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The solvability and explicit solutions of two integral equations via generalized convolutions

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ABSTRACT

This paper presents the necessary and sufficient conditions for the solvability of two integral equations of convolution type; the first equation generalizes from integral equations with the Gaussian kernel, and the second one contains the Toeplitz plus Hankel kernels. Furthermore, the paper shows that the normed rings on $L^1(\mathbb{R}^d)$ are constructed by using the obtained convolutions, and an arbitrary Hermite function and appropriate linear combination of those functions are the weight-function of four generalized convolutions associating F and $\check{F}$. The open question about Hermitian weight-function of generalized convolution is posed at the end of the paper.

1. Introduction and statements of main results

The main aim of this paper is to solve the Fredholm integral equation

$$\lambda\varphi(x) + \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} K(x,y)\varphi(y)\,dy = g(x) \tag{1.1}$$

in the separate cases of kernel: K is of the form

$$K(x,y) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} \big[k_1(u)\Phi_\alpha(x-u-y) + k_2(u)\Phi_\alpha(x+u-y) + k_3(u)\Phi_\alpha(x-u+y) + k_4(u)\Phi_\alpha(x+u+y)\big]\,du, \tag{1.2}$$

where Φ_α is the Hermite function or the appropriate linear combination of those functions, and K is sum of the Toeplitz and Hankel kernels, i.e.

$$K(x,y) = k_1(x-y-h_1) + k_2(x-y+h_2) + k_3(x+y-h_3) + k_4(x+y+h_4), \tag{1.3}$$

where $h_1, h_2, h_3, h_4 \in \mathbb{R}^d$ (called shifts or delays) are given.

The integral equation (1.1) with the kernels (1.2), (1.3) attract attention of many authors as that with (1.2) generalizes from the equations with Gaussian kernel which has applications in radiative wave transmission and in many problems of Medicine and Biology, and that with the kernel (1.3) has many useful applications in such diverse fields as scattering

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theory, fluid dynamics, linear filtering theory, and inverse scattering problems in quantum-mechanics. With respect to the kernel (1.2), known results related to the Gaussian kernel can be considered as a special case when $\Phi_0(x) = e^{-\frac{1}{2}|x|^2}$; also, results exist when $K(x, y) = k_1(x - y - h_1) + k_4(x + y + h_2)$ (see [1–10]). Those results are, however, no more true when Φ_α is an arbitrary Hermite function, and k_1, k_2, k_3, k_4 are generated by different functions with h_1, h_2, h_3, h_4. This is the reason why the results in the case of kernels (1.2), (1.3) cannot be derived from already known results.

Integral equations of convolution type, mathematically, belong to an interesting subject in the theory of integral equations. Our study of the above-mentioned kernels is motivated by a sufficiently long list of materials concerning those equations (see [2,5,7–9,11–14]).

To facilitate the formulation of results we recall certain notations. The Fourier and inverse Fourier transforms are defined as:

$$(Ff)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-i\langle x,y\rangle} f(y)\, dy, \qquad \left(F^{-1} f\right)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i\langle x,y\rangle} f(y)\, dy,$$

where $\langle x, y\rangle$ is denoted the scalar product of $x, y \in \mathbb{R}^d$, and f is a complex-valued function defined on $\mathbb{R}^d$. Write $(\check{f})(x) := f(-x)$, $(\check{F} f)(x) := (F^{-1} f)(x)$. Let $L^1(\mathbb{R}^d)$ denote the space of all complex-valued Lebesgue integrable functions on $\mathbb{R}^d$.

