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New transformation and reduction formulae for double q-Clausen series[☆]

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ABSTRACT

The Sears transformations for non-terminating ${}_3\phi_2$-series are applied to derive several general series transformations for another two double q-Clausen series of type $\Phi^{1:2;\lambda}_{2:0;\mu}$ and $\Phi^{0:3;\lambda}_{2:0;\mu}$. Further new transformations and reduction formulae are established for the basic Clausen hypergeometric series. In particular, several bibasic reduction formulae are obtained.

1. Introduction and motivation

For the two indeterminates x and q, the shifted factorial of x with base q reads as

$$(x;q)_0 = 1 \quad\text{and}\quad (x;q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x) \quad \text{for } n \in \mathbb{N}.$$

When $|q| < 1$, there are two well-defined infinite product expressions:

$$(x;q)_\infty = \prod_{k=0}^{\infty}(1-q^k x) \quad\text{and}\quad (x;q)_n = (x;q)_\infty \,/\, \big(q^n x;q\big)_\infty .$$

The product and the fraction of shifted factorials are abbreviated respectively to

$$[\,\alpha,\ \beta,\ \ldots,\ \gamma;q\,]_n = (\alpha;q)_n\,(\beta;q)_n\cdots(\gamma;q)_n\,,$$
$$\left[\begin{matrix}\alpha, & \beta, & \ldots, & \gamma\\ A, & B, & \ldots, & C\end{matrix}\middle|\, q\right]_n = \frac{(\alpha;q)_n\,(\beta;q)_n\cdots(\gamma;q)_n}{(A;q)_n\,(B;q)_n\cdots(C;q)_n}.$$

Following Bailey [1] and Gasper–Rahman [6], the basic hypergeometric series is defined by

$${}_{1+r}\phi_s\left[\begin{matrix}a_0, & a_1, & \ldots, & a_r\\ & b_1, & \ldots, & b_s\end{matrix}\middle|\, q;z\right] = \sum_{n=0}^{\infty}\left\{(-1)^n q^{\binom{n}{2}}\right\}^{s-r}\left[\begin{matrix}a_0, & a_1, & \ldots, & a_r\\ q, & b_1, & \ldots, & b_s\end{matrix}\middle|\, q\right]_n z^n,$$

where the base q will be restricted to $|q| < 1$ for non-terminating q-series.

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As the q-analogue of the Kampé de Fériet function, Srivastava and Karlsson [11, P. 349] define the generalized bivariate basic hypergeometric function by

$$\Phi^{\lambda:r;s}_{\mu:u;v}\left[\begin{matrix}\alpha_1,\ldots,\alpha_\lambda: & a_1,\ldots,a_r; & c_1,\ldots,c_s; & q:x,y\\ \beta_1,\ldots,\beta_\mu: & b_1,\ldots,b_u; & d_1,\ldots,d_v; & i,j,k\end{matrix}\right]$$
$$=\sum_{m,n=0}^{\infty}\frac{[\alpha_1,\ldots,\alpha_\lambda;q]_{m+n}}{[\beta_1,\ldots,\beta_\mu;q]_{m+n}}\frac{[a_1,\ldots,a_r;q]_m[c_1,\ldots,c_s;q]_n}{[b_1,\ldots,b_u;q]_m[d_1,\ldots,d_v;q]_n}\frac{x^my^nq^{i\binom{m}{2}+j\binom{n}{2}+kmn}}{(q;q)_m(q;q)_n}.$$

It is not hard to check that, when $i,j,k\in\mathbb{N}_0$, this double series $\Phi^{\lambda:r;s}_{\mu:u;v}$ is convergent for $|x|<1$, $|y|<1$ and $|q|<1$.

Two of the most important and useful transformations in the theory of basic hypergeometric series are the q-analogue of Kummer–Thomae–Whipple and the Hall transformations on ${}_3\phi_2$-series (see [6, III-9 and III-10])

$$ {}_3\phi_2\left[\begin{matrix}a, & c, & e\\ & b, & d\end{matrix}\middle| q;\frac{bd}{ace}\right]=\left[\begin{matrix}d/a,\ bd/ce\\ d,\ bd/ace\end{matrix}\middle| q\right]_\infty {}_3\phi_2\left[\begin{matrix}a, & b/c, & b/e\\ & b, & bd/ce\end{matrix}\middle| q;\frac{d}{a}\right]; \tag{1}$$

$$ {}_3\phi_2\left[\begin{matrix}a, & c, & e\\ & b, & d\end{matrix}\middle| q;\frac{bd}{ace}\right]=\left[\begin{matrix}c,\ bd/ac,\ bd/ce\\ b,\ d,\ bd/ace\end{matrix}\middle| q\right]_\infty {}_3\phi_2\left[\begin{matrix}b/c,\ d/c,\ bd/ace\\ bd/ac,\ bd/ce\end{matrix}\middle| q;c\right]. \tag{2}$$

They have been used to investigate systematically summation and reduction formulae for the $\Phi^{0:3;\lambda}_{1:1;\mu}$, $\Phi^{1:2;\lambda}_{1:1;\mu}$ and $\Phi^{2:1;\lambda}_{2:0;\mu}$ series by Jia et al. [2,7,9]. By considering the limit case $n\to\infty$ of Watson's q-Whipple transformation (see [6, III-18]), which results in another ${}_3\phi_2$-series transformation, the first author and Chu [5] recently derived several new transformations and well-poised reduction formulae for the q-Clausen series $\Phi^{0:3;\lambda}_{1:1;\mu}$, $\Phi^{2:1;\lambda}_{2:0;\mu}$, $\Phi^{1:2;\lambda}_{2:0;\mu}$, $\Phi^{1:2;\lambda}_{1:1;\mu}$ and $\Phi^{0:3;\lambda}_{2:0;\mu}$. Jeugt [12] established the invariant transformation group for the double Clausen series with $\lambda+r=3$ and $\mu+u=2$. Further works can be found in Chu–Jia [2,3], Chu–Srivastava [4] and Jia–Wang [8], as well as Singh [10].

To present a full coverage of double q-Clausen series, the objective of this paper is to investigate further the $\Phi^{1:2;\lambda}_{2:0;\mu}$ and $\Phi^{0:3;\lambda}_{2:0;\mu}$ series by Sears transformations. Several new transformations and reduction formulae are established for the basic Clausen hypergeometric series. In particular, four bibasic reduction formulae are obtained. We strongly believe that the new identities obtained in this paper will be useful in the theory of (multivariable) orthogonal polynomials.

