

On the value distribution of some difference polynomials[☆]Xiu-Min Zheng^{a,*}, Zong-Xuan Chen^b^a Institute of Mathematics and Information Science, Jiangxi Normal University, Nanchang, 330022, China^b School of Mathematical Sciences, South China Normal University, Guangzhou, 510631, China

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ABSTRACT

In this paper, we continue to investigate the value distribution of some difference polynomials, which can be considered as difference counterparts of some classic results of Hayman.

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1. Introduction and results

Throughout this paper, we use standard notations in the Nevanlinna theory (see e.g. [12,16,20]). Let $f(z)$ be a meromorphic function. Here and in the following the word “meromorphic” means meromorphic in the whole complex plane. We use notations $\sigma(f)$ and $\lambda(f)$ for the order and the exponent of convergence of zeros of $f(z)$ respectively. Moreover, we denote by $S(r, f)$ any real function of growth $o(T(r, f))$ as $r \rightarrow \infty$ outside of a possible exceptional set of finite logarithmic measure.

Many mathematicians have been interested in the value distribution of different expressions of a meromorphic function and obtained many fruitful results. In [13], Hayman discussed Picard’s values of a meromorphic function and its derivatives. In particular, he proved the following result.

Theorem A. *Let $f(z)$ be a transcendental entire function. Then*

- (a) *for $n \geq 3$ and $a \neq 0$, $\Psi(z) = f'(z) - a(f(z))^n$ assumes all finite values infinitely often.*
- (b) *for $n \geq 2$, $\Phi(z) = f'(z)(f(z))^n$ assumes all finite values except possibly zero infinitely often.*

Recently, a number of papers (including [1–11,14,15,17,18,21,22]) have focused on complex difference equations and difference analogues of Nevanlinna theory. Bergweiler–Langley [2] first investigated the existence of zeros of $\Delta f(z)$ and $\frac{\Delta f(z)}{f(z)}$ and obtained many profound and significant results. These results may be viewed as discrete analogues of the relative existing theorem on the zeros of f' . Later on, many further results were obtained (see e.g. [3–5,17,18]). Also, Halburd–Korhonen [11] posed that the study of zeros distribution of complex difference operator plays an important role in the further study of complex differences and difference equations.

In particular, Laine–Yang [17] proved the following **Theorem B**, which can be considered as a difference counterpart of **Theorem A(b)**.

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* Corresponding author.

E-mail addresses: zhengxiumin2008@sina.com (X.-M. Zheng), chzx@vip.sina.com (Z.-X. Chen).

Theorem B. Let $f(z)$ be a transcendental entire function of finite order, and c be a non-zero complex constant. Then for $n \geq 2$, $\Phi_1(z) = f(z+c)(f(z))^n$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often.

Later on, Chen et al. complemented the case $n = 1$ of Theorem B and obtained the following Theorem C in [3].

Theorem C. Let $f(z)$ be a transcendental entire function of finite order, and let $c \in \mathbb{C} \setminus \{0\}$ be a complex constant. If $f(z)$ has infinitely many multi-order zeros, then $\Phi_2(z) = f(z)f(z+c)$ assumes every value $a \in \mathbb{C}$ infinitely often.

In this paper, we continue to find a difference counterpart of Theorem A(a) and generalize it to more general cases to some extent.

Theorem 1. Let f be a transcendental entire function of finite order, and a, c be non-zero complex constants. Then for any integer $n \geq 3$,

$$\Psi_1(z) = f(z+c) - a(f(z))^n$$

assumes all finite values $b \in \mathbb{C}$ infinitely often.

By using the same reasoning method as Theorem 1, we immediately generalize it to the next Theorem 2.

Theorem 2. Let f be a transcendental entire function of finite order, $a \in \mathbb{C} \setminus \{0\}$, $m, n \in \mathbb{N}^+$ and $c_1, c_2, \dots, c_m$ be complex constants satisfying that at least one of them is non-zero. If $m < \frac{n-1}{2-\frac{1}{n}}$, then

$$\Psi_2(z) = f(z+c_1)f(z+c_2) \cdots f(z+c_m) - a(f(z))^n$$

assumes all finite values $b \in \mathbb{C}$ infinitely often.

