J. Math. Anal. Appl. 418 (2014) 425–443

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Higher integrability for nonlinear elliptic equations with variable growth and discontinuous coefficients ☆

Xia Zhang [a,*], Yan Huo [b], Yongqiang Fu [a]

[a] Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China
[b] College of Aerospace and Civil Engineering, Harbin Engineering University, Harbin 150001, China

ARTICLE INFO

Article history:
Received 21 November 2012
Available online 12 April 2014
Submitted by P.J. McKenna

Keywords:
Variable exponent Sobolev space
Orlicz space
Gradient estimate

ABSTRACT

In this paper, based on the theory of variable exponent spaces, we study the higher integrability for a class of nonlinear elliptic equations with variable growth and discontinuous coefficients. Under suitable assumptions, we obtain a local gradient estimate in Orlicz space for weak solution.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain. We consider the weak solution of the following equation

$$\operatorname{div}\bigl(\bigl(A(x)\nabla u \cdot \nabla u\bigr)^{\frac{p(x)-2}{2}} A(x)\nabla u\bigr) = \operatorname{div}\bigl(|f|^{p(x)-2} f\bigr) \quad \text{in } \Omega. \tag{1.1}$$

The measurable coefficient matrix $A(x) = (a_{ij}(x))_{N\times N}$ is bounded, i.e.

$$\|A\|_\infty = \sup_{x\in\Omega}\bigl\|A(x)\bigr\| < \infty, \tag{1.2}$$

where $\|A(x)\| = \max_{1\leq i,j\leq N} |a_{ij}(x)|$ and satisfies the uniformly elliptic condition: there exist positive constants Λ_1, Λ_2 such that for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^N$,

$$\Lambda_1|\xi|^2 \leq A(x)\xi \cdot \xi \leq \Lambda_2|\xi|^2. \tag{1.3}$$

Here $A(x)$ is not assumed to be symmetric and continuous. In this paper, assume that

☆ This research was supported by NSFC (No. 51206030), NSFC (No. 11371110), the Fundamental Research Funds for the Central Universities (Grant No. HIT.NSRIF.2011005) and the Project sponsored by SRF for ROCS, SEM.

* Corresponding author.
E-mail address: piecesummer1984@163.com (X. Zhang).

http://dx.doi.org/10.1016/j.jmaa.2014.04.010
0022-247X/

$$1 < p_- \le p(x) \le p_+ < +\infty, \tag{1.4}$$

where $p_+ = \sup_{x\in\Omega} p(x)$, $p_- = \inf_{x\in\Omega} p(x)$.

We will study (1.1) in the framework of variable exponent function spaces, the definitions of which will be given in Section 2. Recall that $u \in W^{1,p(x)}_{loc}(\Omega)$ is a weak solution of problem (1.1), if for any $\phi \in W^{1,p(x)}_0(\Omega)$,

$$\int_\Omega \big(A(x)\nabla u \cdot \nabla u\big)^{\frac{p(x)-2}{2}} A(x)\nabla u \cdot \nabla\phi\, dx = \int_\Omega |f|^{p(x)-2} f \nabla\phi\, dx. \tag{1.5}$$

When $p(x)$ is a constant, the regularity for (1.1) has been studied extensively. If A is the unitary matrix, some related results were obtained by DiBenedetto and Manfredi [13] and Iwaniec [17]. Their methods are based on maximal function inequalities and regularity theory for p-harmonic equation. The case of bounded and uniformly continuous coefficients was studied by Morrey [23]. Di Fazio [12], Kinnunen and Zhou [19] proved a local result for (1.1) provided that the coefficients are bounded functions of vanishing mean oscillation. In [6,7], Byun, Wang and Zhou obtained global L^q gradient estimates for weak solution to (1.1) with BMO coefficients. Recently, in [8,9,18,25], local L^q gradient estimates for the corresponding equations were extended to estimates in Orlicz space.

When $p(x)$ is a function, Acerbi and Mingione [1] treated integrability of gradient involving the following equation

$$\operatorname{div}\big(a(x,\nabla u)\big) = \operatorname{div}\big(|f|^{p(x)-2} f\big) \quad \text{in } \Omega, \tag{1.6}$$

where $a(x,\cdot)$ satisfies some continuity assumptions. Under the following geometric condition of $p(x)$:

$$\big|p(x) - p(y)\big| \le \omega\big(|x-y|\big), \tag{1.7}$$

for any $x, y \in \Omega$, where $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is the modulus of continuity of $p(x)$ satisfying

$$\lim_{R\to 0} \omega(R) \ln\left(\frac{1}{R}\right) = 0, \tag{1.8}$$

they proved classical L^q estimates of weak solution u for (1.6): for any $q > 1$,

$$|f|^{p(x)} \in L^q_{loc}(\Omega) \quad \Rightarrow \quad |\nabla u|^{p(x)} \in L^q_{loc}(\Omega).$$

Motivated by their works, by using Calderón and Zygmund type covering arguments and Hardy–Littlewood maximal function, we study higher integrability for (1.1). Under assumptions that $A(x)$ is of vanishing mean oscillation and $|f|^{p(x)} \in L^\phi_{loc}(\Omega)$, we obtain gradient estimates in Orlicz space for weak solutions of (1.1). We assume that $\phi : \mathbb{R} \to [0,\infty)$ is a convex, nondecreasing, even function and satisfies

$$a_1\left(\frac{s}{t}\right)^{\alpha_1} \le \frac{\phi(s)}{\phi(t)} \le a_2\left(\frac{s}{t}\right)^{\alpha_2}, \quad \text{for } 0 < s \le t, \tag{1.9}$$

where $0 < a_1 \le a_2$, $1 < \alpha_2 \le \alpha_1$. Then, ϕ is an N-function satisfying $\triangle_2 \cap \nabla_2$-condition (refer to [3]). Orlicz space $L^\phi(\Omega)$ is the set of all measurable functions $v : \Omega \to \mathbb{R}$ satisfying

$$\int_\Omega \phi\big(|v|\big)\, dx < \infty.$$

For example, we could choose $\phi(t) = t^\alpha \ln^\beta(1+t)$, where $\alpha > 1$, $\beta > 0$.

Denote by $|E|$ the Lebesgue measure of a measurable set E and define the mean value of a locally integrable function $v \in L^1_{loc}(\Omega)$ on a cube $Q_R \subset \Omega$ by

$$v_{Q_R} = \fint_{Q_R} v\,dx = |Q_R|^{-1} \int_{Q_R} v\,dx,$$

where Q_R is a cube with side length $2R$. We recall that a locally integrable function v is of bounded mean oscillation, if

$$\fint_{Q_R} |v - v_{Q_R}|\,dx$$

is uniformly bounded as Q_R ranges over all cubes contained in Ω. If, in addition, we require that these averages tend to zero uniformly as R tends to zero, we say that v is of vanishing mean oscillation and denote $v \in \mathrm{VMO}(\Omega)$.

The main difficulty here is that Eq. (1.1) exhibits the variable growth conditions and possesses more complicated nonlinearities. Thus, some techniques used in the constant exponent case cannot be carried out for the variable exponent case. In order to overcome this difficulty, we will combine with localization technique associated to (1.1) and estimates in $L\log^\beta L$ spaces (refer to [1]). Then, we obtain the following higher integrability for gradient of weak solution:

Theorem 1.1. *Suppose that A is of vanishing mean oscillation, i.e. $a_{ij} \in \mathrm{VMO}(\Omega)$ and $|f|^{p(x)} \in L^{\phi}_{loc}(\Omega)$ with (1.9). Let $u \in W^{1,p(x)}(\Omega)$ be a weak solution to (1.1). Then, under assumptions (1.4), (1.7) and (1.8), there exist $0 < R_1 < R_0$ with $Q_{4R_0} \subset\subset \Omega$ and $\sigma_0 > 0$ such that for any (not necessarily concentric) cube $Q_R \subset Q_{R_1}$,*

$$\int_{Q_R} \phi\bigl(|\nabla u|^{p(x)}\bigr)\,dx \le c \int_{Q_R} \phi\bigl(|f|^{p(x)} + 1\bigr)\,dx,$$

where $c = c(p_+, p_-, \alpha_1, \alpha_2, a_1, a_2, \sigma, N, K, \Lambda_1, \Lambda_2, \|A\|_\infty)$, $K = \int_{Q_{4R_0}} (|\nabla u|^{p(x)} + |f|^{p(x)(1+\sigma)})\,dx + 1$, $0 < \sigma < \min\{\sigma_0, p_- - 1\}$.

2. Preliminaries

In the studies of a class of nonlinear problems with variable exponential growth, see for example [2,4,5,10,15,16,27,21,22,26], variable exponent spaces play an important role. Since they were thoroughly studied by O. Kováčik and J. Rákosník [20], variable exponent spaces have been used to model various phenomena. In [24], M. Růžička presented the mathematical theory for the application of variable exponent Sobolev spaces in electro-rheological fluids. As another application, Chen, Levine and Rao [11] suggested a model for image restoration based on a variable exponent Laplacian.

For the convenience of reader, we recall some definitions and basic properties of variable exponent spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a domain. For a deeper treatment on these spaces, we refer to [14]. Let $\mathbf{P}(\Omega)$ be the set of all Lebesgue measurable functions $p : \Omega \to [1, \infty]$. Denote

$$\rho_{p(x)}(u) = \int_{\Omega \setminus \Omega_\infty} |u|^{p(x)}\,dx + \sup_{x \in \Omega_\infty} \bigl|u(x)\bigr|,$$

where $\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$.

