

# Sup–Inf explicit formulas for minimal Lipschitz extensions for 1-fields on $\mathbb{R}^n$



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## ABSTRACT

We study the relationship between the Lipschitz constant of 1-field introduced in [12] and the Lipschitz constant of the gradient canonically associated with this 1-field. Moreover, we produce two explicit formulas which are two extremal minimal Lipschitz extensions for 1-fields. As a consequence of the previous results, for the problem of minimal extension by Lipschitz continuous functions from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ , we produce explicit formulas similar to those of Bauschke and Wang (see [7]). Finally, we show that Wells's extensions (see [24]) of 1-fields are absolutely minimal Lipschitz extensions when the domain of 1-field to expand is finite. We provide a counter-example showing that this result is false in general.

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## 1. Introduction

Let  $\Omega$  be a subset of Euclidean space  $\mathbb{R}^n$ . We suppose that  $\Omega$  has at least two elements. Let  $\mathcal{P}^1(\mathbb{R}^n, \mathbb{R})$  be the set of first degree polynomials mapping  $\mathbb{R}^n$  to  $\mathbb{R}$ , i.e.

$$\mathcal{P}^1(\mathbb{R}^n, \mathbb{R}) \triangleq \{P : a \in \mathbb{R}^n \mapsto P(a) = p + \langle v, a \rangle, p \in \mathbb{R}, v \in \mathbb{R}^n\}.$$

Let us consider a 1-field  $F$  on domain  $\text{dom}(F) \triangleq \Omega$  defined by

$$\begin{aligned} F : \Omega &\rightarrow \mathcal{P}^1(\mathbb{R}^n, \mathbb{R}) \\ x &\mapsto F(x)(a) \triangleq f_x + \langle D_x f; a - x \rangle, \end{aligned} \quad (1)$$

where  $a \in \mathbb{R}^n$  is the evaluation variable of the polynomial  $F(x)$  and  $f : x \in \Omega \mapsto f_x \in \mathbb{R}$ ,  $Df : x \in \Omega \mapsto D_x f \in \mathbb{R}^n$  are mappings associated with  $F$ . We will always use capital letters to denote the 1-field and small letters to denote these mappings.

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The Lipschitz constant of  $F$  introduced in [12] is

$$\Gamma^1(F; \Omega) \triangleq \sup_{\substack{x, y \in \Omega \\ x \neq y}} \Gamma^1(F; x, y), \quad (2)$$

where

$$\Gamma^1(F; x, y) \triangleq 2 \sup_{a \in \mathbb{R}^n} \frac{|F(x)(a) - F(y)(a)|}{\|x - a\|^2 + \|y - a\|^2}. \quad (3)$$

If  $\Gamma^1(F; \Omega) < +\infty$ , then the Whitney conditions [26], or Glaeser in [10] are satisfied and the 1-field  $F$  can be extended on  $\mathbb{R}^n$ : there exists  $g \in \mathcal{C}^{1,1}(\mathbb{R}^n, \mathbb{R})$  such that  $g(x) = f_x$  and  $\nabla g(x) = D_x f$  for all  $x \in \Omega$  where  $\nabla g$  is the usual gradient. Moreover, from [12, Theorem 2.6] we can find  $g$  which satisfies

$$\Gamma^1(G; \mathbb{R}^n) = \Gamma^1(F; \Omega),$$

where  $G$  is the 1-field associated to  $g$ , i.e.

$$G(x)(y) = g(x) + \langle \nabla g(x), y - x \rangle, \quad x \in \Omega, \quad y \in \mathbb{R}^n.$$

It means that the Lipschitz constant does not increase when extending  $F$  by  $G$ . We say that  $G$  is a minimal Lipschitz extension (MLE for short) of  $F$  and we have

$$\Gamma^1(G; \mathbb{R}^n) = \inf \{ \text{Lip}(\nabla h; \mathbb{R}^n) : h(x) = f_x, \quad \nabla h(x) = D_x f, \quad x \in \Omega, \quad h \in \mathcal{C}^{1,1}(\mathbb{R}^n, \mathbb{R}) \},$$

where the notation  $\text{Lip}(u; \cdot)$  means that

$$\text{Lip}(u; x, y) \triangleq \frac{\|u(x) - u(y)\|}{\|x - y\|}, \quad x, y \in \Omega, \quad x \neq y, \quad \text{and} \quad \text{Lip}(u; \Omega) \triangleq \sup_{x \neq y \in \Omega} \text{Lip}(u; x, y). \quad (4)$$

It is worth asking what is the relationship between  $\Gamma^1(F; \Omega)$  and  $\text{Lip}(Df; \Omega)$ ? From [12], we know that  $\text{Lip}(Df; \Omega) \leq \Gamma^1(F; \Omega)$ . In the special case  $\Omega = \mathbb{R}^n$  we have  $\text{Lip}(Df; \mathbb{R}^n) = \Gamma^1(F; \mathbb{R}^n)$  but in general the formula  $\text{Lip}(Df; \Omega) = \Gamma^1(F; \Omega)$  is untrue. In this paper we will prove that if  $\Omega$  is an open subset of  $\mathbb{R}^n$  then

$$\Gamma^1(F; \Omega) = \max \{ \Gamma^1(F; \partial\Omega), \text{Lip}(Df; \Omega) \}, \quad (5)$$

where  $\partial\Omega$  is a boundary of  $\Omega$ . Moreover, if  $\Omega$  is a convex subset of  $\mathbb{R}^n$  then

$$\Gamma^1(F; \Omega) \leq 2 \text{Lip}(Df; \Omega). \quad (6)$$

To make the connection between  $\Gamma^1(F; \Omega)$  and  $\text{Lip}(Df; \Omega)$ , it is important to know the set of uniqueness of minimal extensions of  $F$  when  $\Omega$  has two elements (this study was performed in [14]). Indeed, many results of Section 3 use this knowledge. For further more details see Section 3.