2. Non-terminating double series $\Phi^{1:2;\lambda}_{2:0;\mu}$

By means of the two transformations for ${}_3\phi_2$-series (1)–(2), in this section we shall establish four general transformations for q-Clausen series $\Phi^{1:2;\lambda}_{2:0;\mu}$, which lead us to two reduction formulae and two symmetric transformations for the basic Clausen hypergeometric series.

Theorem 1. *For an arbitrary sequence $\{\Omega(j)\}$, there holds the transformation*

$$\sum_{i,j=0}^{\infty}\left(\frac{q^jbd}{ace}\right)^i\left[\begin{matrix}a\\ b,d\end{matrix}\middle| q\right]_{i+j}\left[\begin{matrix}c,e\\ q\end{matrix}\middle| q\right]_i\Omega(j) \tag{3a}$$

$$=\left[\begin{matrix}d/a,\ bd/ce\\ d,\ bd/ace\end{matrix}\middle| q\right]_\infty\sum_{n=0}^{\infty}\left[\begin{matrix}a,\ b/c,\ b/e\\ q,\ b,\ bd/ce\end{matrix}\middle| q\right]_n\left(\frac{d}{a}\right)^n \tag{3b}$$

$$\times\sum_{j=0}^{n}\Omega(j)\left[\begin{matrix}q^{-n},\ bd/ace\\ b/c,\ b/e,\ q^nbd/ce\end{matrix}\middle| q\right]_j(-1)^jq^{-\binom{j}{2}}\left(\frac{q^na}{d}\right)^j, \tag{3c}$$

provided that both double series displayed above are absolutely convergent.

Proof. Rewrite the double sum in (3a) as

$$\sum_{i,j=0}^{\infty}\left(\frac{q^jbd}{ace}\right)^i\left[\begin{matrix}a\\ b,d\end{matrix}\middle| q\right]_{i+j}\left[\begin{matrix}c,e\\ q\end{matrix}\middle| q\right]_i\Omega(j) \tag{4a}$$

$$=\sum_{j=0}^{\infty}\Omega(j)\left[\begin{matrix}a\\ b,d\end{matrix}\middle| q\right]_j {}_3\phi_2\left[\begin{matrix}q^ja, & c, & e\\ & q^jb, & q^jd\end{matrix}\middle| q;\frac{q^jbd}{ace}\right]. \tag{4b}$$

Applying the q-analogue of the Kummer–Thomae–Whipple transformation (1) to the inner ${}_3\phi_2$-sum, we have

$$ {}_3\phi_2\left[\begin{matrix} q^ja,\ c,\ e \\ q^jb,\ q^jd \end{matrix}\middle|\, q;\ \frac{q^jbd}{ace}\right] = \left[\begin{matrix} d/a,\ q^{2j}bd/ce \\ q^jd,\ q^jbd/ace \end{matrix}\middle|\, q\right]_\infty {}_3\phi_2\left[\begin{matrix} q^ja, & q^jb/c,\ q^jb/e \\ & q^jb,\ q^{2j}bd/ce \end{matrix}\middle|\, q;\ \frac{d}{a}\right] $$

$$ = \left[\begin{matrix} d/a, & bd/ce \\ d, & bd/ace \end{matrix}\middle|\, q\right]_\infty \left[\begin{matrix} b, & d, & bd/ace \\ a, & b/c, & b/e \end{matrix}\middle|\, q\right]_j \sum_{i=0}^{\infty} \frac{[a, b/c, b/e; q]_{i+j}}{(q;q)_i(b;q)_{i+j}(bd/ce;q)_{i+2j}}\left(\frac{d}{a}\right)^i. $$

Substituting the above relation into (4a)–(4b), we find the following expression:

$$ \left[\begin{matrix} d, & bd/ace \\ d/a, & bd/ce \end{matrix}\middle|\, q\right]_\infty \sum_{i,j=0}^{\infty}\left(\frac{q^jbd}{ace}\right)^i \left[\begin{matrix} a \\ b,\ d \end{matrix}\middle|\, q\right]_{i+j} \left[\begin{matrix} c,\ e \\ q \end{matrix}\middle|\, q\right]_i \Omega(j) $$

$$ = \sum_{i,j=0}^{\infty} \Omega(j) \left[\begin{matrix} bd/ace \\ b/c,\ b/e \end{matrix}\middle|\, q\right]_j \frac{[a, b/c, b/e; q]_{i+j}}{(q;q)_i(b;q)_{i+j}(bd/ce;q)_{i+2j}}\left(\frac{d}{a}\right)^i. $$

Letting $n := i + j$ on the right-hand side of the last equation and then keeping in mind the following relation,

$$ \frac{1}{(q;q)_{n-j}(bd/ce;q)_{n+j}} = \frac{(-1)^j q^{nj-\binom{j}{2}}}{[q, bd/ce; q]_n}\frac{(q^{-n};q)_j}{(q^nbd/ce;q)_j}, \tag{5} $$

we get the general transformation displayed in Theorem 1. $\square$

Alternatively, permuting the parameters of the ${}_3\phi_2$-series displayed in (4b) gives

$$ {}_3\phi_2\left[\begin{matrix} q^ja, & c, & e \\ & q^jb, & q^jd \end{matrix}\middle|\, q;\ \frac{q^jbd}{ace}\right] = {}_3\phi_2\left[\begin{matrix} e, & c, & q^ja \\ & q^jb, & q^jd \end{matrix}\middle|\, q;\ \frac{q^jbd}{ace}\right]. $$

Applying formula (1) to the ${}_3\phi_2$-series on the right-hand side yields

$$ {}_3\phi_2\left[\begin{matrix} e,\ c,\ q^ja \\ q^jb,\ q^jd \end{matrix}\middle|\, q;\ \frac{q^jbd}{ace}\right] = \left[\begin{matrix} q^jd/e,\ q^jbd/ac \\ q^jd,\ q^jbd/ace \end{matrix}\middle|\, q\right]_\infty {}_3\phi_2\left[\begin{matrix} e, & q^jb/c,\ b/a \\ & q^jb,\ q^jbd/ac \end{matrix}\middle|\, q;\ \frac{q^jd}{e}\right] $$

$$ = \left[\begin{matrix} d/e, & bd/ac \\ d, & bd/ace \end{matrix}\middle|\, q\right]_\infty \left[\begin{matrix} b,\ d,\ bd/ace \\ b/c,\ d/e \end{matrix}\middle|\, q\right]_j \sum_{i=0}^{\infty} \left[\begin{matrix} b/c \\ b,\ bd/ac \end{matrix}\middle|\, q\right]_{i+j} \left[\begin{matrix} e,\ b/a \\ q \end{matrix}\middle|\, q\right]_i \left(\frac{q^jd}{e}\right)^i. $$

Substituting them successively into (4a)–(4b), we obtain the following general transformation between two non-terminating double series $\Phi^{1:2;\lambda}_{2:0;\mu}$ and $\Phi^{1:2;\lambda+2}_{2:0;\mu+2}$.