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is the class of all functions u such that $\rho_{p(x)}(tu) < \infty$, for some $t > 0$. $L^{p(x)}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{p(x)} = \inf\{\lambda > 0 : \rho_{p(x)}(\lambda u) \le 1\}.$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is the class of all functions $u \in L^{p(x)}(\Omega)$ such that $|\nabla u| \in L^{p(x)}(\Omega)$. $W^{1,p(x)}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}.$$

By $W_0^{1,p(x)}(\Omega)$ we denote the subspace of $W^{1,p(x)}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{1,p(x)}$. We know that if $\Omega \subset \mathbb{R}^N$ is a bounded domain and $p \in C(\overline{\Omega})$, $\|u\|_{1,p(x)}$ and $\|\nabla u\|_{p(x)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$. Under the condition (1.4), $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are reflexive.

Next, we give some related definitions and notations involving Calderón and Zygmund type coverings and Hardy–Littlewood maximal function.

Let $Q_0 \subset \Omega$ be a cube, we denote by $\mathcal{D}(Q_0)$ the class of those cubes, with sides parallel to those of Q_0, obtained by a positive, finite number of dyadic subdivisions of Q_0. We call $Q' \in \mathcal{D}(Q_0)$ the predecessor of Q if Q is obtained by exactly one dyadic subdivision from Q'.

For $v \in L^1(Q_R)$, the restricted Maximal Function Operator relative to Q_R is defined by

$$M^*_{Q_R}(v)(x) = \sup_{Q \subseteq Q_R, x \in Q} \fint_Q |v(y)|\, dy.$$

For $s > 1$, we define

$$M^*_{s,Q_R}(v)(x) = \sup_{Q \subseteq Q_R, x \in Q} \left(\fint_Q |v(y)|^s\, dy \right)^{\frac{1}{s}}.$$

Finally, we present the following local L^q-gradient estimate established in [1], which is useful in the proof of main result in Section 3.

Theorem 2.1. *Assume that hypotheses (1.4), (1.7), (1.8) are fulfilled and*

$$|a(x,\xi)| \le c\left(1 + |\xi|^2\right)^{\frac{p(x)-1}{2}} \quad \text{and} \quad c|\xi|^{p(x)} - c \le a(x,\xi)\xi,$$

for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^N$. Let $|f|^{p(x)} \in L^q_{loc}(\Omega)$ with $q > 1$ and $u \in W^{1,p(x)}(\Omega)$ be a weak solution of (1.6). Then, there exist $R_0 > 0$ with $Q_{4R_0} \subset\subset \Omega$, $0 < \sigma_0 < q - 1$ such that for any (not necessarily concentric) cube $Q_{2R} \subset Q_{4R_0}$,

$$\left(\fint_{Q_R} |\nabla u|^{p(x)(1+\sigma)}\, dx \right)^{\frac{1}{1+\sigma}} \le C \fint_{Q_{2R}} |\nabla u|^{p(x)}\, dx + C\left(\fint_{Q_{2R}} |f|^{p(x)(1+\sigma)}\, dx + 1 \right)^{\frac{1}{1+\sigma}}, \tag{2.1}$$

where $0 < \sigma \le \sigma_0(p_+, p_-, N, K_0, R_0)$, $C = C(p_+, p_-, N)$, $K_0 = \int_{Q_{4R_0}} (|\nabla u|^{p(x)} + 1)\, dx$.

3. Main results

In this section, assume that hypotheses (1.4), (1.7), (1.8) are fulfilled and A is of vanishing mean oscillation. Let $|f|^{p(x)} \in L^{\phi}_{loc}(\Omega)$ with (1.9) and $u \in W^{1,p(x)}(\Omega)$ be a weak solution of (1.1). We study the regularity for (1.1) and prove a local gradient estimate of u in Orlicz space.

Firstly, we prove a local L^q-gradient estimate of u. Using (1.9), we have

$$L^{\alpha_1}(\Omega) \subset L^{\phi}(\Omega) \subset L^{\alpha_2}(\Omega),$$

then $|f|^{p(x)} \in L^{\alpha_2}_{loc}(\Omega)$. It follows from (1.2), (1.3) that for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^N$,

$$\left|\big(A(x)\xi \cdot \xi\big)^{\frac{p(x)-2}{2}} A(x)\xi\right| \le c|\xi|^{p(x)-1}$$

and

$$c|\xi|^{p(x)} \le \big(A(x)\xi \cdot \xi\big)^{\frac{p(x)-2}{2}} A(x)\xi \cdot \xi,$$

where $c = c(p_+, p_-, \Lambda_1, \Lambda_2, \|A\|_\infty)$. By Theorem 2.1, there exist $R_0 > 0$, $0 < \sigma_0 < \alpha_2 - 1$ such that for any (not necessarily concentric) cube $Q_{2R} \subset Q_{4R_0}$ and $0 < \sigma \le \sigma_0$,

$$\left(\fint_{Q_R} |\nabla u|^{p(x)(1+\sigma)}\,dx\right)^{\frac{1}{1+\sigma}} \le C \fint_{Q_{2R}} |\nabla u|^{p(x)}\,dx + C\left(\fint_{Q_{2R}} |f|^{p(x)(1+\sigma)}\,dx + 1\right)^{\frac{1}{1+\sigma}}. \tag{3.1}$$

Take $0 < \sigma < \min\{\sigma_0, p_- - 1\}$. Then $|f|^{p(x)} \in L^{1+\sigma}_{loc}(\Omega)$, which implies $|\nabla u|^{p(x)} \in L^{1+\sigma}_{loc}(\Omega)$.

For any $\varepsilon \in (0,\sigma)$, by (1.8), there exists $R_1 < \min\{R_0, \frac{1}{2\sqrt{N}}\}$ such that for any $R < R_1$,

$$\omega(2\sqrt{N}R) < \min\left\{\varepsilon, \frac{\sigma}{4}\right\} \quad \text{and} \quad \omega(2\sqrt{N}R)\ln\left(\frac{1}{2\sqrt{N}R}\right) < \varepsilon. \tag{3.2}$$

Denote $A_Q = ((a_{ij})_Q)_{N\times N}$. Using (1.3), we get $\Lambda_1|\xi|^2 \le A_Q\xi \cdot \xi \le \Lambda_2|\xi|^2$ for any $\xi \in \mathbb{R}^N$. As A is of vanishing mean oscillation, we assume that for any $R < R_1$,

$$\fint_{Q_{2R}} \|A - A_{Q_{2R}}\|_\infty\,dx < \varepsilon^{\frac{p_- - 1}{p_-}\frac{\sigma}{1+\sigma}},$$

which implies

$$\left(\fint_{Q_{2R}} \|A - A_{Q_{2R}}\|_\infty^{\frac{p_-}{p_- - 1}\frac{1+\sigma}{\sigma}}\,dx\right)^{\frac{p_- - 1}{p_-}\frac{\sigma}{1+\sigma}} < c\varepsilon, \tag{3.3}$$

where $c = c(p_-, \sigma, \|A\|_\infty)$.

For any $R < R_1$, take $\widetilde{Q} \subset Q_R$ with the side length $2r < R_1$. Next, under the following assumptions: there exist $\lambda > 1$ and $\delta \in (0,1)$ such that

$$\fint_{4\widetilde{Q}} |\nabla u|^{p(x)}\,dx \le \lambda \tag{3.4}$$

and

$$\left(\fint_{4\widetilde{Q}} \left(|f|^{p(x)}+1\right)^{1+\sigma} dx\right)^{\frac{1}{1+\sigma}} \le \delta\lambda, \tag{3.5}$$

where $4\widetilde{Q}$ is a cube with the same center as $\widetilde{Q}$ and the side length $8r$, several technical results will be established. In the following, we derive two kinds of local gradient estimates of u.

Lemma 3.1. *Denote $p_2 = \sup_{x\in 2\widetilde{Q}} p(x)$. Under assumptions (3.4) and (3.5), we obtain*

$$\fint_{2\widetilde{Q}} |\nabla u|^{p_2}\, dx \le c\lambda \tag{3.6}$$

and

$$\int_{2\widetilde{Q}} |\nabla u|^{p_2}\, dx \le c, \tag{3.7}$$

where $c = c(p_+, p_-, \sigma, K, N)$ and $K = \int_{Q_{4R_0}} (|\nabla u|^{p(x)} + |f|^{p(x)(1+\sigma)})\, dx + 1$.

Proof. For any $x \in 2\widetilde{Q}$, using (1.7) we have

$$p_2 \le p(x) + \omega(4\sqrt{N}r) \le p(x)\left(1+\omega(4\sqrt{N}r)\right) \le p(x)(1+\sigma),$$

which implies $|\nabla u| \in L^{p_2}(2\widetilde{Q})$. It follows from (3.1), (3.2) and Hölder inequality that

$$\begin{aligned}
\fint_{2\widetilde{Q}} |\nabla u|^{p_2}\, dx &\le \fint_{2\widetilde{Q}} \left(|\nabla u|^{p(x)(1+\omega(4\sqrt{N}r))} + 1\right) dx \\
&\le c\left\{\left(\fint_{4\widetilde{Q}} |\nabla u|^{p(x)}\, dx\right)^{1+\omega(4\sqrt{N}r)} + \fint_{4\widetilde{Q}} \left(|f|^{p(x)(1+\omega(4\sqrt{N}r)} + 1\right) dx\right\} \\
&= c\left\{\fint_{4\widetilde{Q}} |\nabla u|^{p(x)}\, dx \cdot |4\widetilde{Q}|^{-\omega(4\sqrt{N}r)} \left(\int_{4\widetilde{Q}} |\nabla u|^{p(x)}\, dx\right)^{\omega(4\sqrt{N}r)}\right. \\
&\quad + \left(\fint_{4\widetilde{Q}} \left(|f|^{p(x)(1+\omega(4\sqrt{N}r)} + 1\right) dx\right)^{\frac{1}{1+\omega(4\sqrt{N}r)}} |4\widetilde{Q}|^{-\frac{\omega(4\sqrt{N}r)}{1+\omega(4\sqrt{N}r)}} \\
&\quad \left.\times \left(\int_{4\widetilde{Q}} \left(|f|^{p(x)(1+\omega(4\sqrt{N}r)} + 1\right) dx\right)^{\frac{\omega(4\sqrt{N}r)}{1+\omega(4\sqrt{N}r)}}\right\} \\
&\le cr^{-N\omega(4\sqrt{N}r)}\left\{K^{\omega(4\sqrt{N}r)} \fint_{4\widetilde{Q}} |\nabla u|^{p(x)}\, dx\right. \\
&\quad \left. + \left(\fint_{4\widetilde{Q}} \left(|f|^{p(x)(1+\sigma)} + 1\right) dx\right)^{\frac{1}{1+\sigma}} \left(\int_{4\widetilde{Q}} \left(|f|^{p(x)(1+\sigma)} + 1\right) dx\right)^{\frac{\omega(4\sqrt{N}r)}{1+\omega(4\sqrt{N}r)}}\right\} \\
&\le cr^{-N\omega(4\sqrt{N}r)} K^{\sigma}\lambda,
\end{aligned}$$

where $c = c(p_+, p_-, \sigma, K, N)$. Note that

$$r^{-N\omega(4\sqrt{N}r)} = e^{\ln r^{-N\omega(4\sqrt{N}r)}} = e^{N\omega(4\sqrt{N}r)\ln(4\sqrt{N})} e^{N\omega(4\sqrt{N}r)\ln\frac{1}{4\sqrt{N}r}}.$$

