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On a backward parabolic problem with local Lipschitz source

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ABSTRACT

We consider the regularization of the backward in time problem for a nonlinear parabolic equation in the form $u_t + Au(t) = f(u(t), t)$, $u(1) = \varphi$, where A is a positive self-adjoint unbounded operator and f is a local Lipschitz function. As known, it is ill-posed and occurs in applied mathematics, e.g. in neurophysiological modeling of large nerve cell systems with action potential f in mathematical biology. A new version of quasi-reversibility method is described. We show that the regularized problem (with a regularization parameter $\beta > 0$) is well-posed and that its solution $U_\beta(t)$ converges on $[0, 1]$ to the exact solution $u(t)$ as $\beta \to 0^+$. These results extend some earlier works on the nonlinear backward problem.

1. Introduction

Let H be a Hilbert space with the inner product $\langle .,. \rangle$ and the norm $\|.\|$. In this paper, we consider the backward nonlinear parabolic problem of finding a function $u : [0, 1] \to H$ such that

$$\begin{cases} u_t + Au = f\big(u(t), t\big), & 0 < t < 1, \\ u(1) = \varphi, \end{cases} \tag{1}$$

where the function f is defined later and the operator A is self-adjoint on a dense space $D(A)$ of H such that $-A$ generates a compact contraction semi-group on H. The backward parabolic problems arise in different forms in heat conduction [4,10], material science [16], hydrology [3] and also in many other practical applications of mathematics and engineering sciences. If $H = L^2(0, l)$ for $l > 0$, $A = -\Delta$ and $f(u(t), t) = u\|u\|^2_{L^2(0,l)}$ then a concrete version of problem (1) is given as

$$\begin{cases} u_t - \Delta u = u\|u\|^2_{L^2(0,l)}, & (x, t) \in (0, l) \times (0, 1), \\ u(0, t) = u(l, t) = 0, & t \in (0, 1), \\ u(x, 1) = \varphi(x), & x \in (0, l). \end{cases} \tag{2}$$

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The first equality in problem (2) is a semilinear heat equation with cubic-type nonlinearity and has many applications in computational neurosciences. It occurs in neurophysiological modeling of large nerve cell systems in mathematical biology (see [17]).

Let $u(t)$ be the (unknown) solution of (1), continuous on $t \geqslant 0$ to H with an (unknown) initial value $u(0)$. In practice, $u(1)$ is known only approximately by $\varphi \in H$ with $\|u(1) - \varphi\| \leqslant \beta$, where the constant β is a known small positive number. This problem is well known to be severely ill-posed [15] and regularization methods are required. The homogeneous linear case of problem (1)

$$\begin{cases} u_t + Au = 0, & 0 < t < 1, \\ u(1) = \varphi, \end{cases} \tag{3}$$

has been considered in many papers, such as [2,1,6–9,11–13,18] and references therein. For nonlinear case, there are not many results devoted to backward parabolic equations. In [20,21], under assumptions that $f : H \times \mathbb{R} \to H$ is a global Lipschitz function with respect to the first variable u, i.e. there exists a positive number $k > 0$ independent of $w, v \in H$, $t \in \mathbb{R}$ such that

$$\big\|f(w,t) - f(v,t)\big\| \leqslant k\|w - v\|, \tag{4}$$

we regularized problem (1) and gave some error estimates. To improve the convergence of our method, P.T. Nam [14] gave another method to get the Hölder estimate for regularized solution. More recently, Hetrick and Hughes [5] established some continuous dependence results for nonlinear problem. Their results are also solved under the assumption (4). Until now, to our knowledge, we did not find any papers dealing with the backward parabolic equations included the local Lipschitz source f.

In this paper, we propose a new modified quasi-reversibility method to regularize (1) in case of the local Lipschitz function f. The techniques and methods in previous papers on global Lipschitz function cannot be applied directly to solve the problem (1). The main idea of the paper is of replacing the operator A in (1) by an approximated operator A_β, which will be defined later. Then, using some new techniques, we establish the following approximation problem

$$\begin{cases} v_\beta'(t) + A_\beta v_\beta(t) = f\big(v_\beta(t), t\big), & 0 < t < 1, \\ v_\beta(1) = \varphi, \end{cases} \tag{5}$$

and give an error estimate between the regularized solution of (5) and the exact solution of (1).

Namely, assume that A admits an orthonormal eigenbasis $\{\phi_k\}$ on H corresponding to the eigenvalues $\{\lambda_k\}$ of A; i.e. $A\phi_k = \lambda_k\phi_k$. Without loss of generality, we shall assume that

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots, \quad \lim_{k\to\infty} \lambda_k = \infty.$$

For every v in H having the expansion $v = \sum_{k=1}^\infty \langle v, \phi_k\rangle \phi_k$, we define

$$A_\beta(v) = \sum_{k=1}^\infty \ln^+\left(\frac{1}{\beta\lambda_k + e^{-\lambda_k}}\right)\langle v, \phi_k\rangle\phi_k,$$

where $\ln^+(x) = \max\{\ln x, 0\}$. And for $0 \leqslant t \leqslant s \leqslant T$, we define

$$G_\beta(t,s)(v) = \sum_{k=1}^\infty \max\big\{\big(\beta\lambda_k + e^{-\lambda_k}\big)^{t-s}, 1\big\}\langle v, \phi_k\rangle\phi_k.$$

Then, problem (5) can be rewritten as the following integral equation

$$v_\beta(t) = G_\beta(t,1)\varphi - \int_t^1 G_\beta(t,s) f\bigl(v_\beta(s),s\bigr)\,ds. \tag{6}$$

This paper is organized as follows. In the next section we outline our main results. Its proofs will be given in Sections 3 and 4.