By using a different reasoning from Theorems 1 and 2, we obtain more general results to some extent under additional assumptions.

Theorem 3. Let f be a transcendental entire function of finite order, $a \in \mathbb{C} \setminus \{0\}$, $m, n \in \mathbb{N}^+$ and $c_1, c_2, \dots, c_m$ be complex constants such that at least one of them is non-zero. If $N\left(r, \frac{1}{f}\right) = S(r, f)$, $n \neq m$, then

- (a) for $|n-m| = 1$, $\Psi_2(z)$ has infinitely many zeros;
- (b) for $\min\{n, m\} = d \geq 2$, $\Psi_2(z)$ assumes every non-zero value $b \in \mathbb{C}$ infinitely often.

Theorem 4. Let f be a transcendental entire function of finite order $\sigma(f)$ with a Borel exceptional value s , $a \in \mathbb{C} \setminus \{0\}$, $c_1, c_2, \dots, c_m$ be complex constants satisfying that at least one of them is non-zero. Then for $1 \leq m < n$ and every $b(\neq s^m - as^n) \in \mathbb{C}$, $\Psi_2(z)$ assumes the value b infinitely often and $\lambda(\Psi_2 - b) = \sigma(f)$.

Discussion. We are settled with the case $n = m$ for Theorem 3, the case $n \leq m$ for Theorem 4 and the case of meromorphic functions for Theorems 1–4 as open questions. For example, in [4], Chen–Shon investigated the zeros distribution of $f(z+c_1)f(z+c_2) - (f(z))^2$, where f is a transcendental meromorphic function of order less than 1, without the assumption that $N\left(r, \frac{1}{f}\right) = S(r, f)$ especially.

2. Lemmas for proofs of theorems

Lemma 1 ([7]). Let f be a meromorphic function with order $\sigma = \sigma(f) < \infty$ and let η be a fixed non-zero complex number, then for each $\varepsilon > 0$, we have

$$T(r, f(z+\eta)) = T(r, f) + O(r^{\sigma-1+\varepsilon}) + O(\log r).$$

Remark 1. We immediately have by Lemma 1 that $\sigma(f(z+\eta)) = \sigma = \sigma(f)$, whenever f is of finite order.

Lemma 2 ([10]). Let $T : (0, +\infty) \rightarrow (0, +\infty)$ be a non-decreasing continuous function, $s > 0$, $\alpha < 1$, and let $F \subset \mathbb{R}^+$ be the set of all r such that $T(r) \leq \alpha T(r+s)$. If the logarithmic measure of F is infinite, that is $\int_F \frac{dr}{r} = \infty$, then $\lim_{r \rightarrow \infty} \frac{\log T(r)}{\log r} = \infty$.

Remark 2. We immediately have by Lemma 2 that

$$T(r+s, f) = (1+o(1))T(r, f) \quad \text{and} \quad N(r+s, f) = (1+o(1))N(r, f)$$

hold for $s > 0$ and all r outside of a set with finite logarithmic measure, whenever f is of finite order.

Lemma 3 ([8]). Let $f(z)$ be a non-constant finite order meromorphic solution of

$$f^n P(z, f) = Q(z, f),$$

where $P(z, f)$ and $Q(z, f)$ are difference polynomials in $f(z)$, and let $\delta < 1$. If the degree of $Q(z, f)$ as a polynomial in $f(z)$ and its shifts is at most n , then

$$m(r, P(z, f)) = o\left(\frac{T(r + |c|, f)}{r^\delta}\right) + o(T(r, f))$$

for all r outside of a possible exceptional set with finite logarithmic measure.

By Lemma 3 and Remark 2, we immediately have the following result, which can be viewed as a difference analogue of the Clunie Lemma for the finite order meromorphic function.

Remark 3. If f is a non-constant finite order meromorphic solution of $f^n P(z, f) = Q(z, f)$, where $P(z, f)$ and $Q(z, f)$ are the same as in Lemma 3, then we have $m(r, P(z, f)) = S(r, f)$.

