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A semilinear elliptic PDE not in divergence form via variational methods ☆

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ABSTRACT

In this paper we consider a semilinear equation driven by an operator not in divergence form. Precisely, the principal part of the operator is in divergence form, but it has also a lower order term depending on Du. While the right-hand side of the equation satisfies superlinear and subcritical growth conditions at zero and at infinity. The problem has not a variational structure, but, despite that, we use variational techniques in order to prove an existence and regularity result for the equation.

Contents

1. Introduction

This paper is devoted to semilinear elliptic partial differential equations with homogeneous Dirichlet boundary condition of the form

$$\begin{aligned} &Au = f(x,u) \quad \text{in } \Omega, \\ &u \in H_0^1(\Omega), \end{aligned} \tag{P}$$

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where Ω is a bounded domain of $\mathbb{R}^N$, $N \geqslant 3$, with smooth boundary $\partial\Omega$, Du denotes the gradient of u, $Du = (D_i u)_{i=1,\dots,N}$, $D_i = \partial/\partial x_i$, and A is the elliptic operator, not in divergence form, given by

$$Au = -\sum_{i,j=1}^{N} D_i\big(a_{ij}(x)D_j\, u\big) + \sum_{i=1}^{N} a_i(x)D_i u + a_0(x)u.$$

Problems of this type are studied by many authors with different methods and techniques: truncation and approximation arguments, super- and sub-solutions, fixed points theorems and so on. Among the others, we recall the papers [1,4–6,9,13, 15,16,18] and references therein.

Due to the presence of a lower order term containing Du in A, problem (P) has not a variational structure. Despite that, here we study it performing variational techniques, combined with an iterative scheme. The idea consists of 'freezing' the gradient in A in order to construct a partial differential equation with a variational structure, so that it can be treated via critical points theory. Using this new problem and an iterative technique, it is possible to get an existence and regularity result for (P).

This method was first introduced in [7] (see also [11,12,14]) in order to study a semilinear equation governed by the Laplacian operator $-\Delta$, when the nonlinear term depends also on the gradient of the solution, i.e. when it is of the form $f = f(x, u, Du)$.

Here we adapt this technique in order to consider an equation with an operator more general than the Laplacian. In dimension $N = 3$ and when $f = f(x, u, Du)$, this problem was solved in [14]. The aim of this paper is to extend this result to a general dimension N. We were able to do this only in the semilinear case, that is when $f = f(x, u)$.

The difficulty in treating problem (P) with the method introduced in [7] is mainly related to how getting uniform estimates on the $C^{1,\alpha}$-norm of the solutions of the problem associated with (P). This kind of regularity on the solutions is one of the key point for performing the iteration scheme.

This paper is organized as follows. In Section 2 we give the assumptions on the data of the problem and we state our existence and regularity result (cf. Theorem 1). Section 3 is devoted to the proof of the main result of the paper.

2. Assumptions and statement of the main theorem

2.1. Assumptions on the data

In this paper we consider the semilinear elliptic problem (P) where the elliptic operator A given by

$$Au = -\sum_{i,j=1}^{N} D_i\big(a_{ij}(x)D_j u\big) + \sum_{i=1}^{N} a_i(x)D_i u + a_0(x)u,$$

is such that $a_{ij} : \Omega \to \mathbb{R}$ are functions of class $C(\overline{\Omega})$ with $a_{ij} = a_{ji}$, $i, j = 1, \dots, N$, and

$$\lambda|\xi|^2 \leqslant \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \leqslant \Lambda|\xi|^2 \tag{2.1}$$

a.e. in Ω and for any $\xi \in \mathbb{R}^N$, for some positive constants λ and Λ, while $a_i, a_0 : \Omega \to \mathbb{R}$, $i = 1, \dots, N$, are bounded measurable functions with $a_0(x) \geqslant 0$ a.e. in Ω. In the following we denote by a the vector-valued function $a = (a_i)_{i=1,\dots,N}$ and with $\|a\|_\infty = \max_{i=1,\dots,N} \|a_i\|_\infty$.