In Section 4, we present two MLEs  $U^+$  and  $U^-$  of  $F$  of the form

$$U^+ : x \in \mathbb{R}^n \mapsto U^+(x)(y) \triangleq u^+(x) + \langle D_x u^+; y - x \rangle, \quad y \in \mathbb{R}^n, \quad (7)$$

where

$$u^+(x) \triangleq \sup_{v \in \Lambda_x} \inf_{a \in \Omega} \Psi^+(F, x, a, v), \quad D_x u^+ \triangleq \arg \sup_{v \in \Lambda_x} \inf_{a \in \Omega} \Psi^+(F, x, a, v), \quad (8)$$

and

$$U^- : x \in \mathbb{R}^n \mapsto U^-(x)(y) \triangleq u^-(x) + \langle D_x u^-; y - x \rangle, \quad y \in \mathbb{R}^n, \quad (9)$$

where

$$u^-(x) \triangleq \inf_{v \in \Lambda_x} \sup_{a \in \Omega} \Psi^-(F, x, a, v), \quad D_x u^- \triangleq \arg \inf_{v \in \Lambda_x} \sup_{a \in \Omega} \Psi^-(F, x, a, v), \quad (10)$$

where  $\Lambda_x$  is a non-empty and convex set of  $\mathbb{R}^n$ , defined in Definition 20. The point here is that (7), (8), (9), (10) give explicit sup inf formulas for  $u^\pm$  and their gradients, that is to say, they only depend on  $F$ . In addition,  $u^\pm$  are *extremal*: the first is *over* and the second is *under*, that is,

$$u^-(x) \leq g_x \leq u^+(x), \quad \forall x \in \mathbb{R}^n,$$

for all MLE  $G$  of  $F$ .

We call the real  $\kappa \in \mathbb{R}$ , with  $\kappa > 0$ , to be allowable for  $F$  if  $\kappa$  satisfies the following inequality

$$f_y \leq f_x + \frac{1}{2} \langle D_x f + D_y f, y - x \rangle + \frac{\kappa}{4} \|x - y\|^2 - \frac{1}{4\kappa} \|D_x f - D_y f\|^2, \quad \forall x, y \in \Omega. \quad (11)$$

From [24, Theorem 2], we know that if  $\kappa > 0$  is allowable for  $F$ , then there exists  $w^+ \in \mathcal{C}^{1,1}(\mathbb{R}^n, \mathbb{R})$  such that  $w^+(x) = f_x$ ,  $\nabla w^+(x) = D_x f$  for all  $x \in \Omega$ , and  $\text{Lip}(\nabla w^+, \mathbb{R}^n) \leq \kappa$ . Further, if  $g \in \mathcal{C}^{1,1}(\mathbb{R}^n, \mathbb{R})$  with  $g(x) = f_x$ ,  $\nabla g(x) = D_x f$  for all  $x \in \Omega$  and  $\text{Lip}(\nabla g, \mathbb{R}^n) \leq \kappa$ , then  $g(x) \leq w^+(x)$  for all  $x \in \mathbb{R}^n$ . The construction of Wells  $w^+$  is explicit when  $\Omega$  is finite. It is possible to extend this construction to infinite domain by passing to the limit but there is no explicit formula. During the workshop Whitney problems in 2011, M. Hirn made the link between the constant  $\Gamma^1$  (see (2)) and the constant  $\kappa > 0$  in (11) that is the real  $\kappa$  to be allowable for  $F$  if and only if  $\Gamma^1(F; \Omega) \leq \kappa$ . Moreover, if  $\kappa$  is assigned the Lipschitz constant of the field  $F$ , then  $w^+$  is a MLE. Further, in this case  $w^+$  is an over extremal extension of  $F$ .

In Section 4, we will prove that if  $\kappa$  is assigned the Lipschitz constant of the field  $F$ , then  $u^+ = w^+$ . In a similar way (see Appendix A), we can construct a Wells function  $w^-$  which is an under minimal extension of  $F$  and  $w^- = u^-$ .

We pay attention to the case when  $\Omega$  is finite. In Section 6, using the explicit constructions of  $w^\pm$ , we prove that  $W^\pm$  are absolutely minimal Lipschitz extensions (AMLEs for short) of  $F$ , where  $W^\pm$  are 1-fields associated to  $w^\pm$  respectively. This means that for any bounded open  $D$  satisfying  $\bar{D} \subset \mathbb{R}^n \setminus \Omega$  we have

$$\Gamma^1(W^\pm, D) = \Gamma^1(W^\pm, \partial D). \quad (12)$$

This result gives the existence of AMLEs of  $F$  when  $\Omega$  is finite. In general, we have not uniqueness, since it may happen  $w^- < w^+$ . In fact, we may even have infinity AMLE of  $F$  (see Corollary 36).