2. The main results

From now on, for clarity, we denote the solution of (1) by $u(t)$, and the solution of the problem (5) by $v_\beta(t)$. We shall make the following assumptions

(H_1) For each $p > 0$, there exists a constant K_p such that $f : H \times \mathbb{R} \to H$ satisfies a local Lipschitz condition

$$\bigl\|f(v_1,t) - f(v_2,t)\bigr\| \leqslant K_p\|v_1 - v_2\|,$$

for every $v_1, v_2 \in H$ such that $\|v_i\| \leqslant p, i = 1, 2$.
Noting that if K_p is a positive constant, then f is a global Lipschitz function.

(H_2) There exists a constant $L \geqslant 0$, such that

$$\bigl\langle f(v_1,t) - f(v_2,t), v_1 - v_2\bigr\rangle + L\|v_1 - v_2\|^2 \geqslant 0.$$

(H_3) $f(0,t) = 0$ for $t \in [0,1]$.

We present examples in which f satisfies the assumptions (H_1) and (H_2).

Example 1. If f is a global Lipschitz function, then f satisfies (H_1) and (H_2). In fact, if $K_p = K$ is independent of p then (H_1) is true. And we also have

$$\begin{aligned}\bigl|\bigl\langle f(v_1,t) - f(v_2,t), v_1 - v_2\bigr\rangle\bigr| &\leqslant \bigl\|f(v_1,t) - f(v_2,t)\bigr\|\|v_1 - v_2\| \\ &\leqslant K\|v_1 - v_2\|^2.\end{aligned}$$

So $\langle f(v_1,t) - f(v_2,t), v_1 - v_2\rangle \geqslant -K\|v_1 - v_2\|^2$. This means that (H_2) is true.

Example 2. Let $f(u,t) = u\|u\|^2$. Of course, condition (H_3) holds in this case. We verify condition (H_1). We have

$$\begin{aligned}\bigl\|f(u) - f(v)\bigr\| &= \bigl\|u\|u\|^2 - v\|v\|^2\bigr\| \\ &= \bigl\|\|u\|^2(u - v) + v\bigl(\|u\|^2 - \|v\|^2\bigr)\bigr\| \\ &\leqslant \|u\|^2\|u - v\| + \|v\|\bigl(\|u\| + \|v\|\bigr)\bigl|\|u\| - \|v\|\bigr| \\ &\leqslant \bigl(\|u\|^2 + \|v\|\|u\| + \|v\|^2\bigr)\|u - v\|.\end{aligned}$$

It is easy to check that f is not global Lipschitz. Let $p > 0$. For each v_1, v_2 such that $\|v_i\| \leqslant p$, $i = 1, 2$, $\|\varphi\| < p$, we can choose $K_p = 3p^2$. It follows that condition (H_1) holds. We verify (H_2).

$$
\begin{aligned}
g(u,v) = \langle f(u)-f(v), u-v\rangle &= \langle u\|u\|^2 - v\|v\|^2, u-v\rangle \\
&= \langle (u-v)\|u\|^2 + v(\|u\|^2-\|v\|^2), u-v\rangle \\
&= \|u-v\|^2\|u\|^2 + \langle v(\|u\|^2-\|v\|^2), u-v\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
g(u,v) = \langle f(u)-f(v), u-v\rangle &= \langle u\|u\|^2 - v\|v\|^2, u-v\rangle \\
&= \langle u(\|u\|^2-\|v\|^2) + \|v\|^2(u-v), u-v\rangle \\
&= \|u-v\|^2\|v\|^2 + \langle (\|u\|^2-\|v\|^2), u-v\rangle.
\end{aligned}
$$

Adding two equalities, we get

$$
\begin{aligned}
2g(u,v) &= \|u-v\|^2(\|u\|^2+\|v\|^2) + \langle (u+v)(\|u\|^2-\|v\|^2), u-v\rangle \\
&= \|u-v\|^2(\|u\|^2+\|v\|^2) + (\|u\|^2-\|v\|^2)\langle u+v, u-v\rangle \\
&= \|u-v\|^2(\|u\|^2+\|v\|^2) + (\|u\|^2-\|v\|^2)^2 \geqslant 0.
\end{aligned}
$$

Consequently, we have

$$
g(u,v) \geqslant \frac{\|u\|^2+\|v\|^2}{2}\|u-v\|^2, \tag{7}
$$

for all $u,v\in H$. This implies that $\langle f(u)-f(v), u-v\rangle \geqslant 0$. It follows that (H_2) is true.

Now we state main results of our paper. Its proofs will be given in the next section.

Theorem 1. *Let $0<\beta<1$, $\varphi\in H$ and let $\varphi_\beta\in H$ be a measured data such that $\|\varphi_\beta-\varphi\|\leqslant\beta$. Assume that (H_1), (H_2), (H_3) hold. Then the problem*

$$
\begin{cases}
v'_\beta(t) + A_\beta v_\beta(t) = f(v_\beta(t),t), & 0<t<1,\\
v_\beta(1) = \varphi_\beta
\end{cases} \tag{8}
$$

has uniquely a solution $U_\beta\in C^1([0,1];H)$.

