqslant 1$ these solutions are global. Finally, Farah [15] proved local well-posedness for initial data $(u_0, v_0, v_1) \in H^k(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ provided

(i) $|k| - 1/2 < s < 1/2 + 2k$ for $k \leqslant 0$,
(ii) $k - 1/2 < s < 1/2 + k$ for $k > 0$.

In particular, local well-posedness holds for initial data $(u_0, v_0, v_1) \in H^s(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s > -1/4$. Moreover, when $s = 0$ the solution is shown to be global. It should be mentioned that, in fact, it is possible to obtain global well-posedness for $s \geqslant 0$ in the continuous case. This can be proved using the arguments introduced by Bourgain [7] (see also Angulo et al. [4]). In the proof of Theorem 1.5 below, we also apply these techniques for the periodic SB-system (1)–(2).

The local well-posedness for single dispersive equations with quadratic nonlinearities has been extensively studied in Sobolev spaces. In general, the proofs are based in the Fourier restriction norm approach introduced by Bourgain [6] in his study of the nonlinear Schrödinger (NLS) equation $iu_t + u_{xx} + u|u|^{p-2} = 0$, with $p \geqslant 3$, and the Korteweg–de Vries (KdV) equation $u_t + u_{xxx} + u_x u = 0$. This method was further developed by Kenig, Ponce, and Vega in [23] for the KdV equation, and in [24] for the quadratic nonlinear Schrödinger equations

$$iu_t + u_{xx} + F_j(u, \bar{u}) = 0, \quad j = 1, 2, 3,$$

where $\bar{u}$ denotes the complex conjugate of u and $F_1(u, \bar{u}) = u^2$, $F_2(u, \bar{u}) = u\bar{u}$, $F_3(u, \bar{u}) = \bar{u}^2$.

The original Bourgain method makes extensive use of Strichartz-type inequalities in order to derive the bilinear estimates corresponding to the nonlinearity. On the other hand, Kenig, Ponce, and Vega simplified Bourgain's proof and improved the bilinear estimates using only elementary techniques, such as Cauchy–Schwarz inequality and simple calculus inequalities.

This last technique was used by Farah [16] in the study of the Boussinesq equation. It should be pointed out that the symbol of the Boussinesq equation does not have good cancelations. To overcome this difficulty, the author observed that the dispersion of the Boussinesq equation (given by the symbol $\sqrt{\xi^2 + \xi^4}$) is, in some sense, related with the dispersion of the Schrödinger equation (given by the symbol ξ^2) (see Lemma 3.3 below). Therefore, one can "modify" the symbols and work just with the algebraic relations for the Schrödinger equation already used in Kenig, Ponce, and Vega [24], in order to derive the relevant bilinear estimates.

To describe our results, we first introduce some functional spaces. Given $s \in \mathbb{R}$ and $L > 0$, the periodic Sobolev space $H^s_{per} = H^s_{per}([0, L])$ is the set of all periodic distributions f such that

$$\|f\|_{H^s_{per}} := \left\| \left(1 + |n|\right)^s \hat{f}(n) \right\|_{l^2_n} < \infty.$$

Next, we define the $X^S_{s,b}$ and $X^B_{s,b}$ spaces related, respectively, to the Schrödinger and Boussinesq equations. For the first equation, this spaces were first introduced in [6], whereas for the second one, they were first defined by Fang and Grillakis [14]. By using these spaces and following Bourgain's argument introduced in [6], they proved local well-posedness for the BVP

$$\begin{cases} u_{tt} - u_{xx} + u_{xxxx} + \partial_x^2\left[f(u)\right] = 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = (u_1)_x(x), \end{cases}$$

where $u_0 \in H^s_{per}$, $u_1 \in H^{s-2}_{per}$, $0 \leqslant s \leqslant 1$, and the nonlinearity f satisfies $|f(u)| \leqslant c|u|^p$, with $1 < p < \frac{3-2s}{1-2s}$ if $0 \leqslant s < \frac{1}{2}$, and $1 < p < \infty$ if $\frac{1}{2} \leqslant s \leqslant 1$. Moreover, if $u_0 \in H^1_{per}$, $u_1 \in H^{-1}_{per}$, and $f(u) = \lambda|u|^{q-1}u - |u|^{p-1}u$, $1 < q < p$, $\lambda \in \mathbb{R}$, then the solution is global.

Definition 1.1. Let $\mathcal{Y}$ be the space of functions $F = F(x, t)$ such that

(i) $F : [0, L] \times \mathbb{R} \to \mathbb{C}$.
(ii) $F(x, \cdot) \in \mathcal{S}(\mathbb{R})$ for each $x \in [0, L]$.
(iii) $F(\cdot, t) \in C^\infty([0, L])$ for each $t \in \mathbb{R}$.

For $s, b \in \mathbb{R}$, $X^S_{s,b}$ and $X^B_{s,b}$ denotes, respectively, the completion of $\mathcal{Y}$ with respect to the norm

$$\|F\|_{X^S_{s,b}} = \left\| \left\langle \tau + (2\pi n/L)^2 \right\rangle^b \langle n \rangle^s \widetilde{F} \right\|_{l^2_n L^2_\tau}, \tag{3}$$

$$\|F\|_{X^B_{s,b}} = \left\| \left\langle |\tau| - \gamma_L(n) \right\rangle^b \langle n \rangle^s \widetilde{F} \right\|_{l^2_n L^2_\tau}, \tag{4}$$

where $\widetilde{\ }$ denotes the space-time Fourier transform, $\langle a \rangle \equiv 1 + |a|$ and $\gamma_L(n) \equiv (2\pi/L)^2\sqrt{n^2 + n^4}$.

We will also need the localized $X_{s,b}$ spaces defined as follows.

Definition 1.2. Let I be a time interval. For $s,b\in\mathbb{R}$, $X^{S,I}_{s,b}$ and $X^{B,I}_{s,b}$ denotes the space endowed with the norm

$$\|u\|_{X^{S,I}_{s,b}}=\inf_{w\in X^S_{s,b}}\bigl\{\|w\|_{X^S_{s,b}}:\ w(t)=u(t)\text{ on } I\bigr\},$$
$$\|u\|_{X^{B,I}_{s,b}}=\inf_{w\in X^B_{s,b}}\bigl\{\|w\|_{X^B_{s,b}}:\ w(t)=u(t)\text{ on } I\bigr\}.$$

Now, we state our main results concerning local well-posedness.

Theorem 1.1. *Let $s\geqslant 0$ and $1/4<a<1/2<b$. Then, there exists $c>0$, depending only on a,b,s, such that*

(i) $\|uv\|_{X^S_{s,-a}}\leqslant c\|u\|_{X^S_{s,b}}\|v\|_{X^B_{s,b}}$.
(ii) $\|u_1\bar{u}_2\|_{X^B_{s,-a}}\leqslant c\|u_1\|_{X^S_{s,b}}\|u_2\|_{X^S_{s,b}}$.

Theorem 1.2. *Let $s\geqslant 0$. Then for any $(u_0,v_0,v_1)\in H^s_{per}([0,L])\times H^s_{per}([0,L])\times H^{s-1}_{per}([0,L])$ there exist $T=T(\|u_0\|_{H^s_{per}},\|v_0\|_{H^s_{per}},\|v_1\|_{H^{s-1}_{per}})>0$, $b>1/2$ and a unique solution (u,v) of the BVP (1)–(2), satisfying*

$$u\in C\bigl([0,T]:H^s_{per}\bigl([0,L]\bigr)\bigr)\cap X^{S,[0,T]}_{s,b}\quad\textit{and}\quad v\in C\bigl([0,T]:H^s_{per}\bigl([0,L]\bigr)\bigr)\cap X^{B,[0,T]}_{s,b}.$$

Moreover, the map $(u_0,v_0,v_1)\mapsto(u(t),v(t))$ is locally Lipschitz from $H^s_{per}([0,L])\times H^s_{per}([0,L])\times H^{s-1}_{per}([0,L])$ into $C([0,T]:H^s_{per}([0,L])\times H^s_{per}([0,L]))$.

We also obtain some counter-examples, which show that the bilinear estimates stated in Theorem 1.1 are sharp.

Theorem 1.3.

(i) *The estimate*
$$\|uv\|_{X^S_{k,-a}}\leqslant c\|u\|_{X^S_{k,b}}\|v\|_{X^B_{s,b}}\tag{5}$$
holds only if $k\leqslant s$.
(ii) *The estimate*
$$\|uv\|_{X^S_{k,-a}}\leqslant c\|u\|_{X^S_{k,b}}\|v\|_{X^B_{s,b}}$$
holds only if $k+s\geqslant 0$.
(iii) *The estimate*
$$\|u_1\bar{u}_2\|_{X^B_{s,-a}}\leqslant c\|u_1\|_{X^S_{k,b}}\|u_2\|_{X^S_{k,b}}\tag{6}$$
holds only if $s\leqslant k$.

Theorem 1.3 has an important consequence. It shows that our local well-posedness result in Theorem 1.2 is sharp, in the sense that it cannot be improved using the spaces $X^S_{s,b}$ and $X^B_{s,b}$. This situation is very different from the continuous case obtained in Farah [15], where the author showed local well-posedness for initial data in different Sobolev spaces with negative indices.

Next, we obtain bilinear estimates for the case $s=0$ and $b<1/2$. These estimates will be useful to establish the existence of global solutions.

Theorem 1.4. *Let $a,a_1,b,b_1>1/4$, then there exists $c>0$, depending only on a,a_1,b,b_1, such that*

(i) $\|uv\|_{X^S_{0,-a_1}}\leqslant c\|u\|_{X^S_{0,b_1}}\|v\|_{X^B_{0,b}}$.
(ii) $\|u_1\bar{u}_2\|_{X^B_{0,-a}}\leqslant c\|u_1\|_{X^S_{0,b_1}}\|u_2\|_{X^S_{0,b_1}}$.

The bilinear estimates in Theorem 1.4 are the essential tools to prove our global result. It asserts that the local solution given by Theorem 1.2 is, in fact, a global one, for all $s\geqslant 0$.

Theorem 1.5. *Let $s\geqslant 0$. Then, the BVP (1)–(2) is globally well-posed for data $(u_0,v_0,v_1)\in H^s_{per}([0,L])\times H^s_{per}([0,L])\times H^{s-1}_{per}([0,L])$. Moreover, the solution (u,v) satisfies, for all $t>0$,*

$$\|v(t)\|_{H^s_{per}} + \|(-\Delta)^{-1/2}v_t(t)\|_{H^{s-1}_{per}} \lesssim e^{((\ln 2)\|u_0\|^2_{H^s_{per}} t)} \max\{\|v_0, v_1\|_{\mathfrak{B}^s}, \|u_0\|_{H^s_{per}}\},$$

where

$$\|v_0, v_1\|^2_{\mathfrak{B}^s} \equiv \|v_0\|^2_{H^s_{per}([0,L])} + \|v_1\|^2_{H^{s-1}_{per}([0,L])}.$$

The argument used to prove Theorem 1.5 follows the ideas introduced by Colliander, Holmer, and Tzirakis [10] to deal with the Zakharov system. The intuition for this theorem comes from the fact that the nonlinearity for the second equation in the SB-system (1) depends only on the first equation. Therefore, noting that the bilinear estimates given in Theorem 1.4 hold for $a, a_1, b, b_1 < 1/2$, it is possible to show that the time existence, given in Theorem 1.2, depends only on $\|u_0\|_{L^2_{per}}$. Since this norm is conserved by the flow, we obtain a global solution.

Our second aim in the present paper, is to study existence and orbital (nonlinear) stability of periodic traveling-wave solutions. These two questions are, in general, important in the understanding of the dynamic of the system under consideration.

The stability of traveling waves has been extensively studied for the whole Euclidean space case (solitary waves), whereas the study under periodic boundary conditions has been started quite recently and only a few works are available in the current literature. To cite a few important contributions, in [1] Angulo studied the orbital stability of *dnoidal* wave solutions for the cubic Schrödinger and modified Korteweg–de Vries equations; his method of proofs follows the pioneers ideas of Benjamin, Bona, and Weinstein. In [2], Angulo et al. gave a complete stability study of *cnoidal* wave solutions for the Korteweg–de Vries equation, adapting to the periodic context the abstract theory developed in [18]. For other equations and systems see e.g. [3,4,11,20,29] (and references therein).