When  $\Omega$  is infinite,  $W^+$  and  $W^-$  are extremal MLEs, but in general are not AMLE of  $F$ . To prove this, we present, in Section 6, an example of mapping  $F$  for which  $W^+$  and  $W^-$  are not AMLE of  $F$ . In this particular example, we can check that  $\frac{1}{2}(W^+ + W^-)$  is the unique AMLE of  $F$ . Moreover, this function is not  $\mathcal{C}^2$  although the domain  $\Omega$  of  $F$  is regular and  $F$  is a regular 1-field. The question of the existence of an AMLE remains an open and difficult problem when  $\Omega$  is infinite, see [14] and the references therein.

In Section 5, we explain how to use the previous ideas and methods to construct MLE of mappings from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ , i.e., to solve the Kirszbraun–Valentine extension problem [17,22,23]. Let us define  $\mathcal{Q}_0$  as the problem of the minimum extension for Lipschitz functions, and  $\mathcal{Q}_1$  as the problem of the minimum extension for 1-fields. Curiously, we will show that the problem  $\mathcal{Q}_0$  is a sub-problem of the problem  $\mathcal{Q}_1$ . As a consequence, we obtain two explicit formulas (see (46) and (47)) that solve the problem  $\mathcal{Q}_0$ . The

Bauschke–Wang result [7] gives an explicit formula for the Kirszbraun–Valentine problem from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ . By our approach, we produce analogous formulas. Moreover, when the domain of the function to extend is finite, the result of Wells gives an explicit construction of minimal Lipschitz extensions that we can compute efficiently.

Let  $u : \text{dom}(u) \subset \mathbb{R}^n \rightarrow \mathbb{R}$  be a Lipschitz function. Departing from the work of McShane [18] and Whitney [26] in 1934, it is known that the following extensions

$$m^+(x) \triangleq \inf_{y \in \text{dom}(u)} (u(y) + \text{Lip}(u; \text{dom}(u)) \|x - y\|), \quad (13)$$

$$m^-(x) \triangleq \sup_{y \in \text{dom}(u)} (u(y) - \text{Lip}(u; \text{dom}(u)) \|x - y\|), \quad (14)$$

are two extremal minimal Lipschitz extensions of  $u$ , so that if  $m$  is an arbitrary minimal Lipschitz extension of  $u$ , then  $m^- \leq m \leq m^+$ . Among all minimal Lipschitz extensions of  $u$ , one can search extensions that have good additional properties. In the 1960s, Aronsson published a series of papers [4,3] in which the notion of AMLE appeared, see also [19,16,20] on metric spaces and [2,15,6] on Euclidean spaces. An AMLE has very good stability properties like harmonious extensions (see [13,11]) which is related to “tug of war” game and the infinity Laplacian (see [21]). Moreover, it is “locally best” and this notion is also positively correlated with the infinity harmonic functions, we refer the reader to [5] and the references therein. Note that the formulas which define  $u^+$  and  $u^-$  for 1-fields in this paper are similar to those of Whitney–McShane in the continuous case. The results of this paper allow us to think that the notion of an AMLE of 1-field is not sufficient from all this point of view. Indeed, the minimal extensions  $u^+$  and  $u^-$  are extremal like  $m^+$  and  $m^-$  in the continuous case, but also they are two AMLE when the domain  $\Omega$  is finite. Moreover, they are not the “locally best” extensions since the Lipschitz constant is not local. Despite these disappointing results, one might think that there exist some extensions that have good stability properties and “locally best” like harmonious extensions in the continuous case (for the definition of stability properties for 1-fields see [1]).

For Whitney extension problem for  $\mathcal{C}^m$ , we recommend the reader to consult the seminal article [25] of H. Whitney on  $\mathbb{R}$  and the seminal articles [9,8] of C. Fefferman on  $\mathbb{R}^m$ .

## 2. Preliminaries

In this paper all subsets  $\Omega \subset \mathbb{R}^n$  have at least two elements. If  $\Omega$  is open, we denote by  $\mathcal{C}^{1,1}(\Omega, \mathbb{R})$  the set of all real-valued functions  $f$  that are differentiable on  $\Omega$  and the differential  $\nabla f$  is Lipschitz continuous, that is  $\text{Lip}(\nabla f; \Omega) < +\infty$ . The 1-field  $F$  of domain  $\Omega$  is defined by (see (1))

$$\begin{aligned} F : \Omega &\rightarrow \mathcal{P}^1(\mathbb{R}^n, \mathbb{R}) \\ x &\mapsto F(x)(a) = f_x + \langle D_x f; a - x \rangle, \quad a \in \mathbb{R}^n \end{aligned} \quad (15)$$

with  $f : x \in \Omega \mapsto f_x \in \mathbb{R}$ , and  $Df : x \in \Omega \mapsto D_x f \in \mathbb{R}^n$ .