Using (3.2), we obtain

$$r^{-N\omega(4\sqrt{N}r)} \le c,$$

where $c = c(\sigma, N)$, then

$$\fint_{2\widetilde{Q}} |\nabla u|^{p_2}\,dx \le c\lambda,$$

where $c = c(p_+, p_-, \sigma, K, N)$. Similarly, we get

$$\begin{aligned}
\int_{2\widetilde{Q}} |\nabla u|^{p_2}\,dx &\le \int_{2\widetilde{Q}} \big(|\nabla u|^{p(x)(1+\omega(4\sqrt{N}r)} + 1\big)\,dx \\
&= |2\widetilde{Q}| \fint_{2\widetilde{Q}} \big(|\nabla u|^{p(x)(1+\omega(4\sqrt{N}r))} + 1\big)\,dx \\
&\le c|2\widetilde{Q}| \left\{ \left(\fint_{4\widetilde{Q}} |\nabla u|^{p(x)}\,dx \right)^{1+\omega(4\sqrt{N}r)} + \fint_{4\widetilde{Q}} \big(|f|^{p(x)(1+\omega(4\sqrt{N}r)} + 1\big)\,dx \right\} \\
&\le c\left\{ |2\widetilde{Q}| \cdot |4\widetilde{Q}|^{-1-\omega(4\sqrt{N}r)} \left(\int_{4\widetilde{Q}} |\nabla u|^{p(x)}\,dx \right)^{1+\omega(4\sqrt{N}r)} + \int_{4\widetilde{Q}} \big(|f|^{p(x)(1+\sigma)} + 1\big)\,dx \right\} \\
&\le cr^{-N\omega(4\sqrt{N}r)} K^{1+\sigma} + cK \\
&\le c,
\end{aligned}$$

where $c = c(p_+, p_-, \sigma, K, N)$. Now, we complete the proof. $\square$

Let $v \in W^{1,p_2}(2\widetilde{Q})$ with $v - u \in W_0^{1,p_2}(2\widetilde{Q})$ be the weak solution for the following equation

$$\begin{cases} \operatorname{div}\big((A_{2\widetilde{Q}}\nabla v \cdot \nabla v)^{\frac{p_2-2}{2}} A_{2\widetilde{Q}}\nabla v\big) = 0 & x \in 2\widetilde{Q}, \\ v(x) = u(x) & x \in \partial(2\widetilde{Q}), \end{cases}$$

i.e. for any $\varphi \in W_0^{1,p_2}(2\widetilde{Q})$,

$$\int_{2\widetilde{Q}} (A_{2\widetilde{Q}}\nabla v \cdot \nabla v)^{\frac{p_2-2}{2}} A_{2\widetilde{Q}}\nabla v \nabla\varphi\,dx = 0. \tag{3.8}$$

Using the L^q estimates of the above equation (refer to inequalities (2.2) and (2.3) in [13]), we get

$$\|\nabla v\|_{L^\infty(\frac{3}{2}\widetilde{Q})} \le c\left(\fint_{2\widetilde{Q}} |\nabla v|^{p_2}\,dx \right)^{\frac{1}{p_2}} \le c\left(\fint_{2\widetilde{Q}} |\nabla u|^{p_2}\,dx \right)^{\frac{1}{p_2}},$$

where $c = c(p_+, p_-, \sigma, \Lambda_1, \Lambda_2, N)$. Then, it follows from (3.6) that

$$\|\nabla v\|_{L^\infty(\frac{3}{2}\widetilde{Q})} \le c\bigg(\fint_{2\widetilde{Q}} |\nabla u|^{p_2}\,dx\bigg)^{\frac{1}{p_2}} \le c\lambda^{\frac{1}{p_2}} \triangleq (C_0\lambda)^{\frac{1}{p_2}},$$

where $c = c(p_+, p_-, \sigma, \Lambda_1, \Lambda_2, K, N)$. Thus, for any $x \in \frac{3}{2}\widetilde{Q}$,

$$M^*_{\frac{3}{2}\widetilde{Q}}\big(|\nabla v|^{p_2}\big)(x) \le C_0\lambda. \tag{3.9}$$

Next, based on the above regularity results (3.6), (3.7) and (3.9), we give a comparison estimate associated to u and v in $2\widetilde{Q}$. In the proof of the following comparison result, we will use estimates in $L\log^\beta L$ spaces (refer to [1]) several times.

Lemma 3.2. *There exists $\delta_0 \in (0,1)$ such that for any $0 < \delta < \delta_0$, if hypotheses (3.4) and (3.5) are fulfilled, then*

$$\fint_{2\widetilde{Q}} |\nabla u - \nabla v|^{p_2}\,dx \le c\varepsilon^{\frac{p_2}{p_2-1}}\lambda, \tag{3.10}$$

where $c = c(p_+, p_-, \sigma, K, N, \Lambda_1, \Lambda_2, \|A\|_\infty)$.

Proof. Choosing $\varphi = u - v$ as a test function in (1.5) and (3.8), respectively, we get

$$\begin{aligned}
&\int_{2\widetilde{Q}} \big((A_{2\widetilde{Q}}\nabla u\cdot\nabla u)^{\frac{p_2-2}{2}}A_{2\widetilde{Q}}\nabla u - (A_{2\widetilde{Q}}\nabla v\cdot\nabla v)^{\frac{p_2-2}{2}}A_{2\widetilde{Q}}\nabla v\big)\nabla(u-v)\,dx \\
&\quad= \int_{2\widetilde{Q}} (A_{2\widetilde{Q}}\nabla u\cdot\nabla u)^{\frac{p_2-2}{2}}A_{2\widetilde{Q}}\nabla u\cdot\nabla(u-v)\,dx - \int_{2\widetilde{Q}} (A\nabla u\cdot\nabla u)^{\frac{p(x)-2}{2}}A\nabla u\cdot\nabla(u-v)\,dx \\
&\qquad+ \int_{2\widetilde{Q}} |f|^{p(x)-2}f\nabla(u-v)\,dx \\
&\quad= \int_{2\widetilde{Q}} \big((A_{2\widetilde{Q}}\nabla u\cdot\nabla u)^{\frac{p_2-2}{2}}A_{2\widetilde{Q}}\nabla u - (A_{2\widetilde{Q}}\nabla u\cdot\nabla u)^{\frac{p(x)-2}{2}}A_{2\widetilde{Q}}\nabla u\big)\nabla(u-v)\,dx \\
&\qquad+ \int_{2\widetilde{Q}} \big((A_{2\widetilde{Q}}\nabla u\cdot\nabla u)^{\frac{p(x)-2}{2}}A_{2\widetilde{Q}}\nabla u - (A\nabla u\cdot\nabla u)^{\frac{p(x)-2}{2}}A\nabla u\big)\nabla(u-v)\,dx \\
&\qquad+ \int_{2\widetilde{Q}} |f|^{p(x)-2}f\nabla(u-v)\,dx.
\end{aligned}$$

From the algebraic inequalities (refer to inequalities (3.14) and (3.23) in [19]): for any $\xi, \eta \in \mathbb{R}^N$,

$$|\xi-\eta|^q \le c\big((A_{2\widetilde{Q}}\xi\cdot\xi)^{\frac{q-2}{2}}A_{2\widetilde{Q}}\xi - (A_{2\widetilde{Q}}\eta\cdot\eta)^{\frac{q-2}{2}}A_{2\widetilde{Q}}\eta\big)(\xi-\eta),$$

if $2 \le q < \infty$ and

$$|\xi-\eta|^q \le c\theta^{\frac{q-2}{q}}\big((A_{2\widetilde{Q}}\xi\cdot\xi)^{\frac{q-2}{2}}A_{2\widetilde{Q}}\xi - (A_{2\widetilde{Q}}\eta\cdot\eta)^{\frac{q-2}{2}}A_{2\widetilde{Q}}\eta\big)(\xi-\eta) + \theta|\eta|^q,$$

if $1 < q < 2$, where $\theta \in (0,1)$, we get:

