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A free boundary problem for p-Laplacian in the plane ☆

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ABSTRACT

We consider the following free boundary problem in an unbounded domain Ω in two dimensions: $\Delta_p u = 0$ in Ω, $u = 0$, $\frac{\partial u}{\partial n} = g_0$ on J_0, $u = 1$, $\frac{\partial u}{\partial n} = g_1$ on J_1, where $\partial\Omega = J_0 \cup J_1$. We prove that if $0 < u < 1$ in Ω, J_i is the graph of a function in $C^{1,\alpha}_{loc}(\mathbb{R})$ and g_i is a constant for each $i = 0, 1$, then the free boundary $\partial\Omega$ must be two parallel straight lines and the solution u must be a linear function. The proof is based on maximum principle.

1. Introduction

We consider the following free boundary problem in two dimensions

$$\begin{cases} \Delta_p u = 0, & \text{in } \Omega, \\ u = 0, \quad \dfrac{\partial u}{\partial n} = g_0, & \text{on } J_0, \\ u = 1, \quad \dfrac{\partial u}{\partial n} = g_1, & \text{on } J_1, \end{cases} \tag{1.1}$$

where Δ_p, $1 < p < \infty$, is the p-Laplacian, $g_i, i = 0, 1$ are prescribed functions, $\Omega \subset \mathbb{R}^2$ is unbounded with $\partial\Omega = J_0 \cup J_1$, and n is the unit exterior normal.

This kind of free boundary problems arises from the theory of nonlinear potential flows of power-law types. In (1.1), the p-Laplacian may be seen as an incompressibility condition, the level sets of u correspond to the stream lines, and the conditions about $\frac{\partial u}{\partial n}$ on the free boundary are pressure conditions on the boundary of the fluid due to Bernoulli's law. We refer to [3] for more detailed background of this problem and the research book [2] for general theory of free boundary problems.

In this paper, we will prove the following Liouville-type theorem about the free boundary problem (1.1).

Theorem 1.1. *Assume that $u \in C^{1,\alpha}_{loc}(\bar{\Omega})$ satisfies (1.1) and $0 < u < 1$ in Ω. If J_i is the graph of a function in $C^{1,\alpha}_{loc}(\mathbb{R})$ and g_i is a constant for each $i = 0, 1$, then u is a linear function and J_i, $i = 0, 1$ are two parallel straight lines.*

The above theorem extends the result of [5], where the authors proved similar result for uniformly elliptic equations of the form $F(D^2 u) = 0$.

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Notice that it is not trivial to extend the result of [5] to p-Laplacian. One of the difficulties lies in the fact that, for p-Laplacian, the solution u and the free boundary have at most $C^{1,\alpha}$ regularity. Thus, sometimes we cannot apply the Hopf lemma directly even if $|\nabla u|$ is away from zero. This difficulty is overcome by perturbing the comparison functions appropriately.

Comparing with [5], another different point in our proof is that we do not invoke the Harnack inequality to analyze the level sets of u in Ω. We will perform analysis only on J_0 and J_1 and prove that they are straight lines directly. We sketch the idea of the proof as follows. First, we notice that the blow up limits of the level sets of the fundamental solution are all straight lines. Based on this observation, we can prove that the free boundary is nearly flat in the following sense: there exists a linear function g, such that J_0 lies below the 0-level set of g and J_1 lies below the 1-level set of g. Then, by comparing u with linear functions, we show that the free boundary tends to be more and more flat. We remark that this method can be applied to more general elliptic equations.