Theorem 2. *Let $u\in C^1([0,1];H)$ be a solution of (1). Assume that u has the eigenfunction expansion $u(t)=\sum_{k=1}^\infty\langle u(t),\phi_k\rangle\phi_k$ such that*

$$
E^2 = \int_0^1 \sum_{k=1}^\infty \lambda_k^2 e^{2\lambda_k}\left|\langle u(s),\phi_k\rangle\right|^2 ds < \infty.
$$

Then

$$
\left\|U_\beta(t)-u(t)\right\| \leqslant M\beta^t\left(\ln\frac{e}{\beta}\right)^{t-1}, \quad \forall t\in[0,1], \tag{9}
$$

where $M = 2e^{\frac{(2L+1)}{2}(1-t)}E + e^{L(1-t)}$.

Remark 1.

1. In two recent papers [19,20], under the assumption of global Lipschitz property of f, the error estimate between the exact solution and the approximation solution has the form

$$\left\|u(t)-u_\epsilon(t)\right\| \leqslant C\epsilon^{\frac{t}{T}}. \tag{10}$$

   The estimate in (10) is not good at $t=0$. In our method, we improve it to obtain the error estimate is of order $\beta^t(\ln\frac{e}{\beta})^{t-1}$. If $t\approx 1$, the first term β^t tends to zero quickly, and if $t\approx 0$, the second term $(\ln(\frac{e}{\beta}))^{t-1}$ tends to zero as $\beta\to 0^+$. And if $t=0$, the error (9) becomes

$$\left\|u(0)-U_\beta(0)\right\| \leqslant M\left(\ln\frac{e}{\beta}\right)^{-1}. \tag{11}$$

   We also note that the right hand side of (11) tends to zero when $\beta\to 0^+$.
2. If f is a global Lipschitz function, the error (9) is similar to the one given in [20].

3. Proof of Theorem 1

First, we shall prove some inequalities which will be used in the main part of our proof.

Lemma 3. *For $0<\beta<1$, the equation $\beta x+e^{-x}-1=0$ has uniquely a positive solution $x_\beta>\ln(\frac{1}{\beta})$. We also have*

$$\ln^+\left(\frac{1}{\beta x+e^{-x}}\right)=\ln\left(\frac{1}{\beta x+e^{-x}}\right),\quad 0\leqslant x\leqslant x_\beta,$$
$$\ln^+\left(\frac{1}{\beta x+e^{-x}}\right)=0,\quad x>x_\beta.$$

Moreover, for $x>0$, we have

$$0\leqslant \ln^+\left(\frac{1}{\beta x+e^{-x}}\right)\leqslant \ln\left(\beta^{-1}\left(\ln\left(\frac{e}{\beta}\right)\right)^{-1}\right),$$
$$\max\left\{\frac{1}{\beta x+e^{-x}},1\right\}\leqslant \beta^{-1}\left(\ln\left(\frac{e}{\beta}\right)\right)^{-1}.$$

Proof. First, we consider a function $m(x)=\beta x+e^{-x}-1$. The derivative of the function m is $m'(x)=\beta-e^{-x}$. The solution of $m'(x)=0$ is $x=\ln\frac{1}{\beta}$. The function $m(x)$ is decreasing on $(0,\ln\frac{1}{\beta})$ and is increasing on $(\ln\frac{1}{\beta},+\infty)$. Since $m(0)=0$, $m(\ln(\frac{1}{\beta}))=\beta\ln(\frac{1}{\beta})+\beta-1<0$ and $\lim_{x\to+\infty}m(x)=+\infty$ we conclude that the equation $\beta x+e^{-x}-1=0$ has a unique positive solution $x_\beta>\ln(\frac{1}{\beta})$. Moreover, $\beta x+e^{-x}>1$ holds if and only if $x>x_\beta$. It follows that $\ln(\frac{1}{\beta x+e^{-x}})\leqslant 0$ for $x>x_\beta$. Therefore

$$\ln^+\left(\frac{1}{\beta x+e^{-x}}\right)=0,\quad x>x_\beta. \tag{12}$$

Consider the function

$$g(x)=\ln\left(\frac{1}{\beta x+e^{-x}}\right),\quad \forall x\in(0,x_\beta),$$

for $0<\beta<1$. Computing the derivative of $g(x)$, we get

$$g'(x)=\frac{-\beta+e^{-x}}{\beta x+e^{-x}}.$$

We have $g'(x)=0$ when $x=\ln(\frac{1}{\beta})$. It implies that the function gets its maximum at $x=\ln(\frac{1}{\beta})$. Hence

$$g(x)\leqslant g\left(\ln\frac{1}{\beta}\right)=\ln\left(\beta^{-1}\left(\ln\frac{e}{\beta}\right)^{-1}\right).$$

And, we have

$$\max\left\{\frac{1}{\beta x+e^{-x}},1\right\}=e^{\ln^+(\frac{1}{\beta x+e^{-x}})}\leqslant\beta^{-1}\left(\ln\frac{e}{\beta}\right)^{-1}. \qquad \square$$

Lemma 4. *For any $0<\beta<1$, we have*

$$\|A_\beta\|\leqslant\ln\left(\beta^{-1}\left(\ln\frac{e}{\beta}\right)^{-1}\right).$$

Proof. Let $v\in H$ and let $v=\sum_{k=1}^{\infty}\langle v,\phi_k\rangle\phi_k$ be the eigenfunction expansion of v. Lemma 3 gives

$$\begin{aligned}\|A_\beta v\|^2&=\sum_{k=1}^{\infty}\left|\ln^+\left(\frac{1}{\beta\lambda_k+e^{-\lambda_k}}\right)\right|^2\left|\langle v,\phi_k\rangle\right|^2\\&\leqslant\ln^2\left(\beta^{-1}\left(\ln\frac{e}{\beta}\right)^{-1}\right)\sum_{k=1}^{\infty}\left|\langle v,\phi_k\rangle\right|^2\\&\leqslant\ln^2\left(\beta^{-1}\left(\ln\frac{e}{\beta}\right)^{-1}\right)\|v\|^2.\end{aligned}$$