The nonlinear term $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a function satisfying the following conditions

$$f \text{ is locally Lipschitz continuous in } \overline{\Omega} \times \mathbb{R}, \text{ uniformly with respect to } x \in \overline{\Omega}; \tag{2.2}$$

$$\lim_{t\to 0} \frac{f(x,t)}{t} = 0, \text{ uniformly with respect to } \overline{\Omega} \text{ and to each bounded subset of } \mathbb{R}^N; \tag{2.3}$$

$$\text{there exist } c_1 > 0,\ 1 < s < 4/(N-2) \text{ such that } \big|f(x,t)\big| \leqslant c_1\big(1 + |t|^s\big) \text{ in } \overline{\Omega} \times \mathbb{R}; \tag{2.4}$$

$$\text{there exists } \mu > 2 \text{ such that } 0 < \mu F(x,t) \leqslant t f(x,t) \text{ in } \overline{\Omega} \times \mathbb{R} \setminus \{0\},$$
$$\text{where } F(x,t) = \int_0^t f(x,\tau)\,d\tau \text{ for all } (x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{2.5}$$

In the following we denote by L_R, $R > 0$, the Lipschitz constant of f, i.e.

$$L_R = \sup\left\{ \frac{|f(x,t_1) - f(x,t_2)|}{|t_1 - t_2|},\ x \in \Omega,\ t_i \in \mathbb{R},\ |t_i| \leqslant R,\ i = 1, 2,\ t_1 \neq t_2 \right\}. \tag{2.6}$$

Conditions (2.2)–(2.4) yield that for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\left|f(x,t)\right| \leqslant \varepsilon|t| + \delta(\varepsilon)|t|^s \tag{2.7}$$

and, as a consequence

$$\left|F(x,t)\right| \leqslant \varepsilon|t|^2 + \delta(\varepsilon)|t|^{s+1} \tag{2.8}$$

uniformly in $\overline{\Omega} \times \mathbb{R}$.

While, by (2.5) it easily follows that there exist $c_2, c_3 > 0$ such that

$$F(x,t) \geqslant c_2|t|^\mu - c_3 \quad \text{in } \overline{\Omega} \times \mathbb{R}. \tag{2.9}$$

A model for f is given by the function

$$f(x,t) = b(x)|t|^{s-1}tg(t),$$

with $g \in Lip_{\mathrm{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $g > 0$ in $\mathbb{R}$, while $b \in Lip_{\mathrm{loc}}(\overline{\Omega})$, $b > 0$ in $\overline{\Omega}$ and s is given in assumption (2.4).

2.2. Notations

Throughout the paper we denote by $H_0^1(\Omega)$ the usual Sobolev space equipped with the norm

$$\|u\| = \left(\int_\Omega |Du|^2\,dx\right)^{1/2}$$

and by $L^q(\Omega)$, with $q \in [1,\infty)$, the usual Lebesgue space with the norm defined as

$$\|u\|_q = \left(\int_\Omega |u|^q\,dx\right)^{1/q}.$$

Moreover, $C^{1,\alpha}(\overline{\Omega})$ is equipped with the usual norm $\|\cdot\|_{1,\alpha}$, while $C_R^{1,\alpha}(\overline{\Omega})$ will be the following set

$$C_R^{1,\alpha}(\overline{\Omega}) = \left\{u \in C^{1,\alpha}(\overline{\Omega}):\ \|u\|_{1,\alpha} \leqslant R\right\}$$

with $\alpha \in (0,1)$ and $R > 0$.

Finally, in the following λ_1 will denote the first eigenvalue of the Laplacian operator $-\Delta$ in Ω, that is

$$\lambda_1 = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |Du|^2\,dx}{\int_\Omega |u|^2\,dx},$$

while S_q will be the constant of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for any $1 \leqslant q \leqslant 2^*$.

2.3. Main theorem

Problem (P) has not a variational nature, in the sense that it is not the Euler equation of some functional. Despite that, following [7,11,12], here we study (P) via variational techniques.

The main result of the paper is the following.

Theorem 1. *Assume conditions* (2.2)–(2.5) *hold true. Then there exist two positive constants R and C depending only on a_{ij}, $i, j = 1, 2, 3$, a_0, c_1, c_2, c_3, s, μ, N and Ω, such that if*

$$\|a\|_\infty < C \tag{2.10}$$

and

$$L_R < (\lambda\lambda_1)/2, \tag{2.11}$$

then problem (P) admits a non-trivial solution u belonging to $C^{1,\alpha}(\overline{\Omega})$ for any $\alpha \in (0,1)$.

The constant L_R mentioned in Theorem 1 is defined in formula (2.6), while the explicit formulas for R and C will be given in (2.17) and (2.18), respectively.

2.4. Definition of the constants appearing in Theorem 1

Let us fix a function $\tilde{u}$ in $H_0^1(\Omega)$ with $\|\tilde{u}\| = 1$ and let us define the constant

$$H = \frac{\lambda}{4}\left(\frac{\lambda}{8} \cdot \frac{1}{\delta(\lambda\lambda_1/8)S_{s+1}^{s+1}}\right)^{1/(s-1)} \tag{2.12}$$

and the function

$$h(t) = \left(\frac{\Lambda}{2} + \frac{\|a_0\|_\infty}{2\lambda_1}\right)t^2 - c_2 t^\mu \|\tilde{u}\|_\mu^\mu + c_3|\Omega| + H, \quad t \geqslant 0, \tag{2.13}$$

where δ is given in (2.7). Note that H depends only on a_{ij}, $i, j = 1, \dots, N$, δ, s and Ω and therefore only on a_{ij}, $i, j = 1, \dots, N$, c_1, s and Ω.