**Definition 1.** We call  $F$  to be a Taylorian field on  $\Omega$  if  $F$  is a 1-field on  $\Omega$  and  $\Gamma^1(F, \Omega) < +\infty$ . Denote by  $\mathcal{T}^1(\Omega)$  the set of all Taylorian fields on  $\Omega$ .

**Information and precision for the reader:** Let  $F \in \mathcal{T}^1(\Omega)$ . Let us define the map

$$f(x) \triangleq F(x)(x) = f_x, \quad x \in \Omega.$$

Using [12, Theorem 1.1] there exists  $\tilde{F} \in \mathcal{T}^1(\mathbb{R}^n)$  which extends  $F$ . Moreover  $\tilde{f} \in \mathcal{C}^{1,1}(\mathbb{R}^n, \mathbb{R})$  and  $\nabla f(x) \triangleq \nabla \tilde{f}(x) = D_x f$ ,  $x \in \Omega$ . Therefore, we can canonically associate  $F$  and  $f$ .

Let  $V$  be a subset of  $\mathbb{R}^n$ ,  $V \subset\subset \Omega$  means that  $\bar{V}$  is compact in  $\Omega$ , and  $\bar{V}$  is the closure of  $V$ . Let  $x, y \in \mathbb{R}^n$ . We define  $B(x; r) \triangleq \{y \in \mathbb{R}^n : \|y - x\| < r\}$  and  $B_{1/2}(x, y)$  is the closed ball of center  $\frac{x+y}{2}$  and radius  $\frac{\|x-y\|}{2}$ . The line segment joining two points  $x$  and  $y$  is denoted by  $[x, y]$ , i.e.  $[x, y] \triangleq \{tx + (1-t)y : 0 \leq t \leq 1\}$ .

The  $|$  symbol designates with restriction to.

**Definition 2.** Let  $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^n$  and  $F \in \mathcal{F}^1(\Omega_1)$ . We call  $G \in \mathcal{F}^1(\Omega_2)$  an *extension* of  $F$  on  $\Omega_2$  if  $G(x) = F(x)$  for  $x \in \Omega_1$ . We say that  $G \in \mathcal{F}^1(\Omega_2)$  is a *minimal Lipschitz extension* (MLE) of  $F$  on  $\Omega_2$  if  $G$  is an extension of  $F$  on  $\Omega_2$  and

$$\Gamma^1(G; \Omega_2) = \Gamma^1(F; \Omega_1).$$

We say that  $G_1 \in \mathcal{F}^1(\Omega_2)$  is an *over extremal Lipschitz extension* (over extremal for short) and  $G_2$  is an *under extremal Lipschitz extension* of  $F$  on  $\Omega_2$  if  $G_1$  and  $G_2$  are MLEs of  $F$  on  $\Omega_2$  and

$$g_2(x) \leq k(x) \leq g_1(x), \quad x \in \Omega_2,$$

for all  $K$  MLE of  $F$ . We say that  $G \in \mathcal{F}^1(\Omega_2)$  is an *absolutely minimal Lipschitz extension* (AMLE) of  $F$  on  $\Omega_2$  if  $G$  is a MLE of  $F$  on  $\Omega_2$  and

$$\Gamma^1(G; V) = \Gamma^1(G; \partial V),$$

for any bounded open  $V$  satisfying  $\bar{V} \subset \Omega_2 \setminus \Omega_1$ .

**Definition 3.** Let  $\Omega$  be a subset of  $\mathbb{R}^n$  and let  $F \in \mathcal{F}^1(\Omega)$ . For any  $a \neq b \in \Omega$ , we define

$$A_{a,b}(F) \triangleq \frac{2(f_a - f_b) + \langle D_a f + D_b f, b - a \rangle}{\|a - b\|^2},$$

$$B_{a,b}(F) \triangleq \frac{\|D_a f - D_b f\|}{\|a - b\|}.$$

We recall some results in [12] that will be useful in Sections 3 and 4:

**Proposition 4.** (See [12, Proposition 2.2 and Remark 2.3].) Let  $\Omega$  be a subset of  $\mathbb{R}^n$  and let  $F \in \mathcal{F}^1(\Omega)$ , then for any  $a, b \in \Omega$ ,  $a \neq b$  we have

$$\Gamma^1(F; a, b) = \sqrt{A_{a,b}(F)^2 + B_{a,b}(F)^2} + |A_{a,b}(F)| = 2 \sup_{y \in B_{1/2}(a,b)} \frac{F(a)(y) - F(b)(y)}{\|a - y\|^2 + \|b - y\|^2}.$$

**Theorem 5.** (See [12, Theorem 2.6].) Let  $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^n$  and let  $F \in \mathcal{F}^1(\Omega_1)$ , then there exists a MLE  $G \in \mathcal{F}^1(\Omega_2)$  of  $F$  on  $\Omega_2$ .