One of the main reasons why the stability study in the periodic case has been received little attention, lies on the needed spectral theory associated with the corresponding linearized operator. Indeed, to fix ideas, suppose we have a Schrödinger-type operator $\mathcal{L} = -\frac{d^2}{dx^2} + Q(x)$, where $Q(x)$ is a smooth real function. Assume that Q and ϕ are smooth and rapidly decaying to zero at infinity, and satisfy $\mathcal{L}\phi = 0$. Assume also that ϕ has exactly two zeros on the whole real line. Then it follows immediately, from Sturn–Liouville's theory, that zero is the third eigenvalue of the operator $\mathcal{L}$, and it is a simple eigenvalue. On the other hand, suppose that Q and ϕ are smooth L-periodic functions such that $\mathcal{L}\phi = 0$. If ϕ has exactly two zeros on the interval $[0, L)$, then from Floquet's theory, the eigenvalue zero is the second or the third one (see e.g. [12]). In most cases, it is a hard task to decide between these two alternatives. As a consequence, most of the current papers deal with explicit periodic traveling-wave solutions. This is the case of the present paper.

The explicit solutions are, usually, given in terms of the Jacobian elliptic functions (dnoidal, cnoidal, and snoidal). So, the main idea to obtain the spectral properties for the linearized operator is to reduce matter to some known periodic eigenvalue problem. The most popular one deals with the periodic eigenvalue problem associated with the Lamé operator

$$\mathcal{L}_{Lame} := -\frac{d^2}{dx^2} + n(n+1)sn^2(x; k), \tag{7}$$

for some determined value of $n \in \mathbb{N}$ (see e.g. [1–3,29]).

Here, we consider $\alpha = \beta = -1$ in (1) and look for solutions of the form

$$u(x,t) = e^{i\omega t}\psi_\omega(x), \qquad v(x,t) = \phi_\omega(x), \tag{8}$$

where ω is a real parameter and $\psi_\omega, \phi_\omega : \mathbb{R} \to \mathbb{R}$ are L-periodic functions with a period $L > 0$. Then, substituting this waveform into the system and integrating twice the second equation in the obtained system, we have

$$\begin{cases} \psi''_\omega - \omega\psi_\omega + \psi_\omega\phi_\omega = 0, \\ \phi''_\omega - \phi_\omega + \psi^2_\omega = 0. \end{cases} \tag{9}$$

To solve system (9), we assume $\omega = 1$ and $\psi_\omega = \phi_\omega = \psi$, so that, it admits a periodic solution of the *cnoidal* type, namely,

$$\psi(x) = \beta_2 + (\beta_3 - \beta_2)cn^2\left(\sqrt{\frac{\beta_3 - \beta_1}{6}}x; k\right), \qquad k^2 = \frac{\beta_3 - \beta_2}{\beta_3 - \beta_1}, \tag{10}$$

where $cn(\cdot; k)$ denotes the cnoidal function and $\beta_1, \beta_2, \beta_3$ are real parameters.

Remark 1.1. We point out that existence and stability of hyperbolic-secant-type solitary waves for (1) were recently considered in [19]. The author has proved an orbital stability result by using the abstract theory contained in [18], taking the advantage of the spectral properties established in [27].

Our main theorem concerning the orbital stability of cnoidal waves reads as follows.

Theorem 1.6. *Let ψ be the cnoidal wave solution given in* (10)*. Then, the periodic traveling wave $(e^{it}\psi, \psi)$ is orbitally stable in the energy space $X = H^1_{per}([0, L]) \times H^1_{per}([0, L]) \times L^2_{per}([0, L])$ by the flow of the SB-system* (1)*.*

To prove Theorem 1.6, we shall employ the theory developed by Grillakis, Shatah, and Strauss [18]. To do so, we first observe that system (1) (with $\alpha = \beta = -1$) can be written in a Hamiltonian form (see (51)). We point out that although the operator J in (52) is not onto, along the lines of proofs in [18], the stability result still holds (see also [19,31]).

Our strategy to get the needed spectral properties is to combine the results in [3], which are essentially proved from well-known results for the Lamé operator in (7), with the min–max principle for the characterization of eigenvalues.

Finally, we also obtain periodic traveling waves for $\omega \neq 1$. Our idea is simple: once obtained the cnoidal solution for $\omega = 1$, we employ the Implicit Function Theorem combined with spectral properties related with the linearized operator to extend our range of parameters for ω near 1.

The plan of this paper is as follows: in Section 2, we introduce some notations and state important results that we will use throughout the paper. The proof of the bilinear estimates and the relevant counter-examples are given in Sections 3 and 4, respectively. In Section 5, we prove Theorem 1.5. Finally, the stability questions are treated in Section 6.

2. Notations and preliminaries

In what follows we use $a \lesssim b$ to say that $a \leqslant Cb$ for some constant $C > 0$. Also, we denote $a \sim b$ when $a \lesssim b$ and $b \lesssim a$. We write $a \ll b$ to denote an estimate of the form $a \leqslant cb$ for some small constant $c > 0$. In addition, $a+$ means that there exists $\varepsilon > 0$ such that $a+ = a + \varepsilon$.

Let us recall some properties of L-periodic functions. For a detailed presentation of the spaces of periodic functions and its properties we refer the reader, for instance, to [21]. The Fourier transform of a function $f \in L^1([0, L])$ is defined by

$$\hat{f}(n) = \frac{1}{L} \int_0^L e^{-2\pi i \frac{x}{L} n} f(x)\, dx.$$

For f in an appropriate class of functions we have $f = (\hat{f})^{\vee}$, where for a sequence $s = \{s_n\}_{n \in \mathbb{Z}}$, the symbol $^{\vee}$ denotes the inverse Fourier transform of s given by

$$(s)^{\vee}(x) = \sum_{n \in \mathbb{Z}} e^{2\pi i \frac{x}{L} n} s_n.$$

The Plancherel identity reads as

$$\|f\|_{L^2_{per}} = \|\hat{f}\|_{l^2_n}.$$

The operator $(-\Delta)^{-1/2}$ is defined, via its Fourier transform, by

$$\left[(-\Delta)^{-1/2} f\right]^{\wedge}(n) = |n|^{-1} \hat{f}(n) \quad n \neq 0.$$

Next, we recall some facts on the linear Schrödinger and Boussinesq equations. Consider the free Schrödinger equation

$$iu_t + u_{xx} = 0. \tag{11}$$

It is easy to see that the solution of (11), with initial data $u(0) = u_0$, is given by the formula

$$u(t) = U(t)u_0, \tag{12}$$

where

$$U(t)u_0 = \left(e^{-(2\pi/L)^2 itn^2} \hat{u}_0(n)\right)^{\vee}.$$

On the other hand, for the linear Boussinesq equation

$$v_{tt} - v_{xx} + v_{xxxx} = 0, \tag{13}$$

the solution, with initial data $v(0) = v_0$ and $v_t(0) = (v_1)_x$, is given by

$$u(t) = V_c(t)v_0 + V_s(t)(v_1)_x, \tag{14}$$

where

$$V_c(t)v_0 = \left(\frac{e^{(2\pi/L)^2 it\sqrt{n^2+n^4}} + e^{-(2\pi/L)^2 it\sqrt{n^2+n^4}}}{2} \hat{v}_0(n)\right)^{\vee},$$

$$V_s(t)(v_1)_x = \left(\frac{e^{(2\pi/L)^2 it\sqrt{n^2+n^4}} - e^{-(2\pi/L)^2 it\sqrt{n^2+n^4}}}{2i\sqrt{n^2+n^4}} \widehat{(v_1)_x}(n)\right)^{\vee}.$$

As a consequence, by Duhamel's Principle, the solution of (1)–(2) is equivalent to

$$u(t) = U(t)u_0 - i\int_0^t U(t-t')(\alpha v u)(t')\,dt',$$

$$v(t) = V_c(t)v_0 + V_s(t)(v_1)_x + \int_0^t V_s(t-t')(\beta|u|^2)_{xx}(t')\,dt'. \tag{15}$$

Let θ be a cut-off function satisfying $\theta \in C_0^\infty(\mathbb{R})$, $0 \leqslant \theta \leqslant 1$, $\theta \equiv 1$ in $[-1,1]$, $\mathrm{supp}(\theta) \subseteq [-2,2]$. For $0 < T \leqslant 1$ define $\theta_T(t) = \theta(t/T)$. In order to work in the $X^S_{s,b}$ and $X^B_{s,b}$ spaces, we consider another versions of (15), *viz.*

$$u(t) = \theta(t)U(t)u_0 - i\theta_T(t)\int_0^t U(t-t')(\alpha v u)(t')\,dt',$$

$$v(t) = \theta(t)\big(V_c(t)v_0 + V_s(t)(v_1)_x\big) + \theta_T(t)\int_0^t V_s(t-t')(\beta|u|^2)_{xx}(t')\,dt' \tag{16}$$

and

$$u(t) = \theta_T(t)U(t)u_0 - i\theta_T(t)\int_0^t U(t-t')(\alpha v u)(t')\,dt',$$

$$v(t) = \theta_T(t)\big(V_c(t)v_0 + V_s(t)(v_1)_x\big) + \theta_T(t)\int_0^t V_s(t-t')(\beta|u|^2)_{xx}(t')\,dt'. \tag{17}$$

We will use Eq. (16) (resp. (17)) to study the local (resp. global) well-posedness problem associated to (1)–(2).

Note that the integral equations (16) and (17) are defined for all $(t,x) \in \mathbb{R}^2$. Moreover, if (u,v) is a solution of (16) or (17), then $(\tilde u, \tilde v) = (u|_{[0,T]}, v|_{[0,T]})$ is a solution of (15) in $[0,T]$.

Before proceeding to the group and integral estimates for (16) and (17), we recall that

$$\|v_0, v_1\|^2_{\mathfrak{B}^s} \equiv \|v_0\|^2_{H^s_{per}([0,L])} + \|v_1\|^2_{H^{s-1}_{per}([0,L])}.$$

For simplicity, we denote $\mathfrak{B}^0$ by $\mathfrak{B}$ and, for functions of t, we use the shorthand

$$\big\|v(t)\big\|^2_{\mathfrak{B}^s} \equiv \big\|v(t)\big\|^2_{H^s_{per}([0,L])} + \big\|(-\Delta)^{-1/2}v_t(t)\big\|^2_{H^{s-1}_{per}([0,L])}.$$

The following three lemmas are standard in this context. Although we are studying the periodic case, the proofs are essentially the same ones of the continuous setting. We refer the reader to Farah [15] for the details.

Lemma 2.1 *(Group estimates). Let $L = 2\pi$ and $0 < T \leqslant 1$.*

(a) *Linear Schrödinger equation*:
  (i) $\|U(t)u_0\|_{C(\mathbb{R}:H^s_{per})} = \|u_0\|_{H^s_{per}}$.
  (ii) *If $0 \leqslant b_1 \leqslant 1$, then*

$$\big\|\theta_T(t)U(t)u_0\big\|_{X^S_{s,b_1}} \lesssim T^{1/2-b_1}\|u_0\|_{H^s_{per}}.$$

(b) *Linear Boussinesq equation*:
  (i) $\|V_c(t)v_0 + V_s(t)(v_1)_x\|_{C(\mathbb{R}:H^s_{per})} \leqslant \|v_0\|_{H^s_{per}} + \|v_1\|_{H^{s-1}_{per}}$.
  (ii) $\|V_c(t)v_0 + V_s(t)(v_1)_x\|_{C(\mathbb{R}:\mathfrak{B})} = \|v_0, v_1\|_{\mathfrak{B}}$.
  (iii) *If $0 \leqslant b \leqslant 1$, then*

$$\big\|\theta_T(t)\big(V_c(t)v_0 + V_s(t)(v_1)_x\big)\big\|_{X^B_{s,b}} \lesssim T^{1/2-b}\big(\|v_0\|_{H^s_{per}} + \|v_1\|_{H^{s-1}_{per}}\big).$$

Next, we estimate the integral parts of (15)–(16).