Being $\mu > 2$ by (2.5), it is easily seen that $h(t) \to -\infty$ as $t \to +\infty$. Thus, there exists $T \gg 1$, depending only on a_{ij}, $i, j = 1, \dots, N$, a_0, c_1, c_2, c_3, μ and Ω, such that

$$h(T) < 0. \tag{2.14}$$

Let us define the following constant

$$\sigma = \left(\frac{\Lambda}{2} + \frac{\|a_0\|_\infty}{2\lambda_1}\right)T^2 + H \tag{2.15}$$

and fix $c > 0$ as follows

$$c = \frac{(\mu - 1)H + \sqrt{(\mu - 1)^2 H^2 + 2\lambda\mu(\mu - 2)T^2\sigma}}{\lambda(\mu - 2)T}. \tag{2.16}$$

Note that c depends only on a_{ij}, $i, j = 1, \dots, N$, a_0, c_1, c_2, c_3, s, μ and Ω.

Finally, let $\varepsilon' = 1 - \frac{s(N-2)}{4} \in (0, 1)$ (thanks to the choice of s, cf. assumption (2.4)) and let us define the following constants

$$\tilde{C} = k^{(N-2)/N}\big[N/(N-2)\big]^{(N-2)/2},$$

with k suitable positive constant depending on ε' (see the proof of [17, Theorem 2.4] for more details),

$$K_1 = \tilde{C}^{N/2\varepsilon'}\left[\big(c_1 + \|a_0\|_\infty\big)\big(|\Omega| + S_{2^*}^s c^s\big)|\Omega| + \frac{c}{\sqrt{\lambda_1}}\right],$$
$$K_2 = \tilde{C}^{N/2\varepsilon'}|\Omega|\big(|\Omega| + S_{2^*}^s c^s\big),$$
$$\hat{C} = K_1 + 1,$$

and

$$\bar{C} = c_1|\Omega|\big(1 + \hat{C}^s\big) + |\Omega|\hat{C}\|a_0\|_\infty,$$

all depending only on a_{ij}, $i, j = 1, \dots, N$, a_0, c_1, c_2, c_3, s, μ, N and Ω.

Furthermore, let C_{Mor} be the embedding constant in the Morrey Theorem and C_{CZ} be the constant in the Caldéron–Zygmund Theorem applied in $L^q(\Omega)$, $q \in [1, +\infty)$. It is well known that the constant C_{Mor} depends only on s and Ω, while C_{CZ} depends only on a_{ij}, $i, j = 1, \dots, N$, q and Ω.

Now we can define the constants R and C appearing in Theorem 1. We put

$$R = 2C_{\mathrm{Mor}}C_{\mathrm{CZ}}\bar{C} \tag{2.17}$$

and

$$C = \min\left\{\frac{\sqrt{\lambda_1}H}{\sqrt{3}cT}, \frac{1}{2C_{\mathrm{Mor}}C_{\mathrm{CZ}}|\Omega|}, \frac{1}{K_2 R}, \frac{\lambda\sqrt{\lambda_1}}{2}\right\}. \tag{2.18}$$

Both R and C depend only on a_{ij}, $i, j = 1, \dots, N$, a_0, c_1, c_2, c_3, s, μ, N and Ω. Note that it also holds true that

$$C < \frac{\sqrt{\lambda_1}H}{\sqrt{3}c}, \tag{2.19}$$

since $T \gg 1$.

3. Proof of Theorem 1

Problem (P) is not variational in nature, due to the presence of Du in the lower order term of A. Despite that, we will treat it via variational techniques. Precisely, we associate, in a suitable way, with our problem a semilinear elliptic equation and we essentially adapt to our case the idea introduced in [7] which consists of 'freezing' the gradient of u in the lower order terms of the equation. In this way we have to manage a PDE which can be studied using the classical critical points theorems. In particular in this work we use the Mountain Pass Theorem of Ambrosetti and Rabinowitz [3].

Proof of Theorem 1. Let $\alpha \in (0,1)$ be fixed and let c be as in (2.16), R as in (2.17) and C as in (2.18). Let us fix w in $H_0^1(\Omega) \cap C_R^{1,\alpha}(\overline{\Omega})$, with $\|w\| \leqslant c$.