Theorem 2. *For an arbitrary sequence $\{\Omega(j)\}$, there holds the transformation*

$$ \left[\begin{matrix} d,\ bd/ace \\ d/e,\ bd/ac \end{matrix}\middle|\, q\right]_\infty \sum_{i,j=0}^{\infty}\left(\frac{q^jbd}{ace}\right)^i \left[\begin{matrix} a \\ b,\ d \end{matrix}\middle|\, q\right]_{i+j} \left[\begin{matrix} c,\ e \\ q \end{matrix}\middle|\, q\right]_i \Omega(j) \tag{6a} $$

$$ = \sum_{i,j=0}^{\infty} \Omega(j) \left[\begin{matrix} a,\ bd/ace \\ b/c,\ d/e \end{matrix}\middle|\, q\right]_j \left[\begin{matrix} b/c \\ b,\ bd/ac \end{matrix}\middle|\, q\right]_{i+j} \left[\begin{matrix} e,\ b/a \\ q \end{matrix}\middle|\, q\right]_i \left(\frac{q^jd}{e}\right)^i, \tag{6b} $$

provided that both double series displayed above are absolutely convergent.

Instead, applying the Hall transformation (2), we can reformulate the inner ${}_3\phi_2$-series displayed in (4b) as

$$ {}_3\phi_2\left[\begin{matrix} q^ja,\ c,\ e \\ q^jb,\ q^jd \end{matrix}\middle|\, q;\ \frac{q^jbd}{ace}\right] = \left[\begin{matrix} c,\ q^jbd/ac,\ q^{2j}bd/ce \\ q^jb,\ q^jd,\ q^jbd/ace \end{matrix}\middle|\, q\right]_\infty {}_3\phi_2\left[\begin{matrix} q^jb/c,\ q^jd/c,\ q^jbd/ace \\ q^jbd/ac,\ q^{2j}bd/ce \end{matrix}\middle|\, q;\ c\right] $$

$$ = \left[\begin{matrix} c, & bd/ac, & bd/ce \\ b, & d, & bd/ace \end{matrix}\middle|\, q\right]_\infty \left[\begin{matrix} b, & d \\ b/c, & d/c \end{matrix}\middle|\, q\right]_j \sum_{i=0}^{\infty} \frac{[b/c, d/c, bd/ace; q]_{i+j}}{(q;q)_i(bd/ac;q)_{i+j}(bd/ce;q)_{i+2j}}c^i. $$

Substituting this expression into (4a)–(4b), we get the following transformation:

$$ \left[\begin{matrix} b, & d, & bd/ace \\ c, & bd/ac, & bd/ce \end{matrix}\middle|\, q\right]_\infty \sum_{i,j=0}^{\infty}\left(\frac{q^jbd}{ace}\right)^i \left[\begin{matrix} a \\ b,\ d \end{matrix}\middle|\, q\right]_{i+j} \left[\begin{matrix} c,\ e \\ q \end{matrix}\middle|\, q\right]_i \Omega(j) $$

$$ = \sum_{i,j=0}^{\infty} \Omega(j) \left[\begin{matrix} a \\ b/c,\ d/c \end{matrix}\middle|\, q\right]_j \frac{[b/c, d/c, bd/ace; q]_{i+j}}{(q;q)_i(bd/ac;q)_{i+j}(bd/ce;q)_{i+2j}}c^i. $$

Replacing further the summation indices by $n := i + j$ and then applying relation (5) to the above transformation leads us to the following theorem.

Theorem 3. *For an arbitrary sequence $\{\Omega(j)\}$, there holds the transformation*

$$\sum_{i,j=0}^{\infty}\left(\frac{q^jbd}{ace}\right)^i\left[\begin{matrix}a\\ b,d\end{matrix}\middle|\,q\right]_{i+j}\left[\begin{matrix}c,e\\ q\end{matrix}\middle|\,q\right]_i\Omega(j) \tag{7a}$$

$$=\left[\begin{matrix}c,bd/ac,bd/ce\\ b,\ d,\ bd/ace\end{matrix}\middle|\,q\right]_{\infty}\sum_{n=0}^{\infty}\left[\begin{matrix}b/c, & d/c, & bd/ace\\ q, & bd/ac, & bd/ce\end{matrix}\middle|\,q\right]_n c^n \tag{7b}$$

$$\times\sum_{j=0}^{n}\Omega(j)\left[\begin{matrix}q^{-n}, & a\\ b/c,d/c,q^nbd/ce\end{matrix}\middle|\,q\right]_j(-1)^jq^{-\binom{j}{2}}\left(\frac{q^n}{c}\right)^j, \tag{7c}$$

provided that both double series displayed above are absolutely convergent.

Similarly, exchange the parameters of the ${}_3\phi_2$-series displayed in (4b) as

$${}_3\phi_2\left[\begin{matrix}q^ja, & c, & e\\ & q^jb, & q^jd\end{matrix}\middle|\,q;\frac{q^jbd}{ace}\right]={}_3\phi_2\left[\begin{matrix}c, & q^ja, & e\\ & q^jb, & q^jd\end{matrix}\middle|\,q;\frac{q^jbd}{ace}\right].$$

According to formula (2), the ${}_3\phi_2$-series on the right-hand side can be reformulated as

$${}_3\phi_2\left[\begin{matrix}c,q^ja,e\\ q^jb,q^jd\end{matrix}\middle|\,q;\frac{q^jbd}{ace}\right]=\left[\begin{matrix}q^ja,q^jbd/ac,q^jbd/ae\\ q^jb,\ q^jd,\ q^jbd/ace\end{matrix}\middle|\,q\right]_{\infty}{}_3\phi_2\left[\begin{matrix}b/a,d/a,q^jbd/ace\\ q^jbd/ac,q^jbd/ae\end{matrix}\middle|\,q;q^ja\right]$$

$$=\left[\begin{matrix}a, & bd/ac, & bd/ae\\ b, & d, & bd/ace\end{matrix}\middle|\,q\right]_{\infty}\left[\begin{matrix}b,d\\ a\end{matrix}\middle|\,q\right]_j\sum_{i=0}^{\infty}\left[\begin{matrix}bd/ace\\ bd/ac,bd/ae\end{matrix}\middle|\,q\right]_{i+j}\left[\begin{matrix}b/a,d/a\\ q\end{matrix}\middle|\,q\right]_i(q^ja)^i.$$

Substituting them successively into (4a)–(4b), we obtain another general transformation on both non-terminating double series $\Phi_{2:0;\mu}^{1:2;\lambda}$.