This completes the proof of Lemma 4. $\square$

Lemma 5. *For any $0<\beta<1$ and $0<t\leqslant s\leqslant 1$, we have*

$$\left\|G_\beta(t,s)\right\|\leqslant\beta^{t-s}\left(\ln\frac{e}{\beta}\right)^{t-s}\leqslant\beta^{t-s}.$$

So, if $0\leqslant s-t\leqslant h\leqslant 1$ then

$$\left\|G_\beta(t,s)\right\|\leqslant\beta^{-h}.$$

Proof. First, letting $v\in H$ be as in the proof of Lemma 4, we have

$$\begin{aligned}\left\|G_\beta(t,s)(v)\right\|^2&=\sum_{k=1}^{\infty}\max\left\{\left(\beta\lambda_k+e^{-\lambda_k}\right)^{2t-2s},1\right\}\left|\langle v,\phi_k\rangle\right|^2\\&\leqslant\beta^{2t-2s}\left(\ln\frac{e}{\beta}\right)^{2t-2s}\|v\|^2,\end{aligned}$$

which completes the proof of Lemma 5. $\square$

Lemma 6. *Suppose $v_\beta \in C([0,1];H)$ is the unique solution of the integral equation (6). Then $v_\beta \in C^1([0,1];H)$ and v_β is also a solution of problem (5).*

Proof. In fact, for $0 \leqslant t \leqslant 1$, we have the eigenfunction expansion

$$\begin{aligned} v_\beta(t) = {} & \sum_{k=1}^{\infty} \max\{\left(\beta\lambda_k + e^{-\lambda_k}\right)^{t-1}, 1\}\langle\varphi, \phi_k\rangle\phi_k \\ & - \sum_{k=1}^{\infty}\left(\int_t^1 \max\{\left(\beta\lambda_k + e^{-\lambda_k}\right)^{t-s}, 1\}\langle f\left(v_\beta(s), s\right), \phi_k\rangle\, ds\right)\phi_k. \end{aligned}$$

We get by direct computation

$$\begin{aligned} \frac{d}{dt}v_\beta(t) &= \sum_{k=1}^{\infty} \ln^+\left(\beta\lambda_k + e^{-\lambda_k}\right) \times \max\{\left(\beta\lambda_k + e^{-\lambda_k}\right)^{t-1}, 1\}\langle\varphi, \phi_k\rangle\phi_k \\ &\quad - \sum_{k=1}^{\infty}\left(\int_t^1 \ln^+\left(\beta\lambda_k + e^{-\lambda_k}\right) \max\{\left(\beta\lambda_k + e^{-\lambda_k}\right)^{t-s}, 1\}\langle f\left(v_\beta(s), s\right), \phi_k\rangle ds\right)\phi_k \\ &\quad + \sum_{k=1}^{\infty}\langle f\left(v_\beta(t), t\right), \phi_k\rangle\phi_k \\ &= \sum_{k=1}^{\infty} \ln^+\left(\beta\lambda_k + e^{-\lambda_k}\right)\langle v_\beta(t), \phi_k\rangle\phi_k + f\left(v_\beta(t), t\right) \\ &= -A_\beta v_\beta(t) + f\left(v_\beta(t), t\right). \end{aligned}$$

It is clear that $v_\beta(1) = \sum_{k=1}^{\infty}\langle\varphi, \phi_k\rangle\phi_k = \varphi$. Hence, v_β is the solution of problem (5). $\square$

Lemma 7. *For $0 < \beta < 1$ and $\varphi \in H$, put $P = \|\varphi\|e^{(\|A_\beta\| + L)}$. Letting M be such that $M > 2^{\frac{1}{K_M}}P$, we put*

$$N = \left[\frac{MK_M}{M2^{-\frac{1}{K_M}} - P}\right] + 1 + \left[2\beta^{-1}K_M\right], \qquad h = \frac{1}{N} \tag{13}$$

where $[x]$ is the integer part of the real number x and K_M is the Lipschitz constant in (H_1) with respect to M. For $\varphi_i \in H$, $\|\varphi_i\| \leqslant P$, we define

$$\begin{aligned} & T_i = 1 - ih, \quad i = 0, 1, \ldots, N, \\ & L_i = \Big\{v \in C\left([T_{i+1}, T_i]; H\right),\ v(T_i) = \varphi_i,\ \sup_{T_{i+1} \leqslant t \leqslant T_i} \|v(t)\| \leqslant M\Big\}, \\ & J_i v = G_\beta(t, T_i)\varphi_i - \int_t^{T_i} G_\beta(t, s) f\left(v(s), s\right) ds. \end{aligned}$$

Then the operator J_i has a unique fixed point on L_i.