In order to prove Theorem 1 we proceed by steps:

Step 1: we associate a new PDE (P_w), variational in nature, with problem (P);
Step 2: we prove the existence of a non-trivial weak solution u_w for (P_w) using the Mountain Pass Theorem of Ambrosetti and Rabinowitz [3] and we estimate the $H_0^1(\Omega)$-norm of u_w;
Step 3: we show that u_w is of class $C^{1,\alpha}$ and we give some estimates on the $C^{1,\alpha}$-norm of u_w;
Step 4: we consider a sequence of problems (P_n) and, through an iteration technique, we construct a non-trivial solution u of the equation (P).

Step 1: a variational problem (P_w) associated with (P). Let us consider the following semilinear elliptic PDE

$$\begin{aligned} &A_w u_w = f(x, u_w) \quad \text{in } \Omega, \\ &u_w \in H_0^1(\Omega), \end{aligned} \tag{P_w}$$

where

$$A_w u = -\sum_{i,j=1}^N D_i\big(a_{ij}(x) D_j u\big) + \sum_{i=1}^N a_i(x) D_i w + a_0(x) u. \tag{3.1}$$

Problem (P_w) has a variational nature, indeed its weak solutions can be found as critical points of $I_w : H_0^1(\Omega) \to \mathbb{R}$ defined as

$$I_w(u) = \frac{1}{2} \sum_{i,j=1}^N \int_\Omega a_{ij}(x) D_i u D_j u \, dx + \frac{1}{2} \int_\Omega a_0(x) u^2 \, dx - \int_\Omega F(x,u)\, dx + \sum_{i=1}^N \int_\Omega a_i(x) u D_i w \, dx.$$

The functional I_w is well defined in $H_0^1(\Omega)$ and it is Fréchet differentiable in $H_0^1(\Omega)$, thanks to the Sobolev embedding theorems and (2.4).

Step 2: existence of a non-trivial weak solution u_w for (P_w) and estimates on the $H_0^1(\Omega)$-norm of u_w. It is enough to find a non-trivial critical point for I_w. For this purpose we apply the Mountain Pass Theorem to I_w.

First of all, let us study the geometry of I_w. The non-negativity of a_0, Hölder inequality, (2.8) and the choice of w, yield

$$\begin{aligned} I_w(u) &\geqslant \frac{\lambda}{2} \int_\Omega |Du|^2 dx - \varepsilon \int_\Omega |u|^2 \, dx - \delta(\varepsilon) \int_\Omega |u|^{s+1} \, dx - \frac{\sqrt{3}\|a\|_\infty}{\sqrt{\lambda_1}} \|w\| \|u\| \\ &\geqslant \left(\frac{\lambda}{4} - \frac{\varepsilon}{\lambda_1}\right) \|u\|^2 - \delta(\varepsilon) \|u\|_{s+1}^{s+1} + \frac{\lambda}{4}\|u\|^2 - \frac{\sqrt{3}c}{\sqrt{\lambda_1}} \|a\|_\infty \|u\| \\ &= \left[\left(\frac{\lambda}{4} - \frac{\varepsilon}{\lambda_1}\right) - \delta(\varepsilon) S_{s+1}^{s+1} \|u\|^{s-1}\right] \|u\|^2 + \left[\frac{\lambda}{4}\|u\| - \frac{\sqrt{3}c}{\sqrt{\lambda_1}} \|a\|_\infty\right] \|u\|, \end{aligned}$$

for any $\varepsilon > 0$ and for some positive constant $\delta(\varepsilon)$. Choosing $\varepsilon = \frac{\lambda\lambda_1}{8}$ we obtain

$$I_w(u) \geqslant \left(\frac{\lambda}{8} - \delta(\lambda\lambda_1/8) S_{s+1}^{s+1} \|u\|^{s-1}\right) \|u\|^2 + \left[\frac{\lambda}{4}\|u\| - \frac{\sqrt{3}c}{\sqrt{\lambda_1}} \|a\|_\infty\right] \|u\|.$$

Now, let us take $u \in H_0^1(\Omega)$ with $\|u\| = \rho$. Since (2.10) and (2.19) hold true, we can choose $\rho > 0$ such that $\frac{\lambda}{8} > \delta(\lambda\lambda_1/8) S_{s+1}^{s+1} \rho^{s-1}$ and $\frac{\lambda}{4}\rho > \frac{\sqrt{3}c}{\sqrt{\lambda_1}} \|a\|_\infty$. So that we get

$$I_w(u) \geqslant \beta,$$

for some positive β depending only on a_{ij}, $i, j = 1, \ldots, N$, a_0, c_1, c_2, c_3, s, μ and Ω.