1.1. Integral equations with Hermitian kernel

The multi-dimensional Hermite functions are defined as follows $\Phi_\alpha(x) := (-1)^{|\alpha|} e^{\frac{1}{2}|x|^2} D_x^\alpha e^{-|x|^2}$. Recall that $F\Phi_\alpha = (-i)^{|\alpha|}\Phi_\alpha$, $F^{-1}\Phi_\alpha = (i)^{|\alpha|}\Phi_\alpha$, $\Phi_\alpha(-x) = (-1)^{|\alpha|}\Phi_\alpha(x)$ (see [14–17]). Consider the equation

$$\lambda\varphi(x) + \frac{i^{|\alpha|}}{(2\pi)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big[k_1(u)\Phi_\alpha(x - u - v) + k_2(u)\Phi_\alpha(x + u - v) + k_3(u)\Phi_\alpha(x - u + v) + k_4(u)\Phi_\alpha(x + u + v)\big]\varphi(v)\, du\, dv = p(x), \tag{1.4}$$

where $\lambda \in \mathbb{C}$ is predetermined, k_1, k_2, k_3, k_4, p are given in $L^1(\mathbb{R}^d)$, and φ is to be determined. Put

$$\begin{aligned}
&\boldsymbol{A}(x) := \lambda + \Phi_\alpha(x)\big[(Fk_1)(x) + (\check{F}k_2)(x)\big], \qquad \boldsymbol{B}(x) := \Phi_\alpha(x)\big[(Fk_3)(x) + (\check{F}k_4)(x)\big],\\
&\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}(x) := \boldsymbol{A}(x)\boldsymbol{A}(-x) - \boldsymbol{B}(x)\boldsymbol{B}(-x), \qquad \boldsymbol{D}_{\boldsymbol{F}}(x) := \boldsymbol{A}(-x)(Fp)(x) - \boldsymbol{B}(x)(Fp)(-x),\\
&\boldsymbol{D}_{\check{\boldsymbol{F}}}(x) := \boldsymbol{A}(x)(Fp)(-x) - \boldsymbol{B}(-x)(Fp)(x).
\end{aligned} \tag{1.5}$$

Theorem 1.1. *Assume that one of the following conditions is fulfilled*:

(i) $\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}(x) \neq 0$ *for every* $x \in \mathbb{R}^d$, *and* $\frac{\boldsymbol{D}_{\boldsymbol{F}}}{\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}} \in L^1(\mathbb{R}^d)$.
(ii) $\lambda \neq 0$, $\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}(x) \neq 0$ *for every* $x \in \mathbb{R}^d$, *and* $Fp \in L^1(\mathbb{R}^d)$.

Eq. (1.4) *has a solution in* $L^1(\mathbb{R}^d)$ *if and only if* $F^{-1}(\frac{\boldsymbol{D}_{\boldsymbol{F}}}{\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}}) \in L^1(\mathbb{R}^d)$. *If this is the case, then the solution is given by* $\varphi = F^{-1}(\frac{\boldsymbol{D}_{\boldsymbol{F}}}{\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}})$.

1.2. Integral equations with the mixed Toeplitz–Hankel kernel

Consider the following equation

$$\lambda\varphi(x) + \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \big[k_1(x - y - h_1) + k_2(x - y + h_2) + k_3(x + y - h_3) + k_4(x + y + h_4)\big]\varphi(y)\, dy = p(x), \tag{1.6}$$

where $h_1, h_2, h_3, h_4 \in \mathbb{R}^d$ are given, and $k_1, k_2, k_3, k_4, p \in L^1(\mathbb{R}^d)$ are predetermined. Put

$$\begin{aligned}
&A(x) := \lambda + e^{-i\langle x,h_1\rangle}(Fk_1)(x) + e^{i\langle x,h_2\rangle}(Fk_2)(x), \qquad B(x) := e^{-i\langle x,h_3\rangle}(Fk_3)(x) + e^{i\langle x,h_4\rangle}(Fk_4)(x),\\
&D_{F,\check{F}}(x) := A(x)A(-x) - B(x)B(-x), \qquad D_F(x) := A(-x)(Fp)(x) - B(x)(Fp)(-x),\\
&D_{\check{F}}(x) := A(x)(Fp)(-x) - B(-x)(Fp)(x).
\end{aligned}$$

Theorem 1.2. *Assume that one of the following conditions is fulfilled*:

(i) $D_{F,\check{F}}(x) \neq 0$ *for every* $x \in \mathbb{R}^d$, *and* $\frac{D_F}{D_{F,\check{F}}} \in L^1(\mathbb{R}^d)$.
(ii) $\lambda \neq 0$, $D_{F,\check{F}}(x) \neq 0$ *for every* $x \in \mathbb{R}^d$, *and* $Fp \in L^1(\mathbb{R}^d)$.