Proof. For $v \in L_i$, one has

$$\left\|J_i v(t)\right\| \leqslant \left\|G_\beta(t, T_i)\right\|\|\varphi_i\| + \int_t^{T_i}\left\|G_\beta(t, s)\right\|\left\|f\left(v(s), s\right)\right\| ds.$$

From Lemma 5, one has for $T_i - h \leqslant t \leqslant T_i$

$$\big\|J_i v(t)\big\| \leqslant \beta^{t-T_i}\left(\|\varphi_i\| + \int_t^{T_i} \big\|f\big(v(s),s\big)\big\|\,ds\right).$$

Assumptions (H_2) and (H_3) give

$$\big\|f\big(v(s),s\big)\big\| \leqslant K_M\big\|v(s)\big\| \leqslant MK_M. \tag{14}$$

So, we have for $T_{i+1} \leqslant t \leqslant T_i$

$$\big\|J_i v(t)\big\| \leqslant \beta^{-h}\big(\|\varphi_i\| + MK_M h\big) \leqslant \beta^{-h}(P + MK_M h). \tag{15}$$

From (13), we have $N \geqslant 2\beta^{-1}K_M$ and $N \geqslant \frac{MK_M}{M2^{-\frac{1}{K_M}} - P}$. Hence, $h = \frac{1}{N} \leqslant \frac{\beta}{2K_M}$ and $h = \frac{1}{N} \leqslant \frac{M2^{-\frac{1}{K_M}} - P}{MK_M}$. It implies that $\beta^{-h} \leqslant \beta^{\frac{-\beta}{2K_M}}$ and

$$\beta^{-h}(P + MK_M h) \leqslant \beta^{\frac{-\beta}{2K_M}}\left(P + MK_M\frac{M2^{-\frac{1}{K_M}} - P}{MK_M}\right) = \beta^{\frac{-\beta}{2K_M}}\frac{M}{2^{\frac{1}{K_M}}}. \tag{16}$$

It is easy to prove that for $\beta \in (0,1)$

$$\beta^{\frac{-\beta}{2}} \leqslant e^{\frac{1}{2e}} < 2. \tag{17}$$

Combining (15), (16) and (17), we obtain

$$\big\|J_i v(t)\big\| \leqslant M.$$

It follows that $J_i(L_i) \subset L_i$. Now, for $v_1, v_2 \in L_i$, one has for $T_{i+1} \leqslant t \leqslant T_i$

$$\begin{aligned}
\big\|J_i v_1(t) - J_i v_2(t)\big\| &\leqslant \int_t^{T_i} \big\|G_\beta(t,s)\big\|\big\|f\big(v_1(s),s\big) - f\big(v_2(s),s\big)\big\|\,ds \\
&\leqslant \int_t^{T_i} \beta^{-h}K_M\big\|v_1(s) - v_2(s)\big\|\,ds \\
&\leqslant h\beta^{-h}K_M \sup_{T_{i+1}\leqslant t\leqslant T_i}\big\|v_1(t) - v_2(t)\big\| \\
&\leqslant \frac{\beta}{2K_M}\beta^{-h}K_M \sup_{T_{i+1}\leqslant t\leqslant T_i}\big\|v_1(t) - v_2(t)\big\| \\
&\leqslant \frac{\beta^{1-h}}{2} \sup_{T_{i+1}\leqslant t\leqslant T_i}\big\|v_1(t) - v_2(t)\big\| \\
&\leqslant \frac{1}{2} \sup_{T_{i+1}\leqslant t\leqslant T_i}\big\|v_1(t) - v_2(t)\big\|.
\end{aligned}$$

So J_i is contractive on L_i. Using the Banach fixed point principle, we get Lemma 7. □

Lemma 8. *Assume that f satisfies (H_2), (H_3). Let $0 \leqslant \tau \leqslant 1$ and let $u_\beta \in C^1([\tau,1];H)$ satisfy*

$$\begin{cases} u'_\beta(t) + A_\beta u_\beta(t) = f\big(u_\beta(t),t\big), & \tau < t < 1, \\ u_\beta(1) = \varphi. \end{cases} \tag{18}$$

Then for $\tau \leqslant t \leqslant 1$

$$\big\|u_\beta(t)\big\| \leqslant \|\varphi\| e^{(\|A_\beta\|+L)}.$$

Proof. One has

$$\frac{1}{2}\frac{d}{dt}\|u_\beta\|^2 + \langle A_\beta u_\beta, u_\beta\rangle = \big\langle f\big(u_\beta(t),t\big), u_\beta\big\rangle \geqslant -L\|u_\beta\|^2.$$

It follows that

$$\frac{1}{2}\big\|u_\beta(1)\big\|^2 + \int_t^1 \big\langle A_\beta u_\beta(s), u_\beta(s)\big\rangle\, ds - \frac{1}{2}\big\|u_\beta(t)\big\|^2 \geqslant -L\int_t^1 \big\|u_\beta(s)\big\|^2\, ds.$$

So we have

$$\frac{1}{2}\big\|u_\beta(t)\big\|^2 \leqslant \frac{1}{2}\|\varphi\|^2 + \int_t^1 \big(\big\langle A_\beta u_\beta(s), u_\beta(s)\big\rangle + L\big\|u_\beta(s)\big\|^2\big)\, ds.$$

It follows that

$$\big\|u_\beta(t)\big\|^2 \leqslant \|\varphi\|^2 + 2\big(\|A_\beta\| + L\big)\int_t^1 \big\|u_\beta(s)\big\|^2\, ds.$$

Gronwall's inequality gives

$$\big\|u_\beta(t)\big\|^2 \leqslant \|\varphi\|^2 e^{2(1-t)(\|A_\beta\|+L)} \leqslant \|\varphi\|^2 e^{2(\|A_\beta\|+L)}. \qquad \square$$