Taking into account the choices of T and $\tilde{u}$, (2.9), (2.10) and (2.18) we have

$$I_w(T\tilde{u}) \leqslant \left(\frac{\Lambda}{2} + \frac{\|a_0\|_\infty}{2\lambda_1}\right)T^2 - c_2 T^\mu \|\tilde{u}\|_\mu^\mu + c_3|\Omega| + \frac{\sqrt{3}cT}{\sqrt{\lambda_1}}\|a\|_\infty < h(T) < 0,$$

since (2.14) holds true. Hence, there exists $e = T\tilde{u} \in H_0^1(\Omega)$, depending only on T and therefore only on a_{ij}, $i, j = 1, \ldots, N$, a_0, c_1, c_2, c_3, μ and Ω, such that $I_w(e) < 0$. Thus, I_w has the geometrical structure required by the Mountain Pass Theorem.

It is standard to prove that I_w verifies the Palais–Smale condition. Then, by the Mountain Pass Theorem, the functional I_w has a non-trivial critical point u_w such that

$$I_w(u_w) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_w\big(\gamma(t)\big) \geqslant \beta > 0,$$

where $\Gamma = \{\gamma \in C([0,1]; \mathbb{R}):\ \gamma(0) = 0 \text{ and } \gamma(1) = e\}$.

Now, let us estimate the $H_0^1(\Omega)$-norm of the solution u_w uniformly with respect to w. The Mountain Pass characterization of the critical level gives

$$I_w(u_w) \leqslant \max_{t \in [0,1]} I_w\big(\gamma(t)\big) \quad \text{for all } \gamma \in \Gamma.$$

Taking $\gamma(t) = te$, where $t \in [0,1]$, by (2.5), the definition of e, (2.10) and (2.18) we get

$$\begin{aligned}
I_w(u_w) &\leqslant \max_{t \in [0,1]} \left\{ \frac{\Lambda t^2}{2}\|e\|^2 + \frac{t^2}{2\lambda_1}\|a_0\|_\infty \|e\|^2 - \int_\Omega F(x, te)\,dx + \frac{\sqrt{3}ct}{\sqrt{\lambda_1}}\|a\|_\infty \|e\| \right\} \\
&\leqslant \left(\frac{\Lambda}{2} + \frac{\|a_0\|_\infty}{2\lambda_1}\right)\|e\|^2 + \frac{\sqrt{3}c}{\sqrt{\lambda_1}}\|a\|_\infty \|e\| \\
&< \left(\frac{\Lambda}{2} + \frac{\|a_0\|_\infty}{2\lambda_1}\right)T^2 + H = \sigma,
\end{aligned} \tag{3.2}$$

where σ is the constant defined in (2.15).

Furthermore, the definition of I_w yields

$$\begin{aligned}
I_w(u_w) - \frac{1}{\mu}\big\langle I'_w(u_w), u_w\big\rangle \geqslant{}& \left(\frac{1}{2} - \frac{1}{\mu}\right)\lambda\|u_w\|^2 + \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_\Omega a_0(x)u_w^2 - \int_\Omega F(x, u_w) \\
&+ \frac{1}{\mu}\int_\Omega f(x, u_w)u_w + \left(1 - \frac{1}{\mu}\right)\sum_{i=1}^N \int_\Omega a_i(x)u_w D_i w\,dx.
\end{aligned}$$

Thus, taking into account that u_w is a critical point of I_w, (3.2), (2.5), the non-negativity of a_0 and again (2.10) and (2.18), we get

$$\begin{aligned}
\left(\frac{1}{2} - \frac{1}{\mu}\right)\lambda\|u_w\|^2 &\leqslant \left(\frac{1}{2} - \frac{1}{\mu}\right)\lambda\|u_w\|^2 + \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_\Omega a_0(x)u_w^2 \\
&< \sigma + \left(1 - \frac{1}{\mu}\right)\frac{\sqrt{3}c\|a\|_\infty}{\sqrt{\lambda_1}}\|u_w\| \\
&< \sigma + \left(1 - \frac{1}{\mu}\right)\frac{H}{T}\|u_w\|,
\end{aligned}$$

which, by direct calculations, yields

$$\|u_w\| < c, \tag{3.3}$$

since $\mu > 2$. Here the constant c is the one defined in (2.16). Hence, we have shown that, starting from $w \in H_0^1(\Omega)$ such that $\|w\| \leqslant c$, we can find a non-trivial weak solution $u_w \in H_0^1(\Omega)$ of (P_w) such that $\|u_w\| < c$.