Lemma 4 ([3]). Let $f(z)$ be a meromorphic function of finite order ρ such that, for any given ε ($0 < \varepsilon < 1$), $N\left(r, \frac{1}{f}\right) + N(r, f) = O(r^{\rho-1+\varepsilon}) + S(r, f)$. Suppose that the difference polynomial $G(z) = \sum_{\lambda \in J} b_\lambda(z) \prod_{j=1}^{\tau_\lambda} f(z + \delta_{\lambda,j})^{\mu_{\lambda,j}}$ in $f(z)$ and its shifts with small meromorphic coefficients is of maximal total degree n . If $G(z)$ also satisfies

$$\sum_{\lambda \in J_{n-1}} b_\lambda(z) \prod_{j=1}^{\tau_\lambda} f(z + \delta_{\lambda,j})^{\mu_{\lambda,j}} \not\equiv 0, \quad (2.1)$$

where $J_{n-1} = \{\lambda \in J \mid \sum_{j=1}^{\tau_\lambda} \mu_{\lambda,j} = n - 1\}$. Then $G(z)$ has sufficiently many zeros to satisfy $N\left(r, \frac{1}{G}\right) \neq O(r^{\rho-1+\varepsilon}) + S(r, f)$.

Lemma 5 ([8]). Let $f(z)$ be a non-constant meromorphic function, $c \in \mathbb{C}$, $\delta < 1$ and $\varepsilon > 0$. Then

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r + |c|, f)^{1+\varepsilon}}{r^\delta}\right)$$

for all r outside of a possible exceptional set E with finite logarithmic measure $\int_E \frac{dr}{r} < \infty$.

By Lemma 5 and Remark 2, we immediately have the following result, which can be viewed as a difference analogue of the logarithmic Derivative Lemma for the finite order meromorphic function. Particularly, it deserves mention that some similar results have been established by Chiang–Feng [6,7] independently.

Remark 4. If f is a non-constant finite order meromorphic function, then we have $m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r, f)$.

Lemma 6 ([16] (Valiron-Mohon'ko)). Let $f(z)$ be a meromorphic function, then for all irreducible rational functions in f ,

$$R(z, f(z)) = \frac{\sum_{i=0}^m a_i(z) f(z)^i}{\sum_{j=0}^n b_j(z) f(z)^j},$$

with meromorphic coefficients $a_i(z)$, $b_j(z)$, the characteristic function of $R(z, f(z))$ satisfies

$$T(r, R(z, f(z))) = dT(r, f) + O(\Psi(r)),$$

where $d = \max\{m, n\}$ and $\Psi(r) = \max_{i,j}\{T(r, a_i), T(r, b_j)\}$.

Lemma 7 ([19]). Let $f_j(z)$ ($j = 1, \dots, n$) ($n \geq 2$) be meromorphic functions, $g_j(z)$ ($j = 1, \dots, n$) be entire functions, and satisfy

- (i) $\sum_{j=1}^n f_j(z) e^{g_j(z)} \equiv 0$;
- (ii) when $1 \leq j < k \leq n$, $g_j(z) - g_k(z)$ is not a constant;
- (iii) when $1 \leq j \leq n$, $1 \leq h < k \leq n$,

$$T(r, f_j) = o\left(T(r, e^{g_h - g_k})\right), \quad (r \rightarrow \infty, r \notin E),$$

where $E \subset (1, \infty)$ is of finite linear measure or finite logarithmic measure.

Then $f_j(z) \equiv 0$ ($j = 1, \dots, n$).

3. Proof of Theorem 1

Case (i): We assume first that $0 < \sigma(f) < \infty$. Suppose that the assertion does not hold in this case. Then there exists $b \in \mathbb{C}$ such that $\Psi_1(z) - b$ has finitely many zeros only. Set

$$F(z) = \frac{f(z+c) - b}{a(f(z))^n}. \quad (3.1)$$

Then $F(z)$ has only finite 1-points, that is

$$\bar{N}\left(r, \frac{1}{F-1}\right) = O(\log r). \quad (3.2)$$

We see by (3.1) that the poles of $F(z)$ occur only at zeros of $f(z)$, and those poles which are not simultaneously zeros of $f(z+c) - b$ have multiplicities at least n . Moreover, the zeros of $F(z)$ can only occur at zeros of $f(z+c) - b$ which are not poles of $F(z)$. Thus, we have by Lemma 1 that for each $\varepsilon > 0$,

