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On positive solutions for a fourth order asymptotically linear elliptic equation under Navier boundary conditions ☆

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ABSTRACT

A result on existence of positive solution for a fourth order nonlinear elliptic equation under Navier boundary conditions is established. The nonlinear term involved is asymptotically linear both at the origin and at infinity. We exploit topological degree theory and global bifurcation.

1. Introduction

We discuss existence of positive solutions for the problem

$$\begin{cases} \Delta^2 u = f(x, u, \Delta u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, is a bounded domain with smooth boundary $\partial\Omega$, Δ^2 is the biharmonic operator and $f : \Omega \times [0, \infty) \times (-\infty, 0] \to [0, \infty)$ is a continuous function, asymptotically linear in a suitable sense such that

$$f(x, u, p) > 0, \quad x \in \Omega,\ (u, p) \in \big([0, \infty) \times (-\infty, 0]\big) \setminus \big\{(0, 0)\big\}.$$

A well-known result on asymptotically linear problems establishes that

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

admits a solution if f is continuous, $-\infty < f'(0) < \lambda_1 < f'(\infty) < \infty$, where λ_1 is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$ and

$$f'(0) := \lim_{t\to 0} \frac{f(t)}{t}, \qquad f'(\infty) := \lim_{t\to\infty} \frac{f(t)}{t}.$$

In this work we shall assume that f is asymptotically linear at the origin in the sense of conditions **(H1)** and **(H2)** below.

(**H1**) There are nonnegative constants a_0, b_0, a^0, b^0 with

$$a_0 + b_0 > 0, \qquad a^0 + b^0 > 0,$$

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such that

$$a_0 u - b_0 p - \xi_1(x,u,p) \leqslant f(x,u,p) \leqslant a^0 u - b^0 p + \xi_2(x,u,p),$$

for $(x,u,p) \in \Omega \times [0,\infty) \times (-\infty,0]$, where

$$\xi_1, \xi_2 : \Omega \times [0,\infty) \times (-\infty,0] \to [0,\infty)$$

are continuous functions satisfying

$$\lim_{|(u,p)| \to 0} \frac{\xi_j(x,u,p)}{|(u,p)|} = 0 \quad \text{for each } x \in \Omega,$$

and

$$\frac{|\xi_j(x,u,p)|}{|(u,p)|} \leqslant \gamma_j(x) \quad \text{for each } x \in \Omega \text{ and } \big|(u,p)\big| > 0,$$

where $\gamma_j \in L^\infty(\Omega)$, $j = 1,2$.

Throughout this work,

$$\big|(u,p)\big| := \sqrt{u^2 + p^2}, \quad (u,p) \in \mathbb{R}^2.$$

(**H2**) There are nonnegative constants $c_\infty, d_\infty, c^\infty, d^\infty$ with

$$c_\infty + d_\infty > 0, \qquad c^\infty + d^\infty > 0.$$

such that

$$c_\infty u - d_\infty p - \eta_1(x,u,p) \leqslant f(x,u,p) \leqslant c^\infty u - d^\infty p + \eta_2(x,u,p),$$

for $(x,u,p) \in \Omega \times [0,\infty) \times (-\infty,0]$, where

$$\eta_1, \eta_2 : \Omega \times [0,\infty) \times (-\infty,0] \to [0,\infty)$$

are continuous functions such that

$$\lim_{|(u,p)| \to \infty} \frac{\eta_j(x,u,p)}{|(u,p)|} = 0 \quad \text{for each } x \in \Omega,$$

and

$$\frac{|\eta_j(x,u,p)|}{|(u,p)|} \leqslant \zeta_j(x) \quad \text{for each } x \in \Omega \text{ and } \big|(u,p)\big| > 0,$$

where $\zeta_j \in L^\infty(\Omega)$, $j = 1,2$.

In order to state our main result, we set

$$\mu_1(\alpha,\beta) := \frac{\lambda_1^2}{\beta + \alpha\lambda_1},$$

where α, β are nonnegative numbers such that

$$\alpha + \beta > 0.$$

The main result of this work is:

Theorem 1.1. *Assume* (**H1**), (**H2**) *and*

(**H3**) *there are nonnegative numbers a_1, a_2 with $a_1 + a_2 > 0$ such that*

$$f(x,u,p) \geqslant a_1 u - a_2 p \quad \text{for each } (x,u,p) \in \Omega \times [0,\infty) \times (-\infty,0].$$

If, in addition, either

$$\mu_1(c_\infty, d_\infty) < 1 < \mu_1\big(a^0, b^0\big)$$

or

$$\mu_1(a_0, b_0) < 1 < \mu_1\big(c^\infty, d^\infty\big)$$

then problem (1) *admits at least one positive solution.*

Our main result was motivated by the recent paper [1] by Ruyun Ma and Jia Xu, where the fourth order ODE problem,

$$\begin{cases} u'''' = f(x, u, u'') & \text{in } (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases}$$

is studied. Our Theorem 1.1 improves the main result in [1] in the sense that it also holds in dimension one and in that our assumptions **(H1)**, **(H2)** are less restrictive than the corresponding ones in [1].

We further refer the reader to Champneys and McKenna [13], Micheletti and Pistoia [9,10], Micheletti and Saccon [11], Pao and Wang [12], Ruyun Ma [2] and their references, for boundary value problems for fourth order equations.