Now we claim that

$$\|u_w\| \geqslant b, \tag{3.4}$$

for some positive constant b depending only on a_{ij}, $i, j = 1, \ldots, N$, a_0, c_1, c_2, c_3, s, μ and Ω. Since u_w is a weak solution of (P_w), using the non-negativity of a_0 and (2.7) we get

$$\begin{aligned}
\lambda\|u_w\|^2 &\leqslant \int_\Omega f(x,u_w)u_w\,dx - \sum_{i=1}^N \int_\Omega a_i(x)u_w D_i w\,dx \\
&\leqslant \varepsilon \int_\Omega |u_w|^2\,dx + \delta(\varepsilon)\int_\Omega |u_w|^{s+1}\,dx + \frac{\sqrt{3}c}{\sqrt{\lambda_1}}\|a\|_\infty\|u_w\| \\
&< \frac{\varepsilon}{\lambda_1}\|u_w\|^2 + \delta(\varepsilon)S_{s+1}^{s+1}\|u_w\|^{s+1} + \frac{c}{T}H,
\end{aligned}$$

for any $\varepsilon > 0$, thanks to (2.10), (2.18) and (3.3). Thus, the claim easily follows, being $s > 1$. We remark that the constant b does not depend on w.

Step 3: $C^{1,\alpha}$-regularity of u_w and estimate on the $C^{1,\alpha}$-norm of u_w. Now let us prove that $u_w \in C^{1,\alpha}(\overline{\Omega})$ and $\|u_w\|_{1,\alpha} \leqslant R$.

Since $u_w \in H_0^1(\Omega)$ is a weak solution of problem (P_w), u_w solves the semilinear PDE with homogeneous Dirichlet boundary condition

$$\begin{aligned}
&\tilde{A}u_w = g(x) \quad \text{in } \Omega, \\
&u = 0 \quad \text{on } \partial\Omega,
\end{aligned}$$

where

$$\tilde{A}u = -\sum_{i,j=1}^N D_i\big(a_{ij}(x)D_j u\big)$$

and $g(x) = f(x,u_w(x)) - \sum_{i=1}^N a_i(x)D_i w(x) - a_0(x)u_w(x)$ in Ω.

The term g satisfies the following growth condition

$$\big|g(x)\big| \leqslant c_1\big(1+\big|u_w(x)\big|^s\big) + \|a\|_\infty R + \|a_0\|_\infty\big|u_w(x)\big| \quad \text{a.e. in } \Omega, \tag{3.5}$$

thanks to assumption (2.4) and the regularity of a, a_0 and w.

It is standard to show that u_w is a classical solution of (P_w) (see, for instance, [2]). In particular $u_w \in C(\overline{\Omega})$, and so $u_w \in L^\infty(\Omega)$. Let us estimate the L^∞-norm of u_w. First of all note that $|g(x)| = h(x)(1+|u_w(x)|)$, where

$$\begin{aligned}
h(x) &= \frac{|g(x)|}{1+|u_w(x)|} \\
&\leqslant \frac{c_1(1+|u_w(x)|^s) + \|a\|_\infty R + \|a_0\|_\infty|u_w(x)|}{1+|u_w(x)|} \\
&\leqslant \big(c_1 + \|a\|_\infty R + \|a_0\|_\infty\big)\big(1+\big|u_w(x)\big|^s\big).
\end{aligned}$$

Hence, $h \in L^{2^*/s}(\Omega)$ and

$$\|h\|_{2^*/s} \leqslant \big(c_1 + \|a\|_\infty R + \|a_0\|_\infty\big)\big(|\Omega| + \|u_w\|_{2^*}^s\big). \tag{3.6}$$

Then, using [17, Theorem 2.4] with $\varepsilon' = 1 - \frac{s(N-2)}{4} \in (0,1)$ (thanks to the choice of s), we get the following estimate

$$\big|u_w(x)\big| \leqslant \tilde{C}^{N/2\varepsilon'}\big(\|u_w\|_2 + |\Omega|\|h\|_{2^*/s}\big) \quad \text{in } \Omega, \tag{3.7}$$

where $\tilde{C} = k^{(N-2)/N}[N/(N-2)]^{(N-2)/2}$ with k suitable positive constant depending on ε' (see [17, Theorem 2.4] for more details). Thus, by (3.6) and (3.7), the Sobolev embeddings theorems and the fact that $\|u_w\| \leqslant c$, we have