(i) The case $p_2 \geq 2$. By the Young inequality, for any $\tau \in (0,1)$,

$$\begin{aligned}
&\int_{2\widetilde{Q}} |\nabla u - \nabla v|^{p_2}\,dx \\
&\quad\leq c\int_{2\widetilde{Q}} \big((A_{2\widetilde{Q}}\nabla u\cdot\nabla u)^{\frac{p_2-2}{2}}A_{2\widetilde{Q}}\nabla u - (A_{2\widetilde{Q}}\nabla v\cdot\nabla v)^{\frac{p_2-2}{2}}A_{2\widetilde{Q}}\nabla v\big)\nabla(u-v)\,dx \\
&\quad\leq \tau\int_{2\widetilde{Q}} |\nabla u - \nabla v|^{p_2}\,dx + c(\tau)\int_{2\widetilde{Q}} \big|(A_{2\widetilde{Q}}\nabla u\cdot\nabla u)^{\frac{p_2-2}{2}}A_{2\widetilde{Q}}\nabla u - (A_{2\widetilde{Q}}\nabla u\cdot\nabla u)^{\frac{p(x)-2}{2}}A_{2\widetilde{Q}}\nabla u\big|^{\frac{p_2}{p_2-1}}\,dx \\
&\qquad + c(\tau)\int_{2\widetilde{Q}} |f|^{\frac{p(x)-1}{p_2-1}p_2}\,dx + c(\tau)\int_{2\widetilde{Q}} \big|(A_{2\widetilde{Q}}\nabla u\cdot\nabla u)^{\frac{p(x)-2}{2}}A_{2\widetilde{Q}}\nabla u - (A\nabla u\cdot\nabla u)^{\frac{p(x)-2}{2}}A\nabla u\big|^{\frac{p_2}{p_2-1}}\,dx,
\end{aligned}$$

which implies

$$\begin{aligned}
&\int_{2\widetilde{Q}} |\nabla u - \nabla v|^{p_2}\,dx \\
&\quad\leq c\int_{2\widetilde{Q}} |f|^{\frac{p(x)-1}{p_2-1}p_2}\,dx + c\int_{2\widetilde{Q}} \big|(A_{2\widetilde{Q}}\nabla u\cdot\nabla u)^{\frac{p_2-2}{2}}A_{2\widetilde{Q}}\nabla u - (A_{2\widetilde{Q}}\nabla u\cdot\nabla u)^{\frac{p(x)-2}{2}}A_{2\widetilde{Q}}\nabla u\big|^{\frac{p_2}{p_2-1}}\,dx \\
&\qquad + c\int_{2\widetilde{Q}} \big|(A_{2\widetilde{Q}}\nabla u\cdot\nabla u)^{\frac{p(x)-2}{2}}A_{2\widetilde{Q}}\nabla u - (A\nabla u\cdot\nabla u)^{\frac{p(x)-2}{2}}A\nabla u\big|^{\frac{p_2}{p_2-1}}\,dx. \qquad (3.11)
\end{aligned}$$

(ii) The case $1 < p_2 < 2$. For $\theta, \tau \in (0,1)$, we get

$$\begin{aligned}
&\int_{2\widetilde{Q}} |\nabla u - \nabla v|^{p_2}\,dx \\
&\quad\leq \theta\int_{2\widetilde{Q}} |\nabla u|^{p_2}\,dx + c\theta^{\frac{p_2-2}{p_2}}\int_{2\widetilde{Q}} |f|^{p(x)-2}f\nabla(u-v)\,dx \\
&\qquad + c\theta^{\frac{p_2-2}{p_2}}\int_{2\widetilde{Q}} \big((A_{2\widetilde{Q}}\nabla u\cdot\nabla u)^{\frac{p_2-2}{2}}A_{2\widetilde{Q}}\nabla u - (A_{2\widetilde{Q}}\nabla u\cdot\nabla u)^{\frac{p(x)-2}{2}}A_{2\widetilde{Q}}\nabla u\big)\nabla(u-v)\,dx \\
&\qquad + c\theta^{\frac{p_2-2}{p_2}}\int_{2\widetilde{Q}} \big((A_{2\widetilde{Q}}\nabla u\cdot\nabla u)^{\frac{p(x)-2}{2}}A_{2\widetilde{Q}}\nabla u - (A\nabla u\cdot\nabla u)^{\frac{p(x)-2}{2}}A\nabla u\big)\nabla(u-v)\,dx \\
&\quad\leq \theta\int_{2\widetilde{Q}} |\nabla u|^{p_2}\,dx + c\theta^{\frac{p_2-2}{p_2}}\tau\int_{2\widetilde{Q}} |\nabla u - \nabla v|^{p_2}\,dx + c(\tau)\theta^{\frac{p_2-2}{p_2}}\int_{2\widetilde{Q}} |f|^{\frac{p(x)-1}{p_2-1}p_2}\,dx \\
&\qquad + c(\tau)\theta^{\frac{p_2-2}{p_2}}\int_{2\widetilde{Q}} \big|(A_{2\widetilde{Q}}\nabla u\cdot\nabla u)^{\frac{p_2-2}{2}}A_{2\widetilde{Q}}\nabla u - (A_{2\widetilde{Q}}\nabla u\cdot\nabla u)^{\frac{p(x)-2}{2}}A_{2\widetilde{Q}}\nabla u\big|^{\frac{p_2}{p_2-1}}\,dx \\
&\qquad + c(\tau)\theta^{\frac{p_2-2}{p_2}}\int_{2\widetilde{Q}} \big|(A_{2\widetilde{Q}}\nabla u\cdot\nabla u)^{\frac{p(x)-2}{2}}A_{2\widetilde{Q}}\nabla u - (A\nabla u\cdot\nabla u)^{\frac{p(x)-2}{2}}A\nabla u\big|^{\frac{p_2}{p_2-1}}\,dx.
\end{aligned}$$

Take τ sufficiently small, we obtain

$$\begin{aligned}
&\int_{2\widetilde{Q}} |\nabla u - \nabla v|^{p_2}\,dx \\
&\quad \le 2\theta \int_{2\widetilde{Q}} |\nabla u|^{p_2}\,dx + c(\theta)\int_{2\widetilde{Q}} |f|^{\frac{p(x)-1}{p_2-1}p_2}\,dx \\
&\qquad + c(\theta)\int_{2\widetilde{Q}} \bigl|(A_{2\widetilde{Q}}\nabla u\cdot\nabla u)^{\frac{p_2-2}{2}}A_{2\widetilde{Q}}\nabla u - (A_{2\widetilde{Q}}\nabla u\cdot\nabla u)^{\frac{p(x)-2}{2}}A_{2\widetilde{Q}}\nabla u\bigr|^{\frac{p_2}{p_2-1}}\,dx \\
&\qquad + c(\theta)\int_{2\widetilde{Q}} \bigl|(A_{2\widetilde{Q}}\nabla u\cdot\nabla u)^{\frac{p(x)-2}{2}}A_{2\widetilde{Q}}\nabla u - (A\nabla u\cdot\nabla u)^{\frac{p(x)-2}{2}}A\nabla u\bigr|^{\frac{p_2}{p_2-1}}\,dx. \qquad (3.12)
\end{aligned}$$

By Hölder inequality, we get

$$\begin{aligned}
⨍_{2\widetilde{Q}} |f|^{\frac{p(x)-1}{p_2-1}p_2}\,dx &\le ⨍_{2\widetilde{Q}} \bigl(|f|^{p(x)}+1\bigr)\,dx \\
&\le \biggl(⨍_{2\widetilde{Q}} \bigl(|f|^{p(x)}+1\bigr)^{1+\sigma}\,dx\biggr)^{\frac{1}{1+\sigma}} \le 2^N\delta\lambda. \qquad (3.13)
\end{aligned}$$

In the following, we consider $\int_{2\widetilde{Q}} |(A_{2\widetilde{Q}}\nabla u\cdot\nabla u)^{\frac{p_2-2}{2}}A_{2\widetilde{Q}}\nabla u - (A_{2\widetilde{Q}}\nabla u\cdot\nabla u)^{\frac{p(x)-2}{2}}A_{2\widetilde{Q}}\nabla u|^{\frac{p_2}{p_2-1}}\,dx$. For any $x\in 2\widetilde{Q}$, there exists $\gamma\in(0,1)$ such that

$$\begin{aligned}
&(A_{2\widetilde{Q}}\nabla u\cdot\nabla u)^{\frac{p_2-2}{2}}A_{2\widetilde{Q}}\nabla u - (A_{2\widetilde{Q}}\nabla u\cdot\nabla u)^{\frac{p(x)-2}{2}}A_{2\widetilde{Q}}\nabla u \\
&\quad = \frac{p_2-p(x)}{2}\ln(A_{2\widetilde{Q}}\nabla u\cdot\nabla u)(A_{2\widetilde{Q}}\nabla u\cdot\nabla u)^{\frac{(1-\gamma)(p(x)-2)+\gamma(p_2-2)}{2}}A_{2\widetilde{Q}}\nabla u,
\end{aligned}$$

then

$$\begin{aligned}
&\int_{2\widetilde{Q}} \bigl|(A_{2\widetilde{Q}}\nabla u\cdot\nabla u)^{\frac{p_2-2}{2}}A_{2\widetilde{Q}}\nabla u - (A_{2\widetilde{Q}}\nabla u\cdot\nabla u)^{\frac{p(x)-2}{2}}A_{2\widetilde{Q}}\nabla u\bigr|^{\frac{p_2}{p_2-1}}\,dx \\
&\quad \le c\int_{2\widetilde{Q}} \bigl|\omega(4\sqrt{N}r)|\nabla u|^{(1-\gamma)(p(x)-2)+\gamma(p_2-2)+1}\bigl(|\ln\Lambda_1|+|\ln\Lambda_2|+\bigl|\ln|\nabla u|\bigr|\bigr)\bigr|^{\frac{p_2}{p_2-1}}\,dx, \qquad (3.14)
\end{aligned}$$