The techniques we employ below to prove our Theorem 1.1 apply to the problem

$$\begin{cases} \alpha\Delta^2 u + \beta\Delta u = g(x,u) & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\alpha \geqslant 0$, $-\infty < \beta < \alpha\lambda_1$, the boundary condition $\mathcal{B}u = 0$ on $\partial\Omega$ means that $u = \Delta u = 0$ on $\partial\Omega$ when $\alpha > 0$ and $u = 0$ on $\partial\Omega$ when $\alpha = 0$ and $g(x,u)$ satisfies conditions similar to **(H1)**, **(H2)**, **(H3)**. See, e.g. [8] for remarks on the eigenvalues of the operator $\alpha\Delta^2 u + \beta\Delta u$.

2. Notations, basic results, abstract framework

Consider the space $H := H_0^1(\Omega) \cap H^2(\Omega)$

The Generalized Green Identity establishes that

$$\int_\Omega \nabla u \nabla v \, dx = -\int_\Omega u \Delta v \, dx, \quad u, v \in H.$$

The two inequalities below are well known:

$$\text{(i)} \int_\Omega |\Delta u|^2 \, dx \geqslant \lambda_1 \int_\Omega |\nabla u|^2 \, dx, \qquad \text{(ii)} \int_\Omega |\Delta u|^2 \, dx \geqslant \lambda_1^2 \int_\Omega |u|^2 \, dx, \quad u \in H. \tag{2}$$

Now, the space H endowed with the norm and inner product

$$\|u\|^2 := \int_\Omega |\Delta u|^2 \, dx, \qquad \langle u, v\rangle := \int_\Omega \Delta u \Delta v \, dx, \quad u, v \in H$$

is a Hilbert space.

Let $h \in L^2(\Omega)$ and consider the Dirichlet problem

$$-\Delta u = h \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{3}$$

The solution operator associated to (3), namely

$$S : L^2(\Omega) \to H_0^1(\Omega) \hookrightarrow L^2(\Omega)$$

is linear, compact and symmetric.

The spectral analysis of S gives the principal eigenvalue λ_1 of $(-\Delta, H_0^1(\Omega))$ whose eigenfunction ϕ_1 is positive in Ω.

On the other hand, a function $u \in H$ is a weak solution of

$$\Delta^2 u = h \quad \text{in } \Omega, \qquad u = \Delta u = 0 \quad \text{on } \partial\Omega \tag{4}$$

if

$$\int_\Omega \Delta u \Delta v \, dx = \int_\Omega h v \, dx, \quad v \in H.$$

If $u \in H$ is the solution of (4) it follows by the elliptic a priori estimates, (see e.g. Gupta [3], Gilbarg and Trudinger [7]), that $u \in H_0^1(\Omega) \cap H^4(\Omega)$ and

$$\|u\|_{H^4(\Omega)} \leqslant C\|h\|_{L^2(\Omega)},$$

for some constant $C > 0$. Moreover, if $u \in H$ is a weak solution then $\Delta u = 0$ on $\partial\Omega$ in the trace sense and, in particular, $\Delta u \in H_0^1(\Omega)$.

Next, we recall a version of the Maximum Principle for the biharmonic operator.

Proposition 2.1. *Let $u \in H_0^1(\Omega) \cap H^4(\Omega)$ be a function such that*

$$\begin{cases} \Delta^2 u \geqslant 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then

$$u(x) \geqslant 0, \quad \Delta u(x) \leqslant 0, \quad \text{a.e. } x \in \Omega.$$

Proof. Set $v := \Delta u$. Then

$$v \in H_0^1(\Omega) \cap H^2(\Omega) \quad \text{and} \quad \Delta v \geqslant 0 \quad \text{in } \Omega.$$

By the usual Maximum Principle for the Laplacian, $\sup_\Omega v \leqslant \sup_{\partial\Omega} v^+ = 0$.

Thus we have

$$u \in H_0^1(\Omega) \cap H^2(\Omega) \quad \text{and} \quad \Delta u \leqslant 0 \quad \text{in } \Omega.$$

By the usual Maximum Principle again, we get

$$\inf_\Omega u \geqslant \inf_{\partial\Omega} u^- = 0$$

and hence

$$u \geqslant 0 \quad \text{in } \Omega. \qquad \square$$

The following result on existence of global branches of solutions (see e.g. Rabinowitz [4,5], Schmitt and Thompson [6]), is crucial in this paper.

Theorem 2.2. *Let E be a real Banach space with norm $\|\cdot\|$ and let $T : \mathbb{R} \times E \to E$ be a compact operator such that*

$$T(\lambda, 0) = 0, \quad \lambda \in \mathbb{R}. \tag{5}$$

Assume that there are $a, b \in \mathbb{R}$ with $a < b$ such that $u = 0$ is an isolated solution of $T(\lambda, u) = u$ for $\lambda = a$ and $\lambda = b$ and that $\lambda = a$ and $\lambda = b$ are not bifurcation points of $T(\lambda, u) = u$ with respect to the line of trivial solutions $(\lambda, 0)$. Assume also that

$$\deg\big(I - T(a, .), B_\delta, 0\big) \neq \deg\big(I - T(b, .), B_\delta, 0\big),$$

where $B_\delta = \{u \in E\colon \|u\| < \delta\}$ is an isolating neighborhood of the trivial solution and deg means the Leray–Schauder degree. Set

$$\mathcal{S} = \overline{\big\{(\lambda, u) \in \mathbb{R} \times E\colon u - T(\lambda, u) = 0,\ u \neq 0\big\}} \cup \big\{[a, b] \times \{0\}\big\}.$$