$$\begin{aligned} \bar{N}(r, F) + \bar{N}\left(r, \frac{1}{F}\right) &\leq \frac{1}{n}N(r, F) + \bar{N}\left(r, \frac{1}{f(z+c) - b}\right) \\ &\leq \frac{1}{n}T(r, F) + T(r, f(z+c)) + O(1) \\ &\leq \frac{1}{n}T(r, F) + T(r, f) + O(r^{\sigma(f)-1+\varepsilon}) + O(\log r). \end{aligned} \quad (3.3)$$

By (3.2)–(3.3) and the second fundamental theorem, we have that

$$\begin{aligned} T(r, F) &\leq \bar{N}(r, F) + \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}\left(r, \frac{1}{F-1}\right) + S(r, F) \\ &\leq \frac{1}{n}T(r, F) + T(r, f) + O(r^{\sigma(f)-1+\varepsilon}) + S(r, F) + O(\log r), \end{aligned}$$

that is

$$\left(1 - \frac{1}{n} + o(1)\right)T(r, F) \leq T(r, f) + O(r^{\sigma(f)-1+\varepsilon}) + O(\log r). \quad (3.4)$$

On the other hand, we have that for each $\varepsilon > 0$,

$$\begin{aligned} nT(r, f) &= T(r, f^n) = T\left(r, \frac{f(z+c) - b}{aF}\right) \leq T(r, f(z+c)) + T(r, F) + O(1) \\ &\leq T(r, f) + O(r^{\sigma(f)-1+\varepsilon}) + T(r, F) + O(\log r), \end{aligned}$$

that is

$$(n-1)T(r, f) \leq O(r^{\sigma(f)-1+\varepsilon}) + T(r, F) + O(\log r). \quad (3.5)$$

Noting the assumption that f is transcendental, we immediately have by (3.5) that

$$0 < \sigma(f) \leq \sigma(F). \quad (3.6)$$

Moreover, we have by (3.4) and (3.5) that

$$\begin{aligned} \left(1 - \frac{1}{n} + o(1)\right)T(r, F) &\leq (1 + o(1))T(r, f) + O(r^{\sigma(f)-1+\varepsilon}) \\ &\leq \left(\frac{1}{n-1} + o(1)\right)T(r, F) + O(r^{\sigma(f)-1+\varepsilon}). \end{aligned} \quad (3.7)$$

Since $n \geq 3$, we have $1 - \frac{1}{n} > \frac{1}{n-1}$. Thus we have by (3.6) and (3.7) that

$$\left(1 - \frac{1}{n} - \frac{1}{n-1} + o(1)\right)T(r, F) \leq O(r^{\sigma(f)-1+\varepsilon}) \leq O(r^{\sigma(F)-1+\varepsilon}),$$

from which we deduce a contradiction that $\sigma(F) \leq \max\{0, \sigma(f) - 1\} < \sigma(F)$. Therefore, for any $b \in \mathbb{C}$, $\Psi_1(z) - b$ has infinitely many zeros.

Case (ii): We assume that $\sigma(f) = 0$. Clearly, $\psi_1(z)$ is also of order zero by Remark 1. Furthermore, we assert that $\psi_1(z)$ is transcendental. To this end, we assume on the contrary that $\psi_1(z)$ is a polynomial. On the one hand, we have by Lemma 1 that

$$T(r, f(z+c)) = T(r, f) + O(r^{\sigma(f)-1+\varepsilon}) + O(\log r) = T(r, f) + O(\log r). \quad (3.8)$$

On the other hand, we have

$$T(r, \psi_1) = O(\log r)$$

according to the assumption that $\psi_1(z)$ is a polynomial, consequently by the definition of $\psi_1(z)$ and Lemma 6, we have that

$$T(r, f(z+c)) = T(r, \psi_1 + af^n) = nT(r, f) + O(\log r). \quad (3.9)$$

Hence, we have by (3.8) and (3.9) that $T(r, f) = O(\log r)$, which contradicts with the assumption that f is transcendental.