Lemma 2.2 *(Integral estimates). Let $L = 2\pi$ and $0 < T \leqslant 1$.*

(a) *Nonhomogeneous linear Schrödinger equation*:
  (i) *If $0 \leqslant a_1 < 1/2$, then*

$$\left\| \int_0^t U(t-t')z(t')\,dt' \right\|_{C([0,T]:H^s_{per})} \lesssim T^{1/2-a_1}\|z\|_{X^S_{s,-a_1}}.$$

  (ii) *If $0 \leqslant a_1 < 1/2$, $b_1 \geqslant 0$, and $a_1 + b_1 \leqslant 1$, then*

$$\left\| \theta_T(t)\int_0^t U(t-t')z(t')\,dt' \right\|_{X^S_{s,b_1}} \lesssim T^{1-a_1-b_1}\|z\|_{X^S_{s,-a_1}}.$$

(b) *Nonhomogeneous linear Boussinesq equation*:
  (i) *If $0 \leqslant a < 1/2$, then*

$$\left\| \int_0^t V_s(t-t')z_{xx}(t')\,dt' \right\|_{C([0,T]:\mathfrak{B}^s)} \lesssim T^{1/2-a}\|z\|_{X^B_{s,-a}}.$$

  (ii) *If $0 \leqslant a < 1/2$, $b \geqslant 0$, and $a + b \leqslant 1$, then*

$$\left\| \theta_T(t)\int_0^t V_s(t-t')z_{xx}(t')\,dt' \right\|_{X^B_{s,b}} \lesssim T^{1-a-b}\|z\|_{X^B_{s,-a}}.$$

We also have the following embeddings concerning the $X^S_{s,b}$ and $X^B_{s,b}$ spaces.

Lemma 2.3. *Let $b > 1/2$. There exists $c > 0$, depending only on b, such that*

$$\|u\|_{C(\mathbb{R}:H^s_{per})} \leqslant c\|u\|_{X^B_{s,b}},$$
$$\|u\|_{C(\mathbb{R}:H^s_{per})} \leqslant c\|u\|_{X^S_{s,b}}.$$

We finish this section with the following standard Bourgain–Strichartz estimates.

Lemma 2.4. *Let $u \in L^3_{x,t}$, then*

$$\|u\|_{L^3_{x,t}} \leqslant c\min\bigl\{\|u\|_{X^S_{0,1/4+}}, \|u\|_{X^B_{0,1/4+}}\bigr\}.$$

Proof. This estimate is easily obtained by interpolating between

- $\|u\|_{L^4_{x,t}} \leqslant c\min\{\|u\|_{X^S_{0,3/8+}}, \|u\|_{X^B_{0,3/8+}}\}$ (see Bougain [6], and Fang and Grillakis [14]).
- $\|u\|_{L^2_{x,t}} = \|u\|_{X^S_{0,0}} = \|u\|_{X^B_{0,0}}$ (by definition).

This proves the lemma. □

Remark 2.1. To simplify our well-posedness analysis we will assume $L = 2\pi$. We will return to an arbitrarily $L > 0$ in Section 6, where we study stability questions.

3. Bilinear estimates

First, we state some elementary calculus inequalities that will be useful later.

Lemma 3.1. *For $p, q > 0$ and $r = \min\{p, q, p+q-1\}$ with $p + q > 1$, there exists $c > 0$ such that*

$$\int_{-\infty}^{+\infty} \frac{dx}{\langle x-\alpha\rangle^p\langle x-\beta\rangle^q} \leqslant \frac{c}{\langle\alpha-\beta\rangle^r}. \tag{18}$$

Proof. See Lemma 4.2 in [17]. □

Lemma 3.2. *If $\gamma > 1/2$, then*

$$\sup_{(n,\tau)\in\mathbb{Z}\times\mathbb{R}} \sum_{n_1\in\mathbb{Z}} \frac{1}{(1+|\tau \pm n_1(n-n_1)|)^{\gamma}} < \infty. \tag{19}$$

Proof. See Lemma 5.3 in [24]. □

Lemma 3.3. *There exists $c > 0$ such that*

$$\frac{1}{c} \leqslant \sup_{x,y\geqslant 0} \frac{1+|x-y|}{1+|x-\sqrt{y^2+y}|} \leqslant c. \tag{20}$$

Proof. Since $y \leqslant \sqrt{y^2+y} \leqslant y+1/2$, for all $y \geqslant 0$, a simple computation shows the desired inequalities. □

Remark 3.1. In view of the previous lemma, we have an equivalent way to estimate the $X^B_{s,b}$-norm, *viz.*

$$\|u\|_{X^B_{s,b}} \sim \big\| \big\langle |\tau| - n^2 \big\rangle^b \langle n\rangle^s \tilde{u}(\tau,n) \big\|_{l^2_n L^2_\tau}.$$

This equivalence will be useful in the proof of Theorem 1.1. As we said in the introduction, the Boussinesq symbol $\sqrt{n^2+n^4}$ does not have good cancelations to make use of Lemma 3.1. Therefore, we modify the symbols as above and work only with the algebraic relations for the Schrödinger equation.

Now we are in position to prove the bilinear estimates stated in Theorem 1.1.

Proof of Theorem 1.1. (i) For $u \in X^S_{s,b}$ and $v \in X^B_{s,b}$, we define $f(\tau,n) \equiv \langle \tau + n^2\rangle^b \langle n\rangle^s \tilde{u}(\tau,n)$ and $g(\tau,n) \equiv \langle |\tau| - \gamma(n)\rangle^b \langle n\rangle^s \tilde{v}(\tau,n)$, where $\gamma(n) = \sqrt{n^2+n^4}$. By duality, the desired inequality is equivalent to

$$\big|W(f,g,\phi)\big| \leqslant c\|f\|_{l^2_n L^2_\tau} \|g\|_{l^2_n L^2_\tau} \|\phi\|_{l^2_n L^2_\tau} \tag{21}$$

where

$$W(f,g,\phi) = \sum_{n,n_1} \int_{\mathbb{R}^2} \frac{\langle n\rangle^s}{\langle n_1\rangle^s \langle n_2\rangle^s} \frac{g(\tau_1,n_1) f(\tau_2,n_2) \bar{\phi}(\tau,n)}{\langle\sigma\rangle^a \langle\sigma_1\rangle^b \langle\sigma_2\rangle^b}\, d\tau\, d\tau_1$$

and

$$\begin{aligned} &n_2 = n - n_1, \qquad \tau_2 = \tau - \tau_1, \\ &\sigma = \tau + n^2, \qquad \sigma_1 = |\tau_1| - \gamma(n_1), \qquad \sigma_2 = \tau_2 + n_2^2. \end{aligned} \tag{22}$$

In view of Remark 3.1, we know that $\langle |\tau_1| - \gamma(n_1)\rangle \sim \langle |\tau_1| - n_1^2\rangle$. Therefore, splitting the domain of integration into the regions $\{(n,\tau,n_1,\tau_1) \in \mathbb{R}^4\colon\ \tau_1 < 0\}$ and $\{(n,\tau,n_1,\tau_1) \in \mathbb{R}^4\colon\ \tau_1 \geqslant 0\}$, it is sufficient to prove inequality (21) with $W_1(f,g,\phi)$ and $W_2(f,g,\phi)$ instead of $W(f,g,\phi)$, where

$$W_1(f,g,\phi) = \sum_{n,n_1} \int_{\mathbb{R}^2} \frac{\langle n\rangle^s}{\langle n_1\rangle^s \langle n_2\rangle^s} \frac{g(\tau_1,n_1) f(\tau_2,n_2) \bar{\phi}(\tau,n)}{\langle\sigma\rangle^a \langle\tau_1 + n_1^2\rangle^b \langle\sigma_2\rangle^b}\, d\tau\, d\tau_1$$

and

$$W_2(f,g,\phi) = \sum_{n,n_1} \int_{\mathbb{R}^2} \frac{\langle n\rangle^s}{\langle n_1\rangle^s \langle n_2\rangle^s} \frac{g(\tau_1,n_1) f(\tau_2,n_2) \bar{\phi}(\tau,n)}{\langle\sigma\rangle^a \langle\tau_1 - n_1^2\rangle^b \langle\sigma_2\rangle^b}\, d\tau\, d\tau_1.$$

Applying the Cauchy–Schwarz and Hölder inequalities, it is easy to see that

$$|W_1|^2 \leqslant \|f\|^2_{l^2_n L^2_\tau} \|g\|^2_{l^2_n L^2_\tau} \|\phi\|^2_{l^2_n L^2_\tau} \left\| \frac{\langle n\rangle^{2s}}{\langle\sigma\rangle^{2a}} \sum_{n_1} \int \frac{d\tau_1}{\langle n_1\rangle^{2s} \langle n_2\rangle^{2s} \langle\tau_1 + n_1^2\rangle^{2b} \langle\sigma_2\rangle^{2b}} \right\|_{l^\infty_n L^\infty_\tau}.$$

Note that for $s \geqslant 0$, we have

$$\frac{\langle n\rangle^{2s}}{\langle n_1\rangle^{2s} \langle n_2\rangle^{2s}} \leqslant 1. \tag{23}$$

Therefore, in view of Lemma 3.1, it suffices to get bounds for

$$\sup_{n,\tau} \frac{1}{\langle\sigma\rangle^{2a}} \sum_{n_1} \frac{1}{\langle\tau + n^2 + 2n_1^2 - 2nn_1\rangle^{2b}}.$$

By Lemma 3.2, this expression is bounded provides $a \geqslant 0$ and $b > 1/4$.

Now we turn to the proof of inequality (21) with $W_2(f,g,\phi)$ instead of $W(f,g,\phi)$. Using the Cauchy–Schwarz and Hölder inequalities, the duality argument implies that

$$|W_2|^2 \leqslant \|f\|^2_{l^2_n L^2_\tau} \|g\|^2_{l^2_n L^2_\tau} \|\phi\|^2_{l^2_n L^2_\tau} \left\| \frac{1}{\langle n_2\rangle^{2s}\langle\sigma_2\rangle^{2b}} \sum_{n_1} \int \frac{\langle n_1+n_2\rangle^{2s}\, d\tau_1}{\langle n_1\rangle^{2s}\langle\tau_1 - n_1^2\rangle^{2b}\langle\sigma\rangle^{2a}} \right\|_{l^\infty_{n_2} L^\infty_{\tau_2}}.$$

Therefore, in view of Lemma 3.1 and (23), it suffices to get bounds for

$$\sup_{n_2,\tau_2} \frac{1}{\langle\sigma_2\rangle^{2b}} \sum_{n_1} \frac{1}{\langle\tau_2 + n_2^2 + 2n_1^2 + 2n_1n_2\rangle^{2a}}.$$

By Lemma 3.2, this expression is bounded provides $b \geqslant 0$ and $a > 1/4$.

(ii) For $u_1 \in X^S_{s,b}$ and $u_2 \in X^S_{s,b}$, we define $f(\tau,n) \equiv \langle\tau+n^2\rangle^b\langle n\rangle^s\tilde{u}_1(\tau,n)$ and $g(\tau,n) \equiv \langle\tau+n^2\rangle^b\langle n\rangle^s\tilde{u}_2(\tau,n)$. By duality, the desired inequality is equivalent to

$$\big|Z(f,g,\phi)\big| \leqslant c\|f\|_{l^2_n L^2_\tau}\|g\|_{l^2_n L^2_\tau}\|\phi\|_{l^2_n L^2_\tau} \tag{24}$$

where

$$Z(f,g,\phi) = \sum_{n,n_1}\int_{\mathbb{R}^2} \frac{\langle n\rangle^s}{\langle n_1\rangle^s\langle n_2\rangle^s} \frac{h(\tau_1,n_1)f(\tau_2,n_2)\bar{\phi}(\tau,n)}{\langle\sigma\rangle^a\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b}\, d\tau\, d\tau_1$$

and

$$h(\tau_1,n_1) = \bar{g}(-\tau_1,-n_1), \qquad n_2 = n - n_1, \qquad \tau_2 = \tau - \tau_1,$$
$$\sigma = |\tau| - \gamma(n), \qquad \sigma_1 = \tau_1 - n_1^2, \qquad \sigma_2 = \tau_2 + n_2^2.$$

Therefore, applying Lemma 3.3 and splitting the domain of integration according to the sign of τ, it is sufficient to prove inequality (24) with $Z_1(f,g,\phi)$ and $Z_2(f,g,\phi)$ instead of $Z(f,g,\phi)$, where

$$Z_1(f,g,\phi) = \sum_{n,n_1}\int_{\mathbb{R}^2} \frac{\langle n\rangle^s}{\langle n_1\rangle^s\langle n_2\rangle^s} \frac{h(\tau_1,n_1)f(\tau_2,n_2)\bar{\phi}(\tau,n)}{\langle\tau+n^2\rangle^a\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b}\, d\tau\, d\tau_1$$

and

$$Z_2(f,g,\phi) = \sum_{n,n_1}\int_{\mathbb{R}^2} \frac{\langle n\rangle^s}{\langle n_1\rangle^s\langle n_2\rangle^s} \frac{h(\tau_1,n_1)f(\tau_2,n_2)\bar{\phi}(\tau,n)}{\langle\tau-n^2\rangle^a\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b}\, d\tau\, d\tau_1.$$

Inequality (24) with $Z_1(f,g,\phi)$ instead of $Z(f,g,\phi)$, can be estimate by the same argument as the one used to bound $W_2(f,g,\phi)$.