Let $\mathcal{C} \subset \mathcal{S}$ be the maximal connected component of $\mathcal{S}$ which contains $[a, b] \times \{0\}$. Then, either

(i) *$\mathcal{C}$ is unbounded in $\mathbb{R} \times E$, or*
(ii) *$\mathcal{C} \cap \{\mathbb{R} \setminus \{[a, b] \times \{0\}\}\} \neq \emptyset$.*

Now consider the function $\widetilde{f} : \Omega \times (-\infty, \infty) \times (-\infty, \infty) \to [0, \infty)$,

$$\widetilde{f}(x, u, p) = \begin{cases} f(x, u, p), & x \in \Omega,\ u \geqslant 0,\ p \leqslant 0, \\ f(x, 0, p), & x \in \Omega,\ u \leqslant 0,\ p \leqslant 0, \\ f(x, u, 0), & x \in \Omega,\ u \geqslant 0,\ p \geqslant 0, \\ f(x, 0, 0), & x \in \Omega,\ u \leqslant 0,\ p \geqslant 0, \end{cases}$$

which is a continuous extension of f.

We shall study the family of problems

$$\begin{cases} \Delta^2 u = \lambda \widetilde{f}(x, u, \Delta u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \Omega, \end{cases} \tag{6}$$

where $\lambda > 0$ is a parameter.

Consider the Nemytskii operators $F, \widetilde{F} : H \to L^2(\Omega)$ given by

$$F(u) = f(x, u, \Delta u), \qquad \widetilde{F}(u) = \widetilde{f}(x, u, \Delta u), \quad u \in H.$$

Using **(H1)**, **(H2)** one infers that $F, \widetilde{F}$ are bounded and continuous.

Now consider the nonlinear operator $\Phi : \mathbb{R} \times H \to H$ defined by

$$\Phi_\lambda(u) = u - T(\lambda, u) \quad \text{where } T(\lambda, u) = \lambda S^2 \widetilde{F}(u),\ u \in H,\ \lambda \in \mathbb{R}.$$

Notice that by the continuity, boundedness of $\widetilde{F}$ and the compactness of S^2, $T : \mathbb{R} \times H \to H$ is compact.

We point out that $u \in H$ is a weak solution of problem (6) if and only if u satisfies

$$\Phi_\lambda(u) = 0, \quad u \in H,\ \lambda \in \mathbb{R}. \tag{7}$$

Notice that since, by Sobolev's embeddings, T is a compact operator, Φ_λ is a compact perturbation of the identity which enables us to apply the Leray–Schauder degree theory.

Setting $\lambda = 1$, the positive solutions $u \in H$ of (7) are solutions of (1).

3. Technical lemmata

In this section we shall establish and proof a few usefull technical results.

Lemma 3.1. *Suppose* (**H1**), (**H2**). *Let* $\{u_n\} \subseteq H$ *be a sequence such that* $\|u_n\| > 0$ *and* $\{\frac{u_n}{\|u_n\|}\}$ *converges in* H. *Then*

$$\text{(i)}\ \ o_n^{\xi_j}(1) \equiv \int_\Omega \frac{\xi_j(x, u_n, \Delta u_n)}{\|u_n\|}\phi_1\,dx \overset{n\to\infty}{\longrightarrow} 0 \quad \text{if } \|u_n\| \to 0,\ j = 1, 2,$$

$$\text{(ii)}\ \ o_n^{\eta_j}(1) \equiv \int_\Omega \frac{\eta_j(x, u_n, \Delta u_n)}{\|u_n\|}\phi_1\,dx \overset{n\to\infty}{\longrightarrow} 0 \quad \text{if } \|u_n\| \to \infty,\ j = 1, 2.$$

Proof. Set $v_n := \frac{u_n}{\|u_n\|}$ so that $v_n \overset{H}{\to} v$ for some $v \in H$ with $\|v\| = 1$. It follows by (2) that

$$v_n \to v, \quad \Delta v_n \to \Delta v \quad \text{in } L^2(\Omega).$$

Moreover,

$$v_n \to v, \quad \Delta v_n \to \Delta v \quad \text{a.e. in } \Omega,$$

and there are $h_1, h_2 \in L^2(\Omega)$ such that

$$|v_n| \leqslant h_1, \quad |\Delta v_n| \leqslant h_2 \quad \text{a.e. in } \Omega.$$

We have

$$\begin{aligned} \left|\frac{\xi_j(x, u_n, \Delta u_n)\phi_1}{\|u_n\|}\right| &= \phi_1 \frac{|\xi_j(x, u_n, \Delta u_n)|}{|(u_n, \Delta u_n)|}\,\frac{|(u_n, \Delta u_n)|}{\|u_n\|} \\ &= \phi_1 \frac{|\xi_j(x, u_n, \Delta u_n)|}{|(u_n, \Delta u_n)|}\sqrt{v_n^2 + |\Delta v_n|^2}. \end{aligned} \tag{8}$$

Verification of (i). Setting $H(x) := \max\{h_1(x), h_2(x)\}$ it follows from (8) that

$$\left|\frac{\xi_j(x, u_n, \Delta u_n)\phi_1}{\|u_n\|}\right| \leqslant \phi_1\gamma_j(x)H(x).$$

Notice that $\phi_1\gamma_j H \in L^2(\Omega)$.