Theorem 4. *For an arbitrary sequence $\{\Omega(j)\}$, there holds the transformation*

$$\left[\begin{matrix}b,\ d,\ bd/ace\\ a,bd/ac,bd/ae\end{matrix}\middle|\,q\right]_{\infty}\sum_{i,j=0}^{\infty}\left(\frac{q^jbd}{ace}\right)^i\left[\begin{matrix}a\\ b,d\end{matrix}\middle|\,q\right]_{i+j}\left[\begin{matrix}c,e\\ q\end{matrix}\middle|\,q\right]_i\Omega(j) \tag{8a}$$

$$=\sum_{i,j=0}^{\infty}\Omega(j)\left[\begin{matrix}bd/ace\\ bd/ac,bd/ae\end{matrix}\middle|\,q\right]_{i+j}\left[\begin{matrix}b/a,d/a\\ q\end{matrix}\middle|\,q\right]_i(q^ja)^i, \tag{8b}$$

provided that both double series displayed above are absolutely convergent.

According to Theorems 1–4, by concretely specifying $\Omega(j)$ in different manners, two reduction formulae and two symmetric transformations will be established.

2.1

For the $\Omega(j)$ sequence given by

$$\Omega(j)=(-1)^jq^{\binom{j}{2}}\frac{1-q^{2j-1}bd/ce}{1-q^{j-1}bd/ce}\left[\begin{matrix}\alpha,\ \beta,\ b/c,\ b/e,\ bd/ce\\ q,bd/ace,bd/\alpha ce,bd/\beta ce\end{matrix}\middle|\,q\right]_j\left(\frac{bd^2}{\alpha\beta ace}\right)^j,$$

evaluating the inner sum with respect to j in (3c) by means of the terminating q-Dougall–Dixon formula (see [6, II-21])

$${}_6\phi_5\left[\begin{matrix}a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & q^{-n}\\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & q^{1+n}a\end{matrix}\middle|\,q;\frac{q^{1+n}a}{bc}\right]=\left[\begin{matrix}qa,\ qa/bc\\ qa/b,qa/c\end{matrix}\middle|\,q\right]_n, \tag{9}$$

and then simplifying the corresponding equation displayed in Theorem 1, we obtain the following reduction formula.

Proposition 5 (*Reduction Formula*).

$$\sum_{i,j=0}^{\infty}\frac{1-q^{2j-1}bd/ce}{1-q^{j-1}bd/ce}\left[\begin{matrix}a\\ b,d\end{matrix}\middle|\,q\right]_{i+j}\left[\begin{matrix}c,e\\ q\end{matrix}\middle|\,q\right]_{i}\left(\frac{q^jbd}{ace}\right)^{i}(-1)^jq^{\binom{j}{2}}\left[\begin{matrix}\alpha,\ \beta,\ b/c,\ b/e,\ bd/ce\\ q,\ bd/ace,\ bd/\alpha ce,\ bd/\beta ce\end{matrix}\middle|\,q\right]_{j}\left(\frac{bd^2}{\alpha\beta ace}\right)^{j}$$
$$=\left[\begin{matrix}d/a,\ bd/ce\\ d,\ bd/ace\end{matrix}\middle|\,q\right]_{\infty}{}_4\phi_3\left[\begin{matrix}a,\quad b/c,\ b/e,\ bd/\alpha\beta ce\\ b,\ bd/\alpha ce,\ bd/\beta ce\end{matrix}\middle|\,q;\frac{d}{a}\right].$$

In particular, the limiting case $e\to\infty$ of this proposition results in the following transformation, which is equivalent to the Jackson transformation (cf. [6, III-4]).

Corollary 6 ($e\to\infty$ *in* Proposition 5).

$${}_2\phi_2\left[\begin{matrix}a,\quad c\\ b,\quad d\end{matrix}\middle|\,q;\frac{bd}{ac}\right]=\left[\begin{matrix}d/a\\ d\end{matrix}\middle|\,q\right]_{\infty}{}_2\phi_1\left[\begin{matrix}a,\quad b/c\\ b\end{matrix}\middle|\,q;\frac{d}{a}\right].$$

If taking $e=1$ in this proposition, we shall get another transformation formula, which can be obtained by letting $n\to\infty$ in Watson's q-Whipple transformation (cf. [6, III-18], see also Chu–Zhang [5]).

Corollary 7 ($e=1$ *in* Proposition 5).

$$\sum_{j=0}^{\infty}(-1)^j\frac{1-q^{2j-1}bd/c}{1-q^{j-1}bd/c}q^{\binom{j}{2}}\left[\begin{matrix}bd/c,\quad b/c,\quad a,\quad \alpha,\quad \beta\\ q,\quad d,\quad bd/ac,\quad bd/\alpha c,\quad bd/\beta c\end{matrix}\middle|\,q\right]_{j}\left(\frac{bd^2}{\alpha\beta ac}\right)^{j}$$
$$=\left[\begin{matrix}d/a,\ bd/c\\ d,\ bd/ac\end{matrix}\middle|\,q\right]_{\infty}{}_3\phi_2\left[\begin{matrix}a,\quad b/c,\quad bd/\alpha\beta c\\ bd/\alpha c,\quad bd/\beta c\end{matrix}\middle|\,q;\frac{d}{a}\right].$$

Furthermore, the special case $\alpha\beta=bd/ce$ reduces to the following summation formula.