From now on, we assume that u satisfies the assumptions in Theorem 1.1. That is, we assume that there exist $\phi_i \in C^{1,\alpha}_{loc}(\mathbb{R})$, $i = 0, 1$, such that

$$\Omega = \{(s,t);\ \phi_0(s) < t < \phi_1(s),\ s \in \mathbb{R}\},$$
$$J_i = \{(s,t);\ t = \phi_i(s),\ s \in \mathbb{R}\}, \quad i = 0, 1$$

and $u \in C^{1,\alpha}_{loc}(\bar{\Omega})$ satisfies

$$\begin{cases} 0 < u < 1, \quad \Delta_p u = 0, & \text{in } \Omega, \\ u = 0, \quad |\nabla u| = c_0, & \text{on } J_0, \\ u = 1, \quad |\nabla u| = c_1, & \text{on } J_1, \end{cases}$$

where c_i, $i = 0, 1$ are constants. For convenience, we extend u to the whole of $\mathbb{R}^2$ by setting $u = 1$ above J_1 and $u = 0$ below J_0.

In the proof, we will use some properties of the p-Laplacian. We recall that

$$\Delta_p u = \nabla \cdot \left(|\nabla u|^{p-2} \nabla u\right), \quad 1 < p < \infty.$$

As is well known, the weak comparison principle holds for the p-Laplacian. The strong comparison principle and the Hopf lemma hold as well if the gradient of one of the comparison functions does not vanish (see, e.g., [4,6]).

The proof of Theorem 1.1 is based on maximum principle. Particularly, we will employ the sliding method, for which we refer to [1].

We fix some notations and terminologies which will be used frequently in the proof. For $x, y \in \mathbb{R}^2$, $x \cdot y$ is the inner product of x and y. We define the vertical distance of two sets A and B in the plane to be

$$d_v(A, B) = \inf\{(x - y) \cdot e_2;\ x \in A,\ y \in B,\ x \cdot e_1 = y \cdot e_1\},$$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. We say A lies above B if $d_v(A, B) \geqslant 0$, and A lies strictly above B if $A \cap B = \emptyset$ in addition. The notation that A lies below or strictly below B is defined in a similar way. If A lies above B and $A \cap B \neq \emptyset$, we say A touches B from above. Let $v : A \to \mathbb{R}$ and $w : B \to \mathbb{R}$. We say v touches w from above if $v \geqslant w$ in $A \cap B$ and $v(x) = w(x)$ for some $x \in A \cap B$, and x is called a contact point.

The rest of the paper is organized as follows. In Section 2, we will prove two comparison lemmas. In Section 3, we will present the proof of Theorem 1.1.

2. Two comparison lemmas

In this section, we prove two comparison lemmas, where u is compared with fundamental solution of the p-Laplacian and linear functions respectively.

Denote $\alpha = \frac{p-2}{p-1}$ for $p \neq 2$. We define

$$\psi_{R_0,R_1}(x) = \psi_{R_0,R_1}\left(|x|\right) = \begin{cases} \frac{R_0^\alpha - |x|^\alpha}{R_0^\alpha - R_1^\alpha}, & p \neq 2, \\ \frac{\ln R_0 - \ln |x|}{\ln R_0 - \ln R_1}, & p = 2, \end{cases}$$

where $R_1 \leqslant |x| \leqslant R_0$. Notice that ψ_{R_0,R_1} is the fundamental solution of the p-Laplacian and

$$\psi_{R_0,R_1} = i \quad \text{on } \partial B_{R_i},\ i = 0, 1,$$
$$0 < \psi_{R_0,R_1} < 1 \quad \text{in } B_{R_0} - \bar{B}_{R_1}.$$

Lemma 2.1. *Assume that $c_1 = 1$. Let $0 < \delta < 1$, $k > 1$ and*

$$R_1 = \begin{cases} \frac{\alpha}{\delta(k^\alpha - 1)}, & p \neq 2, \\ \frac{1}{\delta \ln k}, & p = 2, \end{cases}$$
$$R_0 = kR_1.$$

If $B_{R_0}(z)$ lies above J_0, then $B_{R_1}(z)$ lies above J_1 and

$$u(x) \geqslant \psi_{R_0,R_1}(x - z), \quad \forall x \in B_{R_0}(z) - \bar{B}_{R_1}(z).$$

Proof. By a translation, we may assume that $z = 0$.