On the other hand, since

$$u_n \to 0, \quad \Delta u_n \to 0 \quad \text{in } L^2(\Omega),$$
$$u_n \to 0, \quad \Delta u_n \to 0 \quad \text{a.e. in } \Omega,$$

we infer from (8) that

$$\left|\frac{\xi_j(x, u_n, \Delta u_n)\phi_1}{\|u_n\|}\right| \to 0 \quad \text{a.e. in } \Omega.$$

By the Lebesgue Theorem,

$$\int_\Omega \frac{\xi_j(x, u_n, \Delta u_n)\phi_1}{\|u_n\|}\,dx \overset{n}{\to} 0.$$

Verification of (ii). Set

$$\Omega_0 := \{x \in \Omega \mid v(x) = 0\}.$$

It follows (see e.g. Gilbarg and Trudinger [7, Lemma 7.7]) that

$$\Delta v = 0 \quad \text{a.e. in } \Omega_0.$$

We infer through an estimate similar to the one in (8) (for η_j instead of ξ_j) that

$$\left| \frac{\eta_j(x, u_n, \Delta u_n)\phi_1}{\|u_n\|} \right| \to 0 \quad \text{a.e. in } \Omega_0.$$

If $x \in {\Omega_0}^c$ then $u_n(x) = \|u_n\| \ v_n(x) \to \pm\infty$. Estimating as we did in (8),

$$\left| \frac{\eta_j(x, u_n, \Delta u_n)\phi_1}{\|u_n\|} \right| \leqslant \phi_1 \frac{|\eta_j(x, u_n, \Delta u_n)|}{|(u_n, \Delta u_n)|} H \quad \text{a.e. in } {\Omega_0}^c.$$

Thus

$$\left| \frac{\eta_j(x, u_n, \Delta u_n)\phi_1}{\|u_n\|} \right| \to 0 \quad \text{a.e. in } {\Omega_0}^c.$$

Applying the Lebesgue Theorem we infer that

$$\int_\Omega \frac{\eta_j(x, u_n, \Delta u_n)\phi_1}{\|u_n\|} \, dx \overset{n}{\to} 0.$$

This ends the proof of Lemma 3.1. $\square$

The result below establishes an a priori estimate for the bifurcation points of (6).

Lemma 3.2. *Assume* (**H1**). *Let* $(\lambda, 0) \in \mathbb{R}^+ \times H$ *be a bifurcation point of the equation*

$$\Phi_\lambda(u) = 0, \quad u \in H.$$

Then

$$\mu_1\big(a^0, b^0\big) \leqslant \lambda \leqslant \mu_1(a_0, b_0).$$

Proof. We will split the proof into two steps.

Step 1. In order to show that $\lambda \leqslant \mu_1(a_0, b_0)$ pick sequences $\{u_n\} \subseteq H$ and $\{\lambda_n\} \subseteq \mathbb{R}^+$ such that

$$u_n \not\equiv 0, \quad \|u_n\| \to 0, \qquad \lambda_n \to \lambda$$

and

$$\begin{cases} \Delta^2 u_n = \lambda_n \widetilde{f}(x, u_n, \Delta u_n) & \text{in } \Omega, \\ u_n \in H_0^1(\Omega) \cap H^4(\Omega). \end{cases} \tag{9}$$

By Proposition 2.1,

$$u_n \geqslant 0, \quad \Delta u_n \leqslant 0 \quad \text{a.e. } x \in \Omega.$$

These facts allow us to rewrite the equation in (9) as

$$\Delta^2 u_n = \lambda_n f(x, u_n, \Delta u_n) \quad \text{in } \Omega, \tag{10}$$

or equivalently,

$$u_n = \lambda_n S^2 F(u_n), \quad u_n \in H.$$

Set $v_n = \frac{u_n}{\|u_n\|}$. There is $v \in H$ such that

$$v_n \overset{H}{\rightharpoonup} v \quad \text{and} \quad v_n \overset{L^2}{\to} v.$$

Moreover, the equation in (10) rewrites as,

$$\Delta^2 v_n = \lambda_n \frac{f(x, u_n, \Delta u_n)}{\|u_n\|} \quad \text{in } \Omega, \tag{11}$$

or equivalently

$$v_n = \lambda_n S^2 \left(\frac{F(u_n)}{\|u_n\|} \right). \tag{12}$$

On the other hand, estimating, using (**H1**), we get to

$$\begin{aligned}\frac{\xi_j^2(x,u_n,\Delta u_n)}{\|u_n\|^2} &= \frac{\xi_j^2(x,u_n,\Delta u_n)}{|(u_n,\Delta u_n)|^2}\frac{|(u_n,\Delta u_n)|^2}{\|u_n\|^2}\\ &= \frac{\xi_j^2(x,u_n,\Delta u_n)}{|(u_n,\Delta u_n)|^2}\big(v_n^2+|\Delta v_n|^2\big)\\ &\leqslant \gamma_j^2\,\big(v_n^2+|\Delta v_n|^2\big).\end{aligned}$$

Since $\gamma_j \in L^\infty(\Omega)$ we infer that

$$\left\{\frac{\xi_j(x,u_n,\Delta u_n)}{\|u_n\|}\right\} \quad \text{is bounded in } \in L^2(\Omega).$$

As a consequence, using **(H1)** again,

$$\left\{\frac{F(u_n)}{\|u_n\|}\right\}$$

is bounded in $L^2(\Omega)$.

By **(H1)**, it follows that $F : H \to L^2(\Omega)$ is bounded and continuous and so $S^2F : H \to H$ is compact.