Thus, we see that for any $b \in \mathbb{C}$, $\psi_1(z) - b$ is a transcendental entire function with order zero, consequently has infinitely many zeros.

4. Proof of Theorem 3

(a) By the assumption that $|n-m| = 1$, we see that (2.1) holds clearly and then $\psi_2(z)$ must have infinitely many zeros. Otherwise, we have

$$N\left(r, \frac{1}{\psi_2(z)}\right) = O(\log r) = S(r, f),$$

which contradicts with Lemma 4.

(b) Suppose that the assertion does not hold. Then there is a non-zero value $b \in \mathbb{C}$ such that

$$\psi_2(z) - b = f(z+c_1)f(z+c_2)\cdots f(z+c_m) - a(f(z))^n - b = p(z)e^{q(z)}, \quad (4.1)$$

where $p(z)$, $q(z)$ are polynomials, not vanishing identically.

Differentiating (4.1) and eliminating $e^{q(z)}$, we obtain

$$f^d P(z, f) = Q(z, f), \quad (4.2)$$

where $P(z, f)$, $Q(z, f)$ are two difference polynomials in f and its shifts as follows,

$$\begin{aligned} P(z, f) &= \sum_{j=1}^m p(z) \frac{f'(z+c_j)}{f(z+c_j)} \prod_{v=1}^d \frac{f(z+c_v)}{f(z)} \prod_{v=d+1}^m f(z+c_v) \\ &\quad - anp(z) \frac{f'(z)}{f(z)} (f(z))^{n-d} - p^*(z) \prod_{j=1}^m \frac{f(z+c_j)}{f(z)} (f(z))^{m-d} + ap^*(z) (f(z))^{n-d}, \end{aligned} \quad (4.3)$$

$$Q(z, f) = -bp^*(z), \quad (4.4)$$

and

$$p^*(z) = p'(z) + p(z)q'(z). \quad (4.5)$$

Clearly, $p^*(z) \not\equiv 0$. Indeed, if $p^*(z) \equiv 0$, then we know by (4.5) that $p(z)$, $q(z)$ are constants. Thus, by (4.1), there exists $b^* \in \mathbb{C}$ such that

$$f(z+c_1)f(z+c_2)\cdots f(z+c_m) - a(f(z))^n = b^*,$$

consequently,

$$\frac{f(z+c_1)}{f(z)} \frac{f(z+c_2)}{f(z)} \cdots \frac{f(z+c_m)}{f(z)} = \frac{a(f(z))^n + b^*}{(f(z))^m}. \quad (4.6)$$

On the one hand, by Remark 4 and the assumption that $N\left(r, \frac{1}{f}\right) = S(r, f)$, we have that for $j = 1, \dots, m$,

$$m\left(r, \frac{f(z+c_j)}{f(z)}\right) = S(r, f), \quad N\left(r, \frac{f(z+c_j)}{f(z)}\right) \leq N\left(r, \frac{1}{f}\right) = S(r, f), \quad (4.7)$$

consequently,

$$T\left(r, \frac{f(z+c_1)}{f(z)} \frac{f(z+c_2)}{f(z)} \cdots \frac{f(z+c_m)}{f(z)}\right) = S(r, f). \quad (4.8)$$

On the other hand, by Lemma 6 and the assumptions that $n \neq m$, $\min\{n, m\} = d \geq 2$, we have that

$$T\left(r, \frac{a(f(z))^n + b^*}{(f(z))^m}\right) \geq T(r, f) + O(1). \quad (4.9)$$

We can immediately deduce a contradiction by (4.6) and (4.8)–(4.9). Thus, we have $p^*(z) \neq 0$, consequently $P(z, f) \neq 0$ since $b \neq 0$.