We slide ψ_{R_0,R_1} up–down.

Denote

$$V_t = B_{R_0} - \bar{B}_{R_1} + te_2, \quad t \in \mathbb{R},$$
$$v_t(x) = \psi_{R_0,R_1}(x - te_2), \quad x \in \overline{V}_t.$$

Let

$$t^* = \inf\{t \in \mathbb{R};\ v_s < u \text{ in } V_s \cap \Omega \text{ or } V_s \cap \Omega = \emptyset, \text{ for all } s > t\}.$$

We claim that $t^* \leqslant 0$. Clearly, if this holds, then we are done.

We prove by contradiction. Suppose that $t^* > 0$. We will derive a contradiction.

Since $u = 1$ on J_1 and $\psi_{R_0,R_1} = 0$ on ∂B_{R_0}, we have $(\partial B_{R_0} + t^* e_2) \cap \Omega \neq \emptyset$, and hence $V_{t^*} \cap \Omega \neq \emptyset$. By definition of t^*, u touches v_{t^*} from above at some point in $\overline{V_{t^*} \cap \Omega}$.

Let $x^* \in \overline{V_{t^*} \cap \Omega}$ be a contact point. By the strong comparison principle, we may assume that $x^* \in \partial(V_{t^*} \cap \Omega)$. Then, we have either $x^* \in J_0 \cap (\partial B_{R_0} + t^* e_2)$ or $x^* \in J_1 \cap (\partial B_{R_1} + t^* e_2)$.

Since $t^* > 0$, $\partial B_{R_0} + t^* e_2$ lies strictly above J_0. Hence

$$u > 0 \quad \text{on } \partial B_{R_0} + t^* e_2.$$

Thus, we must have $x^* \in J_1 \cap (\partial B_{R_1} + t^* e_2)$. Since u touches v_{t^*} from above at x^* and $0 < u < 1$ in Ω, we conclude that

$$\left|\nabla u\left(x^*\right)\right| \leqslant \left|\nabla v_{t^*}\left(x^*\right)\right| = \delta < 1.$$

This contradicts the assumption that $c_1 = 1$. Hence, we must have $t^* \leqslant 0$. This proves the lemma. □

Let $y, z \in \mathbb{R}^2$. We use $[y, z]$ to denote the line segment jointing y and z which contains both y and z. We denote $[y, z) = [y, z] - \{z\}$ and $(y, z) = [y, z] - \{y, z\}$.

Lemma 2.2. *Assume that $c_0 = c_1 = 1$. Let $y_0, z_0 \in \mathbb{R}^2$ be such that*

$$(y_0 - z_0) \cdot e_1 \neq 0.$$

Denote by $\nu = (\nu_1, \nu_2)$ the unit normal of $[y_0, z_0]$ with $\nu_2 > 0$ and let

$$f(x) = (x - y_0) \cdot \nu, \quad x \in \mathbb{R}^2,$$
$$y_1 = y_0 + e_2/\nu_2, \quad z_1 = z_0 + e_2/\nu_2,$$
$$D = \left\{x \in \mathbb{R}^2;\ x = \theta y_0 + (1 - \theta) z_0 + \lambda e_2,\ 0 < \theta < 1,\ 0 < \lambda < 1/\nu_2\right\}.$$

If

$$u \geqslant f \quad \text{on } [y_0, y_1] \cup [z_0, z_1], \tag{2.1}$$

then the parallelogram D lies above J_0 and

$$u \geqslant f \quad \text{in } D.$$

The proof of Lemma 2.2 is more complex than that of Lemma 2.1. When we slide f up–down, we cannot apply the Hopf lemma if the contact point lies on J_1 since J_1 has only $C^{1,\alpha}_{loc}$ regularity. To obtain the desired conclusion, we will perturb f and slide the perturbed function up–down.