### 3. Relations between $\Gamma^1(F; \Omega)$ and $\text{Lip}(Df; \Omega)$

In this section  $\Omega$  is an open subset of  $\mathbb{R}^n$ . Let  $F \in \mathcal{F}^1(\Omega)$ . From Proposition 4, we have  $\Gamma^1(F; \Omega) \geq \text{Lip}(Df; \Omega)$ . When  $\Omega = \mathbb{R}^n$ , we know that (see [12, Proposition 2.4])

$$\text{Lip}(Df; \mathbb{R}^n) = \Gamma^1(F; \mathbb{R}^n),$$

but in general  $\Gamma^1(F; \Omega)$  may be strictly bigger than  $\text{Lip}(Df; \Omega)$ . For example, let  $A$  and  $B$  be open sets in  $\mathbb{R}^n$  such that  $A \cap B = \emptyset$ . Let  $\Omega = A \cup B$  and  $F \in \mathcal{F}^1(\Omega)$  such that  $f_x = 0$  if  $x \in A$ ,  $f_x = 1$  if  $x \in B$ , and

$D_x f = 0, \forall x \in \Omega$ . Then  $\text{Lip}(Df; \Omega) = 0$  and from Proposition 4 we have

$$\Gamma^1(F; \Omega) = \sup_{x \in A} \sup_{y \in B} \frac{4}{\|x - y\|^2} > 0.$$

We now give two new results where we have  $\Gamma^1(F, \Omega) = \text{Lip}(Df, \Omega)$ .

**Proposition 6.** Let  $F \in \mathcal{F}^1(\Omega)$ . Suppose there exist  $a, b \in \Omega$ ,  $a \neq b$  such that  $\Gamma^1(F; a, b) = \Gamma^1(F; \Omega)$ . Then  $\Gamma^1(F; \Omega) = \text{Lip}(Df; \Omega)$ .

**Proof.** It is enough to prove that  $\Gamma^1(F; \Omega) \leq \text{Lip}(Df; \Omega)$ .

Let  $G = F|_{\{a, b\}}$  be a Taylorian field on  $\text{dom}(G) = \{a, b\}$  with  $G(a) = F(a)$  and  $G(b) = F(b)$ . Let  $U$  be a MLE of  $F$  on  $\mathbb{R}^n$ . We have  $U(a) = F(a) = G(a)$ ,  $U(b) = F(b) = G(b)$  and  $\Gamma^1(U; \mathbb{R}^n) = \Gamma^1(F; \Omega) = \Gamma^1(F; a, b) = \Gamma^1(G; \text{dom}(G))$ . Therefore  $U$  is a MLE of  $G$  on  $\mathbb{R}^n$ .

Using [14, Lemma 8 and Lemma 10], there exists a point  $c \in B_{1/2}(a, b)$  such that

$$\text{Lip}(Du; x, y) = \text{Lip}(Du; s, t) = \Gamma^1(G; a, b) = \Gamma^1(F; a, b), \quad (16)$$

for all  $x, y \in [a, c]$  ( $x \neq y$ ) and  $s, t \in [b, c]$  ( $s \neq t$ ).

Since  $a \neq b$ , we have  $c \neq a$  or  $c \neq b$ . We can assume  $c \neq a$ . Because  $\Omega$  is open, there exists  $x \neq y \in [a, c] \cap \Omega$ , thus from (16) we have

$$\text{Lip}(Df; \Omega) \geq \text{Lip}(Df; x, y) = \text{Lip}(Du; x, y) = \Gamma^1(F; a, b) = \Gamma^1(F; \Omega). \quad \square$$

**Proposition 7.** Let  $F \in \mathcal{F}^1(\Omega)$ . Suppose there exists  $\Omega' \subset \subset \Omega$  such that  $\Gamma^1(F; \Omega') = \Gamma^1(F; \Omega)$ . Then  $\Gamma^1(F; \Omega) = \text{Lip}(Df; \Omega)$ .

**Proof.** It is enough to prove that  $\Gamma^1(F; \Omega) \leq \text{Lip}(Df; \Omega)$ . Letting  $h > 0$ , we define  $\Lambda_h = \{(a, b) \in \overline{\Omega'} \times \overline{\Omega'} : |a - b| \geq h\}$  and  $\Gamma_h^1(F; \overline{\Omega'}) = \sup_{(a, b) \in \Lambda_h} \Gamma^1(F; a, b)$ . Applying Proposition 4, the mapping  $(a, b) \mapsto \Gamma^1(F; a, b)$  is continuous on  $\Lambda_h$ . Moreover,  $\Lambda_h$  is compact, thus there exists  $(a_h, b_h) \in \Lambda_h$  such that

$$\Gamma^1(F; a_h, b_h) = \Gamma_h^1(F; \overline{\Omega'}). \quad (17)$$

**Case 1.** There exists  $h > 0$  such that

$$\Gamma_h^1(F; \overline{\Omega'}) = \Gamma^1(F; \overline{\Omega'}). \quad (18)$$

From (17), (18) and the condition  $\Gamma^1(F; \Omega') = \Gamma^1(F; \Omega)$ , we have  $\Gamma^1(F; a_h, b_h) = \Gamma^1(F; \Omega)$ . Applying Proposition 6 we have  $\Gamma^1(F; \Omega) = \text{Lip}(Df; \Omega)$ .