where $c=c(p_+,p_-,\Lambda_2,N,\|A\|_\infty)$. Note that

$$\begin{aligned}
&\int_{2\widetilde{Q}} \bigl||\nabla u|^{(1-\gamma)(p(x)-2)+\gamma(p_2-2)+1}\bigl(|\ln\Lambda_1|+|\ln\Lambda_2|+\bigl|\ln|\nabla u|\bigr|\bigr)\bigr|^{\frac{p_2}{p_2-1}}\,dx \\
&\quad = \int_{\{x\in 2\widetilde{Q}:|\nabla u(x)|>e\}} \bigl||\nabla u|^{(1-\gamma)(p(x)-2)+\gamma(p_2-2)+1}\bigl(|\ln\Lambda_1|+|\ln\Lambda_2|+\bigl|\ln|\nabla u|\bigr|\bigr)\bigr|^{\frac{p_2}{p_2-1}}\,dx \\
&\qquad + \int_{\{x\in 2\widetilde{Q}:|\nabla u(x)|\le e\}} \bigl||\nabla u|^{(1-\gamma)(p(x)-2)+\gamma(p_2-2)+1}\bigl(|\ln\Lambda_1|+|\ln\Lambda_2|+\bigl|\ln|\nabla u|\bigr|\bigr)\bigr|^{\frac{p_2}{p_2-1}}\,dx
\end{aligned}$$

$$\le c\int_{2\widetilde{Q}} |\nabla u|^{p_2} \ln^{\frac{p_2}{p_2-1}}\left(e+|\nabla u|^{p_2}\right) dx + c|2\widetilde{Q}|$$

$$\le c|2\widetilde{Q}| ⨍_{2\widetilde{Q}} |\nabla u|^{p_2} \ln^{\frac{p_2}{p_2-1}}\left(e+|\nabla u|^{p_2}\big\||\nabla u|^{p_2}\big\|_1^{-1}\right) dx$$

$$+ c|2\widetilde{Q}| ⨍_{2\widetilde{Q}} |\nabla u|^{p_2} \ln^{\frac{p_2}{p_2-1}}\left(e+\big\||\nabla u|^{p_2}\big\|_1\right) dx + c|2\widetilde{Q}|, \tag{3.15}$$

where $c = c(p_+, p_-, \Lambda_1, \Lambda_2)$, $\||\nabla u|^{p_2}\|_1 = ⨍_{2\widetilde{Q}} |\nabla u|^{p_2}\, dx$.

For any $x \in 2\widetilde{Q}$, we get

$$\begin{aligned} p_2\left(1+\frac{\sigma}{4}\right) &\le \left(p(x)+\omega(4\sqrt{N}r)\right)\left(1+\frac{\sigma}{4}\right) \\ &\le p(x)\left(1+\frac{\sigma}{4}\right)+\omega(4\sqrt{N}r)p_- \\ &\le p(x)\left(1+\frac{\sigma}{4}+\omega(4\sqrt{N}r)\right). \end{aligned}$$

Using estimate in $L\log^\beta L$ (refer to (28) in [1]), we obtain

$$⨍_{2\widetilde{Q}} |\nabla u|^{p_2} \ln^{\frac{p_2}{p_2-1}}\left(e+|\nabla u|^{p_2}\big\||\nabla u|^{p_2}\big\|_1^{-1}\right) dx \le c\left(⨍_{2\widetilde{Q}} |\nabla u|^{p_2(1+\frac{\sigma}{4})}\, dx\right)^{\frac{1}{1+\frac{\sigma}{4}}}.$$

Then, similarly to the proof of (3.6) in Lemma 3.1,

$$\begin{aligned} &⨍_{2\widetilde{Q}} |\nabla u|^{p_2} \ln^{\frac{p_2}{p_2-1}}\left(e+|\nabla u|^{p_2}\big\||\nabla u|^{p_2}\big\|_1^{-1}\right) dx \\ &\quad\le c\left(⨍_{2\widetilde{Q}} |\nabla u|^{p_2(1+\frac{\sigma}{4})}\, dx\right)^{\frac{1}{1+\frac{\sigma}{4}}} \\ &\quad\le c\left(⨍_{2\widetilde{Q}} |\nabla u|^{p(x)(1+\frac{\sigma}{4}+\omega(4\sqrt{N}r))}\, dx+1\right)^{\frac{1}{1+\frac{\sigma}{4}}} \\ &\quad\le c\lambda, \end{aligned} \tag{3.16}$$

where $c = c(p_+, p_-, \sigma, K, N)$ and further

$$\begin{aligned} &⨍_{2\widetilde{Q}} |\nabla u|^{p_2} \ln^{\frac{p_2}{p_2-1}}\left(e+\big\||\nabla u|^{p_2}\big\|_1\right) dx \\ &\quad= |2\widetilde{Q}|^{-1}\int_{2\widetilde{Q}} |\nabla u|^{p_2}\, dx \cdot \ln^{\frac{p_2}{p_2-1}}\left(|2\widetilde{Q}|^{-1}\cdot|2\widetilde{Q}|e+|2\widetilde{Q}|^{-1}\int_{2\widetilde{Q}} |\nabla u|^{p_2}\, dx\right) \\ &\quad\le c|2\widetilde{Q}|^{-1}\left\{\int_{2\widetilde{Q}} |\nabla u|^{p_2}\, dx\cdot\left|\ln\left(|2\widetilde{Q}|^{-1}\right)\right|^{\frac{p_2}{p_2-1}} + \int_{2\widetilde{Q}} |\nabla u|^{p_2}\, dx\cdot\left|\ln\left(|2\widetilde{Q}|e+\int_{2\widetilde{Q}} |\nabla u|^{p_2}\, dx\right)\right|^{\frac{p_2}{p_2-1}}\right\} \end{aligned}$$

$$\le c \fint_{2\widetilde{Q}} |\nabla u|^{p_2}\,dx \cdot \left|\ln\left(|2\widetilde{Q}|^{-1}\right)\right|^{\frac{p_2}{p_2-1}} + c\left(|2\widetilde{Q}|e + \int_{2\widetilde{Q}} |\nabla u|^{p_2}\,dx\right)^{1+\sigma}$$

$$\le c \fint_{2\widetilde{Q}} |\nabla u|^{p_2}\,dx \cdot \left|\ln\left(|2\widetilde{Q}|^{-1}\right)\right|^{\frac{p_2}{p_2-1}} + c\left(|2\widetilde{Q}|e + \int_{2\widetilde{Q}} |\nabla u|^{p_2}\,dx\right)^{\sigma} \cdot \left(e + \fint_{2\widetilde{Q}} |\nabla u|^{p_2}\,dx\right),$$

where $c = c(p_+, p_-, \sigma)$. It follows from Lemma 3.1 that

$$\fint_{2\widetilde{Q}} |\nabla u|^{p_2} \ln^{\frac{p_2}{p_2-1}}\left(e + \left\||\nabla u|^{p_2}\right\|_1\right)dx \le cK^\sigma\lambda \ln^{\frac{p_2}{p_2-1}}\left(|2\widetilde{Q}|^{-1}\right) + cK^{\sigma(1+\sigma)}\left(e + cK^\sigma\lambda\right)$$
$$\le cK^\sigma\lambda \ln^{\frac{p_2}{p_2-1}}\left(|2\widetilde{Q}|^{-1}\right) + cK^{\sigma(2+\sigma)}\lambda, \tag{3.17}$$

where $c = c(p_+, p_-, \sigma, N)$. Then, using (3.14)–(3.17) we get

$$\fint_{2\widetilde{Q}} \left|(A_{2\widetilde{Q}}\nabla u \cdot \nabla u)^{\frac{p_2-2}{2}} A_{2\widetilde{Q}}\nabla u - (A_{2\widetilde{Q}}\nabla u \cdot \nabla u)^{\frac{p(x)-2}{2}} A_{2\widetilde{Q}}\nabla u\right|^{\frac{p_2}{p_2-1}} dx$$
$$\le c\omega(4\sqrt{N}r)^{\frac{p_2}{p_2-1}}\left(K^\sigma \ln^{\frac{p_2}{p_2-1}}\left(|2\widetilde{Q}|^{-1}\right) + K^{\sigma(2+\sigma)}\right)\lambda$$
$$\le c\varepsilon^{\frac{p_2}{p_2-1}}\lambda, \tag{3.18}$$

where $c = c(p_+, p_-, \sigma, K, N)$.

Next, we consider $\fint_{2\widetilde{Q}} |(A_{2\widetilde{Q}}\nabla u \cdot \nabla u)^{\frac{p(x)-2}{2}} A_{2\widetilde{Q}}\nabla u - (A\nabla u \cdot \nabla u)^{\frac{p(x)-2}{2}} A\nabla u|^{\frac{p_2}{p_2-1}}\,dx$. For any $\xi \in \mathbb{R}^N$, we get

$$\left|(A_{2\widetilde{Q}}\xi \cdot \xi)^{\frac{p(x)-2}{2}} A_{2\widetilde{Q}}\xi - (A\xi \cdot \xi)^{\frac{p(x)-2}{2}} A\xi\right| \le \|A - A_{2\widetilde{Q}}\|_\infty |\xi|^{p(x)-1}.$$

Using Hölder inequality and (3.3), we have

$$\fint_{2\widetilde{Q}} \left|(A_{2\widetilde{Q}}\nabla u \cdot \nabla u)^{\frac{p(x)-2}{2}} A_{2\widetilde{Q}}\nabla u - (A\nabla u \cdot \nabla u)^{\frac{p(x)-2}{2}} A\nabla u\right|^{\frac{p_2}{p_2-1}} dx$$
$$\le c \fint_{2\widetilde{Q}} \|A - A_{2\widetilde{Q}}\|_\infty^{\frac{p_2}{p_2-1}} |\nabla u|^{(p(x)-1)\frac{p_2}{p_2-1}}\,dx$$
$$\le c \fint_{2\widetilde{Q}} \|A - A_{2\widetilde{Q}}\|_\infty^{\frac{p_2}{p_2-1}} \left(|\nabla u|^{p(x)} + 1\right)dx$$
$$\le c\left(\fint_{2\widetilde{Q}} \|A - A_{2\widetilde{Q}}\|_\infty^{\frac{p_2}{p_2-1}\frac{1+\sigma}{\sigma}}\,dx\right)^{\frac{\sigma}{1+\sigma}} \left(\fint_{2\widetilde{Q}} \left(|\nabla u|^{p(x)} + 1\right)^{1+\sigma} dx\right)^{\frac{1}{1+\sigma}}$$
$$\le c\varepsilon^{\frac{p_2}{p_2-1}} \left\{\fint_{4\widetilde{Q}} |\nabla u|^{p(x)}\,dx + \left(\fint_{4\widetilde{Q}} \left(|f|^{p(x)(1+\sigma)} + 1\right)dx\right)^{\frac{1}{1+\sigma}}\right\}$$
$$\le c\varepsilon^{\frac{p_2}{p_2-1}}\lambda, \tag{3.19}$$

where $c = c(p_+, p_-, N, \Lambda_1, \Lambda_2, \|A\|_\infty)$.