Eq. (1.6) *has a solution in* $L^1(\mathbb{R}^d)$ *if and only if* $F^{-1}(\frac{D_F}{D_{F,\check{F}}}) \in L^1(\mathbb{R}^d)$. *If this is the case, then the solution is given by* $\varphi = F^{-1}(\frac{D_F}{D_{F,\check{F}}})$.

In Section 3, by using the convolutions constructed in Section 2 the normed rings on $L^1(\mathbb{R}^d)$ are constructed; some of them are commutative.

In Section 4, Theorem 4.1 states that arbitrary Hermite function and appropriate finite linear combination of those functions are the weight-functions of four generalized convolutions for F, $\check{F}$. Finally, the open question about generalized convolution with Hermitian weight-function is posed.

2. Proof of the results

In recent years, many works involving in the applications of generalized convolutions have been appeared (see [17–23]). This paper is in directions of those studies. For briefness of the formulation of Lemma 2.1 and Theorem 2.1 below, it is referred the concept of generalized convolutions with weight-function of integral transforms in [15,18,23].

Lemma 2.1. *If $f, g \in L^1(\mathbb{R}^d)$, then each of the integral expressions* (2.1)–(2.4) *below defines the generalized convolution:*

$$\Big(f \overset{\Phi_\alpha}{\underset{F}{*}} g\Big)(x) = \frac{i^{|\alpha|}}{(2\pi)^d} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} f(v)g(u)\Phi_\alpha(x-u-v)\,du\,dv, \tag{2.1}$$

$$\Big(f \overset{\Phi_\alpha}{\underset{F,F,\check{F}}{*}} g\Big)(x) = \frac{i^{|\alpha|}}{(2\pi)^d} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} f(v)g(u)\Phi_\alpha(x+u-v)\,du\,dv, \tag{2.2}$$

$$\Big(f \overset{\Phi_\alpha}{\underset{F,\check{F},F}{*}} g\Big)(x) = \frac{i^{|\alpha|}}{(2\pi)^d} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} f(v)g(u)\Phi_\alpha(x-u+v)\,du\,dv, \tag{2.3}$$

$$\Big(f \overset{\Phi_\alpha}{\underset{F,\check{F},\check{F}}{*}} g\Big)(x) = \frac{i^{|\alpha|}}{(2\pi)^d} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} f(v)g(u)\Phi_\alpha(x+u+v)\,du\,dv. \tag{2.4}$$

Proof. By $f, g, \Phi_\alpha \in L^1(\mathbb{R}^d)$, we have

$$\int\limits_{\mathbb{R}^d} \Big(f \overset{\Phi_\alpha}{\underset{F}{*}} g\Big)(x)\,dx \leqslant \frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^d} \big|f(u)\big|\,du \int\limits_{\mathbb{R}^d} \big|g(v)\big|\,dv \int\limits_{\mathbb{R}^d} \big|\Phi_\alpha(x-u-v)\big|\,dx < \infty.$$

This means that $f \overset{\Phi_\alpha}{\underset{F}{*}} g \in L^1(\mathbb{R}^d)$. We shall prove the factorization identity of (2.1). By $\Phi_\alpha = (i)^{|\alpha|} F\Phi_\alpha$, we have

$$\begin{aligned}
\Phi_\alpha(x)(Ff)(x)(Fg)(x) &= \frac{i^{|\alpha|}}{(2\pi)^{\frac{3d}{2}}} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} e^{-i\langle x, u+v+t\rangle}\Phi_\alpha(t)f(v)g(u)\,du\,dv\,dt \\
&= \frac{i^{|\alpha|}}{(2\pi)^{\frac{3d}{2}}} \int\limits_{\mathbb{R}^d} e^{-i\langle x,y\rangle}\,dy \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \Phi_\alpha(y-u-v)f(v)g(u)\,du\,dv = F\Big(f \overset{\Phi_\alpha}{\underset{F}{*}} g\Big)(x).
\end{aligned}$$

The convolutions (2.2)–(2.4) may be proved analogously. □

Proof of Theorem 1.1. Let us first prove item (i).