**Case 2.** For all  $h > 0$ , we always have

$$\Gamma_h^1(F; \overline{\Omega'}) < \Gamma^1(F; \overline{\Omega'}). \quad (19)$$

Let  $h = 1/n$ , then for any  $n \in \mathbb{N}$  there exists  $(a_n, b_n) \in \Lambda_{1/n}$  such that  $\Gamma^1(F; a_n, b_n) = \Gamma_{1/n}^1(F; \overline{\Omega'})$ . Since  $(a_n), (b_n) \subset \overline{\Omega'}$  and  $\overline{\Omega'}$  is compact, there exist a subsequence  $(a_{n_k})$  of  $(a_n)$  and a subsequence  $(b_{n_k})$  of  $(b_n)$  such that  $(a_{n_k})$  converges to an element  $a$  of  $\overline{\Omega'}$  and  $(b_{n_k})$  converges to an element  $b$  of  $\overline{\Omega'}$ .

If  $a \neq b$  then

$$\Gamma^1(F; \overline{\Omega'}) = \lim_{k \rightarrow \infty} \Gamma_{1/n_k}^1(F; \overline{\Omega'}) = \lim_{k \rightarrow \infty} \Gamma^1(F; a_{n_k}, b_{n_k}) = \Gamma^1(F; a, b).$$

But this is not possible because for  $l = |a - b| > 0$  we deduce from (19) that  $\Gamma^1(F; a, b) \leq \Gamma_l^1(F; \overline{\Omega'}) < \Gamma^1(F; \overline{\Omega'})$ . Therefore, we must have  $a = b$ . From the proof of [12, Proposition 2.4], we see that if  $B_{1/2}(x, y) \subset \Omega$  then  $\Gamma^1(F; x, y) \leq \text{Lip}(Df; \Omega)$ . We will use this property for proving in the case  $a = b$ . For any  $\varepsilon > 0$ , since  $(a_{n_k})$  and  $(b_{n_k})$  both converge to  $a \in \overline{\Omega'} \subset \Omega$ , there exists  $k \in \mathbb{N}$  such that  $B_{1/2}(a_{n_k}, b_{n_k}) \subset \Omega$  and

$$\varepsilon + \Gamma^1(F; a_{n_k}, b_{n_k}) \geq \Gamma^1(F; \overline{\Omega'}) = \Gamma^1(F; \Omega).$$

Since  $B_{1/2}(a_{n_k}, b_{n_k}) \subset \Omega$  we have

$$\Gamma^1(F; a_{n_k}, b_{n_k}) \leq \text{Lip}(Df; \Omega).$$

Therefore

$$\varepsilon + \text{Lip}(Df; \Omega) \geq \Gamma^1(F; \Omega).$$

This inequality holds for any  $\varepsilon > 0$ , so that we have  $\text{Lip}(Df; \Omega) \geq \Gamma^1(F; \Omega)$ .  $\square$

**Proposition 8.** Let  $\Omega$  be an open and convex set in  $\mathbb{R}^n$  and let  $F \in \mathcal{F}^1(\Omega)$ . Then

$$\Gamma^1(F; \Omega) \leq 2 \text{Lip}(Df; \Omega).$$

**Proof.** Let  $f$  be the canonical associate to  $F$ . We can write

$$f(x) - f(y) = \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle dt.$$

For any  $x, y \in \Omega$  and  $z \in \mathbb{R}^n$  we have

$$\begin{aligned} F(x)(z) - F(y)(z) &= f(x) - f(y) + \langle \nabla f(x), z - x \rangle - \langle \nabla f(y), z - y \rangle \\ &= \int_0^1 \langle \nabla f(y + t(x - y)) - \nabla f(x), x - z \rangle dt + \int_0^1 \langle \nabla f(y + t(x - y)) - \nabla f(y), z - y \rangle dt. \end{aligned}$$

Hence

$$\begin{aligned} |F(x)(z) - F(y)(z)| &\leq \int_0^1 \text{Lip}(\nabla f; \Omega) \|x - y\| \|x - z\| (1 - t) dt + \int_0^1 \text{Lip}(\nabla f; \Omega) \|(x - y)\| \|z - y\| t dt \\ &= \frac{1}{2} \text{Lip}(\nabla f; \Omega) \|x - y\| (\|x - z\| + \|z - y\|) \\ &\leq \frac{1}{2} \text{Lip}(\nabla f; \Omega) (\|x - z\| + \|z - y\|)^2 \\ &\leq \text{Lip}(\nabla f; \Omega) (\|x - z\|^2 + \|z - y\|^2). \end{aligned}$$

Therefore  $\Gamma^1(F; \Omega) \leq 2 \text{Lip}(\nabla f; \Omega) = 2 \text{Lip}(Df; \Omega)$ .  $\square$

From the proof of [Proposition 8](#), we obtain

**Corollary 9.** *Let  $\Omega$  be an open and convex set in  $\mathbb{R}^n$  and  $f \in \mathcal{C}^{1,1}(\Omega, \mathbb{R})$ . Then  $F \in \mathcal{F}^1(\Omega)$  where  $F$  is the 1-field associated to  $f$ .*

**Lemma 10.** *Let  $u \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ , then*

$$\text{Lip}(\nabla u; x, y) \leq \inf_{z \in [x, y]} \max\{\text{Lip}(\nabla u; x, z), \text{Lip}(\nabla u; z, y)\}, \quad \text{for all } x \neq y \in \mathbb{R}^n.$$