Proof of Theorem 1. We shall prove by induction that the equation

$$v'(t) + A_\beta v(t) = f\big(v(t),t\big) \tag{19}$$

subject to the condition $v(1) = \varphi$ has a unique solution on $[T_i, 1]$ for $i = 0,1,2,\ldots,N$. In fact, for $i = 0$, we put $\varphi_0 = \varphi$. From Lemma 7, we can find $u_0 \in C([T_1,1];H)$ such that $J_0 u_0 = u_0$. Using Lemma 6, we can verify that u_0 satisfies (19) on $[T_1,1]$. From (14), we have

$$\big\|u'_0(t)\big\| = \big\|A_\beta u_0(t)\big\| + \big\|f\big(u_0(t),t\big)\big\| \leqslant \|A_\beta\|M + MK_M.$$

This implies that $u_0 \in C^1([T_1,1];H)$. Now, we assume that (19) has a unique solution $u \in C^1([T_k,1];H)$, $1 \leqslant k \leqslant N-1$ with $u(1) = \varphi$. We shall prove that we can extend this solution to the interval $[T_{k+1},1]$. In fact, from Lemma 8, we have

$$\big\|u(t)\big\| \leqslant P = \|\varphi\| e^{(\|A_\beta\|+L)}$$

for $T_k \leqslant t \leqslant 1$. Put $\varphi_k = u(t_k)$. From Lemma 7, we can find $u_k \in C([T_{k+1}, T_k]; H)$ such that $J_k u_k = u_k$. Using Lemma 6, we can verify that u_k satisfies (19) on $[T_{k+1}, T_k]$ with $u_k(T_k) = u(T_k)$. And we also obtain $u_k \in C^1([T_{k+1}, T_k]; H)$. So we can extend the solution u on $[T_{k+1}, 1]$ by putting $u(t) = u_k(t)$ for $T_{k+1} \leqslant t \leqslant T_k$. By induction, we can complete the proof of Theorem 1. □

4. Proof of Theorem 2

Lemma 9. *Let u_β and v_β be two solutions of problem (5) corresponding to final values φ and ω respectively, then*

$$\left\|u_\beta(t) - v_\beta(t)\right\| \leqslant e^{L(1-t)}\beta^{t-1}\left(\ln\frac{e}{\beta}\right)^{t-1}\|\varphi - \omega\|. \tag{20}$$

Here we recall that the constant L is in (H_2).

Proof. For $a > 0$, we put

$$w_\beta(t) = e^{a(t-1)}\big(u_\beta(t) - v_\beta(t)\big).$$

By direct computation, we get

$$\frac{d}{dt}w_\beta(t) + A_\beta w_\beta(t) - a w_\beta(t) = e^{a(t-1)}\big(f\big(u_\beta(t), t\big) - f\big(v_\beta(t), t\big)\big).$$

It follows that

$$\big\langle w_\beta'(t) + A_\beta w_\beta(t) - a w_\beta(t), w_\beta(t)\big\rangle = \big\langle e^{a(t-1)}\big(f\big(u_\beta(t), t\big) - f\big(v_\beta(t), t\big)\big), w_\beta(t)\big\rangle.$$

We obtain in view of condition (H_2)

$$\begin{aligned}\big\langle e^{a(t-1)}\big(f\big(u_\beta(t), t\big) - f\big(v_\beta(t), t\big)\big), w_\beta(t)\big\rangle &= e^{2a(t-1)}\big\langle f\big(u_\beta(t), t\big) - f\big(v_\beta(t), t\big), u_\beta(t) - v_\beta(t)\big\rangle \\ &\geqslant -Le^{2a(t-1)}\left\|u_\beta(t) - v_\beta(t)\right\|^2 = -L\|w_\beta\|^2.\end{aligned}$$

From Lemma 4, we have

$$\big|\big\langle -A_\beta w_\beta(t), w_\beta(t)\big\rangle\big| \leqslant \ln\left(\beta^{-1}\left(\ln\frac{e}{\beta}\right)^{-1}\right)\left\|w_\beta(t)\right\|^2,$$

which gives

$$\big\langle -A_\beta w_\beta(t), w_\beta(t)\big\rangle \geqslant -\ln\left(\beta^{-1}\left(\ln\frac{e}{\beta}\right)^{-1}\right)\left\|w_\beta(t)\right\|^2.$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\left\|w_\beta(t)\right\|^2 \geqslant a\left\|w_\beta(t)\right\|^2 - L\left\|w_\beta(t)\right\|^2 - \ln\left(\beta^{-1}\left(\ln\frac{e}{\beta}\right)^{-1}\right)\left\|w_\beta(t)\right\|^2.$$

Then, we get

$$\left\|w_\beta(1)\right\|^2 - \left\|w_\beta(t)\right\|^2 \geqslant 2\int_t^1\left(a - L - \ln\left(\beta^{-1}\left(\ln\frac{e}{\beta}\right)^{-1}\right)\right)\left\|w_\beta(s)\right\|^2 ds.$$

We choose $a = L + \ln(\beta^{-1}(\ln \frac{e}{\beta})^{-1})$ then

$$\left\|w_\beta(t)\right\|^2 \leqslant \left\|w_\beta(1)\right\|^2 = \|\varphi - \omega\|^2.$$

Hence, we get

$$\left\|u_\beta(t) - v_\beta(t)\right\| \leqslant e^{a(1-t)}\|\varphi - \omega\| = e^{L(1-t)}\beta^{t-1}\left(\ln \frac{e}{\beta}\right)^{t-1}\|\varphi - \omega\|. \qquad \square$$

Lemma 10. *Let v_β be a solution of problem (5). Then*

$$\left\|u(t) - v_\beta(t)\right\| \leqslant e^{\frac{(2L+1)}{2}(1-t)}\beta^t\left(\ln \frac{e}{\beta}\right)^{t-1} E, \quad \forall t \in (0,1], \tag{21}$$

where we recall

$$E = \sqrt{\int_0^1 \sum_{k=1}^\infty \lambda_k^2 e^{2\lambda_k}\left|\left\langle u(s), \phi_k\right\rangle\right|^2 ds}.$$