By Remarks 1, 2 and 4, we have that

$$\begin{aligned} m\left(r, \frac{f'(z + c_j)}{f(z + c_j)}\right) &= O(\log r) = S(r, f), \\ N\left(r, \frac{f'(z + c_j)}{f(z + c_j)}\right) &= \bar{N}\left(r, \frac{1}{f(z + c_j)}\right) \leq \bar{N}\left(r + |c_j|, \frac{1}{f}\right) = (1 + o(1))\bar{N}\left(r, \frac{1}{f}\right) = S(r, f), \\ m\left(r, \frac{f'(z)}{f(z)}\right) &= O(\log r) = S(r, f), \quad N\left(r, \frac{f'(z)}{f(z)}\right) = \bar{N}\left(r, \frac{1}{f}\right) = S(r, f), \end{aligned}$$

and (4.7) hold for $j = 1, \dots, m$. Consequently, we have that for $j = 1, \dots, m$,

$$T\left(r, \frac{f'(z + c_j)}{f(z + c_j)}\right) = S(r, f), \quad T\left(r, \frac{f(z + c_j)}{f(z)}\right) = S(r, f), \quad T\left(r, \frac{f'(z)}{f(z)}\right) = S(r, f),$$

which means by (4.3)–(4.4) that all the coefficients of $P(z, f)$ and $Q(z, f)$ are of growth $S(r, f)$. Thus, by (4.2), Lemma 3 and Remark 3, we have that

$$m(r, P(z, f)) = S(r, f). \quad (4.10)$$

Similarly, since $d \geq 2$, we have that

$$m(r, fP(z, f)) = S(r, f). \quad (4.11)$$

Moreover, we have by (4.2)–(4.4) that

$$N(r, fP(z, f)) = N(r, P(z, f)) = N\left(r, \frac{Q(z, f)}{f^d}\right) \leq dN\left(r, \frac{1}{f}\right) = S(r, f). \quad (4.12)$$

Thus, we have by (4.10)–(4.12) that

$$T(r, f) \leq T(r, fP(z, f)) + T\left(r, \frac{1}{P(z, f)}\right) = S(r, f),$$

a contradiction.

5. Proof of Theorem 4

We prove first that $\sigma(\Psi_2) = \sigma(f)$. On the one hand, we clearly have by Remark 1 that

$$\sigma(\Psi_2) \leq \sigma(f). \quad (5.1)$$

On the other hand, we have by Lemma 1 that

$$\begin{aligned} T(r, f) &= \frac{1}{n}T(r, af^n) + O(1) = \frac{1}{n}T(r, f(z + c_1) \cdots f(z + c_m) - \Psi_2(z)) + O(1) \\ &\leq \frac{1}{n} \sum_{j=1}^m T(r, f(z + c_j)) + \frac{1}{n}T(r, \Psi_2(z)) + O(1) \\ &= \frac{m}{n}T(r, f) + \frac{1}{n}T(r, \Psi_2(z)) + O(r^{\sigma(f)-1+\varepsilon}) + O(\log r). \end{aligned} \quad (5.2)$$

Since $1 \leq m < n$, we deduce by (5.2) that

$$\sigma(f) \leq \sigma(\Psi_2). \quad (5.3)$$

Thus, we have by (5.1) and (5.3) that

$$\sigma(\Psi_2) = \sigma(f). \quad (5.4)$$

Next, we suppose to the contrary that $\lambda(\Psi_2 - b) < \sigma(f)$ and deduce a contradiction.

Since s is the Borel exceptional value of $f(z)$, we can rewrite $f(z)$ as the form

$$f(z) = s + P(z)e^{\alpha z^k}, \quad (5.5)$$

where $P(z)$ is an entire function with $\sigma(P) < \sigma(f) = k$, α is a non-zero complex constant, and k is a positive integer. Thus, we have that

$$f(z + c_j) = s + P(z + c_j)P_j(z)e^{\alpha z^k}, \quad j = 1, \dots, m, \quad (5.6)$$

where

$$P_j(z) = e^{\alpha k c_j z^{k-1} + \dots + \alpha c_j^k}, \quad \sigma(P_j) \leq k - 1, \quad j = 1, \dots, m.$$