Next, we proof inequality (24) with $Z_2(f,g,\phi)$ replacing $Z(f,g,\phi)$. First, we make the change of variables $\tau_2 = \tau - \tau_1$, $n_2 = n - n_1$ to obtain

$$Z_2(f,g,\phi) = \sum_{n,n_2}\int_{\mathbb{R}^2} \frac{\langle n\rangle^s}{\langle n-n_2\rangle^s\langle n_2\rangle^s} \frac{h(\tau-\tau_2,n-n_2)f(\tau_2,n_2)\bar{\phi}(\tau,n)}{\langle\tau-n^2\rangle^a\langle(\tau-\tau_2)-(n-n_2)^2\rangle^b\langle\tau_2+n_2^2\rangle^b}\, d\tau\, d\tau_2.$$

Then, changing the variables $(n,\tau,n_2,\tau_2) \mapsto -(n,\tau,n_2,\tau_2)$, we can rewrite $Z_2(f,g,\phi)$ as

$$Z_2(f,g,\phi) = \sum_{n,n_2}\int_{\mathbb{R}^2} \frac{\langle n\rangle^s}{\langle n-n_2\rangle^s\langle n_2\rangle^s} \frac{k(\tau-\tau_2,n-n_2)l(\tau_2,n_2)\bar{\psi}(\tau,n)}{\langle\tau+n^2\rangle^a\langle\tau-\tau_2+(n-n_2)^2\rangle^b\langle\tau_2-n_2^2\rangle^b}\, d\tau\, d\tau_2$$

where

$$k(a,b) = h(-a,-b), \qquad l(a,b) = f(-a,-b) \quad \text{and} \quad \psi(a,b) = \phi(-a,-b).$$

Since the L^2-norm is preserved under the reflection operation, the result follows from the estimate for $Z_1(f,g,\phi)$. This completes the proof of the theorem. □

Remark 3.2. Once the bilinear estimates in Theorem 1.1 are established, it is a standard matter to conclude the local well-posedness statement of Theorem 1.2. We refer the reader to [24,5,17,15] for further details.

Finally, we should remark that Theorem 1.4 can be obtained easily using Lemma 2.4 (see Farah [15]).

Before leaving this section, we state a slightly modified bilinear estimates that will be useful in the proof of Theorem 1.5.

Corollary 3.1. *Let $a, a_1, b, b_1 > 1/4$ and $s \geqslant 0$, then there exists $c > 0$, depending only on a, a_1, b, b_1, s, such that*

(i) $\|uv\|_{X^S_{s,-a_1}} \lesssim \|u\|_{X^S_{s,b_1}} \|v\|_{X^B_{0,b}} + \|u\|_{X^S_{0,b_1}} \|v\|_{X^B_{s,b}}$.
(ii) $\|u_1\bar{u}_2\|_{X^B_{s,-a}} \lesssim \|u_1\|_{X^S_{s,b_1}} \|u_2\|_{X^S_{0,b_1}} + \|u_1\|_{X^S_{0,b_1}} \|u_2\|_{X^S_{s,b_1}}$.

Proof. The above estimates are direct consequence of Theorem 1.4 and the fact that, for all $s > 0$, the following inequality holds

$$\langle\xi\rangle^s \leqslant \langle\xi_1\rangle^s + \langle\xi - \xi_1\rangle^s. \qquad \square$$

4. Counter-examples to the bilinear estimates

Proof of Theorem 1.3. (i) For $u \in X^S_{k,b}$ and $v \in X^B_{s,b}$, we define $f(\tau,n) \equiv \langle\tau + n^2\rangle^b \langle n\rangle^k \tilde{u}(\tau,n)$ and $g(\tau,n) \equiv \langle|\tau| - \gamma(n)\rangle^b \langle n\rangle^s \tilde{v}(\tau,n)$. By Lemma 3.3, inequality (5) is equivalent to

$$\left\| \frac{\langle n\rangle^k}{\langle\sigma\rangle^a} \sum_{n_1} \int \frac{f(\tau_1,n_1)g(\tau_2,n_2)\,d\tau_1}{\langle n_1\rangle^k \langle n_2\rangle^s \langle\sigma_1\rangle^b \langle\sigma_2\rangle^b} \right\|_{l^2_n L^2_\tau} \lesssim \|f\|_{l^2_n L^2_\tau} \|g\|_{l^2_n L^2_\tau}, \tag{25}$$

where

$$n_2 = n - n_1, \qquad \tau_2 = \tau - \tau_1,$$
$$\sigma = \tau + n^2, \qquad \sigma_1 = \tau_1 + n_1^2, \qquad \sigma_2 = |\tau_2| - n_2^2.$$

For $N \in \mathbb{Z}$ define

$$f_N(\tau,n) = a_n \chi\big((\tau + n^2)/2\big), \quad \text{with } a_n = \begin{cases} 1, & n = 0, \\ 0, & \text{elsewhere,} \end{cases}$$

and

$$g_N(\tau,n) = b_n \chi\big((\tau + n^2)/2\big), \quad \text{with } b_n = \begin{cases} 1, & n = N, \\ 0, & \text{elsewhere,} \end{cases}$$

where $\chi(\cdot)$ denotes the characteristic function of the interval $[-1,1]$. Thus,

$$a_{n_1} b_{n-n_1} \neq 0 \quad \text{if and only if} \quad n_1 = 0 \text{ and } n = N.$$

Consequently, for N large

$$\int \chi\big((\tau_1 + n_1^2)/2\big)\chi\big((\tau - \tau_1 + (n-n_1)^2)/2\big) \gtrsim \chi\big((\tau + (n-n_1)^2 + n_1^2)\big) \gtrsim \chi\big((\tau + N^2)\big).$$

Therefore, using the fact that $||\tau_2| - n_2^2| \leqslant |\tau_2 + n_2^2|$, inequality (25) implies

$$1 \gtrsim \big\|N^{k-s}\chi\big((\tau + N^2)\big)\big\|_{L^2_\tau} \gtrsim N^{k-s}.$$

Letting $N \to \infty$, this inequality is possible only when $k \leqslant s$.

(ii) Here, we define

$$f_N(\tau,n) = a_n \chi\big((\tau + n^2)/2\big), \quad \text{with } a_n = \begin{cases} 1, & n = -N, \\ 0, & \text{elsewhere,} \end{cases}$$

and

$$g_N(\tau,n) = b_n \chi\big((\tau - n^2)/2\big), \quad \text{with } b_n = \begin{cases} 1, & n = N, \\ 0, & \text{elsewhere.} \end{cases}$$

Then

$$a_{n_1} b_{n-n_1} \neq 0 \quad \text{if and only if} \quad n_1 = 0 \text{ and } n = N.$$

Thus, for N large

$$\int \chi\left(\left(\tau_1+n_1^2\right)/2\right)\chi\left(\left(\tau-\tau_1-(n-n_1)^2\right)/2\right) \gtrsim \chi\left(\left(\tau+n^2-2nn_1\right)\right) \gtrsim \chi\left((\tau)\right).$$

Therefore, using the fact that $||\tau_2|-n_2^2| \leqslant |\tau_2-n_2^2|$, inequality (25) implies

$$1 \gtrsim \left\|N^{-(k+s)}\chi\left((\tau)\right)\right\|_{L^2_\tau} \gtrsim N^{-(k+s)}.$$

Letting $N\to\infty$, this inequality is possible only when $k+s\geqslant 0$.

(iii) For $u_1 \in X^S_{k,b}$ and $u_2 \in X^S_{k,b}$, we define $f(\tau,n)\equiv\langle\tau+n^2\rangle^b\langle n\rangle^k\tilde{u}_1(\tau,\xi)$ and $g(\tau,n)\equiv\langle\tau+n^2\rangle^b\langle n\rangle^k\tilde{u}_2(\tau,\xi)$. By Lemma 3.3, inequality (6) is equivalent to

$$\left\|\frac{\langle n\rangle^s}{\langle\sigma\rangle^a}\sum_{n_1}\int\frac{f(\tau_1,n_1)h(\tau_2,n_2)\,d\tau_1}{\langle n_1\rangle^k\langle n_2\rangle^k\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b}\right\|_{l^2_nL^2_\tau} \lesssim \|f\|_{l^2_nL^2_\tau}\|g\|_{l^2_nL^2_\tau}, \tag{26}$$

where

$$\begin{aligned}
&h(\tau_2,n_2)=\bar{g}(-\tau_2,-n_2), \qquad n_2=n-n_1, \qquad \tau_2=\tau-\tau_1,\\
&\sigma=|\tau|-n^2, \qquad \sigma_1=\tau_1+n_1^2, \qquad \sigma_2=\tau_2-n_2^2.
\end{aligned}$$

For $N\in\mathbb{Z}$, define

$$f_N(\tau,n)=a_n\chi\left(\left(\tau+n^2\right)/2\right), \quad \text{with } a_n=\begin{cases}1, & n=N,\\ 0, & \text{elsewhere},\end{cases}$$

and

$$h_N(\tau,n)=b_n\chi\left(\left(\tau-n^2\right)/2\right), \quad \text{with } b_n=\begin{cases}1, & n=0,\\ 0, & \text{elsewhere},\end{cases}$$

where $\chi(\cdot)$ denotes the characteristic function of the interval $[-1,1]$. Thus

$$a_{n_1}b_{n-n_1}\neq 0 \quad \text{if and only if} \quad n_1=N \text{ and } n=N.$$

Moreover,

$$\int \chi\left(\left(\tau_1+n_1^2\right)/2\right)\chi\left(\left(\tau-\tau_1-(n-n_1)^2\right)/2\right) \gtrsim \chi\left(\left(\tau-(n-n_1)^2+n_1^2\right)\right) \gtrsim \chi\left(\left(\tau+N^2\right)\right).$$

Therefore, using the fact that $||\tau|-n^2| \leqslant |\tau+n^2|$, inequality (26) implies

$$1 \gtrsim \left\|N^{s-k}\chi\left(\left(\tau+N^2\right)\right)\right\|_{L^2_\tau} \gtrsim N^{s-k}.$$

Letting $N\to\infty$, this inequality is possible only when $s\leqslant k$. □

5. Global well-posedness

We divide our analysis in two cases. The proof of Theorem 1.5 for $s=0$ follows the same lines as in Farah [15, Theorem 1.4]. For the convenience of the reader, we repeat the proof of this case below. The case $s>0$ can be proved using the arguments introduced by Bourgain [7] for the Schrödinger equation, and further developed by Angulo et al. [4] for the Schrödinger–Benjamin–Ono system.

Proof of Theorem 1.5. Case $s=0$.