Since $\{\lambda_n\}$ converges we infer from (12) that $\{v_n\}$ admits a convergent subsequence, still denoted $\{v_n\}$, that is

$$v_n \xrightarrow{H} v, \quad \|v\| = 1.$$

Multiplying the equation in (11) by ϕ_1, integrating, using **(H1)** we have,

$$\begin{aligned}\lambda_1^2\int_\Omega v_n\phi_1\,dx = \int_\Omega \Delta^2 v_n\phi_1\,dx &\geqslant \lambda_n\int_\Omega (a_0v_n - b_0\Delta v_n)\phi_1\,dx - o_n^{\xi_1}(1)\\ &= \lambda_n\int_\Omega (a_0\phi_1 - b_0\Delta\phi_1)v_n\,dx - o_n^{\xi_1}(1)\\ &= \lambda_n(a_0+b_0\lambda_1)\int_\Omega v_n\phi_1\,dx - o_n^{\xi_1}(1).\end{aligned}$$

Passing to the limit, in the inequality above, using Lemma 3.1, we get

$$\left(\frac{\lambda_1^2}{a_0+b_0\lambda_1} - \lambda\right)\int_\Omega v\phi_1\,dx \geqslant 0.$$

Since the integral above is strictly positive it follows that,

$$\lambda \leqslant \mu_1(a_0,b_0).$$

This ends the proof in **Step 1**.

Step 2. The verification that $\lambda \geqslant \mu_1(a^0,b^0)$ follows by arguments similar to those in **Step 1**.

This ends the proof of Lemma 3.2. □

The result below is about non-existence of solutions of the Banach space equation:

$$\Phi_\lambda(u) = 0, \quad (\lambda,u) \in \mathbb{R}^+ \times H.$$

Lemma 3.3. *Assume* (**H1**). *If $\Lambda \subset \mathbb{R}^+$ is compact and*

$$\big[\mu_1\big(a^0,b^0\big),\mu_1(a_0,b_0)\big] \cap \Lambda = \emptyset$$

then there is $\delta_1 > 0$ such that

$$\Phi_\lambda(u) \neq 0 \quad \text{if } 0 < \|u\| \leqslant \delta,\ \lambda \in \Lambda,\ 0 < \delta \leqslant \delta_1.$$

Proof. Arguing by contradiction, there are sequences $\{u_n\} \subseteq H$ and $\{\lambda_n\} \subseteq \Lambda$ such that

$$\begin{aligned} &\Phi_{\lambda_n}(u_n) = 0, \\ &0 < \|u_n\| \leqslant \frac{1}{n}. \end{aligned} \tag{13}$$

By eventually taking subsequences we have, $\lambda_n \to \lambda \in \Lambda$.

Set $v_n = \frac{u_n}{\|u_n\|}$. There is a function $v \in H$ such that

$$v_n \rightharpoonup v \quad \text{in } H, \qquad v_n \to v \quad \text{in } L^2(\Omega) \quad \text{and} \quad v_n \to v \quad \text{a.e. in } \Omega.$$

We have

$$\Delta^2 v_n = \lambda_n \frac{f(x, u_n, \Delta u_n)}{\|u_n\|} \quad \text{in } \Omega.$$

Arguing as in the proof of Lemma 3.2,

$$v_n \stackrel{H}{\to} v, \quad \|v\| = 1.$$

Multiplying by ϕ_1 and integrating, using **(H1)** we get to

$$\begin{aligned} \lambda_1^2 \int_\Omega v_n \phi_1 \, dx = \int_\Omega \Delta^2 v_n \phi_1 \, dx &\geqslant \lambda_n \int_\Omega (a_0 v_n - b_0 \Delta v_n) \phi_1 \, dx - o_n^{\xi_1}(1) \\ &= \lambda_n \int_\Omega (a_0 \phi_1 - b_0 \Delta \phi_1) v_n \, dx - o_n^{\xi_1}(1) \\ &= \lambda_n (a_0 + b_0 \lambda_1) \int_\Omega v_n \phi_1 \, dx - o_n^{\xi_1}(1). \end{aligned}$$

Passing to the limit, applying Lemma 3.1,

$$\left(\frac{\lambda_1^2}{a_0 + b_0 \lambda_1} - \lambda \right) \int_\Omega v \phi_1 \, dx \geqslant 0.$$

Since v is nontrivial and nonnegative the integral just above is positive. Hence,

$$\lambda \leqslant \mu_1(a_0, b_0).$$

By a similar argument we obtain, using by **(H1)** again,

$$\lambda \geqslant \mu_1(a^0, b^0).$$

This contradicts $\lambda \in \Lambda$ obtained as before. This ends the proof of Lemma 3.3. □

Next we will use the previous lemma to compute the Leray–Schauder degree of Φ_λ for $\lambda \in (0, \mu_1(a^0, b^0))$.

Lemma 3.4. *Assume* (**H1**). *Then*

$$\deg(\Phi_\lambda, B_\delta, 0) = 1,$$

where $0 < \delta \leqslant \delta_1$, $0 < \lambda < \mu_1(a^0, b^0)$.