Necessity. Suppose that Eq. (1.4) has a solution $\varphi \in L^1(\mathbb{R}^d)$, i.e. φ fulfills (1.4). Applying F to both sides of (1.4) and using the factorization identities of the convolutions from (2.1) to (2.4) and noticing that $(Ff)(x) = (\check{F}f)(-x)$, we obtain the system of two equations

$$\begin{cases} \boldsymbol{A}(x)(F\varphi)(x) + \boldsymbol{B}(x)(\check{F}\varphi)(x) = (Fp)(x), \\ \boldsymbol{B}(-x)(F\varphi)(x) + \boldsymbol{A}(-x)(\check{F}\varphi)(x) = (\check{F}p)(x), \end{cases} \tag{2.2b}$$

where $\boldsymbol{A}(x)$, $\boldsymbol{B}(x)$ are defined as in (1.5), and $F\varphi, \check{F}\varphi$ are the unknown functions. The determinants of (2.2b): $\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}(x), \boldsymbol{D}_{\boldsymbol{F}}(x), \boldsymbol{D}_{\check{\boldsymbol{F}}}(x)$ are defined as in (1.5). As $\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}(x) \neq 0$ for every $x \in \mathbb{R}^d$, $F\varphi = \frac{\boldsymbol{D}_{\boldsymbol{F}}}{\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}}$, and $\check{F}\varphi = \frac{\boldsymbol{D}_{\check{\boldsymbol{F}}}}{\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}}$. Since $\frac{\boldsymbol{D}_{\boldsymbol{F}}}{\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}} \in L^1(\mathbb{R}^d)$, we apply the inverse Fourier transform to obtain $\varphi = F^{-1}(\frac{\boldsymbol{D}_{\boldsymbol{F}}}{\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}})$. The necessity is proved.

Sufficiency. Clearly, $\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}(x) \equiv \boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}(-x)$, and $\boldsymbol{D}_{\boldsymbol{F}}(x) \equiv \boldsymbol{D}_{\check{\boldsymbol{F}}}(-x)$. Then, $F^{-1}(\frac{\boldsymbol{D}_{\boldsymbol{F}}}{\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}}) = F(\frac{\boldsymbol{D}_{\check{\boldsymbol{F}}}}{\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}})$. Put $\varphi := F^{-1}(\frac{\boldsymbol{D}_{\boldsymbol{F}}}{\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}}) = F(\frac{\boldsymbol{D}_{\check{\boldsymbol{F}}}}{\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}}) \in L^1(\mathbb{R}^d)$. We then have $F\varphi = \frac{\boldsymbol{D}_{\boldsymbol{F}}}{\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}}$, $\check{F}\varphi = \frac{\boldsymbol{D}_{\check{\boldsymbol{F}}}}{\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}}$. This implies that $F\varphi$ and $\check{F}\varphi$ satisfy (2.2b). Hence, $\boldsymbol{A}(x)(F\varphi)(x) + \boldsymbol{B}(x)(\check{F}\varphi)(x) = (Fp)(x)$. Equivalently,

$$F\Bigg(\lambda\varphi(x)+\frac{i^{|\alpha|}}{(2\pi)^d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\big[k_1(u)\Phi_\alpha(x-u-v)+k_2(u)\Phi_\alpha(x+u-v)$$
$$+k_3(u)\Phi_\alpha(x-u+v)+k_4(u)\Phi_\alpha(x+u+v)\big]\varphi(v)\,du\,dv\Bigg)=(Fp)(x).$$

By the inverse Fourier transform, φ fulfills (1.4) for almost every $x\in\mathbb{R}^d$. Item (i) is proved.

Instead of the proof of item (ii), we shall prove the following claim.