Let $(u_0,v_0,v_1)\in L^2_{per}([0,1])\times L^2_{per}([0,1])\times H^{-1}_{per}([0,1])$ and $0<T\leqslant 1$. Based on the integral formulation (17), we define the integral operators

$$G^S_T(u,v)(t)=\theta_T(t)U(t)u_0-i\theta_T(t)\int_0^t U\left(t-t'\right)(\alpha vu)\left(t'\right)dt',$$

$$G^B_T(u,v)(t)=\theta_T(t)\left(V_c(t)v_0+V_s(t)(v_1)_x\right)+\theta_T(t)\int_0^t V_s\left(t-t'\right)\left(\beta|u|^2\right)_{xx}\left(t'\right)dt'. \tag{27}$$

Therefore, applying Lemmas 2.1–2.2 and Theorem 1.3, we obtain

$$\begin{aligned}
\left\|G_T^S(u,v)\right\|_{X^S_{0,b_1}} &\leqslant cT^{1/2-b_1}\|u_0\|_{L^2_{per}} + cT^{1-(a_1+b_1)}\|uv\|_{X^S_{0,-a_1}} \\
&\leqslant cT^{1/2-b_1}\|u_0\|_{L^2_{per}} + cT^{1-(a_1+b_1)}\|u\|_{X^S_{0,b_1}}\|v\|_{X^B_{0,b}}, \\
\left\|G_T^B(u,v)\right\|_{X^B_{0,b}} &\leqslant cT^{1/2-b}\|v_0,v_1\|_{\mathfrak{B}} + cT^{1-(a+b)}\|u\bar{u}\|_{X^B_{0,-a}} \\
&\leqslant cT^{1/2-b}\|v_0,v_1\|_{\mathfrak{B}} + cT^{1-(a+b)}\|u\|^2_{X^S_{0,b_1}}.
\end{aligned} \tag{28}$$

Also,

$$\begin{aligned}
\left\|G_T^S(u,v)-G_T^S(z,w)\right\|_{X^S_{0,b_1}} &\leqslant cT^{1-(a_1+b_1)}\left(\|u\|_{X^S_{0,b_1}}\|v-w\|_{X^B_{0,b}} + \|u-z\|_{X^S_{0,b_1}}\|w\|_{X^B_{0,b}}\right), \\
\left\|G_T^B(u,v)-G_T^B(z,w)\right\|_{X^B_{0,b}} &\leqslant cT^{1-(a+b)}\left(\|u\|_{X^S_{0,b_1}} + \|z\|_{X^S_{0,b_1}}\right)\|u-z\|_{X^S_{0,b_1}}.
\end{aligned} \tag{29}$$

We define

$$\begin{aligned}
X^S_{0,b_1}(d_1) &= \left\{u\in X^S_{0,b_1}\colon \|u\|_{X^S_{0,b_1}} \leqslant d_1\right\}, \\
X^B_{0,b}(d) &= \left\{v\in X^B_{0,b}\colon \|v\|_{X^B_{0,b}} \leqslant d\right\},
\end{aligned}$$

where $d_1 = 2cT^{1/2-b_1}\|u_0\|_{L^2_{per}}$ and $d = 2cT^{1/2-b}\|v_0,v_1\|_{\mathfrak{B}}$.

For (G_T^S, G_T^B) to be a contraction in $X^S_{0,b_1}(d_1)\times X^B_{0,b}(d)$, it needs to satisfy

$$d_1/2 + cT^{1-(a_1+b_1)}d_1 d \leqslant d_1 \quad\Leftrightarrow\quad T^{3/2-(a_1+b_1+b)}\|v_0,v_1\|_{\mathfrak{B}} \lesssim 1, \tag{30}$$

$$d/2 + cT^{1-(a+b)}d_1^2 \leqslant d \quad\Leftrightarrow\quad T^{3/2-(a+2b_1)}\|u_0\|^2_{L^2_{per}} \lesssim \|v_0,v_1\|_{\mathfrak{B}}, \tag{31}$$

$$2cT^{1-(a+b)}d_1 \leqslant 1/2 \quad\Leftrightarrow\quad T^{3/2-(a+b+b_1)}\|u_0\|_{L^2_{per}} \lesssim 1, \tag{32}$$

$$2cT^{1-(a_1+b_1)}d_1 \leqslant 1/2 \quad\Leftrightarrow\quad T^{3/2-(a_1+2b_1)}\|u_0\|_{L^2_{per}} \lesssim 1. \tag{33}$$

Therefore, we conclude that there exists a solution $(u,v)\in X^S_{0,b_1}\times X^B_{0,b}$ satisfying

$$\|u\|_{X^{S,[0,T]}_{0,b_1}} \leqslant 2cT^{1/2-b_1}\|u_0\|_{L^2_{per}} \quad\text{and}\quad \|v\|_{X^{B,[0,T]}_{0,b}} \leqslant 2cT^{1/2-b}\|v_0,v_1\|_{\mathfrak{B}}. \tag{34}$$

On the other hand, applying Lemmas 2.1–2.2, we have that, in fact, $(u,v)\in C([0,T]:L^2)\times C([0,T]:L^2)$. Moreover, since the L^2-norm of u is conserved by the flow, we have $\|u(T)\|_{L^2_{per}} = \|u_0\|_{L^2_{per}}$.

Now, we need to control the growth of $\|v(t)\|_{\mathfrak{B}}$ in each time step. If, for all $t>0$, $\|v(t)\|_{\mathfrak{B}} \lesssim \|u_0\|^2_{L^2_{per}}$ we can repeat the local well-posedness argument and extend the solution globally in time. Thus, without loss of generality, we suppose that after some number of iterations we reach a time $t^*>0$ where $\|v(t^*)\|_{\mathfrak{B}} \gg \|u_0\|^2_{L^2_{per}}$. Hence, since $0<T\leqslant 1$, condition (31) is automatically satisfied and conditions (30)–(33) imply that we can select a time increment of size

$$T \sim \left\|v\left(t^*\right)\right\|_{\mathfrak{B}}^{-1/(3/2-(a_1+b_1+b))}. \tag{35}$$

Therefore, applying Lemmas 2.1(b)–2.2(b) to $v = G_T^B(u,v)$ we have

$$\left\|v\left(t^*+T\right)\right\|_{\mathfrak{B}} \leqslant \left\|v\left(t^*\right)\right\|_{\mathfrak{B}} + cT^{3/2-(a+2b_1)}\left(\|u_0\|^2_{L^2_{per}}+1\right).$$

Thus, we can carry out m iterations on time intervals, each of length (35), before the quantity $\|v(t)\|_{\mathfrak{B}}$ doubles, where m is given by

$$mT^{3/2-(a+2b_1)}\left(\|u_0\|^2_{L^2_{per}}+1\right) \sim \left\|v\left(t^*\right)\right\|_{\mathfrak{B}}.$$

The total time of existence we obtain after these m iterations is

$$\Delta T = mT \sim \frac{\|v(t^*)\|_{\mathfrak{B}}}{T^{1/2-(a+2b_1)}(\|u_0\|^2_{L^2_{per}}+1)} \sim \frac{\|v(t^*)\|_{\mathfrak{B}}}{\|v(t^*)\|_{\mathfrak{B}}^{-(1/2-(a+2b_1))/(3/2-(a_1+b_1+b))}(\|u_0\|^2_{L^2_{per}}+1)}.$$

Taking a,b,a_1,b_1 such that

$$\frac{a+2b_1-1/2}{(3/2-(a_1+b_1+b))} = 1$$

(for instance, $a = b = a_1 = b_1 = 1/3$), we have that ΔT depends only on $\|u_0\|_{L^2_{per}}$, which is conserved by the flow. Hence, we can repeat this entire argument and extend the solution (u, v) globally in time. Moreover, since in each step of time ΔT the size of $\|v(t)\|_{\mathfrak{B}}$ will at most double, it is easy to see that, for all $\widetilde{T} > 0$,

$$\|v(\widetilde{T})\|_{\mathfrak{B}} \lesssim \exp\big((\ln 2)\|u_0\|^2_{L^2_{per}} \widetilde{T}\big) \max\big\{\|v_0, v_1\|_{\mathfrak{B}}, \|u_0\|_{L^2_{per}}\big\}. \tag{36}$$

<u>*Case* $s > 0$.</u>

Let $(u_0, v_0, v_1) \in H^s_{per} \times H^s_{per} \times H^{s-1}_{per}$. By the previous case, there exists a global solution $(u, v) \in C([0, +\infty); L^2_{per}) \times C([0, +\infty); L^2_{per})$. Moreover, (u, v) is a solution of the integral equation (27) in the time interval $[0, \Delta T]$, with $\Delta T \sim \frac{1}{\|u_0\|^2_{L^2_{per}} + 1}$, satisfying

$$\max\big\{\|u\|_{X^{S,[0,\Delta T]}_{0,1/3}}, \|v\|_{X^{B,[0,\Delta T]}_{0,1/3}}\big\} \lesssim C\big(\|u_0\|_{L^2_{per}}, \|v_0, v_1\|_{\mathfrak{B}}\big), \tag{37}$$

where the constant $C(\|u_0\|_{L^2_{per}}, \|v_0, v_1\|_{\mathfrak{B}}) > 0$ depends only on $\|u_0\|_{L^2_{per}}$ and $\|v_0, v_1\|_{\mathfrak{B}}$.

We claim that the solution (u, v), in fact, belongs to $X^{S,[0,T_0]}_{s,1/3} \times X^{B,[0,T_0]}_{s,1/3}$ for all $0 < T_0 \leqslant \Delta T$. Indeed, applying Lemmas 2.1–2.2 and Corollary 3.1, with $a = b = a_1 = b_1 = 1/3$, we obtain

$$\|u\|_{X^{S,[0,T_0]}_{s,1/3}} \lesssim \|u_0\|_{H^s_{per}} + T_0^{1/3}\big(\|u\|_{X^{S,[0,T_0]}_{s,1/3}}\|v\|_{X^{B,[0,T_0]}_{0,1/3}} + \|u\|_{X^{S,[0,T_0]}_{0,1/3}}\|v\|_{X^{B,[0,T_0]}_{s,1/3}}\big) \tag{38}$$

and

$$\|v\|_{X^{B,[0,T_0]}_{s,1/3}} \lesssim \|v_0, v_1\|_{\mathfrak{B}^s} + T_0^{1/3}\big(\|u\|_{X^{B,[0,T_0]}_{s,1/3}}\|u\|_{X^{B,[0,T_0]}_{0,1/3}}\big), \tag{39}$$

where $0 < T_0 \leqslant \Delta T$. Inserting inequality (39) into (38), and using (37), we conclude that

$$\|u\|_{X^{S,[0,T_0]}_{s,1/3}} \lesssim \|u_0\|_{H^s_{per}} + C\big(\|u_0\|_{L^2_{per}}, \|v_0, v_1\|_{\mathfrak{B}}\big)\|v_0, v_1\|_{\mathfrak{B}^s} + T_0^{1/3} C\big(\|u_0\|_{L^2_{per}}, \|v_0, v_1\|_{\mathfrak{B}}\big)\|u\|_{X^{S,[0,T_0]}_{s,1/3}}.$$

Let

$$T_0 \sim \frac{1}{(1 + C(\|u_0\|_{L^2_{per}}, \|v_0, v_1\|_{\mathfrak{B}}))^3}.$$

Hence, from the choice of T_0, we deduce the following a priori estimates

$$\|u\|_{X^{S,[0,T_0]}_{s,1/3}} \lesssim \|u_0\|_{H^s_{per}} + C\big(\|u_0\|_{L^2_{per}}, \|v_0, v_1\|_{\mathfrak{B}}\big)\|v_0, v_1\|_{\mathfrak{B}^s}$$

and

$$\|v\|_{X^{B,[0,T_0]}_{s,1/3}} \lesssim \|v_0, v_1\|_{\mathfrak{B}^s} + C\big(\|u_0\|_{L^2_{per}}, \|v_0, v_1\|_{\mathfrak{B}}\big)\big(\|v_0, v_1\|_{\mathfrak{B}^s} + \|u_0\|_{L^2_{per}}\big).$$

Thus, applying Lemmas 2.1–2.2, we get that $(u, v) \in C([0, T_0]; H^s_{per}) \times C([0, T_0]; H^s_{per})$. The preceding statement remains valid for any bounded interval $[0, T]$, since T_0 depends only on $\|u_0\|_{L^2_{per}}$ and $\|v_0, v_1\|_{\mathfrak{B}}$. Therefore, we can iterate the above argument a finite number of times to deduce that

$$(u, v) \in C\big([0, T]; H^s_{per}\big) \times C\big([0, T]; H^s_{per}\big), \quad \text{for all } T > 0.$$

This completes the proof of Theorem 1.5. □

6. Stability of periodic traveling waves

As we said in the Introduction, here we will consider system (1) with $\alpha = \beta = -1$, namely

$$\begin{cases} iu_t + u_{xx} + uv = 0, \\ v_{tt} - v_{xx} + v_{xxxx} + \big(|u|^2\big)_{xx} = 0, \end{cases} \tag{40}$$

and look for traveling waves of the form

$$u(x, t) = e^{i\omega t}\psi_\omega(x), \qquad v(x, t) = \phi_\omega(x), \tag{41}$$

where ω is a real parameter (to be determined later), and $\psi_\omega, \phi_\omega : \mathbb{R} \to \mathbb{R}$ are smooth periodic functions with the same fixed period $L > 0$. Then, substituting (41) into (40), integrating twice the second equation in the obtained system, and assuming that the integration constants are zero, we obtain

$$\begin{cases} \psi_\omega'' - \omega\psi_\omega + \psi_\omega\phi_\omega = 0, \\ \phi_\omega'' - \phi_\omega + \psi_\omega^2 = 0. \end{cases} \tag{42}$$

In order to solve system (42), we assume $\omega = 1$ and $\psi_1 = \phi_1$, so that it reduces to a single ordinary differential equation, namely,

$$\psi_1'' - \psi_1 + \psi_1^2 = 0. \tag{43}$$

As we will see later in our stability analysis, it is necessary to construct a smooth branch of periodic wave solutions (depending on ω) passing through solution ψ_1 of (43). Then, we will consider the family of equations

$$\psi_\omega'' - \omega\psi_\omega + \psi_\omega^2 = 0, \tag{44}$$

so that, at $\omega = 1$, we obtain a solution for (43).