Corollary 8 ($\alpha\beta=bd/ce$ *in* Proposition 5).

$$\sum_{i,j=0}^{\infty}\frac{1-q^{2j-1}bd/ce}{1-q^{j-1}bd/ce}\left[\begin{matrix}a\\ b,d\end{matrix}\middle|\,q\right]_{i+j}\left[\begin{matrix}c,e\\ q\end{matrix}\middle|\,q\right]_{i}\left(\frac{q^jbd}{ace}\right)^{i}(-1)^jq^{\binom{j}{2}}\left[\begin{matrix}b/c,b/e,bd/ce\\ q,\quad bd/ace\end{matrix}\middle|\,q\right]_{j}\left(\frac{d}{a}\right)^{j}=\left[\begin{matrix}d/a,\ bd/ce\\ d,\ bd/ace\end{matrix}\middle|\,q\right]_{\infty}.$$

2.2

Specializing the $\Omega(j)$ sequence in Theorem 2 by

$$\Omega(j)=\left[\begin{matrix}d/e\\ a\end{matrix}\middle|\,q\right]_{j}\frac{1}{(q;q)_j},$$

we obtain a transformation between two double q-Clausen series $\Phi^{1:2;1}_{2:0;1}$. After making the parameter replacements $a\to ab,\ b\to abcd,\ c\to bd,\ d\to be$, the following symmetric transformation can be established.

Proposition 9 (*Symmetric Transformation*).

$$[b,ce;q]_\infty\sum_{i,j=0}^{\infty}\left[\begin{matrix}ac\\ abcd,ce\end{matrix}\middle|\,q\right]_{i+j}\left[\begin{matrix}e,cd\\ q\end{matrix}\middle|\,q\right]_{i}\left[\begin{matrix}c\\ ac\end{matrix}\middle|\,q\right]_{j}\frac{q^{ij}b^i}{(q;q)_j}$$
$$=[c,be;q]_\infty\sum_{i,j=0}^{\infty}\left[\begin{matrix}ab\\ abcd,be\end{matrix}\middle|\,q\right]_{i+j}\left[\begin{matrix}e,bd\\ q\end{matrix}\middle|\,q\right]_{i}\left[\begin{matrix}b\\ ab\end{matrix}\middle|\,q\right]_{j}\frac{q^{ij}c^i}{(q;q)_j}.$$

2.3

In Theorem 3, specify the $\Omega(j)$ sequence by

$$\Omega(j)=(-1)^jq^{\binom{j}{2}}\frac{1-q^{2j-1}bd/ce}{1-q^{j-1}bd/ce}\left[\begin{matrix}\alpha,\beta,b/c,d/c,bd/ce\\ q,a,bd/\alpha ce,bd/\beta ce\end{matrix}\middle|\,q\right]_{j}\left(\frac{bd}{\alpha\beta e}\right)^{j}.$$

According to the q-Dougall–Dixon formula (9), evaluating the inner sum with respect to j displayed in (7c) and simplifying the corresponding equation gives rise to the following reduction formula.

Proposition 10 (*Reduction Formula*).

$$\sum_{i,j=0}^{\infty}\frac{1-q^{2j-1}bd/ce}{1-q^{j-1}bd/ce}\left[\begin{matrix}a\\b,d\end{matrix}\middle|q\right]_{i+j}\left[\begin{matrix}c,e\\q\end{matrix}\middle|q\right]_i\left(\frac{q^jbd}{ace}\right)^i(-1)^jq^{\binom{j}{2}}\left[\begin{matrix}\alpha,\beta,b/c,d/c,bd/ce\\q,a,bd/\alpha ce,bd/\beta ce\end{matrix}\middle|q\right]_j\left(\frac{bd}{\alpha\beta e}\right)^j$$
$$=\left[\begin{matrix}c,bd/ac,bd/ce\\b,\ d,\ bd/ace\end{matrix}\middle|q\right]_\infty{}_4\phi_3\left[\begin{matrix}b/c,\quad d/c,bd/ace,bd/\alpha\beta ce\\bd/ac,bd/\alpha ce,bd/\beta ce\end{matrix}\middle|q;c\right].$$

The limiting case $e\to\infty$ of this proposition yields another transformation between the ${}_2\phi_2$ and ${}_2\phi_1$ series.

Corollary 11 ($e\to\infty$ *in* Proposition 10).

$${}_2\phi_2\left[\begin{matrix}a,&c\\b,&d\end{matrix}\middle|q;\frac{bd}{ac}\right]=\left[\begin{matrix}c,&bd/ac\\b,&d\end{matrix}\middle|q\right]_\infty{}_2\phi_1\left[\begin{matrix}b/c,&d/c\\&bd/ac\end{matrix}\middle|q;c\right].$$

Combining with Corollary 6, we can derive from the last corollary the following formula on both ${}_2\phi_1$-series, which is equivalent to the Heine transformation (cf. [6, III-2]).

$${}_2\phi_1\left[\begin{matrix}a,&b/c\\&b\end{matrix}\middle|q;\frac{d}{a}\right]=\left[\begin{matrix}c,&bd/ac\\b,&d/a\end{matrix}\middle|q\right]_\infty{}_2\phi_1\left[\begin{matrix}b/c,&d/c\\&bd/ac\end{matrix}\middle|q;c\right].$$

In particular, the special case $\alpha\beta=bd/ce$ of this proposition reduces to the following summation formula.

Corollary 12 ($\alpha\beta=bd/ce$ *in* Proposition 10).

$$\sum_{i,j=0}^{\infty}\frac{1-q^{2j-1}bd/ce}{1-q^{j-1}bd/ce}\left[\begin{matrix}a\\b,d\end{matrix}\middle|q\right]_{i+j}\left[\begin{matrix}c,e\\q\end{matrix}\middle|q\right]_i\left(\frac{q^jbd}{ace}\right)^i(-1)^jq^{\binom{j}{2}}\left[\begin{matrix}b/c,d/c,bd/ce\\q,\quad a\end{matrix}\middle|q\right]_jc^j$$
$$=\left[\begin{matrix}c,bd/ac,bd/ce\\b,\ d,\ bd/ace\end{matrix}\middle|q\right]_\infty.$$

2.4

Specialize the $\Omega(j)$ sequence in Theorem 4 by

$$\Omega(j)=\frac{1}{(q;q)_j}.$$

Under the parameter replacements $a\to ac$, $b\to abcx$, $c\to ax$, $d\to acdx$, $e\to cx$, we obtain another symmetric transformation on both double q-Clausen series $\Phi^{1:2;0}_{2:0;0}$.