$$\|u_w\|_\infty \leqslant K_1 + K_2\|a\|_\infty R, \tag{3.8}$$

where $K_1 = \tilde{C}^{N/2\varepsilon'}[(c_1+\|a_0\|_\infty)(|\Omega| + S_{2^*}^s c^s)|\Omega| + \frac{c}{\sqrt{\lambda_1}}]$ and $K_2 = \tilde{C}^{N/2\varepsilon'}|\Omega|(|\Omega| + S_{2^*}^s c^s)$ depend only on a_{ij}, $i,j = 1,\dots,N$, a_0, c_1, c_2, c_3, s, μ, N and Ω. Then, by (2.10) and (2.18) we get

$$\|u\|_\infty \leqslant K_1 + 1 =: \hat{C}, \tag{3.9}$$

where $\hat{C}$ depends only on a_{ij}, $i,j = 1,\dots,N$, a_0, c_1, c_2, c_3, s, μ, N and Ω.

Hence, by (3.5) $g \in L^\infty(\Omega)$ and so $g \in L^q(\Omega)$ for all $q \in [1,+\infty)$. Using the Sobolev embedding theorems, (3.3) and (3.5), we get

$$\begin{aligned}\|g\|_q &\leqslant c_1|\Omega|\big(1+\|u_w\|_\infty^s\big)+|\Omega|\|a\|_\infty R+|\Omega|\|a_0\|_\infty\|u_w\|_\infty\\ &\leqslant c_1|\Omega|\big(1+\hat{C}^s\big)+|\Omega|\|a\|_\infty R+|\Omega|\hat{C}\|a_0\|_\infty\\ &\leqslant \overline{C}+|\Omega|\|a\|_\infty R,\end{aligned} \tag{3.10}$$

where $\overline{C}=c_1|\Omega|(1+\hat{C}^s)+|\Omega|\hat{C}\|a_0\|_\infty$ depends only on a_{ij}, $i,j=1,\dots,N$, a_0, c_1, c_2, c_3, s, μ, N and Ω. Hence $\overline{C}$ is independent of w and R.

Being $a_{ij} \in C(\overline{\Omega})$, $i,j=1,\dots,N$, by the Caldéron–Zygmund Theorem (see, for instance [10, Lemma 9.17]) we also have that $u \in H^{2,q}(\Omega)$ and

$$\|u_w\|_{2,q} \leqslant C_{\mathrm{CZ}}\|g\|_q, \tag{3.11}$$

where C_{CZ} is a positive constant depending only on Ω, q and the coefficients a_{ij}, $i,j=1,\dots,N$.

Taking $q>N$, by Morrey Theorem (see, for instance [8, Section 5.6, Theorem 5]), we easily deduce that $u \in C^{1,\alpha}(\overline{\Omega})$, for any $\alpha \in (0,1)$, and

$$\|u_w\|_{1,\alpha} \leqslant C_{\mathrm{Mor}}\|u_w\|_{2,q}, \tag{3.12}$$

where C_{Mor} is a positive constant depending only on α and Ω.

Combining (3.10)–(3.12) we get

$$\|u_w\|_{1,\alpha} \leqslant C_{\mathrm{Mor}}C_{\mathrm{CZ}}\big[\overline{C}+|\Omega|\|a\|_\infty R\big], \tag{3.13}$$

and so, by (2.10) and (2.18) we have

$$\|u_w\|_{1,\alpha} \leqslant R/2+C_{\mathrm{Mor}}C_{\mathrm{CZ}}|\Omega|\|a\|_\infty R < R/2+R/2=R.$$

In conclusion we have shown that

$$\text{if } \|w\|_{1,\alpha} \leqslant R \quad \text{then } \|u_w\|_{1,\alpha} < R,$$

that is on the $C^{1,\alpha}$-norm of u_w we have the same control as for $\|w\|_{1,\alpha}$. This will be important in order to perform an iterative technique.

Step 4: the iterative scheme. Let us fix $u_0 \in C_R^{1,\alpha}(\overline{\Omega}) \cap H_0^1(\Omega)$ with $\|u_0\| \leqslant c$ and consider the following sequence of semilinear elliptic PDE

$$\begin{aligned}&A_n u_n = f(x,u_n),\\ &u_n \in H_0^1(\Omega),\end{aligned} \tag{P_n}$$

where $A_n u = A_{u_{n-1}}u$ and $n \in \mathbb{N}$ (cf. the definition (3.1)).