It follows from (3.11)–(3.13), (3.18) and (3.19) that

$$\begin{aligned}\fint_{2\widetilde{Q}} |\nabla u - \nabla v|^{p_2}\,dx &\le 2\theta \fint_{2\widetilde{Q}} |\nabla u|^{p_2}\,dx + c(\theta)2^N\delta\lambda + c(\theta)c\varepsilon^{\frac{p_2}{p_2-1}}\lambda + c\varepsilon^{\frac{p_2}{p_2-1}}\lambda \\ &\le c\theta\lambda + c(\theta)2^N\delta\lambda + c(\theta)\varepsilon^{\frac{p_2}{p_2-1}}\lambda + c\varepsilon^{\frac{p_2}{p_2-1}}\lambda,\end{aligned}$$

where $c = c(p_+, p_-, \sigma, K, N, \Lambda_1, \Lambda_2, \|A\|_\infty)$. Take θ, δ sufficiently small, we obtain

$$\fint_{2\widetilde{Q}} |\nabla u - \nabla v|^{p_2}\,dx \le c\varepsilon^{\frac{p_2}{p_2-1}}\lambda,$$

where $c = c(p_+, p_-, \sigma, K, N, \Lambda_1, \Lambda_2, \|A\|_\infty)$. Now, we complete the proof. $\square$

From Lemma 3.2, we get the following result, which will be used in the proof of Theorem 1.1.

Lemma 3.3. *Assume that hypotheses (3.4) and (3.5) with $0 < \delta < \delta_0$ are fulfilled, where δ_0 is from Lemma 3.2. For any cube $Q \in \mathcal{D}(Q_R)$, if $\widetilde{Q} \in \mathcal{D}(Q_R)$ and it is the predecessor of Q, then*

$$\left|\left\{x \in Q : M^*_{\frac{3}{2}\widetilde{Q}}\left(|\nabla u|^{p(\cdot)}\right)(x) > 2^{p_+ + 2}C_0\lambda\right\}\right| \le \varepsilon_1|Q|,$$

where $\varepsilon_1 = c(p_+, p_-, \sigma, K, N, \Lambda_1, \Lambda_2, \|A\|_\infty)\varepsilon^{\frac{p_2}{p_2-1}}$.

Proof. Note that

$$|\nabla u|^{p_2} \le 2^{p_+}|\nabla u - \nabla v|^{p_2} + 2^{p_+}|\nabla v|^{p_2}.$$

Take $Q \in \mathcal{D}(Q_R)$. Then, for any $x \in Q$,

$$M^*_{\frac{3}{2}\widetilde{Q}}\left(|\nabla u|^{p_2}\right)(x) \le 2^{p_+}M^*_{\frac{3}{2}\widetilde{Q}}\left(|\nabla u - \nabla v|^{p_2}\right)(x) + 2^{p_+}M^*_{\frac{3}{2}\widetilde{Q}}\left(|\nabla v|^{p_2}\right)(x).$$

As $M^*_{\frac{3}{2}\widetilde{Q}}$ is weak $(1,1)$ type, it follows from (3.9) and (3.10) that

$$\begin{aligned}&\left|\left\{x \in Q : M^*_{\frac{3}{2}\widetilde{Q}}\left(|\nabla u|^{p_2}\right)(x) > 2^{p_+ + 1}C_0\lambda\right\}\right| \\ &\quad\le \left|\left\{x \in Q : 2^{p_+}M^*_{\frac{3}{2}\widetilde{Q}}\left(|\nabla u - \nabla v|^{p_2}\right)(x) > 2^{p_+}C_0\lambda\right\}\right| + \left|\left\{x \in Q : 2^{p_+}M^*_{\frac{3}{2}\widetilde{Q}}\left(|\nabla v|^{p_2}\right)(x) > 2^{p_+}C_0\lambda\right\}\right| \\ &\quad= \left|\left\{x \in Q : 2^{p_+}M^*_{\frac{3}{2}\widetilde{Q}}\left(|\nabla u - \nabla v|^{p_2}\right)(x) > 2^{p_+}C_0\lambda\right\}\right| \\ &\quad\le \frac{c(N)}{C_0\lambda}\int_{\frac{3}{2}\widetilde{Q}} |\nabla u - \nabla v|^{p_2}\,dx \\ &\quad\le \frac{c(N)}{C_0\lambda}c\varepsilon^{\frac{p_2}{p_2-1}}\lambda|2\widetilde{Q}| \\ &\quad\le c\varepsilon^{\frac{p_2}{p_2-1}}|Q|,\end{aligned}$$

where $c = c(p_+, p_-, \sigma, K, N, \Lambda_1, \Lambda_2, \|A\|_\infty)$ and $c(N)$ is the constant in weak $(1,1)$ type estimate for $M^*_{\frac{3}{2}\widetilde{Q}}$.

For any $y \in 2\widetilde{Q}$, $1 \le p(y) \le p_2$, then $|\nabla u|^{p(y)} \le |\nabla u|^{p_2} + 1$. By using the definition of the Maximal Function Operator in Section 2, we have

$$M^*_{\frac{3}{2}\widetilde{Q}}\big(|\nabla u|^{p(\cdot)}\big)(x) \le M^*_{\frac{3}{2}\widetilde{Q}}\big(|\nabla u|^{p_2}+1\big)(x) \le M^*_{\frac{3}{2}\widetilde{Q}}\big(|\nabla u|^{p_2}\big)(x)+1.$$

If $M^*_{\frac{3}{2}\widetilde{Q}}(|\nabla u|^{p(\cdot)})(x) > 2^{p_+ +2}C_0\lambda$, we have

$$2^{p_+ +2}C_0\lambda < M^*_{\frac{3}{2}\widetilde{Q}}\big(|\nabla u|^{p(\cdot)}\big)(x) \le M^*_{\frac{3}{2}\widetilde{Q}}\big(|\nabla u|^{p_2}\big)(x)+1.$$

Then,

$$M^*_{\frac{3}{2}\widetilde{Q}}\big(|\nabla u|^{p_2}\big)(x) > 2^{p_+ +2}C_0\lambda - 1 = 2^{p_+ +1}C_0\lambda + 2^{p_+ +1}C_0\lambda - 1.$$

As $C_0 > 1$, $\lambda > 1$, we get $2^{p_+ +1}C_0\lambda - 1 > 0$. Then,

$$M^*_{\frac{3}{2}\widetilde{Q}}\big(|\nabla u|^{p_2}\big)(x) > 2^{p_+ +1}C_0\lambda,$$

which implies

$$\big|\big\{x\in Q : M^*_{\frac{3}{2}\widetilde{Q}}\big(|\nabla u|^{p(\cdot)}\big)(x) > 2^{p_+ +2}C_0\lambda\big\}\big| \le \big|\big\{x\in Q : M^*_{\frac{3}{2}\widetilde{Q}}\big(|\nabla u|^{p_2}\big)(x) > 2^{p_+ +1}C_0\lambda\big\}\big|.$$

Thus

$$\big|\big\{x\in Q : M^*_{\frac{3}{2}\widetilde{Q}}\big(|\nabla u|^{p(\cdot)}\big)(x) > 2^{p_+ +2}C_0\lambda\big\}\big| \le c\varepsilon^{\frac{p_2}{p_2-1}}|Q| \triangleq \varepsilon_1|Q|.$$

Now, we complete the proof. □

Finally, by using a consequence of a Calderón and Zygmund type covering argument (refer to Proposition 1 in [1]), we complete the proof of the main result.

Proof of Theorem 1.1. Take $\lambda_0 = \frac{c(N)4^N}{\varepsilon_1}⨍_{Q_{4R}}|\nabla u|^{p(x)}\,dx + 1$, where ε_1 is from Lemma 3.3 and $c(N)$ is the constant in weak $(1,1)$ type estimate for $M^*_{\frac{3}{2}\widetilde{Q}}$. Denote

$$\mu_1(t) = \big|\big\{x\in Q_R : M^*_{Q_{4R}}\big(|\nabla u|^{p(\cdot)}\big)(x) > t\big\}\big|,$$
$$\mu_2(t) = \big|\big\{x\in Q_R : M^*_{1+\sigma,Q_{4R}}\big(\big|f(\cdot)\big|^{p(\cdot)}+1\big)(x) > t\big\}\big|.$$

Then

$$\mu_1(\lambda_0) \le \frac{c(N)}{\lambda_0}\int_{Q_{4R}}|\nabla u|^{p(x)}\,dx \le \varepsilon_1|Q_R|.$$

Take $C_1 = 2^{p_+ +2}C_0 + 5^N + a_2 > 1$, we get $\mu_1(C_1\lambda_0) \le \mu_1(\lambda_0) \le \varepsilon_1|Q_R|$.