Proof of Lemma 2.2. First, we observe that $|\nabla f| = 1$ and

$$[y_0, z_0] \subset \{f = 0\}, \qquad [y_1, z_1] \subset \{f = 1\}.$$

The assumption (2.1) implies that $\{y_0, z_0\}$ lies above J_0, and $\{y_1, z_1\}$ lies above J_1.

We slide f up–down.

Denote

$$V_t = D + te_2, \quad t \in \mathbb{R},$$
$$v_t(x) = f(x - te_2), \quad x \in \overline{V}_t.$$

Let

$$t^* = \inf\{t \in \mathbb{R};\ v_s < u \text{ in } V_s \cap \Omega \text{ or } V_s \cap \Omega = \emptyset,\ \text{for all } s > t\}.$$

If we can show that $t^* \leqslant 0$, then we are done. We prove this by the method of contradiction. Suppose that $t^* > 0$. We will derive a contradiction.

Since $u = 1$ on J_1 and $f = 0$ on $[y_0, z_0]$, we have $V_{t^*} \cap \Omega \neq \emptyset$. By definition of t^*, u touches v_{t^*} from above at some point in $\overline{V_{t^*} \cap \Omega}$. Let x^* be an arbitrary contact point.

Claim 1. $x^* \in J_1 \cap ((y_1, z_1) + t^* e_2)$.

Proof. Since $t^* > 0$, by assumption (2.1), we have

$$u > v_{t^*} \quad \text{on } [y_0, y_1) \cup [z_0, z_1) + t^* e_2. \tag{2.2}$$

Since $\{y_1, z_1\}$ lies above J_1 and $t^* > 0$, $\{y_1 + t^* e_2, z_1 + t^* e_2\}$ lies strictly above J_1. Hence

$$y_1 + t^* e_2, z_1 + t^* e_2 \notin \overline{V_{t^*} \cap \Omega}.$$

So we have $x^* \notin [y_0, y_1] \cup [z_0, z_1] + t^* e_2$.

If $x^* \in V_{t^*} \cap \Omega$, then, by the strong comparison principle, we have $u \equiv v_{t^*}$ in $V_{t^*} \cap \Omega$. This implies that $V_{t^*} \cap \Omega = V_{t^*}$ and $u \equiv v_{t^*}$ on V_{t^*}, which contradicts (2.2). Hence, we have $x^* \notin V_{t^*} \cap \Omega$ and $u > v_{t^*}$ in $V_{t^*} \cap \Omega$.

If $x^* \in J_0 \cap ((y_0, z_0) + t^* e_2)$, then

$$\left|\nabla f\left(x^*\right)\right| = \left|\nabla u\left(x^*\right)\right| = 1.$$

This contradicts the Hopf lemma. Hence, we also have $x^* \notin J_0 \cap ((y_0, z_0) + t^* e_2)$.

Thus, we must have $x^* \in J_1 \cap ((y_1, z_1) + t^* e_2)$. □

Claim 1 implies that $[y_1, z_1] + t^* e_2$ touches J_1 from above and $[y_0, z_0] + t^* e_2$ lies strictly above J_0.

We perturb f a little and slide the perturbed function up–down again.

Let $0 < \epsilon < 1$ be a small constant to be chosen later and $\delta = 1 - \epsilon$. Denote

$$\tilde{f} = \delta f,$$
$$\tilde{y}_1 = y_0 + e_2/(\delta \nu_2), \qquad \tilde{z}_1 = z_0 + e_2/(\delta \nu_2),$$
$$\tilde{D} = \left\{x \in \mathbb{R};\ x = \theta y_0 + (1 - \theta) z_0 + \lambda e_2,\ 0 < \theta < 1,\ 0 < \lambda < 1/(\delta \nu_2)\right\}.$$

Notice that $|\nabla f| = \delta < 1$ and

$$[y_0, z_0] \subset \{\tilde{f} = 0\}, \qquad [\tilde{y}_1, \tilde{z}_1] \subset \{\tilde{f} = 1\}.$$