Every Eq. (P_n) admits a non-trivial solution $u_n \in H_0^1(\Omega)$ such that $\|u_n\| \geqslant b$ for any $n \in \mathbb{N}$, where b is independent of n. Moreover, $u_n \in C_R^{1,\alpha}(\overline{\Omega})$, that is $u_n \in C^{1,\alpha}(\overline{\Omega})$ and $\|u_n\|_{1,\alpha} \leqslant R$ for any $n \in \mathbb{N}$. In particular we have that

$$\big|u_n(x)\big| \leqslant R \quad \text{a.e. in } \Omega \text{ for any } n \in \mathbb{N}. \tag{3.14}$$

Moreover, the following equations hold true for any $v \in H_0^1(\Omega)$ and $n \in \mathbb{N}$

$$\sum_{i,j=1}^N \int_\Omega a_{ij}(x)D_i u_n D_j v\,dx+\sum_{i=1}^N \int_\Omega a_i(x)D_i u_{n-1} v\,dx+\int_\Omega a_0(x)u_n v\,dx=\int_\Omega f(x,u_n)v\,dx,$$

$$\sum_{i,j=1}^N \int_\Omega a_{ij}(x)D_i u_{n+1} D_j v\,dx+\sum_{i=1}^N \int_\Omega a_i(x)D_i u_n v\,dx+\int_\Omega a_0(x)u_{n+1} v\,dx=\int_\Omega f(x,u_{n+1})v\,dx.$$

Taking $v=u_{n+1}-u_n$ as a test function in both these equations, subtracting one equation from the other and using the non-negativity of a_0, we get

$$\begin{aligned}\lambda\|u_{n+1}-u_n\|^2 &\leqslant \int_\Omega \big[f(x,u_{n+1})-f(x,u_n)\big](u_{n+1}-u_n)\,dx-\sum_{i=1}^N\int_\Omega a_i(x)D_i(u_n-u_{n-1})(u_{n+1}-u_n)\,dx\\ &\leqslant L_R\int_\Omega |u_{n+1}-u_n|^2\,dx+\|a\|_\infty\int_\Omega |Du_n-Du_{n-1}|(u_{n+1}-u_n)\,dx\\ &\leqslant \frac{L_R}{\lambda_1}\|u_{n+1}-u_n\|^2+\frac{\|a\|_\infty}{\sqrt{\lambda_1}}\|u_n-u_{n-1}\|\|u_{n+1}-u_n\|,\end{aligned}$$

for any $n\in\mathbb{N}$, being (2.2), (2.6) and (3.14) valid. Then, by (2.10) and (2.18), it follows that

$$\lambda\|u_{n+1}-u_n\|<\frac{L_R}{\lambda_1}\|u_{n+1}-u_n\|+\frac{\lambda}{2}\|u_n-u_{n-1}\|,\quad \forall n\in\mathbb{N}.$$

Now (2.11) yields

$$0<\left(\lambda-\frac{L_R}{\lambda_1}\right)\|u_{n+1}-u_n\|<\frac{\lambda}{2}\|u_n-u_{n-1}\|,\quad \forall n\in\mathbb{N},$$

so that $(u_n)_n$ is a Cauchy sequence in $H^1_0(\Omega)$, again thanks to assumption (2.11). Thus

$$u_n\to u\quad \text{in } H^1_0(\Omega) \text{ as } n\to\infty \tag{3.15}$$

for some $u\in H^1_0(\Omega)$. Passing to the limit in the weak form of (P_n) as $n\to\infty$ it is easy to see that

$$\sum_{i,j=1}^N\int_\Omega a_{ij}(x)D_iuD_jv\,dx+\sum_{i=1}^N\int_\Omega a_i(x)D_iuv\,dx+\int_\Omega a_0(x)uv\,dx=\int_\Omega f(x,u)v\,dx,$$

that is, u is a weak solution of Eq. (P).

We claim that $u\not\equiv 0$ in Ω. By (3.4) and (3.15), it easily follows that $\|u\|\geqslant b>0$, so that the claim is proved. Hence u is a non-trivial weak solution of Eq. (P).

Finally, since $u_n\in C^{1,\alpha}_R(\overline{\Omega})$ for any $n\in\mathbb{N}$, the sequences $(u_n)_n$ and $(Du_n)_n$ are equicontinuous and equibounded in $\overline{\Omega}$. The Ascoli–Arzelà Theorem implies that $u_n\to u$ and $Du_n\to Du$ uniformly in $\overline{\Omega}$ as $n\to\infty$, so that $u\in C^1(\overline{\Omega})$. With the same arguments we also deduce that $u\in C^{1,\alpha}_R(\overline{\Omega})$.

Then, Theorem 1 is completely proved. □

Remark. In the proof of Theorem 1 it is crucial to show that $(u_n)_n$ is a Cauchy sequence in $H^1_0(\Omega)$. Indeed, as a consequence, we have strong convergence of the whole sequences $(u_n)_n$ and $(Du_n)_n$ in $L^2(\Omega)$ and not only up to subsequences.

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