Proof. Setting $\Lambda = [0, \lambda]$, we have

$$\Lambda \cap \left[\mu_1\left(a^0, b^0\right), \mu_1(a_0, b_0)\right] = \emptyset.$$

Consider the homotopy $N : [0, 1] \times B_\delta \to B_\delta$ defined by

$$N(t, u) = u - t\lambda S^2 \widetilde{F}(u), \quad t \in [0, 1], \ u \in B_\delta \subset H.$$

We claim that

$$N(t, u) \neq 0, \quad t \in [0, 1], \ u \in \partial B_\delta.$$

Indeed, the case $t = 0$ is obvious, so let $0 < t \leqslant 1$. Notice that the equation

$$N(t, u) = 0, \quad u \in \partial B_\delta,$$

is equivalent to

$$\Phi_{\lambda t}(u) = 0, \quad u \in \partial B_\delta,$$

which is not solvable by Lemma 3.3, showing the claim.

By the homotopy invariance property of the Leray–Schauder degree,

$$\deg\big(N(1,.), B_\delta, 0\big) = \deg\big(N(0,.), B_\delta, 0\big) = \deg(I, B_\delta, 0) = 1,$$

showing that

$$\deg(\Phi_\lambda, B_\delta, 0) = 1. \qquad \square$$

Next, we state a non-existence result for the Banach space equation $\Phi_\lambda(u) = \tau\phi_1$ with λ big enough, which will be useful in the computation of some topological degrees.

Lemma 3.5. *Assume* (**H1**) *and* $\lambda > \mu_1(a_0, b_0)$. *Then there is* $\delta_2 > 0$ *such that*

$$\Phi_\lambda(u) \neq \tau\phi_1, \quad \tau \in [0,1],\ 0 < \|u\| \leqslant \delta,\ 0 < \delta \leqslant \delta_2.$$

Proof. Assume, on the contrary, that there are sequences $\{u_n\} \subseteq H$ and $\{\tau_n\} \subseteq [0,1]$ such that

$$\Phi_\lambda(u_n) = \tau_n\phi_1 \quad \text{and} \quad 0 < \|u_n\| \leqslant \frac{1}{n}.$$

We have,

$$u_n \to 0 \quad \text{in } H, \qquad u_n \to 0 \quad \text{in } L^2(\Omega), \qquad u_n \to 0 \quad \text{a.e. in } \Omega \text{ and } |u_n| \leqslant h,\ h \in L^2(\Omega).$$

The equation above can be rewritten as

$$u_n = \lambda S^2 \widetilde{F}(u_n) + \tau_n\phi_1 \quad \text{in } \Omega.$$

It follows that

$$\begin{cases} \Delta^2 u_n = \lambda f(x, u_n, \Delta u_n) + \tau_n \Delta^2\phi_1 & \text{in } \Omega, \\ u_n \in H, \quad u_n \geqslant 0, \quad \Delta u_n \leqslant 0 & \text{in } \Omega. \end{cases}$$

Now using the continuity of the operator S^2F and the fact that $\|u_n\| \to 0$,

$$S^2F(u_n) \to 0 \quad \text{in } H.$$

Dividing the equation above by $\|u_n\|$ and setting $v_n = \frac{u_n}{\|u_n\|}$ we have

$$v_n = \lambda S^2\left(\frac{F(u_n)}{\|u_n\|}\right) + \frac{\tau_n}{\|u_n\|}\phi_1, \quad \text{in } \Omega. \tag{14}$$

By standard arguments there is a function $v \in H$ such that

$$v_n \rightharpoonup v \quad \text{in } H, \qquad v_n \to v \quad \text{in } L^2(\Omega), \qquad v_n \to v \quad \text{a.e. in } \Omega.$$

Arguing as in the proof of Lemma 3.2 one infers that

$$\left\{S^2\left(\frac{F(u_n)}{\|u_n\|}\right)\right\} \quad \text{is bounded.}$$

Since also $\{v_n\}$ is bounded, it follows by (14) that $\{\frac{\tau_n}{\|u_n\|}\}$ is bounded.

Using the compactness of S^2F, it follows from (14) that $v_n \to v$ in H.

From (14),

$$\Delta^2 v_n = \lambda\frac{f(x, u_n, \Delta u_n)}{\|u_n\|} + \frac{\tau_n}{\|u_n\|}\Delta^2\phi_1 \quad \text{in } \Omega$$

Multiplying by ϕ_1 in the equation above, integrating, using **(H1)**, we have

$$\begin{aligned} \lambda_1^2 \int_\Omega v_n\phi_1\,dx = \int_\Omega \Delta^2 v_n\phi_1\,dx &\geqslant \lambda_n \int_\Omega (a_0 v_n - b_0\Delta v_n)\phi_1\,dx - o_n^{\xi_1}(1) \\ &= \lambda_n \int_\Omega (a_0\phi_1 - b_0\Delta\phi_1)v_n\,dx - o_n^{\xi_1}(1) \\ &= \lambda_n(a_0 + b_0\lambda_1)\int_\Omega v_n\phi_1\,dx - o_n^{\xi_1}(1). \end{aligned}$$

Passing to the limit in the set of inequalities above, and applying Lemma 3.1

$$\left(\frac{\lambda_1^2}{a_0 + b_0\lambda_1} - \lambda\right)\int_\Omega v\phi_1\,dx \geqslant 0.$$

Since $\|v\| = 1$, $v \geqslant 0$ and $\phi_1 > 0$ it follows that

$$\lambda \leqslant \mu_1(a_0, b_0),$$

which is a contradiction. This ends the proof of Lemma 3.5. □

In the next lemma we will use the preceding result to compute the Leray–Schauder degree of Φ_λ, for $\lambda \in (\mu_1(a_0, b_0), +\infty)$.