Proposition 13 (*Symmetric Transformation*).

$$[bd,abcx,acdx;q]_\infty\sum_{i,j=0}^{\infty}\left[\begin{matrix}ac\\abcx,acdx\end{matrix}\middle|q\right]_{i+j}\left[\begin{matrix}ax,cx\\q\end{matrix}\middle|q\right]_i\frac{q^{ij}(bd)^i}{(q;q)_j}$$
$$=[ac,abdx,bcdx;q]_\infty\sum_{i,j=0}^{\infty}\left[\begin{matrix}bd\\abdx,bcdx\end{matrix}\middle|q\right]_{i+j}\left[\begin{matrix}bx,dx\\q\end{matrix}\middle|q\right]_i\frac{q^{ij}(ac)^i}{(q;q)_j}.$$

3. Non-terminating double series $\Phi^{0:3;\lambda}_{2:0;\mu}$

The Sears transformations (1)–(2) will be utilized further to establish the general transformations for the double q-Clausen series $\Phi^{0:3;\lambda}_{2:0;\mu}$ in this section. By specializing the parameters and the $\Omega(j)$ sequence, four bibasic reduction formulae will be obtained.

Theorem 14. *For an arbitrary sequence $\{\Omega(j)\}$, there holds the transformation*

$$\begin{bmatrix} d, & bd/ace \\ d/a, & bd/ce \end{bmatrix} q \Bigg]_\infty \sum_{i,j=0}^{\infty} \frac{\Omega(j)}{[b,d;q]_{i+j}} \begin{bmatrix} a,c,e \\ q \end{bmatrix} q \Bigg]_i \left(\frac{q^{2j}bd}{ace}\right)^i \tag{10a}$$

$$= \sum_{i,j=0}^{\infty} \frac{\Omega(j)}{(bd/ce;q)_{i+2j}} \frac{(bd/ace;q)_{2j}}{[b/c,b/e,d/a;q]_j} \begin{bmatrix} a \\ q \end{bmatrix} q \Bigg]_i \begin{bmatrix} b/c,b/e \\ b \end{bmatrix} q \Bigg]_{i+j} \left(\frac{q^j d}{a}\right)^i, \tag{10b}$$

provided that both double series displayed above are absolutely convergent.

Proof. Rewrite the double sum in (10a) as

$$\sum_{i,j=0}^{\infty} \frac{\Omega(j)}{[b,d;q]_{i+j}} \begin{bmatrix} a,c,e \\ q \end{bmatrix} q \Bigg]_i \left(\frac{q^{2j}bd}{ace}\right)^i = \sum_{j=0}^{\infty} \frac{\Omega(j)}{[b,d;q]_j} {}_3\phi_2 \left[\begin{matrix} a, & c, & e \\ & q^jb, & q^jd \end{matrix} \Bigg| q; \frac{q^{2j}bd}{ace}\right]. \tag{11}$$

According to (1), we can reformulate the above ${}_3\phi_2$-series as

$$\begin{aligned} {}_3\phi_2 \left[\begin{matrix} a, & c, & e \\ & q^jb, & q^jd \end{matrix} \Bigg| q; \frac{q^{2j}bd}{ace}\right] &= \begin{bmatrix} q^jd/a, q^{2j}bd/ce \\ q^jd, q^{2j}bd/ace \end{bmatrix} q \Bigg]_\infty {}_3\phi_2 \left[\begin{matrix} a, & q^jb/c, q^jb/e \\ & q^jb, q^{2j}bd/ce \end{matrix} \Bigg| q; \frac{q^jd}{a}\right] \\ &= \begin{bmatrix} d/a, & bd/ce \\ d, & bd/ace \end{bmatrix} q \Bigg]_\infty \begin{bmatrix} b, & d \\ b/c, b/e, d/a \end{bmatrix} q \Bigg]_j (bd/ace;q)_{2j} \sum_{i=0}^{\infty} \begin{bmatrix} a \\ q \end{bmatrix} q \Bigg]_i \begin{bmatrix} b/c,b/e \\ b \end{bmatrix} q \Bigg]_{i+j} \frac{1}{(bd/ce;q)_{i+2j}} \left(\frac{q^jd}{a}\right)^i. \end{aligned}$$

Substituting the last expression into (11) and then simplifying the resulting equation, we get the transformation displayed in Theorem 14. $\square$

Instead, applying the Hall transformation (2) to the inner ${}_3\phi_2$-series displayed in (11) yields

$$\begin{aligned} {}_3\phi_2 \left[\begin{matrix} a, c, e \\ q^jb, q^jd \end{matrix} \Bigg| q; \frac{q^{2j}bd}{ace}\right] &= \begin{bmatrix} c, q^{2j}bd/ac, q^{2j}bd/ce \\ q^jb, q^jd, q^{2j}bd/ace \end{bmatrix} q \Bigg]_\infty {}_3\phi_2 \left[\begin{matrix} q^jb/c, q^jd/c, q^{2j}bd/ace \\ q^{2j}bd/ac, q^{2j}bd/ce \end{matrix} \Bigg| q; c\right] \\ &= \begin{bmatrix} c, & bd/ac, & bd/ce \\ b, & d, & bd/ace \end{bmatrix} q \Bigg]_\infty \begin{bmatrix} b, & d \\ b/c, & d/c \end{bmatrix} q \Bigg]_j \sum_{i=0}^{\infty} \frac{[b/c,d/c;q]_{i+j}}{(q;q)_i} \begin{bmatrix} bd/ace \\ bd/ac, bd/ce \end{bmatrix} q \Bigg]_{i+2j} c^i. \end{aligned}$$

Substituting this expression into (11) and then reordering the factors, we obtain another transformation formula.

Theorem 15. *For an arbitrary sequence $\{\Omega(j)\}$, there holds the transformation*

$$\begin{bmatrix} b, d, bd/ace \\ c, bd/ac, bd/ce \end{bmatrix} q \Bigg]_\infty \sum_{i,j=0}^{\infty} \frac{\Omega(j)}{[b,d;q]_{i+j}} \begin{bmatrix} a,c,e \\ q \end{bmatrix} q \Bigg]_i \left(\frac{q^{2j}bd}{ace}\right)^i \tag{12a}$$

$$= \sum_{i,j=0}^{\infty} \Omega(j) \frac{[b/c,d/c;q]_{i+j}}{(q;q)_i[b/c,d/c;q]_j} \begin{bmatrix} bd/ace \\ bd/ac, bd/ce \end{bmatrix} q \Bigg]_{i+2j} c^i, \tag{12b}$$

provided that both double series displayed above are absolutely convergent.