Denote

$$W_t = \tilde{D} + te_2,$$
$$w_t(x) = \tilde{f}(x - te_2), \quad x \in \overline{W}_t,$$

and let

$$\tilde{t} = \inf\{t \in \mathbb{R};\ w_s < u \text{ in } W_s \cap \Omega \text{ or } W_s \cap \Omega = \emptyset,\ \text{for all } s > t\}.$$

Claim 2. *If ϵ is small enough, then $0 < \tilde{t} \leqslant t^*$ and $[y_0, z_0] + \tilde{t} e_2$ lies strictly above J_0.*

Proof. By definition, we have $\tilde{f} < f$ in $\tilde{D}$. Hence, $\tilde{t} \leqslant t^*$. Since $t^* > 0$, we may choose ϵ small enough such that $\tilde{t} > 0$.

As ϵ tends to zero, we have

$$d_v\big([\tilde{y}_1, \tilde{z}_1] + t^*e_2, [y_1, z_1] + t^*e_2\big) = d_v\big([\tilde{y}_1, \tilde{z}_1], [y_1, z_1]\big) \to 0.$$

Since $d_v([y_0, z_0] + t^*e_2, J_0) > 0$, we may choose ϵ small enough such that

$$d_v\big([\tilde{y}_1, \tilde{z}_1] + t^*e_2, [y_1, z_1] + t^*e_2\big) < d_v\big([y_0, z_0] + t^*e_2, J_0\big).$$

It follows that $[y_0, z_0] + \tilde{t}e_2$ lies strictly above J_0. □

Now, we may argue as before. By definition of $\tilde{t}$, u touches $w_{\tilde{t}}$ from above. Let $\tilde{x}$ be an arbitrary contact point.

Claim 3. $\tilde{x} \in J_1 \cap ((\tilde{y}_1, \tilde{z}_1) + \tilde{t}e_2)$.

Proof. Since $\tilde{t} > 0$, by assumption (2.1), we have $\tilde{x} \notin [y_0, \tilde{y}_1] \cup [z_0, \tilde{z}_1]$. Invoking to the strong comparison principle, we derive $\tilde{x} \notin W_{\tilde{t}} \cap \Omega$. Since $[y_0, z_0] + \tilde{t}e_2$ lies strictly above J_0, we have $\tilde{x} \notin J_0 \cap ([y_0, z_0] + \tilde{t}e_2)$. Thus, we obtain $\tilde{x} \in J_1 \cap ((\tilde{y}_1, \tilde{z}_1) + \tilde{t}e_2)$. □

As a consequence of Claim 3, we have

$$\big|\nabla u(\tilde{x})\big| \leqslant \big|\nabla w_{\tilde{t}}(\tilde{x})\big| = \delta < 1.$$

This contradicts the assumption that $|\nabla u| = 1$ on J_1. Hence, we must have $t^* \leqslant 0$. This completes the proof. □

3. Proof of Theorem 1.1

In this section, we present the proof of Theorem 1.1. The proof consists of three steps. First, we prove that $c_0 = c_1 \neq 0$ (Lemmas 3.1, 3.2). Then, we show that there exists a linear function g, such that J_i lies below the i-level set of g for $i = 0, 1$ respectively (Lemmas 3.3–3.5). At last, we prove that $u \equiv g$ in Ω by the sliding method (Lemmas 3.6, 3.7).

Lemma 3.1. $c_i > 0$, $i = 0, 1$.