Claim 2.1. *Assume that $\lambda\neq 0$. Then*:

(i) $\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}(x)\neq 0$ *for every x outside a ball with finite radius.*
(ii) *If $\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}(x)\neq 0$ $\forall x\in\mathbb{R}^d$, and if $Fp\in L^1(\mathbb{R}^d)$, then $\frac{\boldsymbol{D}_{\boldsymbol{F}}}{\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}}\in L^1(\mathbb{R}^d)$.*

(i) By the Riemann–Lebesgue lemma, the function $\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}$ is continuous on $\mathbb{R}^d$ and $\lim_{|x|\to\infty}\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}(x)=\lambda^2$ (see [24, Theorem 7.5]). Now item (i) follows from $\lambda\neq 0$ and the continuity of $\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}$.

(ii) By the continuity of $\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}$ and $\lim_{|x|\to\infty}\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}(x)=\lambda^2\neq 0$, there exist $R>0$, $\epsilon_1>0$ so that $\inf_{|x|>R}|\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}(x)|>\epsilon_1$. Since the function $\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}$ is continuous and not vanished in the compact set $S(0,R)=\{x\in\mathbb{R}^d\colon\ |x|\leqslant R\}$, there exists $\epsilon_2>0$ so that $\inf_{|x|\leqslant R}|\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}(x)|>\epsilon_2$. We then have $\sup_{x\in\mathbb{R}^d}\frac{1}{|\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}(x)|}\leqslant\max\{\frac{1}{\epsilon_1},\frac{1}{\epsilon_2}\}<\infty$. Hence, the function $\frac{1}{|\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}(x)|}$ is continuous and bounded on $\mathbb{R}^d$. Note that the functions $\boldsymbol{A}$, $\boldsymbol{B}$ are continuous and bounded on $\mathbb{R}^d$. As $Fp\in L^1(\mathbb{R}^d)$, each of two terms defining $\boldsymbol{D}_{\boldsymbol{F}}$ as in (1.5) belongs to $L^1(\mathbb{R}^d)$, i.e. $\boldsymbol{D}_{\boldsymbol{F}}\in L^1(\mathbb{R}^d)$. Due to the continuity and boundary of $\frac{1}{|\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}(x)|}$, $\frac{\boldsymbol{D}_{\boldsymbol{F}}}{\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}}\in L^1(\mathbb{R}^d)$. The claim is proved.

The proof of Theorem 1.1 is complete. □

Remark 2.1. (a) In the general theory of integral equations, the requirement that $\boldsymbol{D}_{\boldsymbol{F},\check{\boldsymbol{F}}}(x)\neq 0$ for every $x\in\mathbb{R}^d$ is the normally solvable condition of the equation (see [12,13]). If $\lambda\neq 0$, then the assumptions in item (ii) of Theorem 1.1 are simpler and easier to check than that in item (i); these assumptions are well fair.

(b) If $|\alpha|=0$, then Φ_0 is the Gaussian function. It is possible to prove that if k_1,k_2,k_3,k_4 are the Gaussian functions, so $K(x,y)$ is. Many mathematical models of the problems in Physics, Medicine and Biology were constructed that reduced integral equations with the Gaussian kernel (see [7–9,14]).

Put $\theta(x)=e^{-i\langle x,h\rangle}$. We recall the theorem for proving Theorem 1.2.

Theorem 2.1. *(See [6].) If $f,g\in L^1(\mathbb{R}^d)$, then each one of the integral expressions from* (2.5) *to* (2.8) *defines the generalized convolution*:

$$\Big(f\overset{\theta}{\underset{F}{*}}g\Big)(x)=\frac{1}{(2\pi)^{\frac{d}{2}}}\int\limits_{\mathbb{R}^n}f(x-y-h)g(y)\,dy,\tag{2.5}$$

$$\Big(f\overset{\theta}{\underset{F,F,\check{F}}{*}}g\Big)(x)=\frac{1}{(2\pi)^{\frac{d}{2}}}\int\limits_{\mathbb{R}^n}f(x+y-h)g(y)\,dy,\tag{2.6}$$

$$\Big(f\overset{\theta}{\underset{\check{F}}{*}}g\Big)(x)=\frac{1}{(2\pi)^{\frac{d}{2}}}\int\limits_{\mathbb{R}^n}f(x-y+h)g(y)\,dy,\tag{2.7}$$

$$\Big(f\overset{\theta}{\underset{\check{F},\check{F},F}{*}}g\Big)(x)=\frac{1}{(2\pi)^{\frac{d}{2}}}\int\limits_{\mathbb{R}^n}f(x+y+h)g(y)\,dy.\tag{2.8}$$

One interesting fact possessed by the factorization identities in this theorem is that the shift (or delay) h in the left-side moved only into the weight-function in the right-side. This is the main key for solving integral equations with different shifts or delays.