Take $0<\delta<\delta_0$ and denote

$$X = \big\{x\in Q_R : M^*_{Q_{4R}}\big(|\nabla u|^{p(\cdot)}\big)(x) > C_1\lambda_0\big\},$$
$$Y = \big\{x\in Q_R : M^*_{Q_{4R}}\big(|\nabla u|^{p(\cdot)}\big)(x) > \lambda_0\big\} \cup \big\{x\in Q_R : M^*_{1+\sigma,Q_{4R_0}}\big(|f(\cdot)|^{p(\cdot)}+1\big)(x) > \delta\lambda_0\big\},$$

where δ_0 is from Lemma 3.2. Then

$$X\subset Y\subset Q_R \quad\text{and}\quad |X| < \varepsilon_1|Q_R|. \tag{3.20}$$

In the following, we verify that for any $Q \in \mathcal{D}(Q_R)$,

$$\text{if } |X \cap Q| > \varepsilon_1 |Q|, \quad \text{its predecessor } \widetilde{Q} \subset Y. \tag{3.21}$$

Indeed, if the conclusion is not satisfied, we may assume that there exits $x_0 \in \widetilde{Q}$ such that $x_0 \notin Y$, which implies

$$M^*_{Q_{4R}}\big(|\nabla u|^{p(\cdot)}\big)(x_0) \le \lambda_0 \quad \text{and} \quad M^*_{1+\sigma,Q_{4R}}\big(\big|f(\cdot)\big|^{p(\cdot)} + 1\big)(x_0) \le \delta\lambda_0.$$

Thus, by the definition of $M^*_{Q_{4R}}(|\nabla u|^{p(\cdot)})(x_0)$ and $M^*_{1+\sigma,Q_{4R}}(|f(\cdot)|^{p(\cdot)} + 1)(x_0)$, for any $\widehat{Q} \subseteq Q_{4R}$ with $x_0 \in \widehat{Q}$, we get

$$\fint_{\widehat{Q}} |\nabla u|^{p(x)}\,dx \le \lambda_0 \quad \text{and} \quad \bigg(\fint_{\widehat{Q}} \big(|f|^{p(x)} + 1\big)^{1+\sigma}\,dx\bigg)^{\frac{1}{1+\sigma}} \le \delta\lambda_0.$$

Especially, we have

$$\fint_{4\widetilde{Q}} |\nabla u|^{p(x)}\,dx \le \lambda_0 \quad \text{and} \quad \bigg(\fint_{4\widetilde{Q}} \big(|f|^{p(x)} + 1\big)^{1+\sigma}\,dx\bigg)^{\frac{1}{1+\sigma}} \le \delta\lambda_0.$$

Then, similarly to the proof of Lemma 2, step 4 in [1], we will verify that

$$\big|\big\{x \in Q : M^*_{Q_{4R}}\big(|\nabla u|^{p(\cdot)}\big)(x) > C_1\lambda_0\big\}\big| \le \varepsilon_1|Q|.$$

For any $x \in Q$, if $Q' \subseteq Q_{4R}$ with $x \in Q'$ and its length $r' \ge \frac{r}{2}$, it follows from $Q' \cap \widetilde{Q} \ne \emptyset$ that there exists $\widehat{Q} \subseteq Q_{4R}$ containing both Q' and $\widetilde{Q}$ and its length $r'' \le 2r + r' \le 5r'$, then

$$\fint_{Q'} |\nabla u|^{p(x)}\,dx \le \frac{1}{|Q'|}\int_{\widehat{Q}} |\nabla u|^{p(x)}\,dx = \frac{|\widehat{Q}|}{|Q'|}\fint_{\widehat{Q}} |\nabla u|^{p(x)}\,dx \le 5^N\lambda_0.$$

If $Q' \subseteq Q_{4R}$ with $x \in Q'$ and its length $r' \le \frac{r}{2}$, then $Q' \subseteq \frac{3}{2}\widetilde{Q}$ and

$$\fint_{Q'} |\nabla u|^{p(x)}\,dx \le M^*_{\frac{3}{2}\widetilde{Q}}\big(|\nabla u|^{p(\cdot)}\big)(x).$$

Therefore, for any $x \in Q$, $M^*_{Q_{4R}}(|\nabla u|^{p(\cdot)})(x) \le \max\{M^*_{\frac{3}{2}\widetilde{Q}}(|\nabla u|^{p(\cdot)})(x), 5^N\lambda_0\}$. By Lemma 3.3,

$$\big|\big\{x \in Q : M^*_{\frac{3}{2}\widetilde{Q}}\big(|\nabla u|^{p(\cdot)}\big)(x) > 2^{p_+ +2}C_0\lambda\big\}\big| \le \varepsilon_1|Q|,$$

then

$$|X \cap Q| = \big|\big\{x \in Q : M^*_{Q_{4R}}\big(|\nabla u|^{p(\cdot)}\big)(x) > C_1\lambda_0\big\}\big| \le \big|\big\{x \in Q : M^*_{\frac{3}{2}\widetilde{Q}}\big(|\nabla u|^{p(\cdot)}\big)(x) > 2^{p_+ +2}C_0\lambda\big\}\big| \le \varepsilon_1|Q|,$$

that is a contradiction.

Then, it follows from (3.20), (3.21) and Proposition 1 in [1] that

$$|X| < \varepsilon_1|Y|.$$

Thus, we get

$$\mu_1(C_1\lambda_0) \le \varepsilon_1\mu_1(\lambda_0) + \varepsilon_1\mu_2(\delta\lambda_0).$$

Next, by induction, we verify that for any $n \in \mathbb{N}$,

$$\mu_1\big(C_1^n\lambda_0\big) \le \varepsilon_1^n\mu_1(\lambda_0) + \sum_{i=1}^{n} \varepsilon_1^i\mu_2\big(\delta C_1^{n-i}\lambda_0\big). \tag{3.22}$$

Firstly, we assume that $\mu_1(C_1^k\lambda_0) \le \varepsilon_1^k\mu_1(\lambda_0) + \sum_{i=1}^k \varepsilon_1^i\mu_2(\delta C_1^{k-i}\lambda_0)$. Then, repeating the above process, we get

$$\begin{aligned}\mu_1\big(C_1^{k+1}\lambda_0\big) = \mu_1\big(C_1C_1^k\lambda_0\big) &\le \varepsilon_1\mu_1\big(C_1^k\lambda_0\big) + \varepsilon_1\mu_2\big(\delta C_1^k\lambda_0\big)\\ &\le \varepsilon_1^{k+1}\mu_1(\lambda_0) + \sum_{i=1}^{k} \varepsilon_1^{i+1}\mu_2\big(\delta C_1^{k-i}\lambda_0\big) + \varepsilon_1\mu_2\big(\delta C_1^k\lambda_0\big)\\ &= \varepsilon_1^{k+1}\mu_1(\lambda_0) + \sum_{i=1}^{k+1} \varepsilon_1^{i}\mu_2\big(\delta C_1^{k+1-i}\lambda_0\big).\end{aligned}$$

Note that

$$\begin{aligned}\int_{Q_R} \phi\big(|\nabla u|^{p(x)}\big)\,dx &\le \int_{Q_R} \phi\big(M_{Q_{4R}}\big(|\nabla u|^{p(\cdot)}\big)(x)\big)\,dx\\ &= \int_0^\infty \mu_1(\lambda)\,d\phi(\lambda)\\ &= \int_0^{C_1\lambda_0} \mu_1(\lambda)\,d\phi(\lambda) + \sum_{n=1}^{\infty} \int_{C_1^n\lambda_0}^{C_1^{n+1}\lambda_0} \mu_1(\lambda)\,d\phi(\lambda).\end{aligned}$$

Using (1.9) and (3.22), we have

$$\int_0^{C_1\lambda_0} \mu_1(\lambda)\,d\phi(\lambda) \le |Q_R|\phi(C_1\lambda_0) \le |Q_R|a_1^{-1}(C_1\lambda_0)^{\alpha_1}\phi(1)$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} \int_{C_1^n\lambda_0}^{C_1^{n+1}\lambda_0} \mu_1(\lambda)\,d\phi(\lambda) &\le \sum_{n=1}^{\infty} \phi\big(C_1^{n+1}\lambda_0\big)\mu_1\big(C_1^n\lambda_0\big)\\ &\le \sum_{n=1}^{\infty} \phi\big(C_1^{n+1}\lambda_0\big)\left\{\varepsilon_1^n\mu_1(\lambda_0) + \sum_{i=1}^{n} \varepsilon_1^i\mu_2\big(\delta C_1^{n-i}\lambda_0\big)\right\}.\end{aligned}$$

Take $\varepsilon_1 < \frac{1}{2}C_1^{-\alpha_1}$. It follows from (1.9) that

$$\sum_{n=1}^{\infty} \phi\big(C_1^{n+1}\lambda_0\big)\varepsilon_1^n\mu_1(\lambda_0) \le \sum_{n=1}^{\infty} a_1^{-1}\phi(1)\big(C_1^{n+1}\lambda_0\big)^{\alpha_1}\varepsilon_1^n\mu_1(\lambda_0) \le a_1^{-1}\phi(1)(C_1\lambda_0)^{\alpha_1}|Q_R|$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{i=1}^{n}\phi\big(C_1^{n+1}\lambda_0\big)\varepsilon_1^i\mu_2\big(\delta C_1^{n-i}\lambda_0\big)\\
&\quad=\sum_{i=1}^{\infty}\sum_{n=i}^{\infty}\phi\big(C_1^{n+1}\lambda_0\big)\varepsilon_1^i\mu_2\big(\delta C_1^{n-i}\lambda_0\big)\\
&\quad=\sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\phi\big(C_1^{k+i+1}\lambda_0\big)\varepsilon_1^i\mu_2\big(\delta C_1^{k}\lambda_0\big)\\
&\quad\le a_1^{-1}C_1^{\alpha_1}\delta^{-\alpha_1}\sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\big(\varepsilon_1C_1^{\alpha_1}\big)^i\phi\big(\delta C_1^k\lambda_0\big)\mu_2\big(\delta C_1^k\lambda_0\big)\\
&\quad\le a_1^{-1}C_1^{\alpha_1}\delta^{-\alpha_1}\sum_{k=0}^{\infty}\phi\big(\delta C_1^k\lambda_0\big)\mu_2\big(\delta C_1^k\lambda_0\big).
\end{aligned}$$