Proof. Let $x_0 \in \Omega, x_1 \in J_0$ and $r > 0$ be such that $B_r(x_0) \subset \Omega$ and $\partial B_r(x_0) \cap \partial\Omega = \{x_1\}$. Denote

$$v(x) = m\psi_{r, \frac{r}{2}}(x - x_0)$$

where $m = \min_{\partial B_{\frac{r}{2}}(x_0)} u > 0$. By the weak comparison principle, we have

$$u \geqslant v \quad \text{in } B_r(x_0) - \bar{B}_{\frac{r}{2}}(x_0).$$

It follows that

$$c_0 = \big|\nabla u(x_1)\big| \geqslant \big|\nabla v(x_1)\big| > 0.$$

Let $w = 1 - u$. We observe that w satisfies a similar free boundary problem as u with J_0 and J_1 interchanged. Hence, according to the above conclusion, on the 0-level set of w, we have

$$|\nabla w| = |\nabla u| = c_1 > 0.$$

This completes the proof. □

Lemma 3.2. $c_0 = c_1$.

Proof. By a dilation, we assume that $c_1 = 1$.

Let δ, k, R_0 and R_1 be as in Lemma 2.1. There exists $t \in \mathbb{R}$, such that $\bar{B}_{R_0}(te_2)$ touches J_0 from above at some point x_0. By Lemma 2.1, we have

$$c_0 = \big|\nabla u(x_0)\big| \geqslant \big|\nabla\psi_{R_0,R_1}(R_0)\big|. \tag{3.1}$$

Take $\delta, k \to 1$ in (3.1). Noticing that

$$\lim_{\delta,k\to 1}\big|\nabla\psi_{R_0,R_1}(R_0)\big| = 1,$$

we get $c_0 \geqslant c_1$.

Consider $w = 1 - u$. By the above conclusion, we have $c_1 \geqslant c_0$ as well. Hence, $c_0 = c_1$. □

By a dilation, we will assume that $c_0 = c_1 = 1$ from now on.

Let

$$\Omega_1 = \{x \in \mathbb{R}^2;\ x \text{ lies strictly above } J_1\},$$
$$\Omega_0 = \{x \in \mathbb{R}^2;\ x \text{ lies strictly below } J_0\}.$$

We denote by $COV(\Omega_i)$ the convex envelope of Ω_i, $i = 0, 1$.

Lemma 3.3. *J_0 lies below $COV(\Omega_1)$ and J_1 lies above $COV(\Omega_0)$.*

Proof. Let $y_0, z_0 \in \Omega_1$ and $y_0 \neq z_0$. We claim that $[y_0, z_0]$ lies above J_0. If $(y_0 - z_0) \cdot e_1 = 0$, the claim holds obviously. If $(y_0 - z_0) \cdot e_1 \neq 0$, then, we invoke to Lemma 2.2 to obtain the claim. This proves the first assertion of the lemma.

We consider $w = 1 - u$ and obtain the second assertion of the lemma. $\square$

Lemma 3.4. *The boundaries of $COV(\Omega_i)$, $i = 0, 1$ are two parallel straight lines.*

Proof. The boundaries $\tilde{J}_i$ of $COV(\Omega_i)$, $i = 0, 1$ are graphs of Lipschitz functions, which we denote by $\tilde{\phi}_i$, $i = 0, 1$ respectively. Suppose by contradiction that they are not parallel straight lines. Then

$$\lim_{s \to \infty} \big(\tilde{\phi}_1(s) - \tilde{\phi}_0(s)\big) = \infty. \tag{3.2}$$

Let δ, k, R_0, R_1 be as in Lemma 2.1. By (3.2), there exists $s_0 \in \mathbb{R}$, such that

$$\tilde{\phi}_1(s) - \tilde{\phi}_0(s) > 2R_0, \quad \forall s > s_0. \tag{3.3}$$

Denote $s_1 = s_0 + 2R_0$. There exists $t \in \mathbb{R}$, such that $\bar{B}_{R_0}(s_1 e_1 + t e_2)$ touches $\tilde{J}_0$ from above. Then, by Lemma 2.1, we conclude that $\bar{B}_{R_1}(s_1 e_1 + t e_2)$ lies above $\tilde{J}_1$. This contradicts (3.3). Hence, $\tilde{J}_i$, $i = 0, 1$ are two parallel straight lines. $\square$