Proof. Since v_β and u are two solutions of problems (5) and (1) respectively, we have

$$\begin{cases} \dfrac{d}{dt}v_\beta(t) + A_\beta v_\beta(t) = f\big(v_\beta(t), t\big), \\ v_\beta(1) = \varphi, \end{cases}$$

and

$$\begin{cases} \dfrac{d}{dt}u(t) + A_\beta u(t) = (A_\beta - A)u(t) + f\big(u(t), t\big), \\ u(1) = \varphi. \end{cases}$$

For any $b > 0$, let

$$z_\beta(t) = e^{b(t-1)}\big(v_\beta(t) - u(t)\big).$$

Then by direct calculation

$$\begin{aligned} \frac{d}{dt}z_\beta(t) &= be^{b(t-1)}\big(v_\beta(t) - u(t)\big) + e^{b(t-1)}\big(v_\beta'(t) - u'(t)\big) \\ &= bz_\beta(t) + e^{b(t-1)}\big(-A_\beta v_\beta(t) + f\big(v_\beta(t), t\big) + A_\beta u(t) - f\big(u(t), t\big)\big) - e^{b(t-1)}(A_\beta - A)u(t) \\ &= bz_\beta(t) - A_\beta z_\beta(t) + e^{b(t-1)}\big(f\big(v_\beta(t), t\big) - f\big(u(t), t\big) - (A_\beta - A)u(t)\big). \end{aligned} \tag{22}$$

By taking the inner product two sides of the latter equality with z_β, we get

$$\begin{aligned} \left\langle z_\beta'(t) + A_\beta z_\beta(t) - bz_\beta(t), z_\beta(t)\right\rangle &= \left\langle e^{b(t-1)}\big(f\big(v_\beta(t), t\big) - f\big(u(t), t\big)\big), z_\beta(t)\right\rangle \\ &\quad - e^{b(t-1)}\left\langle (A_\beta - A)u(t), z_\beta(t)\right\rangle. \end{aligned} \tag{23}$$

This means that

$$\begin{aligned}\frac{d}{dt}\big\|z_\beta(t)\big\|^2 &= 2\big\langle -A_\beta z_\beta(t), z_\beta(t)\big\rangle + 2b\big\|z_\beta(t)\big\|^2 \\ &\quad + 2\big\langle e^{b(t-1)}\big(f\big(v_\beta(t),t\big) - f\big(u(t),t\big)\big), z_\beta(t)\big\rangle \\ &\quad - 2e^{b(t-1)}\big\langle (A_\beta - A)u(t), z_\beta(t)\big\rangle.\end{aligned}$$

Estimating as in the proof of Lemma 9, we get

$$\big\langle e^{b(t-1)}\big(f\big(v_\beta(t),t\big) - f\big(u(t),t\big)\big), z_\beta(t)\big\rangle \geqslant -Le^{2b(t-1)}\big\|v_\beta(t) - u(t)\big\|^2 = -L\|z_\beta\|^2.$$

and

$$\big\langle -A_\beta z_\beta(t), z_\beta(t)\big\rangle \geqslant -\ln\left(\beta^{-1}\left(\ln\frac{e}{\beta}\right)^{-1}\right)\big\|z_\beta(t)\big\|^2.$$

This implies that for $t \in [0,1]$ and $s \in [t,1]$

$$\begin{aligned}\frac{d}{dt}\big\|z_\beta(s)\big\|^2 + 2e^{b(s-1)}\big\langle (A_\beta - A)u(s), z_\beta(s)\big\rangle &\geqslant 2b\big\|z_\beta(s)\big\|^2 - 2L\big\|z_\beta(s)\big\|^2 \\ &\quad - 2\ln\left(\beta^{-1}\left(\ln\frac{e}{\beta}\right)^{-1}\right)\big\|z_\beta(s)\big\|^2.\end{aligned}$$

Choosing $b = \ln(\beta^{-1}(\ln\frac{e}{\beta})^{-1})$ and integrating the latter inequality from t to 1, we obtain

$$\big\|z_\beta(1)\big\|^2 - \big\|z_\beta(t)\big\|^2 + \int_t^1 2e^{b(s-1)}\big\langle (A_\beta - A)u(s), z_\beta(s)\big\rangle\, ds \geqslant -2L\int_t^1 \big\|z_\beta(s)\big\|^2\, ds.$$

This can be rewritten as

$$\big\|z_\beta(t)\big\|^2 \leqslant 2L\int_t^1 \big\|z_\beta(s)\big\|^2\, ds + \int_t^1 2e^{b(s-1)}\big\langle (A_\beta - A)u(s), z_\beta(s)\big\rangle\, ds.$$

Applying Hölder's inequality and $e^{2b(s-1)} \leqslant 1$ for $s \in [t,1]$, we have

$$\begin{aligned}\int_t^1 2e^{b(s-1)}\big\langle (A_\beta - A)u(s), z_\beta(s)\big\rangle\, ds &\leqslant \int_t^1 e^{2b(s-1)}\big\|(A_\beta - A)u(s)\big\|^2\, ds + \int_t^1 \big\|z_\beta(s)\big\|^2\, ds \\ &\leqslant \int_t^1 \big\|(A_\beta - A)u(s)\big\|^2\, ds + \int_t^1 \big\|z_\beta(s)\big\|^2\, ds. \qquad (24)\end{aligned}$$