Then

$$\begin{aligned}
\int_{Q_R}\phi\big(|\nabla u|^{p(x)}\big)\,dx &\le |Q_R|a_1^{-1}(C_1\lambda_0)^{\alpha_1}\phi(1)+|Q_R|a_1^{-1}\phi(1)(C_1\lambda_0)^{\alpha_1}\\
&\quad+a_1^{-1}C_1^{\alpha_1}\delta^{-\alpha_1}\sum_{k=0}^{\infty}\phi\big(\delta C_1^k\lambda_0\big)\mu_2\big(\delta C_1^k\lambda_0\big). \qquad (3.23)
\end{aligned}$$

Define $\widetilde{\phi}(t)=\phi(t^{\frac{1}{1+\sigma}})$ for any $t>0$. By (1.9), there exists $a>1$ such that $\widetilde{\phi}(t)<\frac{1}{2a}\widetilde{\phi}(at)$. Then, as a direct consequence of Theorem 2 in [18], we get

$$\int_{Q_R}\widetilde{\phi}\big(M^*_{Q_{4R}}\big(\big|f(\cdot)\big|^{p(\cdot)(1+\sigma)}+1\big)(x)\big)\,dx\le c\int_{Q_R}\widetilde{\phi}\big(\big|f(x)\big|^{p(x)(1+\sigma)}+1\big)\,dx.$$

As $M^*_{1+\sigma,Q_{4R}}(|f(\cdot)|^{p(\cdot)}+1)(x)=(M^*_{Q_{4R}}(|f(\cdot)|^{p(\cdot)(1+\sigma)}+1)(x))^{\frac{1}{1+\sigma}}$, we have

$$\int_{Q_R}\phi\big(M^*_{1+\sigma,Q_{4R}}\big(\big|f(\cdot)\big|^{p(\cdot)}+1\big)(x)\big)\,dx=\int_{Q_R}\widetilde{\phi}\big(M^*_{Q_{4R}}\big(\big|f(\cdot)\big|^{p(\cdot)(1+\sigma)}+1\big)(x)\big)\,dx.$$

Thus,

$$\begin{aligned}
c\int_{Q_R}\phi\big(|f|^{p(x)}+1\big)\,dx &\ge \int_{Q_R}\phi\big(M^*_{1+\sigma,Q_{4R}}\big(\big|f(\cdot)\big|^{p(\cdot)}+1\big)(x)\big)\,dx\\
&=\int_0^{\infty}\mu_2(\lambda)\,d\phi(\lambda)\\
&=\int_0^{\delta\lambda_0}\mu_2(\lambda)\,d\phi(\lambda)+\sum_{n=0}^{\infty}\int_{\delta C_1^n\lambda_0}^{\delta C_1^{n+1}\lambda_0}\mu_2(\lambda)\,d\phi(\lambda).
\end{aligned}$$

Note that

$$\int_0^{\delta\lambda_0} \mu_2(\lambda)\, d\phi(\lambda) \geq \mu_2(\delta\lambda_0)\phi(\delta\lambda_0),$$

and using (1.9), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \int_{\delta C_1^n \lambda_0}^{\delta C_1^{n+1}\lambda_0} \mu_2(\lambda)\, d\phi(\lambda) &\geq \sum_{n=0}^{\infty} \mu_2\big(\delta C_1^{n+1}\lambda_0\big)\big(\phi\big(\delta C_1^{n+1}\lambda_0\big) - \phi\big(\delta C_1^n \lambda_0\big)\big) \\ &\geq \sum_{n=0}^{\infty} \big(1 - c_2 C_1^{-\alpha_1}\big)\mu_2\big(\delta C_1^{n+1}\lambda_0\big)\phi\big(\delta C_1^{n+1}\lambda_0\big).\end{aligned}$$

Thus

$$c \int_{Q_R} \phi\big(|f|^{p(x)} + 1\big)\, dx \geq \sum_{n=0}^{\infty} \big(1 - c_2 C_1^{-\alpha_1}\big)\mu_2\big(\delta C_1^n \lambda_0\big)\phi\big(\delta C_1^n \lambda_0\big).$$

It follows from (3.23) that

$$\begin{aligned}\int_{Q_R} \phi\big(|\nabla u|^{p(x)}\big)\, dx &\leq 2|Q_R|\phi(1)a_1^{-1}(C_1\lambda_0)^{\alpha_1} + a_1^{-1}C_1^{\alpha_1}\delta^{-\alpha_1}\big(1 - a_2 C_1^{-\alpha_1}\big)^{-1} \int_{Q_R} \phi\big(|f|^{p(x)} + 1\big)\, dx \\ &\leq \big\{2\phi(1)a_1^{-1}(C_1\lambda_0)^{\alpha_1} + a_1^{-1}C_1^{\alpha_1}\delta^{-\alpha_1}\big(1 - a_2 C_1^{-\alpha_1}\big)^{-1}\big\} \int_{Q_R} \phi\big(|f|^{p(x)} + 1\big)\, dx.\end{aligned}$$

Now, we complete the proof. □

References

[1] E. Acerbi, G. Mingione, Gradient estimates for the $p(x)$-Laplacean system, J. Reine Angew. Math. 584 (2005) 117–148.
[2] T. Adamowicz, P. Hästö, Harnack's inequality and the strong $p(x)$-Laplacian, J. Differential Equations 250 (3) (2011) 1631–1649.
[3] R.A. Adams, J.J.F. Fournier, Sobolev Spaces, second edition, Academic Press, New York, 2003.
[4] C.O. Alves, M.A.S. Souto, Existence of solutions for a class of problems in $\mathbb{R}^N$ involving $p(x)$-Laplacian, in: Contributions to Nonlinear Analysis, in: Progr. Nonlinear Differential Equations Appl., vol. 66, Birkhäuser, Basel, 2006, pp. 17–32.
[5] S. Antontsev, M. Chipot, Y. Xie, Uniquenesss results for equation of the $p(x)$-Laplacian type, Adv. Math. Sci. Appl. 17 (1) (2007) 287–304.
[6] S.S. Byun, Elliptic equations with BMO coefficients in Lipschitz domains, Trans. Amer. Math. Soc. 357 (3) (2005) 1025–1046.
[7] S.S. Byun, L.H. Wang, S.L. Zhou, Nonlinear elliptic equations with small BMO coefficients in Reifenberg domains, J. Funct. Anal. 250 (1) (2007) 167–196.
[8] S.S. Byun, F.P. Yao, S.L. Zhou, Gradient estimates in Orlicz space for nonlinear elliptic equations, J. Funct. Anal. 255 (8) (2008) 1851–1873.
[9] S.S. Byun, S.J. Ryu, Gradient estimates for higher order elliptic equations on nonsmooth domains, J. Differential Equations 250 (1) (2011) 243–263.
[10] J. Chabrowski, Y.Q. Fu, Existence of solutions for $p(x)$-Laplacian problems on a bounded domain, J. Math. Anal. Appl. 306 (2) (2005) 604–618. Erratum in: J. Math. Anal. Appl. 323 (2006) 1483.
[11] Y. Chen, S. Levine, M. Rao, Variable exponent, linear growth functionals in image restoration, SIAM J. Appl. Math. 66 (4) (2006) 1383–1406.
[12] G. Di Fazio, L^p estimates for divergence form elliptic equations with discontinuous coefficients, Boll. Unione Mat. Ital. Sez. A 10 (2) (1996) 409–420.
[13] E. DiBenedetto, J. Manfredi, On the higher integrability of the gradient of weak solutions of certain degenerate elliptic systems, Amer. J. Math. 115 (5) (1993) 1107–1134.
[14] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, Legesgue and Sobolev Spaces with Variable Exponents, Lecture Notes in Math., vol. 2017, Springer-Verlag, Heidelberg, 2011.
[15] X.L. Fan, X.Y. Han, Existence and multiplicity of solutions for $p(x)$-Laplacian equations in $\mathbb{R}^N$, Nonlinear Anal. 59 (1–2) (2004) 173–188.

[16] Y.Q. Fu, X. Zhang, Multiple solutions for a class of $p(x)$-Laplacian equations in $\mathbb{R}^N$ involving the critical exponent, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 466 (2118) (2010) 1667–1686.
[17] T. Iwaniec, Projections onto gradient fields and L^p-estimates for degenerated elliptic operators, Studia Math. 75 (3) (1983) 293–312.
[18] H.L. Jia, D.S. Li, L.H. Wang, Regularity of Orlicz spaces for the Poisson equation, Manuscripta Math. 122 (3) (2007) 265–275.
[19] J. Kinnunen, S.L. Zhou, A local estimate for nonlinear equations with discontinuous coefficients, Comm. Partial Differential Equations 24 (11–12) (1999) 2043–2068.
[20] O. Kováčik, J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, Czechoslovak Math. J. 41 (4) (1991) 592–618.
[21] M. Mihăilescu, V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 462 (2073) (2006) 2625–2641.
[22] M. Mihăilescu, V. Rădulescu, On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent, Proc. Amer. Math. Soc. 135 (9) (2007) 2929–2937.
[23] C.B. Morrey, Multiple Integrals in the Calculus of Variations, Springer-Verlag, Berlin, 1966.
[24] M. Růžička, Electro-Rheological Fluids: Modeling and Mathematical Theory, Springer-Verlag, Berlin, 2000.
[25] F.P. Yao, Y. Sun, S.L. Zhou, Gradient estimates in Orlicz spaces for quasilinear elliptic equation, Nonlinear Anal. 69 (8) (2008) 2553–2565.
[26] C. Zhang, S.L. Zhou, Renormalized and entropy solutions for nonlinear parabolic equations with variable exponents and L^1 data, J. Differential Equations 248 (6) (2010) 1376–1400.
[27] X. Zhang, Y.Q. Fu, Solutions for nonlinear elliptic equations with variable growth and degenerate coercivity, Ann. Mat. Pura Appl. 193 (1) (2014) 133–161.