Specifying the parameters and the $\Omega(j)$ sequence such that the double sum (10a) and (12a) can be expressed as single series, we can prove the following four bibasic reduction formulae.

3.1

Letting $d = -b = \alpha$ and replacing e by $-e$, we can rewrite the double sum in (10a) as

$$\sum_{i=0}^{\infty} \begin{bmatrix} a, & c, & -e \\ q, & \alpha, & -\alpha \end{bmatrix} q \Bigg]_i \left(\frac{\alpha^2}{ace}\right)^i \sum_{j=0}^{\infty} \frac{\Omega(j)}{(q^{2i}\alpha^2;q^2)_j} q^{2ij}. \tag{13}$$

Specifying further the $\Omega(j)$ sequence in Theorem 14 by

$$\Omega(j) = \begin{bmatrix} \beta, \gamma \\ q^2 \end{bmatrix} q^2 \Bigg]_j \left(\frac{\alpha^2}{\beta\gamma}\right)^j$$

and then evaluating the inner sum with respect to j displayed in (13) by means of the q-Gauss sum (see [6, II-8])

$$ {}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}\middle| q; \frac{c}{ab}\right] = \left[\begin{matrix} c/a, c/b \\ c, \ c/ab \end{matrix}\middle| q\right]_\infty, \tag{14} $$

we get the following bibasic reduction formula.

Proposition 16 (*Reduction Formula*)**.**

$$ \sum_{i,j=0}^{\infty} \frac{q^{ij}(\alpha^2/ace;q)_{2j}}{[\alpha/a,-\alpha/c,\alpha/e;q]_j} \left[\begin{matrix} a \\ q \end{matrix}\middle| q\right]_i \left[\begin{matrix} \beta,\gamma \\ q^2 \end{matrix}\middle| q^2\right]_j \left[\begin{matrix} -\alpha/c,\alpha/e \\ -\alpha \end{matrix}\middle| q\right]_{i+j} \frac{(\alpha/a)^i(\alpha^2/\beta\gamma)^j}{(\alpha^2/ce;q)_{i+2j}} $$
$$ = \left[\begin{matrix} \alpha, \ \alpha^2/ace \\ \alpha/a, \alpha^2/ce \end{matrix}\middle| q\right]_\infty \left[\begin{matrix} \alpha^2/\beta, \alpha^2/\gamma \\ \alpha^2, \ \alpha^2/\beta\gamma \end{matrix}\middle| q^2\right]_\infty \sum_{i=0}^{\infty} \frac{[a,c,-e;q]_i(\alpha^2/\beta\gamma;q^2)_i}{(q;q)_i[\alpha^2/\beta,\alpha^2/\gamma;q^2]_i} \left(\frac{\alpha^2}{ace}\right)^i. $$

In particular, taking $c = 1$ in this proposition reduces to the following summation formula.

Corollary 17 ($c = 1$ *in* Proposition 16)**.**

$$ \sum_{i,j=0}^{\infty} \frac{q^{ij}(\alpha^2/ae;q)_{2j}}{[\alpha/a,-\alpha,\alpha/e;q]_j} \left[\begin{matrix} a \\ q \end{matrix}\middle| q\right]_i \left[\begin{matrix} \beta,\gamma \\ q^2 \end{matrix}\middle| q^2\right]_j \frac{(\alpha/e;q)_{i+j}}{(\alpha^2/e;q)_{i+2j}} \left(\frac{\alpha}{a}\right)^i \left(\frac{\alpha^2}{\beta\gamma}\right)^j $$
$$ = \left[\begin{matrix} \alpha, \ \alpha^2/ae \\ \alpha/a, \alpha^2/e \end{matrix}\middle| q\right]_\infty \left[\begin{matrix} \alpha^2/\beta, \alpha^2/\gamma \\ \alpha^2, \ \alpha^2/\beta\gamma \end{matrix}\middle| q^2\right]_\infty. $$

Instead, specifying the $\Omega(j)$ sequence in Theorem 14 by

$$ \Omega(j) = \left[\begin{matrix} \beta, & q^2/\beta \\ q^2, & -q^2 \end{matrix}\middle| q^2\right]_j (q^{j-1}\alpha^2)^j $$

and then evaluating the inner sum with respect to j displayed in (13) by means of the q-analogue of Bailey's ${}_2F_1(\frac{1}{2})$ sum (see [6, II-10])

$$ {}_2\phi_2\left[\begin{matrix} a, & q/a \\ -q, & b \end{matrix}\middle| q; -b\right] = \frac{[ab, qb/a; q^2]_\infty}{(b;q)_\infty}, \tag{15} $$

we obtain another reduction formula.

Proposition 18 (*Reduction Formula*)**.**

$$ \sum_{i,j=0}^{\infty} \frac{q^{ij}(\alpha^2/ace;q)_{2j}}{[\alpha/a,-\alpha/c,\alpha/e;q]_j} \frac{[\beta,q^2/\beta;q^2]_j}{(\alpha^2/ce;q)_{i+2j}} \left[\begin{matrix} a \\ q \end{matrix}\middle| q\right]_i \left[\begin{matrix} -\alpha/c,\alpha/e \\ -\alpha \end{matrix}\middle| q\right]_{i+j} \frac{q^{j(j-1)}\alpha^{2j}}{(q^4;q^4)_j} \left(\frac{\alpha}{a}\right)^i $$
$$ = \left[\begin{matrix} \alpha^2/\beta \\ \alpha^2 \end{matrix}\middle| q^2\right]_\infty \left[\begin{matrix} \alpha, \ \alpha^2/ace \\ \alpha/a, \alpha^2/ce \end{matrix}\middle| q\right]_\infty \sum_{i=0}^{\infty} \frac{[a,c,-e;q]_i}{(q;q)_i(\alpha^2/\beta;q^2)_i} \left[\begin{matrix} q^{2i}\alpha^2\beta \\ q^{2i}\alpha^2/\beta \end{matrix}\middle| q^4\right]_\infty \left(\frac{\alpha^2}{ace}\right)^i. $$

The special case $c = 1$ of this proposition leads us to a summation formula.