Proof of Theorem 1.2. Note that the h in the convolutions (2.5)–(2.8) is separate. Therefore, by arguments similar to that in the proof of item (i) in Theorem 1.1 we can prove easily item (i) of this theorem (see also [6]).

The assertion in the item (ii) of Theorem 1.2 is an immediate consequence of the following claim, the proof of which is similar to that of Claim 2.1. □

Claim 2.2. *Assume that* $\lambda \neq 0$. *Then*:

(i) $D_{F,\check{F}}(x) \neq 0$ *for every* x *outside a ball with finite radius.*
(ii) *If* $D_{F,\check{F}}(x) \neq 0$ *for every* $x \in \mathbb{R}^d$, *and* $Fp \in L^1(\mathbb{R}^d)$, *then* $\frac{D_F}{D_{F,\check{F}}} \in L^1(\mathbb{R}^d)$.

Remark 2.2. If $\lambda \neq 0$, then the assumptions in the item (ii) of Theorem 1.2 are also well fair.

Comparison 2.1. Other works presented an alternative manner for approaching to integral equations of convolution type, that is the use of an appropriate convolution and the Wiener–Lèvy theorem (see [12,21]). However, those works obtained only sufficient conditions for the solvability and implicit solutions of equations.

The second term on the left-side of Eq. (1.6), (1.4) is not any known generalized convolution. For this reason, it is impossible to use the constructed convolutions in a normal manner [12]. Nevertheless, the groups of four generalized convolutions associating F and $\check{F}$ work out the necessary and sufficient conditions for the solvability and the explicit solutions of the considered equations.

3. Normed ring structures

Convolution transforms deserve special interest of a great number of authors as they have many applications in pure and applied mathematics (see [14,17,25–27]). Practically, generalized convolution is considered as a tool for the multi-dimensional filtering tasks; theoretically, it is a new transform which can be an object of study. Namely, for any f (or g) fixed in $L^1(\mathbb{R}^d)$ the above-mentioned convolution transforms are continuous operators from $L^1(\mathbb{R}^d)$ into itself. In this section, we present the normed ring structures on the space $X := L^1(\mathbb{R}^d)$; X, therefore, is a Banach $*$-algebra (see [28]). We recall the concept of normed ring.

Definition 3.1. (See [29].) A vector space V with a ring structure and a vector norm is called the normed ring if $\|vw\| \leqslant \|v\|\|w\|$, for all $v, w \in V$.

If V has a multiplicative unit element e, it is also required that $\|e\| = 1$.

Through Section 3, we put $N_\alpha := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |\Phi_\alpha(x)|\,dx$ for given Hermite function Φ_α, and define the norm of $f \in X$ as $\|f\| := \frac{\sqrt{N_\alpha}}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |f(x)|\,dx$.

Theorem 3.1. *X, equipped with each of the convolution multiplications mentioned in Lemma* 2.1, *becomes a normed ring having no unit. Moreover,*

(i) *for the convolution multiplications* (2.1), (2.4), *X is commutative*;
(ii) *for the convolution multiplications* (2.2), (2.3), *X is non-commutative.*

Proof. The proof is divided into two steps.

Step 1. *X has a normed ring structure.* It is clear that X, equipped with each of the convolution multiplications listed above, has the ring structure. We will prove the multiplicative inequalities of the convolution (2.1), and the proofs for the others are similar. We have

$$\int_{\mathbb{R}^d} \left| f \overset{\Phi_\alpha}{\underset{F}{*}} g \right|(x)\,dx \leqslant \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left|f(u)\right|\left|g(v)\right|\left|\Phi_\alpha(x-u-v)\right| du\,dv\,dx$$
$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left|f(u)\right| du \int_{\mathbb{R}^d} \left|g(v)\right| dv \int_{\mathbb{R}^d} \left|\Phi_\alpha(x)\right| dx = (2\pi)^{\frac{d}{2}} \|f\|.\|g\|.$$

Hence, $\|f \overset{\Phi_\alpha}{\underset{F}{*}} g\| \leqslant \|f\|.\|g\|$.