Lemma 3.6. *Assume* (**H1**) *and* $\lambda \in (\mu_1(a_0, b_0), \infty)$. *Then*

$$\deg(\Phi_\lambda, B_\delta, 0) = 0, \quad 0 < \delta \leqslant \delta_2.$$

Proof. Consider the homotopy $M : [0, 1] \times B_\delta \to B_\delta$ given by

$$M(t, u) = \Phi_\lambda(u) - t\phi_1, \quad u \in B_\delta,\ t \in [0, 1].$$

It follows by Lemma 3.5 that

$$M(t, u) \neq 0, \quad u \in \partial B_\delta,\ 0 \leqslant t \leqslant 1.$$

By property of homotopy invariance of the Leray–Schauder degree,

$$\deg\big(M(0, .), B_\delta, 0\big) = \deg\big(M(1, .), B_\delta, 0\big) = 0.$$

As a consequence,

$$\deg(\Phi_\lambda, B_\delta, 0) = 0.$$

This finishes the proof of the lemma. □

4. Proof of the main result

As a first step we establish and prove a result on existence of a *continuum* of positive solutions of Eq. (7).

Pick n big enough such that

$$\mu_1\big(a^0, b^0\big) - \frac{1}{n} > 0.$$

Consider the numbers

$$a_n = \mu_1\big(a^0, b^0\big) - \frac{1}{n}, \qquad b_n = \mu_1(a_0, b_0) + \frac{1}{n}.$$

The lemma below is based on Proposition 3.6 of Ruyun Ma and Jia Xu in [1].

Lemma 4.1. *Suppose* (**H1**). *Then there is an unbounded connected component* $\mathcal{C}_n$ *of positive solutions of Eq.* (7) *such that* $[a_n, b_n] \times \{0\} \subseteq \mathcal{C}_n$ *and*

$$\mathcal{C}_n \cap \big\{\big\{\mathbb{R}\backslash[a_n, b_n]\big\} \times \{0\}\big\} = \emptyset.$$

Proof. By Lemma 3.2, a_n and b_n are not bifurcation points of

$$\Phi_\lambda(u) = 0,$$

and $u = 0$ is an isolated solution of this equation for both $\lambda = a_n$ and $\lambda = b_n$.

Let $\widetilde{\delta} = \min(\delta_1, \delta_2)$. By Lemmas 3.4 and 3.6, we infer that

$$\deg(\Phi_{a_n}, B_{\widetilde{\delta}}, 0) = 1 \quad \text{and} \quad \deg(\Phi_{b_n}, B_{\widetilde{\delta}}, 0) = 0.$$

Set

$$\mathcal{S}_n = \overline{\big\{(\lambda, u) \in \mathbb{R} \times H \;\big|\; \Phi_\lambda(u) = 0,\ u \neq 0\big\}} \cup \big\{[a_n, b_n] \times \{0\}\big\}$$

and denote by $\mathcal{C}_n \subset \mathcal{S}_n$ the connected component which contains

$$[a_n, b_n] \times \{0\},$$

given by Theorem 2.2. Hence either

(i) $\mathcal{C}_n$ is unbounded in $\mathbb{R} \times H$, or
(ii) $\mathcal{C}_n \cap \{\{\mathbb{R} \setminus [a_n, b_n]\} \times \{0\}\} \neq \emptyset$.

We claim that (ii) does not hold.

Indeed, let

$$\Lambda \subset \mathbb{R} \setminus [a_n, b_n]$$

be a compact set. By Lemma 3.3,

$$\Phi_\lambda(u) \neq 0, \quad 0 < \|u\| \leqslant \widetilde{\delta},\ u \in H,\ \lambda \in \Lambda.$$

It follows that (ii) does not hold. Thus the continuum $\mathcal{C}_n$ is unbounded and in addition $\mathcal{C}_n \cap \{\{\mathbb{R} \setminus [a_n, b_n]\} \times \{0\}\} = \emptyset$. This ends the proof of the lemma. □

4.1. Proof of Theorem 1.1

It is enough is to show that the unbounded component of positive solutions $\mathcal{C} = \mathcal{C}_n$ given by Lemma 4.1 meets $\{1\} \times H$.

In order to do that, pick a sequence $\{(\sigma_k, u_k)\} \in \mathcal{C}$ such that

$$\sigma_k + \|u_k\| \to +\infty.$$

Using the fact that

$$\mathcal{C} \cap \big\{\big\{\mathbb{R} \backslash [a_n, b_n]\big\} \times \{0\}\big\} = \emptyset,$$

we infer that, there is a subsequence still denoted $\{(\sigma_k, u_k)\}$ such that $u_k \neq 0$.

We claim that $\sigma_k > 0$. Indeed, notice that each (σ_k, u_k) satisfies

$$\Delta^2 u_k = \sigma_k \widetilde{f}(x, u_k, \Delta u_k) \quad \text{in } \Omega,\ u_k \in H_0^1(\Omega) \cap H^4(\Omega). \tag{15}$$

If some $\sigma_k = 0$, then by (15), $u_k = 0$, impossible.

On the other hand, if some σ_k is negative then $\mathcal{C}$ would cross $\{0\} \times H$, impossible. In conclusion, each σ_k is positive.