Corollary 19 ($c = 1$ *in* Proposition 18)**.**

$$ \sum_{i,j=0}^{\infty} \frac{q^{ij}(\alpha^2/ae;q)_{2j}}{[\alpha/a,-\alpha,\alpha/e;q]_j} \frac{[\beta,q^2/\beta;q^2]_j}{(\alpha^2/e;q)_{i+2j}} \left[\begin{matrix} a \\ q \end{matrix}\middle| q\right]_i \frac{(\alpha/e;q)_{i+j}}{(q^4;q^4)_j} \left(\frac{\alpha}{a}\right)^i q^{j(j-1)}\alpha^{2j} = \left[\begin{matrix} \alpha^2/\beta \\ \alpha^2 \end{matrix}\middle| q^2\right]_\infty \left[\begin{matrix} \alpha, \ \alpha^2/ae \\ \alpha/a, \alpha^2/e \end{matrix}\middle| q\right]_\infty. $$

3.2

Making the parameter replacements $b = -d = \sqrt{\alpha}$ and replacing c by $-c$, the double sum in (12a) can be reformulated as

$$ \sum_{i=0}^{\infty} \left[\begin{matrix} a, & -c, & e \\ q, & \sqrt{\alpha}, & -\sqrt{\alpha} \end{matrix}\middle| q\right]_i \left(\frac{\alpha}{ace}\right)^i \sum_{j=0}^{\infty} \frac{\Omega(j)}{(q^{2i}\alpha;q^2)_j} q^{2ij}. \tag{16} $$

Specializing further the $\Omega(j)$ sequence in Theorem 15 by

$$\Omega(j) = \left[\begin{matrix} \beta, \gamma \\ q^2 \end{matrix}\middle| q^2\right]_j \left(\frac{\alpha}{\beta\gamma}\right)^j$$

and then computing the inner sum with respect to j displayed in (16) by means of the q-Gauss sum (14), we derive the following bibasic reduction formula.

Proposition 20 (*Reduction Formula*)**.**

$$\sum_{i,j=0}^{\infty} \frac{(\alpha/c^2;q^2)_{i+j}}{(q;q)_i} \left[\begin{matrix} \beta, & \gamma \\ q^2, & \alpha/c^2 \end{matrix}\middle| q^2\right]_j \left[\begin{matrix} \alpha/ace \\ \alpha/ac, \alpha/ce \end{matrix}\middle| q\right]_{i+2j} \left(\frac{\alpha}{\beta\gamma}\right)^j (-c)^i$$
$$= \left[\begin{matrix} c, \alpha/ace \\ \alpha/ac, \alpha/ce \end{matrix}\middle| q\right]_\infty \left[\begin{matrix} \alpha/\beta, \alpha/\gamma \\ c^2, \alpha/\beta\gamma \end{matrix}\middle| q^2\right]_\infty \sum_{i=0}^{\infty} \frac{[a,-c,e;q]_i(\alpha/\beta\gamma;q^2)_i}{(q;q)_i[\alpha/\beta,\alpha/\gamma;q^2]_i} \left(\frac{\alpha}{ace}\right)^i.$$

For the special case $e = 1$ of this proposition, we get the following summation formula.

Corollary 21 ($e = 1$ *in* Proposition 20)**.**

$$\sum_{i,j=0}^{\infty} \frac{(\alpha/\beta\gamma)^j}{(q;q)_i} \left[\begin{matrix} \beta, & \gamma \\ q^2, & \alpha/c^2 \end{matrix}\middle| q^2\right]_j \frac{(\alpha/c^2;q^2)_{i+j}}{(\alpha/c;q)_{i+2j}} (-c)^i = \left[\begin{matrix} c \\ \alpha/c \end{matrix}\middle| q\right]_\infty \left[\begin{matrix} \alpha/\beta, \alpha/\gamma \\ c^2, \alpha/\beta\gamma \end{matrix}\middle| q^2\right]_\infty.$$

Alternatively, given the $\Omega(j)$ sequence in Theorem 15 by

$$\Omega(j) = \left[\begin{matrix} \beta, & q^2/\beta \\ q^2, & -q^2 \end{matrix}\middle| q^2\right]_j (q^{j-1}\alpha)^j$$

according to the q-analogue of Bailey's ${}_2F_1(\frac{1}{2})$ sum (15), we can evaluate the inner sum with respect to j displayed in (16) and establish further the following reduction formula.

Proposition 22 (*Reduction Formula*)**.**

$$\sum_{i,j=0}^{\infty} \frac{(\alpha/c^2;q^2)_{i+j}}{(q;q)_i(q^4;q^4)_j} \left[\begin{matrix} \beta, q^2/\beta \\ \alpha/c^2 \end{matrix}\middle| q^2\right]_j \left[\begin{matrix} \alpha/ace \\ \alpha/ac, \alpha/ce \end{matrix}\middle| q\right]_{i+2j} \left(q^{j-1}\alpha\right)^j (-c)^i$$
$$= \left[\begin{matrix} c, \alpha/ace \\ \alpha/ac, \alpha/ce \end{matrix}\middle| q\right]_\infty \left[\begin{matrix} \alpha/\beta \\ c^2 \end{matrix}\middle| q^2\right]_\infty \sum_{i=0}^{\infty} \frac{[a,-c,e;q]_i}{(q;q)_i(\alpha/\beta;q^2)_i} \left[\begin{matrix} q^{2i}\alpha\beta \\ q^{2i}\alpha/\beta \end{matrix}\middle| q^4\right]_\infty \left(\frac{\alpha}{ace}\right)^i.$$

In particular, the special case $e = 1$ of this proposition reduces to the following summation.

Corollary 23 ($e = 1$ *in* Proposition 22)**.**

$$\sum_{i,j=0}^{\infty} \frac{(\alpha/c^2;q^2)_{i+j}}{(q;q)_i(q^4;q^4)_j} \left[\begin{matrix} \beta, q^2/\beta \\ \alpha/c^2 \end{matrix}\middle| q^2\right]_j \frac{\left(q^{j-1}\alpha\right)^j}{(\alpha/c;q)_{i+2j}} (-c)^i = \left[\begin{matrix} c \\ \alpha/c \end{matrix}\middle| q\right]_\infty \left[\begin{matrix} \alpha/\beta \\ c^2 \end{matrix}\middle| q^2\right]_\infty.$$

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