Step 2. *X has no unit.* For briefness of the proof, let us use the common symbol $*$ for all the convolutions (2.1)–(2.4). Suppose that there exists an element $e \in X$ such that $f = f * e = e * f$ for every $f \in X$. Choosing $\delta(x) := e^{-\frac{1}{2}|x|^2}$ we get $F\delta = \check{F}\delta = \delta$ (see [24, Theorem 7.6]). By $\delta = \delta * e = e * \delta$ and the factorization identities of those convolutions, we obtain $F(\delta) = \Phi_\alpha \mathcal{F}_k(\delta)\mathcal{F}_\ell(e)$, where $\mathcal{F}_k, \mathcal{F}_\ell \in \{F, \check{F}\}$ (note that it may be $\mathcal{F}_k = F_\ell = F$, etc.). We then have $\delta = \Phi_\alpha \delta \mathcal{F}_\ell(e)$. Since $\delta(x) \neq 0$ for every $x \in \mathbb{R}^d$, $\Phi_\alpha(x)(\mathcal{F}_\ell e)(x) = 1$ for every $x \in \mathbb{R}^d$. But, this contradicts to the fact: $\lim_{|x|\to\infty} \Phi_\alpha(x)(\mathcal{F}_\ell e)(x) = 0$ which deduced from the Riemann–Lebesgue lemma. Hence, X has no unit.

Evidently, the convolutions (2.2), (2.3) are commutative. It is sufficient to prove the non-commutativity for (2.2), as that for (2.3) might be proved analogously. Consider the Hermite functions: $\Phi_0(x) = e^{-\frac{1}{2}|x|^2}$, $\Phi_1(x) = -2x_1 e^{-\frac{1}{2}|x|^2}$. By $F\Phi_\alpha = (-i)^{|\alpha|}\Phi_\alpha$, $\check{F}\Phi_\alpha = (i)^{|\alpha|}\Phi_\alpha$, we have

$$F\Big(\Phi_0 \underset{F,F,\check{F}}{\overset{\Phi_\alpha}{*}} \Phi_1\Big) = i\Phi_\alpha\Phi_0\Phi_1, \qquad F\Big(\Phi_1 \underset{F,F,\check{F}}{\overset{\Phi_\alpha}{*}} \Phi_0\Big) = -i\Phi_\alpha\Phi_0\Phi_1.$$

This implies that (2.2) is not commutative. The theorem is proved. □

4. Claim and open question on Hermitian weight-function

In this small section, we deal with generalized convolutions with another weight-functions. Let $r \in \{0, 1, 2, 3\}$ be given, and let Ψ be an arbitrary linear combination of the Hermite functions Φ_{α_k} as $\Psi = \sum_{k=1}^{N} a_k\Phi_{\alpha_k}$, where $a_k \in \mathbb{C}$, $|\alpha_k| = r \pmod 4$ for every $k = 1, \dots, N$. Since $F\Phi_{\alpha_k} = (-i)^{|\alpha_k|}\Phi_{\alpha_k}$, the following theorem is an immediate consequence of Lemma 2.1.

Theorem 4.1.

(i) *The convolutions in Lemma* 2.1 *remain valid whenever the function Φ_α is replaced with Ψ.*
(ii) *Theorem* 3.1 *works for those convolutions with the weight-function Ψ.*

So, the appropriate linear combination of Hermite functions is the weight-function of four explicit convolutions for $F, \check{F}$. In other words, an infinite number of generalized convolutions are constructed. Especially, if $|\alpha| = 0$, then Φ_0 is the Gaussian function and the convolutions mentioned in Lemma 2.1 become convolutions for the well-known Weierstrass transform.

Open question. Let Ψ be an arbitrary linear combination of Hermite functions. Does there exists a generalized convolution with the weight-function Ψ for any appropriate transforms?

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