By (5.4) and the assumption that $\lambda(\Psi_2 - b) < \sigma(f)$, we have that $\lambda(\Psi_2 - b) < \sigma(f) = \sigma(\Psi_2) = \sigma(\Psi_2 - b) = k$. Consequently, we can rewrite $\Psi_2(z) - b$ as the form

$$\Psi_2(z) - b = q(z)e^{\beta z^k}, \quad (5.7)$$

where β is a non-zero complex constant, $q(z)$ is an entire function satisfying $\sigma(q) = \lambda(\Psi_2 - b) < k$. By (5.5)–(5.7), we have that

$$\begin{aligned} s^m + s^{m-1} \left(\sum_{j=1}^m P(z + c_j)P_j(z) \right) e^{\alpha z^k} + s^{m-2} \left(\sum_{1 \leq i < j \leq m} P(z + c_i)P(z + c_j)P_i(z)P_j(z) \right) e^{2\alpha z^k} \\ + \dots + \left(\prod_{j=1}^m P(z + c_j)P_j(z) \right) e^{m\alpha z^k} - a \left(s^n + ns^{n-1}P(z)e^{\alpha z^k} + \dots + (P(z))^n e^{n\alpha z^k} \right) \\ - b - q(z)e^{\beta z^k} = 0. \end{aligned} \quad (5.8)$$

Since $a(P(z))^n \neq 0$, $q(z) \neq 0$ and $1 \leq m < n$, we have $\beta = n\alpha$. Moreover, since we have

$$\max\{\sigma(P), \sigma(P(z + c_j)), \sigma(P_j), \sigma(q)\} < k = \sigma(e^{i\alpha z^k})$$

for $i = 1, \dots, n$, $j = 1, \dots, m$, we may apply Lemma 7 to (5.8). Thus, we have $b = s^m - as^n$, which contradicts with our assumption that $b \neq s^m - as^n$. Hence, we have that $\lambda(\Psi_2 - b) = \sigma(f)$.

6. Examples for theorems

Example 1. The assumption that f is of finite order cannot be omitted in Theorems 1–4. We may investigate functions $f = e^{e^z}$ and $g = e^{\sin^2 z}$, which are clearly of infinite order. Nevertheless, the constant 0 is a Picard exceptional value of $\Psi_1(z) = f(z + \log 3) + (f(z))^3 = 2e^{3e^z}$, where $c = \log 3$, $n = 3$ and $a = -1$, showing that Theorems 1 and 2 fail in this infinite order case. Moreover, the constant e is a Picard exceptional value of $\Psi_2(z) = g(z)g(z + \frac{\pi}{2}) + (g(z))^4 = e^{4\sin^2 z} + e$, where $c_1 = 0$, $c_2 = \frac{\pi}{2}$, $m = 2 < 4 = n$ and $a = -1$, showing that Theorems 3(b) and 4 fail.

Example 2. The results of Theorems 1 and 2 are best possible. In fact, if $n = 1$ and $f(z) = e^z + b$, then $\Psi_1(z) = f(z + c) - af(z) = (e^c - a)e^z + b(1 - a) \neq b(1 - a)$, provided that $a \neq e^c$. If $n = 2$ and $g(z) = e^z + \frac{e^c}{2a}$, then $\Psi_1(z) = g(z + c) - a(g(z))^2 = -ae^{2z} + \frac{e^c(2 - e^c)}{4a} \neq \frac{e^c(2 - e^c)}{4a}$.

Example 3. For the entire function $f(z) = e^z$, and for $c = \log 2$, $a = 1$, $n = 3$, we have $\Psi_1(z) = f(z + c) - a(f(z))^3 = 2e^z - e^{3z} = -e^z(e^{2z} - 2)$. Then $\Psi_1(z)$ can assume all finite values $b \in \mathbb{C}$ infinitely often by Nevanlinna's second fundamental theorem and Lemma 6. This example shows that Theorems 1–2 and 4 may hold.

Example 4. For the same function as above and for $c_1 = -c_2 = c$, $a = 1$, $m = 2$, $n = 3$, we have $\Psi_2(z) = f(z + c_1)f(z + c_2) - a(f(z))^3 = e^{2z} - e^{3z} = e^{2z}(1 - e^z)$. Then $\Psi_2(z)$ can also assume all finite values $b \in \mathbb{C}$ infinitely often, showing that Theorems 3(a), (b) and 4 may hold.

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