**Proof.** Let  $x, y \in \Omega$  and  $z \in [x, y]$ , then we have  $\|x - y\| = \|x - z\| + \|z - y\|$ . It follows

$$\begin{aligned} \text{Lip}(\nabla u; x, y) &\leq \frac{\|x - z\|}{\|x - z\| + \|y - z\|} \text{Lip}(\nabla u; x, z) + \frac{\|z - y\|}{\|x - z\| + \|y - z\|} \text{Lip}(\nabla u; z, y) \\ &\leq \max\{\text{Lip}(\nabla u; x, z), \text{Lip}(\nabla u; z, y)\}. \quad \square \end{aligned}$$

**Proposition 11.** *There exist an open convex  $\Omega$  and  $F \in \mathcal{F}^1(\Omega)$  such that*

$$\text{Lip}(\nabla f; \Omega) < \Gamma^1(F; \Omega).$$

**Proof.** Suppose  $n = 2$ . Let  $a = (-1, 0)$ ,  $b = (1, 0)$ . We define  $U \in \mathcal{F}^1(\{a, b\})$  as

$$D_a u = (0, 1), \quad D_b u = (0, -1), \quad u_a = \frac{1}{\sqrt{3}}, \quad u_b = -\frac{1}{\sqrt{3}}.$$

Let us define  $\kappa \triangleq \Gamma^1(U; \{a, b\})$ ,  $h \triangleq \frac{b-a}{2}$ ,  $v \triangleq \frac{D_a u - D_b u}{2\kappa}$  and  $\beta \triangleq \|v\| \in (0, 1)$ .

Using [\[14, Lemmas 7, 8\]](#), we define

$$\begin{aligned} c &= \frac{a+b}{2} + v, \\ \tilde{u}_c &= U(a)(c) - \frac{\kappa}{2} \|a - c\|^2 = U(b)(c) + \frac{\kappa}{2} \|b - c\|^2, \\ D_c \tilde{u} &= D_a u + \kappa(a - c) = D_b u - \kappa(b - c), \\ \tilde{U}(c)(z) &= \tilde{u}_c + \langle D_c \tilde{u}, z - c \rangle, \quad z \in \mathbb{R}^n, \end{aligned}$$

and

$$f(z) \triangleq \begin{cases} \tilde{U}(c)(z) - \frac{\kappa}{2} \frac{\langle z-c, a-c \rangle^2}{\|a-c\|^2}, & \text{if } p(z) \geq 0 \text{ and } q(z) \leq 0, \\ \tilde{U}(c)(z) + \frac{\kappa}{2} \frac{\langle z-c, b-c \rangle^2}{\|b-c\|^2}, & \text{if } p(z) \leq 0 \text{ and } q(z) \geq 0, \\ \tilde{U}(c)(z), & \text{if } p(z) \leq 0 \text{ and } q(z) \leq 0, \\ \tilde{U}(c)(z) - \frac{\kappa}{2} \frac{\langle z-c, a-c \rangle^2}{\|a-c\|^2} + \frac{\kappa}{2} \frac{\langle z-c, b-c \rangle^2}{\|b-c\|^2}, & \text{if } p(z) \geq 0 \text{ and } q(z) \geq 0, \end{cases}$$

where  $p(z) = \langle a - c, z - c \rangle$  and  $q(z) = \langle b - c, z - c \rangle$ . Then the 1-field  $F$  which is associated with  $f$  is a MLE of  $U$ .

Since  $\|h\| = 1$ ,  $\beta = \|v\| \in (0, 1)$ ,  $c - a = h + v$ , and  $c - b = -h + v$ , we have

$$\langle c - a, w_a \rangle = 0, \quad \langle c - b, w_b \rangle = 0, \tag{20}$$



If  $x \in \Delta_a$ ,  $x' \in \Delta_b$ , then

$$\begin{aligned} \frac{1}{\kappa} \text{Lip}(\nabla f; x, x') &= \left\| \frac{-\langle x - x', c - a \rangle (c - a)}{\|x - x'\| \|c - a\|^2} + \frac{\langle x - x', c - b \rangle (c - b)}{\|x - x'\| \|c - b\|^2} \right\| \\ &= 1 - \frac{(\beta - 1)^2}{1 + \beta^2}. \end{aligned} \quad (24)$$

If  $x = c + \alpha w_b \in \Delta_a$ , then

$$\begin{aligned} \frac{1}{\kappa} \text{Lip}(\nabla f; x, a) &= \frac{|\langle x - a, a - c \rangle|}{\|x - a\| \|a - c\|} \\ &= 1 - \frac{\alpha^2 (\beta^2 - 1)^2}{(1 + \beta^2)(\beta^2(1 - \alpha)^2 + (\beta^2 - \alpha)^2)}. \end{aligned} \quad (25)$$

If  $x = c + \alpha w_a \in \Delta_b$ , then

$$\begin{aligned} \frac{1}{\kappa} \text{Lip}(\nabla f; x, b) &= \frac{|\langle x - b, b - c \rangle|}{\|x - b\| \|b - c\|} \\ &= 1 - \frac{\alpha^2 (\beta^2 - 1)^2}{(1 + \beta^2)(\beta^2(1 - \alpha)^2 + (\beta^2 - \alpha)^2)}. \end{aligned} \quad (26)$$