Let l_0 be the boundary of $COV(\Omega_0)$, and $\mu = (\mu_1, \mu_2)$ be its unit normal with $\mu_2 > 0$. By a translation, we assume that $0 \in l_0$. Then

$$l_0 = \{x \in \mathbb{R}^2;\ x \cdot \mu = 0\}.$$

We denote

$$g(x) = x \cdot \mu, \quad x \in \mathbb{R}^2$$

and let l_1 be the 1-level set of g. Define T_1^0 to be the truncation function at levels 0 and 1, that is

$$T_1^0(t) = \min\{\max\{t, 0\}, 1\}, \quad t \in \mathbb{R}.$$

We wish to prove $u \equiv T_1^0(g)$. Clearly, this will establish Theorem 1.1.

Lemma 3.5. *l_1 lies above J_1 and $u \geqslant T_1^0(g)$.*

Proof. Let δ, k, R_0, R_1 be as in Lemma 2.1. Since $\bar{B}_{R_0}(R_0\mu)$ lies above l_0, it lies above J_0. In view of Lemma 2.1, we have

$$u(x) \geqslant \psi_{R_0,R_1}(x - R_0\mu), \quad \forall x \in B_{R_0}(R_0\mu) - \bar{B}_{R_1}(R_0\mu).$$

Taking $\delta, k \to 1$, we get the desired conclusion. $\square$

Lemma 3.6. *For each $z \in l_0$, we have either $z \in J_0$ or $z + e_2/\mu_2 \in J_1$.*

Proof. Without loss of generality, we assume that $z = 0$.

We prove by contradiction. Suppose that $0 \notin J_0$ and $e_2/\mu_2 \notin J_1$. Then, by Lemma 3.5, we have

$$u(x) > g(x), \quad \forall x \in [0, e_2/\mu_2).$$

So there exists $\epsilon > 0$, such that

$$u(x) > g(x + \epsilon e_2), \quad \forall x \in \big[-\epsilon e_2, (1/\mu_2 - \epsilon) e_2\big). \tag{3.4}$$

Denote $y_0 = -\epsilon e_2$. Let $s \in \mathbb{R}, s \neq 0$. Then, $z_0 = (s, (1 - s\mu_1)/\mu_2) \in l_1$. Denote by ν the unit normal of $[y_0, z_0]$ with $\nu \cdot e_2 \geqslant 0$. We observe that

$$\lim_{|s| \to \infty} (\nu - \mu) = 0. \tag{3.5}$$

We wish to employ Lemma 2.2. Let y_0, z_0 be defined as above and y_1, z_1, f be as in Lemma 2.2.

First, we check (2.1). Since z_0 lies above J_1, we have

$$u \geqslant f \quad \text{on } [z_0, z_1].$$

According to (3.4) and (3.5), there exists $s_0 > 0$, such that

$$u \geqslant f \quad \text{on } [y_0, y_1]$$

if $|s| > s_0$. Hence, (2.1) holds if $|s| > s_0$.

Now it follows from Lemma 2.2 that J_0 lies below $[y_0, z_0]$ if $|s| > s_0$. Taking $|s| \to \infty$, we find that J_0 lies below $l_0 - \epsilon e_2$. But this contradicts the fact that l_0 is the boundary of $COV(\Omega_0)$. Hence, we must have either $0 \in J_0$ or $e_2/\mu_2 \in J_1$. $\square$

Lemma 3.7. $u \equiv T_1^0(g)$.

Proof. If l_0 touches J_0 from above or $l_1 = J_1$, we get $u \equiv T_1^0(g)$ by the Hopf lemma. Otherwise, there exists $z \in l_0$, such that $z \notin J_0$ and $z + e_2/\mu_2 \notin J_1$. But this contradicts Lemma 3.6. This proves the lemma. $\square$

The proof of Theorem 1.1 is completed.

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