Since $u(s) = \sum_{k=1}^\infty \langle u(s), \phi_k\rangle \phi_k$, it follows that

$$\begin{aligned}\int_t^1 \big\|(A_\beta - A)u(s)\big\|^2\, ds &= \int_t^1 \sum_{k=1}^\infty \left(\lambda_k - \ln^+\left(\frac{1}{\beta\lambda_k + e^{-\lambda_k}}\right)\right)^2 \big|\big\langle u(s), \phi_k\big\rangle\big|^2\, ds \\ &= \int_t^1 \sum_{\lambda_k < x_\beta} \ln^2\big(1 + \beta\lambda_k e^{\lambda_k}\big)\big|\big\langle u(s), \phi_k\big\rangle\big|^2\, ds + \int_t^1 \sum_{\lambda_k \geqslant x_\beta} \lambda_k^2\big|\big\langle u(s), \phi_k\big\rangle\big|^2\, ds. \qquad (25)\end{aligned}$$

Here the positive number x_β is defined in Lemma 3. We have

$$\begin{aligned}\int_t^1 \sum_{\lambda_k<x_\beta} \ln^2\left(1+\beta\lambda_k e^{\lambda_k}\right)\left|\left\langle u(s),\phi_k\right\rangle\right|^2 ds &\leqslant \int_t^1 \sum_{k=1}^\infty \beta^2\lambda_k^2 e^{2\lambda_k}\left|\left\langle u(s),\phi_k\right\rangle\right|^2 ds \\ &\leqslant \beta^2\int_0^1 \sum_{k=1}^\infty \lambda_k^2 e^{2\lambda_k}\left|\left\langle u(s),\phi_k\right\rangle\right|^2 ds \\ &= \beta^2 E^2. \end{aligned} \tag{26}$$

Using the inequality $x_\beta > \ln\frac{1}{\beta}$, we get

$$\begin{aligned}\int_t^1 \sum_{\lambda_k\geqslant x_\beta} \lambda_k^2\left|\left\langle u(s),\phi_k\right\rangle\right|^2 ds &= \int_t^1 \sum_{\lambda_k\geqslant x_\beta} e^{-2\lambda_k}\lambda_k^2 e^{2\lambda_k}\left|\left\langle u(s),\phi_k\right\rangle\right|^2 ds \\ &\leqslant e^{-2x_\beta}\int_t^1 \sum_{\lambda_k\geqslant x_\beta} \lambda_k^2 e^{2\lambda_k}\left|\left\langle u(s),\phi_k\right\rangle\right|^2 ds \\ &\leqslant \beta^2\int_0^1 \sum_{k=1}^\infty \lambda_k^2 e^{2\lambda_k}\left|\left\langle u(s),\phi_k\right\rangle\right|^2 ds = \beta^2 E^2. \end{aligned} \tag{27}$$

From (24), (25), (26) and (27), we obtain

$$e^{2b(t-1)}\left\|v_\beta(t)-u(t)\right\|^2 \leqslant (2L+1)\int_t^1 e^{2b(s-1)}\left\|v_\beta(s)-u(s)\right\|^2 ds + 2\beta^2 E^2.$$

It implies that

$$\begin{aligned} e^{2bt}\left\|v_\beta(t)-u(t)\right\|^2 &\leqslant (2L+1)\int_t^1 e^{2bs}\left\|v_\beta(s)-u(s)\right\|^2 ds + \beta^2 e^{2b}E^2 \\ &= (2L+1)\int_t^1 e^{2bs}\left\|v_\beta(s)-u(s)\right\|^2 ds + E^2\left(\ln\frac{e}{\beta}\right)^{-2}. \end{aligned} \tag{28}$$

Here we recall that $e^b = \beta^{-1}(\ln\frac{e}{\beta})^{-1}$. Using Gronwall's inequality, we obtain

$$e^{2bt}\left\|v_\beta(t)-u(t)\right\|^2 \leqslant 2e^{(2L+1)(1-t)}\left(\ln\frac{e}{\beta}\right)^{-2} E^2.$$

It can be rewritten as

$$\left\|v_\beta(t)-u(t)\right\|^2 \leqslant 2e^{(2L+1)(1-t)}\beta^{2t}\left(\ln\frac{e}{\beta}\right)^{2t-2} E^2.$$

Thus

$$\left\|v_\beta(t)-u(t)\right\| \leqslant 2e^{\frac{(2L+1)}{2}(1-t)}\beta^t\left(\ln\frac{e}{\beta}\right)^{t-1}E.$$

This ends the proof of Lemma 10. □

Now, we shall finish the proof of Theorem 2.

Proof of Theorem 2. Let v_β, U_β be the solutions of problem (5) corresponding to φ and φ_β respectively. Using Lemma 9 and Lemma 10, we have

$$\begin{aligned}\left\|U_\beta(t)-u(t)\right\| &\leqslant \left\|U_\beta(t)-v_\beta(t)\right\|+\left\|v_\beta(t)-u(t)\right\| \\ &\leqslant e^{L(1-t)}\beta^{t-1}\left(\ln\frac{e}{\beta}\right)^{t-1}\|\varphi-\varphi_\beta\|+2e^{\frac{(2L+1)}{2}(1-t)}\beta^t\left(\ln\frac{e}{\beta}\right)^{t-1}E \\ &\leqslant \beta^t\left(\ln\frac{e}{\beta}\right)^{t-1}\left(2e^{\frac{(2L+1)}{2}(1-t)}E+e^{L(1-t)}\right),\end{aligned}$$

for every $t\in[0,1]$.

This completes the proof of Theorem 2. □

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