6.1. Existence of traveling waves

Along this subsection, we review the theory of finding solutions for (44). Indeed, Eq. (44) can be solved by using the standard *direct integration method* (for details, we refer to [3]). As a matter of fact, Eq. (44) has a *strictly positive* solution of the form

$$\psi_\omega(x) = \beta_2 + (\beta_3 - \beta_2)cn^2\left(\sqrt{\frac{\beta_3 - \beta_1}{6}}x; k\right), \qquad k^2 = \frac{\beta_3 - \beta_2}{\beta_3 - \beta_1}, \tag{45}$$

where $cn(\cdot; k)$ denotes the Jacobian elliptic function of *cnoidal* type, k is the elliptic modulus, and $\beta_1, \beta_2, \beta_3$ are real constants satisfying

$$\frac{3\omega}{2} = \sum_{i=1}^{3} \beta_i, \qquad 0 = \sum_{i<j} \beta_i\beta_j, \qquad \beta_1\beta_2\beta_3 = 3A_\psi, \tag{46}$$

where A_ψ is an integration constant. Moreover, it must be the case that

$$\beta_1 < 0 < \beta_2 < \omega < \beta_3 < \frac{3\omega}{2}.$$

The first question concerning solution (45) is the following. Fixed $L > 0$, can we choose $\beta_1, \beta_2, \beta_3$ such that the function in (45) has fundamental period L? The answer is yes. To prove so, one first note since $cn^2(\cdot; k)$ has fundamental period $2K(k)$, where K is the complete elliptic integral of the first kind defined by (see e.g. [9])

$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},$$

the function ψ_ω given in (45) has fundamental period

$$T_{\psi_\omega} = \frac{2\sqrt{6}}{\sqrt{\beta_3 - \beta_1}} K(k). \tag{47}$$

Next, we observe that T_{ψ_ω} can be rewritten as a function depending only on β_2 (and $\omega > 0$ fixed). In fact, by defining $\omega_0 = \omega/2$, we readily see, from (46), that

$$T_{\psi_\omega}(\beta_2; \omega_0) = \frac{2\sqrt{6}}{\sqrt{\rho(\beta_2; \omega_0)}} K\big(k(\beta_2; \omega_0)\big), \tag{48}$$

where

$$\rho(\beta_2; \omega_0) = \sqrt{9\omega_0^2 - 3\beta_2^2 + 6\omega_0\beta_2}, \qquad k^2(\beta_2; \omega_0) = \frac{1}{2} + \frac{3(\omega_0 - \beta_2)}{2\rho(\beta_2; \omega_0)}. \tag{49}$$

Moreover, from (48), it is easy to see that $T_{\psi_\omega} \to +\infty$, as $\beta_2 \to 0$ and $T_{\psi_\omega} \to \sqrt{2}\pi/\sqrt{\omega_0}$, as $\beta_2 \to 2\omega_0$. Since the function $\beta_2 \in (0, 2\omega_0) \to T_{\psi_\omega}(\beta_2; \omega_0)$ is strictly decreasing (this will be proved in the next theorem) we see that, fixed $L > 0$ and choosing $\omega_0 > 2\pi^2/L^2$, there exists a unique $\beta_2 \equiv \beta_2(\omega_0) \in (0, 2\omega_0)$ such that the corresponding cnoidal wave given by (45) has fundamental period $T_{\psi_\omega}(\beta_2; \omega_0) = L$.

In supplement to the above analysis, fixed $L > 0$, we can construct a smooth curve (depending on ω) of cnoidal waves solutions for (44) such that each one of its elements have fundamental period L. This is the content of the next theorem.

Theorem 6.1. *Let $L > 2\pi$ be fixed. Choose arbitrarily $\omega_0 > 2\pi^2/L^2$, and consider the unique $\beta_{2,0} = \beta_2(\omega_0) \in (0, 2\omega_0)$ such that*

$$L = \frac{2\sqrt{6}}{\sqrt{\rho(\beta_{2,0};\omega_0)}} K\big(k(\beta_{2,0};\omega_0)\big).$$

Then,

(i) *there exist an interval $J_1(\omega_0)$ around ω_0, an interval $J_2(\beta_{2,0})$ around $\beta_{2,0}$ and a unique smooth function $\Lambda : J_1(\omega_0) \to J_2(\beta_{2,0})$ such that $\Lambda(\omega_0) = \beta_{2,0}$ and*

$$L = \frac{2\sqrt{6}}{\sqrt{\rho(\beta_2;\eta)}} K\big(k(\beta_2;\eta)\big),$$

where $\eta \in J_1(\omega_0)$, $\beta_2 = \Lambda(\eta)$, and $k(\beta_2;\eta)$, $\rho(\beta_2;\eta)$ are defined in (49) *with ω_0 replaced with η. Moreover, the interval $J_1(\omega_0)$ can be chosen to be $\mathcal{I} = (2\pi^2/L^2, +\infty)$ and the modulus $k = k(\eta)$, where*

$$k^2(\eta) := \frac{1}{2} + \frac{3(\eta - \Lambda(\eta))}{2\rho(\Lambda(\eta);\eta)}, \tag{50}$$

is a strictly increasing function.

(ii) *For $\omega \in (4\pi^2/L^2, +\infty)$ and $\eta(\omega) = \omega/2$, the cnoidal wave solution $\psi_\omega(\cdot) = \psi_{\eta(\omega)}(\cdot;\beta_2(\eta(\omega)))$ has fundamental period L and satisfies* (44)*. In addition, the mapping*

$$\omega \in \left(\frac{4\pi^2}{L^2}, +\infty\right) \mapsto \psi_\omega \in H^k_{per}\big([0,L]\big), \quad k = 0, 1, \ldots$$

is a smooth function.

Sketch of the proof. The proof is an application of the Implicit Function Theorem. Here, we give only the main steps (for details see [3]). Define $\Omega = \{(\beta_2,\eta) \in \mathbb{R}^2;\ \eta > 2\pi^2/L^2,\ \beta_2 \in (0, 2\eta)\}$ and $\Gamma : \Omega \to \mathbb{R}$ by

$$\Gamma(\beta_2,\eta) = \frac{2\sqrt{6}}{\sqrt{\rho(\beta_2;\eta)}} K\big(k(\beta_2;\eta)\big) - L.$$

By our assumptions, we have $\Gamma(\beta_{2,0},\omega_0) = 0$. Moreover, taking into account the properties of the complete elliptic integrals and the definitions of k and ρ, one infers that $\partial\Gamma/\partial\beta_2 < 0$, for all $(\beta_2,\eta) \in \Omega$. So, an application of the Implicit Function Theorem gives the desired statements. The fact that $J_1(\omega_0)$ can be chosen to be $\mathcal{I}$ follows from the fact that ω_0 can be arbitrarily chosen in $\mathcal{I}$ and the uniqueness of the function arising in the Implicit Function Theorem.

To see that $k(\eta)$ is a strictly increasing function one just take the derivative with respect to η in (50) and note that $dk/d\eta > 0$. □

Remark 6.1. We have assumed $L > 2\pi$ in Theorem 6.1 because we want to get a smooth curve of cnoidal waves (defined in an open interval) passing through $\omega = 1$. Otherwise, that is, if $L \leqslant 2\pi$ then such a curve does not exist.

6.2. Spectral analysis

To obtain our stability results, we will use the Grillakis, Shatah, and Strauss theory [18]. As it is well known in such approach, we need to study the spectrum of some linearized operators.

First, we note that introducing a new variable w defined by $v_t = w_x$, system (40) can be written as a Hamiltonian system of the form

$$\frac{d}{dt}U(t) = J\mathcal{E}'\big(U(t)\big), \tag{51}$$

where $U = (P, v, Q, w)$, $P = \mathrm{Re}(u)$, $Q = \mathrm{Im}(u)$, J is the skew-symmetric matrix

$$J = \begin{pmatrix} 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & \partial_x \\ -1/2 & 0 & 0 & 0 \\ 0 & \partial_x & 0 & 0 \end{pmatrix}, \tag{52}$$

and $\mathcal{E}$ is the energy functional given by

$$\mathcal{E}(U) = \int_0^L \left\{ P_x^2 + Q_x^2 + \frac{v_x^2}{2} + \frac{v^2}{2} + \frac{w^2}{2} - v\big(P^2 + Q^2\big) \right\} dx. \tag{53}$$

Next, we consider the linearized operator we need to study. We first remind that system (40) preserves the L^2 norm of u and so, in the above notation,

$$\mathcal{F}(U) = \int_0^L \{P^2 + Q^2\}\,dx$$

is a conserved quantity of system (40).

To simplify our exposition, we denote $\Psi_\omega = (\psi_\omega, \psi_\omega, 0, 0)$, where ψ_ω is a cnoidal wave given in Theorem 6.1. By a direct computation, we see that Ψ_ω is a critical point of the functional $\mathcal{E} + \omega\mathcal{F}$ at $\omega = 1$, that is,

$$\mathcal{E}'(\Psi_1) + \mathcal{F}'(\Psi_1) = 0. \tag{54}$$

Consider the operator

$$\mathcal{A} := \mathcal{E}''(\Psi_1) + \mathcal{F}''(\Psi_1) = \begin{pmatrix} \mathcal{A}_R & 0 \\ 0 & \mathcal{A}_I \end{pmatrix}, \tag{55}$$

where $\mathcal{A}_R$ and $\mathcal{A}_I$ are the self-adjoint 2×2 matrix differential operators defined by

$$\mathcal{A}_R = \begin{pmatrix} 2(-\partial_x^2 + 1 - \psi_1) & -2\psi_1 \\ -2\psi_1 & -\partial_x^2 + 1 \end{pmatrix} \tag{56}$$

and

$$\mathcal{A}_I = \begin{pmatrix} 2(-\partial_x^2 + 1 - \psi_1) & 0 \\ 0 & 1 \end{pmatrix}. \tag{57}$$

Let us study the spectrum of the operator $\mathcal{A}$. In what follows, we use the notation $\sigma(\mathcal{L})$ to represent the spectrum of the linear operator $\mathcal{L}$. We first remind that if $\sigma_{ess}(\mathcal{L})$ and $\sigma_{disc}(\mathcal{L})$ denote, respectively, the essential and discrete spectra of $\mathcal{L}$, then $\sigma(\mathcal{L}) = \sigma_{ess}(\mathcal{L}) \cup \sigma_{disc}(\mathcal{L})$.

To begin our analysis, we observe that since $\mathcal{A}$ is a diagonal operator, we have $\sigma(\mathcal{A}) = \sigma(\mathcal{A}_R) \cup \sigma(\mathcal{A}_I)$. Moreover, since $\mathcal{A}$ has a compact resolvent, we obtain $\sigma(\mathcal{A}) = \sigma_{disc}(\mathcal{A})$ (see e.g. [30]).

Before studying the spectra of operators $\mathcal{A}_R$ and $\mathcal{A}_I$, we recall the following lemma.

Lemma 6.1. *Let $\psi = \psi_1$ be the cnoidal wave given by Theorem* 6.1. *Then the following spectral properties hold.*

(i) *The operator*

$$\mathcal{L}_1 := -\partial_x^2 + 1 - 2\psi$$

defined in $L^2_{per}([0, L])$ with domain $H^2_{per}([0, L])$ has exactly one negative eigenvalue which is simple; zero is an eigenvalue which is simple with eigenfunction ψ'. Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues.