Using the fact that $\widetilde{f} \geqslant 0$, it follows by the maximum principle that

$$u_k \geqslant 0 \quad \text{and} \quad \Delta u_k \leqslant 0.$$

We claim that the sequence $\{\sigma_k\}$ is bounded. Indeed, estimating using **(H3)**, we have

$$\Delta^2 u_k \geqslant \sigma_k (a_1 u_k - a_2 \Delta u_k) \quad \text{in } \Omega.$$

Multiplying this equation by ϕ_1 and integrating, we obtain

$$\begin{aligned} \lambda_1^2 \int_\Omega u_k \phi_1\, dx = \int_\Omega \Delta^2 u_k \phi_1\, dx \geqslant \sigma_k \int_\Omega (a_1 u_k - a_2 \Delta u_k)\phi_1\, dx \\ = \sigma_k \int_\Omega (a_1 \phi_1 - a_2 \Delta \phi_1) u_k\, dx \\ = \sigma_k (a_1 + a_2 \lambda_1) \int_\Omega u_k \phi_1\, dx. \end{aligned} \tag{16}$$

Since $\int_\Omega u_k \phi_1 > 0$, we obtain from (16) that

$$0 < \sigma_k \leqslant \mu_1(a_1, a_2) < \infty.$$

It follows that $\sigma_k \to \sigma$ and $\|u_k\| \to \infty$, up to a subsequence.

Let

$$v_k = \frac{u_k}{\|u_k\|}.$$

There is a function $v \in H$ such that

$$v_k \rightharpoonup v \quad \text{in } H, \qquad v_k \to v \quad \text{in } L^2(\Omega), \qquad v_k \to v \quad \text{a.e. in } \Omega.$$

Dividing the equation in (15) by $\|u_k\|$, we have

$$\begin{cases} \Delta^2 v_k = \sigma_k \dfrac{f(x, u_k, \Delta u_k)}{\|u_k\|} & \text{in } \Omega, \\ v_k \in H, \quad v_k \geqslant 0, \quad \Delta v_k \leqslant 0 & \text{on } \Omega. \end{cases}$$

Arguing as in the proof of Lemma 3.2 we infer that

$$v_k \xrightarrow{H} v, \quad \|v\| = 1.$$

At this point we recall that $\{(\sigma_k, u_k)\} \in \mathcal{C}$, $0 < \sigma_k \leqslant \mu_1(a_1, a_2) < \infty$ and $\|u_k\| \to \infty$. We distinguish between two cases:

Case 1. $\mu_1(c_\infty, d_\infty) < 1 < \mu_1(a^0, b^0)$.

Using **(H2)** we have

$$\Delta^2 v_k \geqslant \sigma_k(a_0 v_k - b_0 \Delta v_k) - \sigma_k \frac{\eta_1(x, u, \Delta u_k)}{\|u_k\|}.$$

Multiplying by ϕ_1 and integrating, we obtain

$$\begin{aligned} \lambda_1^2 \int\limits_\Omega v_k \phi_1\, dx = \int\limits_\Omega \Delta^2 v_k \phi_1\, dx &\geqslant \sigma_k \int\limits_\Omega (c_\infty v_k - d_\infty \Delta v_k)\phi_1\, dx - \sigma_k o_k^{\eta_1}(1) \\ &= \sigma_k \int\limits_\Omega (c_\infty \phi_1 - d_\infty \Delta \phi_1) v_k\, dx - \sigma_k o_k^{\eta_1}(1) \\ &= \sigma_k(c_\infty + d_\infty \lambda_1) \int\limits_\Omega \phi_1 v_k\, dx - \sigma_k o_k^{\eta_1}(1). \end{aligned}$$

Taking limits and applying Lemma 3.1 we get

$$\left(\frac{\lambda_1^2}{c_\infty + d_\infty \lambda_1} - \sigma\right) \int\limits_\Omega v\phi_1\, dx \geqslant 0.$$

As a consequence,

$$\sigma \leqslant \mu_1(c_\infty, d_\infty) < 1 < \mu_1\big(a^0, b^0\big),$$

and so

$$\big(\sigma, \mu_1\big(a^0, b^0\big)\big) \subseteq \mathrm{Proj}_{\mathbb{R}^+}\mathcal{C}.$$

Thus $\mathcal{C}$ meets $\{1\} \times H$.

Case 2. $\mu_1(a_0, b_0) < 1 < \mu_1(c^\infty, d^\infty)$.

It follows using **(H2)**, that

$$\begin{aligned} \lambda_1^2 \int\limits_\Omega v_k \phi_1\, dx = \int\limits_\Omega \Delta^2 v_k \phi_1\, dx &\leqslant \sigma_k \int\limits_\Omega \big(c^\infty v_k - d^\infty \Delta v_k\big)\phi_1\, dx + \sigma_k o_k^{\eta_2}(1) \\ &= \sigma_k \int\limits_\Omega \big(c^\infty \phi_1 - d^\infty \Delta \phi_1\big) v_k\, dx + \sigma_k o_k^{\eta_2}(1) \\ &= \sigma_k\big(c^\infty + d^\infty \lambda_1\big) \int\limits_\Omega \phi_1 v_k\, dx + \sigma_k o_k^{\eta_2}(1). \end{aligned}$$

Passing to the limit, applying Lemma 3.1 we get to

$$\sigma \geqslant \mu_1\big(c^\infty, d^\infty\big) > 1 > \mu_1(a_0, b_0).$$

As a consequence,

$$\big(\mu_1(a_0, b_0), \sigma\big) \subseteq \mathrm{Proj}_{\mathbb{R}^+}\mathcal{C},$$

showing that $\mathcal{C}$ meets $\{1\} \times H$.

This ends the proof of Theorem 1.1. □

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