(ii) *The operator*

$$\mathcal{L}_2 := -\partial_x^2 + 1 - \psi$$

defined in $L^2_{per}([0, L])$ with domain $H^2_{per}([0, L])$ has no negative eigenvalues; zero is a simple eigenvalue with eigenfunction ψ. Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues.

Proof. For the first part, see Theorem 4.1 in [3]. The second part follows immediately from Floquet's theory. Indeed, in view of (43), we have that 0 is an eigenvalue to $\mathcal{L}_2$ with eigenfunction ψ. Moreover, since ψ has no zeros in the interval $[0, L]$, 0 must be the first eigenvalue (see e.g. [12, Chapter 3]). $\square$

With Lemma 6.1 in hands, we are able to prove some spectral properties to the operators $\mathcal{A}_R$ and $\mathcal{A}_I$.

Theorem 6.2. *Let $\psi = \psi_1$ be the* cnoidal *wave solution given by Theorem* 6.1.

(i) *The operator $\mathcal{A}_R$ in* (56) *defined in $L^2_{per}([0, L]) \times L^2_{per}([0, L])$ with domain $H^2_{per}([0, L]) \times H^2_{per}([0, L])$ has its first three eigenvalues simple, being the eigenvalue zero the second one with eigenfunction (ψ', ψ'). Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues.*

(ii) *The operator $\mathcal{A}_I$ in* (57) *defined in $L^2_{per}([0, L]) \times L^2_{per}([0, L])$ with domain $H^2_{per}([0, L]) \times L^2_{per}([0, L])$ has no negative eigenvalues; zero is the first eigenvalue which is simple with eigenfunction $(\psi, 0)$. Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues.*

Proof. (i) First we observe that, from (44), it is easy to see that zero is an eigenvalue with eigenfunction (ψ', ψ'). Now, we consider the quadratic form associated with $\mathcal{A}_R$. Let $Y = H^1_{per}([0,L]) \times H^1_{per}([0,L])$. Then, for $(f,g) \in Y$,

$$\begin{aligned} Q_R(f,g) &:= \langle \mathcal{A}_R(f,g), (f,g)\rangle \\ &= \int_0^L \big\{2\big(-\partial_x^2 + 1 - \psi\big) f^2 - 4\psi f g + \big(-\partial_x^2 + 1\big) g^2\big\}\, dx \\ &= 2\langle \mathcal{L}_1 f, f\rangle + \langle \mathcal{L}_1 g, g\rangle + 2\int_0^L \psi (f-g)^2\, dx. \end{aligned} \tag{58}$$

In order to prove that $\mathcal{A}_R$ has at least one negative eigenvalue, let us prove that there exists a pair $(f,g) \in Y$ such that $Q_R(f,g) < 0$. Indeed, from Lemma 6.1, there exist $\mu_0 < 0$ and $f_0 \in H^2_{per}([0,L])$ satisfying $\mathcal{L}_1 f_0 = \mu_0 f_0$, and so that $\langle \mathcal{L}_1 f_0, f_0\rangle < 0$. Thus, by choosing $f = g = f_0$ in (58), we obtain

$$Q_R(f_0, f_0) = 3\langle \mathcal{L}_1 f_0, f_0\rangle < 0.$$

This implies that the first eigenvalue of $\mathcal{A}_R$, say λ_1, is negative. We prove next that the second eigenvalue of $\mathcal{A}_R$ is zero. To do so, we use the *min–max characterization* of eigenvalues (see e.g. [30, Theorem XIII.1]). Thus, if λ_2 denotes the second eigenvalue of $\mathcal{A}_R$, we have

$$\lambda_2 = \max_{(\phi_1,\phi_2)\in Y} \min_{\substack{(f,g)\in Y\setminus\{(0,0)\}\\ f\perp\phi_1,\, g\perp\phi_2}} \frac{Q_R(f,g)}{\|(f,g)\|_Y^2}. \tag{59}$$

By taking $\phi_1 = \phi_2 = f_0$, we see that

$$\lambda_2 \geqslant \min_{\substack{(f,g)\in Y\setminus\{(0,0)\}\\ f\perp f_0,\, g\perp f_0}} \frac{Q_R(f,g)}{\|(f,g)\|_Y^2}.$$

Now, if $f \perp f_0$ and $g \perp f_0$, we obtain $\langle \mathcal{L}_1 f, f\rangle + \langle \mathcal{L}_1 g, g\rangle \geqslant 0$ (recall that, from Lemma 6.1, $\mathcal{L}_1$ has a unique negative eigenvalue). Moreover, since ψ is a strictly positive function (and thus, the last integral in (58) is non-negative), we obtain $Q_R(f,g) \geqslant 0$, which implies $\lambda_2 \geqslant 0$.

Finally, to prove that the third eigenvalue is strictly positive, we use the min–max principle again, taking into account that $\mathcal{L}_1$ has a unique negative eigenvalue and zero is a simple eigenvalue. This proves part (i).

(ii) In this case, if Q_I denotes the quadratic form associated with $\mathcal{A}_I$, we have

$$\begin{aligned} Q_I(f,g) := \langle \mathcal{A}_I(f,g), (f,g)\rangle &= \int_0^L \big\{2\big(-\partial_x^2 + 1 - \psi\big) f^2 + g^2\big\}\, dx \\ &= 2\langle \mathcal{L}_2 f, f\rangle + \|g\|^2. \end{aligned} \tag{60}$$

Therefore, since $\mathcal{L}_2$ has no negative eigenvalue (see Lemma 6.1), we have $\langle \mathcal{L}_2 f, f\rangle \geqslant 0$. Then, from (60), we deduce that $Q_I(f,g) \geqslant 0$. This implies that $\mathcal{A}_I$ has no negative eigenvalue. Moreover, it is easy to see, from (44), that zero is an eigenvalue with eigenfunction $(\psi, 0)$. This completes the proof of the theorem. □

6.3. Orbital stability

In this subsection, we prove our orbital stability result for the periodic wave $(e^{it}\psi, \psi)$, where $\psi = \psi_1$ is the cnoidal wave given in Theorem 6.1. To make clear our notion of orbital stability, we point out that system (40) has translation and phase symmetries, i.e., if $(u(x,t), v(x,t))$ is a solution for (40), so is

$$\big(e^{i\theta} u(x + x_0, t), v(x + x_0, t)\big), \tag{61}$$

for any $\theta, x_0 \in \mathbb{R}$. Thus, our notion of orbital stability is modulo such symmetries. To be more precise, we have the following definition.

Definition 6.1. A traveling wave solution for (40), of the form $(e^{i\omega t}\psi_\omega(x), \phi_\omega(x))$, is said to be orbitally stable in $X = H^1_{per}([0,L]) \times H^1_{per}([0,L]) \times L^2_{per}([0,L])$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $(u_0, v_0, v_1) \in X$ satisfies $\|(u_0, v_0, v_1) - (\psi_\omega, \phi_\omega, 0)\|_X < \delta$, then the solution $\vec{u}(t) = (u, v, v_t)$ of (40) with $\vec{u}(0) = (u_0, v_0, v_1)$ exists for all t and satisfies

$$\sup_{t\geqslant 0} \inf_{s,y\in\mathbb{R}} \big\| \vec{u}(t) - \big(e^{is}\psi_\omega(\cdot + y), \phi_\omega(\cdot + y), 0\big)\big\|_X < \varepsilon.$$

Otherwise, $(e^{i\omega t}\psi_\omega(x), \phi_\omega(x))$ is said to be orbitally unstable in X.

From Theorem 6.2 we obtain the following properties.

(i) The operator $\mathcal{A}$ has exactly one negative eigenvalue, that is, the negative eigenspace of $\mathcal{A}$, say $\mathcal{N}$, is one-dimensional.
(ii) For $\vec{f} = (\psi', \psi', 0, 0)$ and $\vec{g} = (0, 0, \psi, 0)$, the set $\mathcal{Z} = \{r_1\vec{f} + r_2\vec{g};\ r_1, r_2 \in \mathbb{R}\}$ is the kernel of the operator $\mathcal{A}$.
(iii) There exists a closed subspace, say $\mathcal{P}$, such that $\langle \mathcal{A}u, u\rangle \geqslant \delta_0\|u\|_X$, for all $u \in \mathcal{P}$ and some $\delta_0 > 0$.

Therefore, from (i)–(iii), we obtain the following orthogonal decomposition of the real Hilbert space $X_{\mathbb{R}} = H^1_{per}([0, L]) \times H^1_{per}([0, L]) \times H^1_{per}([0, L]) \times L^2_{per}([0, L])$:

$$X_{\mathbb{R}} = \mathcal{N} \oplus \mathcal{Z} \oplus \mathcal{P}. \tag{62}$$

Next, for $\omega \in \mathcal{I} = (4\pi^2/L^2, +\infty)$ and ψ_ω the cnoidal wave given by Theorem 6.1, we define $d : \mathcal{I} \to \mathbb{R}$ by

$$d(\omega) = \mathcal{E}(\Psi_\omega) + \omega\mathcal{F}(\Psi_\omega), \tag{63}$$

where, as before, $\Psi_\omega = (\psi_\omega, \psi_\omega, 0, 0)$.

In the present setting, our orbital stability result in Theorem 1.6 can be rephrased as follows.

Theorem 6.3. *Let $\psi = \psi_1$ be the cnoidal wave given in Theorem* 6.1*. Then, the periodic traveling wave $(e^{it}\psi, \psi)$ is orbitally stable in X.*

Proof. Since the periodic boundary value problem associated with (40) is globally well-posed in X (see Theorem 1.5), $X_{\mathbb{R}}$ admits the decomposition (62), and $\mathcal{N}$ is one-dimensional, the proof of the theorem follows from the *Abstract Stability Theorem* in Grillakis, Shatah, and Strauss [18], provided we are able to show that $d''(1) > 0$, where d is the function defined in (63).

First, from (63), we have

$$\begin{aligned} d'(\omega) &= \left\langle \mathcal{E}'(\Psi_\omega) + \omega\mathcal{F}'(\Psi_\omega), \frac{d}{d\omega}\Psi_\omega \right\rangle + \mathcal{F}(\Psi_\omega) \\ &= 2\left\langle -\psi_\omega'' + \omega\psi_\omega - \psi_\omega^2, \frac{d}{d\omega}\psi_\omega \right\rangle + \left\langle -\psi_\omega'' + \psi_\omega - \psi_\omega^2, \frac{d}{d\omega}\psi_\omega \right\rangle + \mathcal{F}(\Psi_\omega). \end{aligned}$$

But, since ψ_ω satisfies (44), we see that the first term in the last equality must be zero. Thus,

$$d'(\omega) = \left\langle -\psi_\omega'' + \psi_\omega - \psi_\omega^2, \frac{d}{d\omega}\psi_\omega \right\rangle + \mathcal{F}(\Psi_\omega). \tag{64}$$

Taking the derivative with respect to ω in (64), we deduce

$$d''(\omega) = \left\langle -\psi_\omega'' + \psi_\omega - \psi_\omega^2, \frac{d^2}{d\omega^2}\psi_\omega \right\rangle + \left\langle \left(-\partial_x^2 + 1 - 2\psi_\omega\right)\frac{d}{d\omega}\psi_\omega, \frac{d}{d\omega}\psi_\omega \right\rangle + \frac{d}{d\omega}\mathcal{F}(\Psi_\omega). \tag{65}$$

But, again from (44), at $\omega = 1$, the first term in (65) must be zero. Then,

$$d''(1) = \left\langle \left(-\partial_x^2 + 1 - 2\psi_1\right)\frac{d}{d\omega}\psi_\omega\bigg|_{\omega=1}, \frac{d}{d\omega}\psi_\omega\bigg|_{\omega=1} \right\rangle + \frac{d}{d\omega}\mathcal{F}(\Psi_\omega)\bigg|_{\omega=1}.$$

Now, taking the derivative with respect to ω in (44), and evaluating at $\omega = 1$, we get

$$\left(-\partial_x^2 + 1 - 2\psi_1\right)\frac{d}{d\omega}\psi_\omega\bigg|_{\omega=1} + \psi_1 = 0.$$

Hence,

$$d''(1) = -\left\langle \psi_1, \frac{d}{d\omega}\psi_\omega\bigg|_{\omega=1} \right\rangle + \frac{d}{d\omega}\|\psi_\omega\|^2_{L^2_{per}}\bigg|_{\omega=1}.$$

Note that

$$\frac{d}{d\omega}\|\psi_\omega\|^2_{L^2_{per}}\bigg|_{\omega=1} = \frac{d}{d\omega}\langle\psi_\omega, \psi_\omega\rangle\bigg|_{\omega=1} = 2\left\langle \psi_1, \frac{d}{d\omega}\psi_\omega\bigg|_{\omega=1} \right\rangle. \tag{66}$$

Therefore, (66) implies that

$$d''(1) = \frac{1}{2}\frac{d}{d\omega}\|\psi_\omega\|^2_{L^2_{per}}\bigg|_{\omega=1}.$$

As a consequence, our task reduces to show that $\frac{d}{d\omega}\|\psi_\omega\|^2_{L^2_{per}}|_{\omega=1} > 0$. Actually, as we will see below, we can prove that $\frac{d}{d\omega}\|\psi_\omega\|^2_{L^2_{per}} > 0$, for all $\omega \in \mathcal{I}$. This was essentially proved in [3], but for the sake of completeness, we bring the main steps here. Integrating (44) over $[0, L]$, we get

$$\int_0^L \psi_\omega^2(x)\,dx = \omega \int_0^L \psi_\omega(x)\,dx.$$

Then, for the positivity of $\frac{d}{d\omega}\|\psi_\omega\|^2_{L^2_{per}}$, it suffices to show that the function $G(\omega) = \omega \int_0^L \psi_\omega(x)\,dx$ is strictly increasing.

In what follows we replace (up to a multiplicative positive constant) η with ω in the definition of k and ρ in Theorem 6.1. Using that

$$\int_0^K cn^2(x;k)\,dx = \frac{[E(k) - (1-k^2)K(k)]}{k^2},$$

where $E(k)$ is the complete elliptic integral of the second kind, $L = 2\sqrt{6}K/\sqrt{\beta_3 - \beta_1}$, and $k^2 = (\beta_3 - \beta_2)/(\beta_3 - \beta_1)$, we deduce

$$\int_0^L \psi_\omega(x)\,dx = \beta_2 L + 24\frac{K}{L}\big[E - \big(1-k^2\big)K\big].$$

Moreover, in view of the definitions of k and ρ, we infer that

$$\beta_2 = \frac{8K^2}{L}\big[\sqrt{k^4 - k^2 + 1} + 1 - 2k^2\big].$$

As a consequence,

$$\int_0^L \psi_\omega(x)\,dx = \frac{8K^2}{L}\big[\sqrt{k^4 - k^2 + 1} - 2 + k^2\big] + 24\frac{KE}{L} \equiv H\big(k(\omega)\big).$$

Finally,

$$\frac{d}{d\omega}G(\omega) = \int_0^L \psi_\omega(x)\,dx + \omega\frac{dH}{dk}\frac{dk}{d\omega} > 0,$$

where we have used that $k \mapsto H(k)$ is a strictly increasing function and $dk/d\omega > 0$ (see Theorem 6.1). This completes the proof of the theorem. □

6.4. Existence of non-explicit solutions

In Subsection 6.1, we proved that system (42) admits a periodic wave solution for $\omega = 1$ and $\psi_\omega = \phi_\omega$, where ψ_ω is given explicitly by the formula in (45). The advantage in that case, is the reduction of system (42) to a single ordinary differential equation. However, one can naturally ask if the system also admits a periodic solution for $\omega \neq 1$. In this regard, we shall prove that for ω sufficiently close to 1, system (42) does admit an *even* periodic solution such that, at $\omega = 1$, this solution is the aforementioned one. We shall employ the Implicit Function Theorem combined with the spectral results given in Theorem 6.2.

Let $H^s_{per,e}([0, L])$ be the subspace of $H^s_{per}([0, L])$ constituted by the even distributions. Let $X_e = H^2_{per,e}([0, L]) \times H^2_{per,e}([0, L])$ and $Y_e = L^2_{per,e}([0, L]) \times L^2_{per,e}([0, L])$. Define the function $\Phi : \mathbb{R} \times X_e \to Y_e$ by

$$\Phi(\omega, \psi, \phi) = \big(-\psi'' + \omega\psi - \psi\phi, -\phi'' + \phi - \psi^2\big).$$

In view of Theorem 6.1, we deduce that $\Phi(1, \psi_1, \psi_1) = (0, 0)$. Moreover, if $\Phi_{(\psi,\phi)}$ denotes the Fréchet derivative of Φ at (ψ, ϕ), it is easy to check that

$$\Phi_{(\psi,\phi)}(\omega, \psi, \phi) = \begin{pmatrix} -\partial_x^2 + \omega - \phi & -\psi \\ -2\psi & -\partial_x^2 + 1 \end{pmatrix}.$$

Thus, at $\omega = 1$ and $\psi = \phi = \psi_1$, we obtain

$$\mathcal{B} := \Phi_{(\psi,\phi)}(1, \psi_1, \psi_1) = \begin{pmatrix} -\partial_x^2 + 1 - \psi_1 & -\psi_1 \\ -2\psi_1 & -\partial_x^2 + 1 \end{pmatrix}.$$

Let us prove that $\mathcal{B}$ is a bijection from X_e into Y_e. In fact, it is sufficient to show that 0 does not belong to $\sigma(\mathcal{B})$. An elementary calculation shows us that $(f, g) \in \operatorname{Ker}(\mathcal{B})$ if and only if $(f, g) \in \operatorname{Ker}(\mathcal{A}_R)$, where $\mathcal{A}_R$ is the operator given by (56). But, from Theorem 6.2, we have $\operatorname{Ker}(\mathcal{A}_R) = [(\psi_1', \psi_1')]$ (as an operator on $L^2_{per}([0, L]) \times L^2_{per}([0, L])$). However, since ψ_1 is an even function, it follows that $\psi_1' \notin L^2_{per,e}([0, L])$ and so $0 \notin \sigma(\mathcal{B})$ (as an operator on Y_e).

Consequently, from the Implicit Function Theorem there exist an $\varepsilon > 0$ and a unique smooth function $\digamma : (1 - \varepsilon, 1 + \varepsilon) \to X_e$,

$$\digamma(\omega) = (\psi_\omega, \phi_\omega),$$

such that $\digamma(1) = (\psi_1, \psi_1)$ and $\Phi(\omega, \digamma(\omega)) = (0, 0)$, for all $\omega \in (1 - \varepsilon, 1 + \varepsilon)$, that is, the pair $(\psi_\omega, \phi_\omega)$ is a solution for system (42).

Remark 6.2. The periodic solution we found here are also orbitally stable. This can be proved by using classical perturbation theory (see [22]) to show that the linearized operators arising in this context have the same spectral properties as those ones in Theorem 6.2 (for related references see e.g. [4,29]). This will appear elsewhere.

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References

[1] J. Angulo, Non-linear stability of periodic traveling-wave solutions to the Schrödinger and modified Korteweg–de Vries equations, J. Differential Equations 235 (1) (2007) 1–30.
[2] J. Angulo, J. Bona, M. Scialom, Stability of cnoidal waves, Adv. Differential Equations 11 (12) (2006) 1321–1374.
[3] J. Angulo, F. Linares, Periodic pulses of coupled nonlinear Schrödinger equations in optics, Indiana Univ. Math. J. 56 (2) (2007) 847–877.
[4] J. Angulo, C. Matheus, D. Pilod, Global well-posedness and nonlinear stability of periodic traveling waves for a Schrödinger–Benjamin–Ono system, Commun. Pure Appl. Anal. 8 (32) (2009).
[5] D. Bekiranov, T. Ogawa, G. Ponce, Interaction equations for short and long dispersive waves, J. Funct. Anal. 158 (2) (1998) 357–388.
[6] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I and II. The KdV-equation, Geom. Funct. Anal. 3 (3) (1993) 107–156, 209–262.
[7] J. Bourgain, Global Solutions of Nonlinear Schrödinger Equations, Amer. Math. Soc. Colloq. Publ., vol. 46, Amer. Math. Soc., Providence, RI, 1999.
[8] J. Boussinesq, Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide continu dans 21 ce canal des vitesses sensiblement pareilles de la surface au fond, J. Math. Pures Appl. 17 (2) (1872) 55–108.
[9] P.F. Byrd, M.D. Friedman, Handbook of Elliptic Integrals for Enginners and Scientists, Springer-Verlag, New York, 1971.
[10] J. Colliander, J. Holmer, N. Tzirakis, Low regularity global well-posedness for the Zakharov and Klein–Gordon–Schrödinger systems, Trans. Amer. Math. Soc. 360 (9) (2008) 4619–4638.
[11] B. Deconinck, T. Kapitula, On the orbital (in)stability of spatially periodic stationary solutions of generalized Korteweg–de Vries equations, preprint, 2009.
[12] M.S.P. Eastham, The Spectral Theory of Periodic Differential Equations, Scottish Academic Press, 1973.
[13] F. Falk, E. Laedke, K. Spatschek, Stability of solitary-wave pulses in shape-memory alloys, Phys. Rev. B 36 (6) (1987) 3031–3041.
[14] Y.-F. Fang, M.G. Grillakis, Existence and uniqueness for Boussinesq type equations on a circle, Comm. Partial Differential Equations 21 (7–8) (1996) 1253–1277.
[15] L.G. Farah, Local and global solutions for the non-linear Schrödinger–Boussinesq system, Differential Integral Equations 21 (2008) 743–770.
[16] L.G. Farah, Local solutions in Sobolev spaces with negative indices for the "good" Boussinesq equation, Comm. Partial Differential Equations 34 (2009) 52–73.
[17] J. Ginibre, Y. Tsutsumi, G. Velo, On the Cauchy problem for the Zakharov system, J. Funct. Anal. 151 (2) (1997) 384–436.
[18] M. Grillakis, J. Shatah, W. Strauss, Stability theory of solitary waves in the presence of symmetry. II, J. Funct. Anal. 94 (2) (1990) 308–348.
[19] S. Hakkaev, Orbital stability of solitary waves of the Schrödinger–Boussinesq equations, Commun. Pure Appl. Anal. 6 (4) (2007) 1043–1050.
[20] S. Hakkaev, I. Iliev, K. Kirchev, Stability of periodic travelling shallow-water waves determined by newton's equation, J. Phys. A 41 (8) (2008) 1–31.
[21] R. Iorio, V. Iorio, Fourier Analysis and Partial Differential Equations, Cambridge University Press, 2001.
[22] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, 1976.
[23] C.E. Kenig, G. Ponce, L. Vega, A bilinear estimate with applications to the KdV equation, J. Amer. Math. Soc. 9 (2) (1996) 573–603.
[24] C.E. Kenig, G. Ponce, L. Vega, Quadratic forms for the 1-D semilinear Schrödinger equation, Trans. Amer. Math. Soc. 348 (8) (1996) 3323–3353.
[25] F. Linares, A. Navas, On Schrödinger–Boussinesq equations, Adv. Differential Equations 9 (1–2) (2004) 159–176.
[26] F. Linares, G. Ponce, Introduction to Nonlinear Dispersive Equations, Universitext, Springer, New York, 2009.
[27] O. Lopes, Stability of solitary waves of some coupled systems, Nonlinearity 19 (2007) 95–113.
[28] V. Makhankov, On stationary solutions of Schrödinger equation with a self-consistent potential satisfying Boussinesq's equations, Phys. Lett. A 50 (A) (1974) 42–44.
[29] F. Natali, A. Pastor, On periodic traveling waves for the Klein–Gordon–Schrödinger system with Yukawa interaction, preprint, 2009.
[30] M. Reed, B. Simon, Methods of Modern Mathematical Physics: Analysis Operator, vol. IV, Academic Press, 1975.
[31] W. Strauss, Nonlinear Wave Equations, vol. 73, Amer. Math. Soc., Providence, 1989.
[32] N. Yajima, J. Satsuma, Soliton solutions in a diatomic lattice system, Prog. Theor. Phys. 62 (2) (1979) 370–378.
[33] H. Yongqian, The Cauchy problem of nonlinear Schrödinger–Boussinesq equations in $H^s(R^d)$, J. Partial Differential Equations 18 (1) (2005) 1–20.
[34] V. Zakharov, On stochastization of one-dimensional chains of nonlinear oscillators, Sov. Phys. JETP 38 (1974) 108–110.