From (21)–(26), Proposition 4 and noting that

$$1 - \frac{\alpha^2 (\beta^2 - 1)^2}{(1 + \beta^2)(\beta^2(1 - \alpha)^2 + (\beta^2 - \alpha)^2)} \leq 1 - \frac{\alpha_0^2 (\beta^2 - 1)^2}{(1 + \beta^2)(\beta^2(1 - \alpha_0)^2 + (\beta^2 - \alpha_0)^2)},$$

for all  $\alpha \geq \alpha_0$ , we obtain

$$\max_{i=1,2,3} \Gamma^1(F; \omega_i) \leq k\kappa, \quad (27)$$

where

$$k = \sup \left\{ 1 - \frac{(\beta - 1)^2}{1 + \beta^2}, 1 - \frac{\alpha_0^2 (\beta^2 - 1)^2}{(1 + \beta^2)(\beta^2(1 - \alpha_0)^2 + (\beta^2 - \alpha_0)^2)} \right\}. \quad (28)$$

**Step 2.** We will define an open convex  $\Omega$  and a 1-field  $G \in \mathcal{F}^1(\Omega)$  such that

$$\text{Lip}(\nabla g; \Omega) < \Gamma^1(G; \Omega).$$

Let us use the notation

$$R_{x,y} = \{x + t(y - x) : t \in \mathbb{R}^+\}.$$

Define (see Fig. 1)

$$A_1 = \text{convex hull}(\Delta_a \cup R_{x_a,a}), \quad A_2 = \text{convex hull}(\Delta_a \cup \Delta_b), \quad A_3 = \text{convex hull}(R_{x_b,b} \cup \Delta_b),$$

and let  $\Omega$  be the interior of  $A_1 \cup A_2 \cup A_3$ . Then  $\Omega$  is open and convex.

For each  $i \in \{1, 2, 3\}$ , let us consider  $G_i$  as a MLE of  $F|_{\omega_i}$ . We define a 1-field  $G$  on  $\overline{\Omega}$  by

$$G(x) = G_i(x), \quad \text{for } x \in A_i, \text{ for } i \in \{1, 2, 3\}.$$

Let us show that  $g \in \mathcal{C}^1(\Omega, \mathbb{R})$ . Indeed, for all  $x \in \Omega$ , there exists  $r > 0$  such that  $B(x, r) \subset \Omega$ . For all  $h \in B(x, r)$ , we have

$$\begin{aligned} |g_{x+h} - g_x - \langle D_x g; h \rangle| &= |G(x+h)(x+h) - G(x)(x+h)| \\ &\leq \frac{1}{2} \max_{i=1,2,3} \Gamma^1(G_i; A_i) \|h\|^2. \end{aligned}$$

Hence  $g \in \mathcal{C}^1(\Omega, \mathbb{R})$ . Thus, by applying [Lemma 10](#), [Proposition 4](#) and [\(27\)](#) we have

$$\text{Lip}(\nabla g; \Omega) \leq \max_{i=1,2,3} \text{Lip}(\nabla g; A_i) \leq \max_{i=1,2,3} \Gamma^1(F; \omega_i) \leq k\kappa. \quad (29)$$

Therefore, by applying [Corollary 9](#) we have  $G \in \mathcal{F}^1(\Omega)$ . On the other hand, we have

$$\Gamma^1(G; \Omega) = \Gamma^1(G; \overline{\Omega}) \geq \Gamma^1(G; a, b) = \kappa. \quad (30)$$

From [\(29\)](#) and [\(30\)](#), we have

$$\text{Lip}(\nabla g; \Omega) \leq k\kappa \leq k\Gamma^1(G; \Omega) < \Gamma^1(G; \Omega). \quad \square$$

**Remark 12.** With the same notation as in the proof of [Proposition 11](#). There exists an open strictly convex subset  $\Omega'$  of  $\Omega$  such that  $a, b \in \overline{\Omega'}$ , and we have

$$\text{Lip}(\nabla g; \Omega') \leq \text{Lip}(\nabla g; \Omega) \leq k\kappa \leq k\Gamma^1(G; \overline{\Omega'}) = k\Gamma^1(G; \Omega') < \Gamma^1(G; \Omega').$$

Thus in [Proposition 11](#) we can replace  $\Omega$  open convex by  $\Omega$  open strictly convex.

Moreover, when we let  $x_a, x_b$  be such that  $\text{dist}(x_a, [a, b])$  and  $\text{dist}(x_b, [a, b])$  converge to 0, then the constant  $k$  satisfying [\(28\)](#) converges to  $\frac{\sqrt{3}}{2}$ . An interesting question is that what is the optimal constant  $c$  that is the largest constant and satisfies  $\text{Lip}(\nabla g, \Omega) \geq c\Gamma^1(F, \Omega)$  for all  $\Omega$  open convex set and for all  $F \in \mathcal{F}^1(\Omega)$ ? We do not know the accurate value for the optimal constant  $c$ , but from above consideration and [Proposition 8](#), we obtain  $c \in [\frac{1}{2}, \frac{\sqrt{3}}{2}]$ .

**Theorem 13.** Let  $\Omega$  be an open set in  $\mathbb{R}^n$  and let  $F \in \mathcal{F}^1(\overline{\Omega})$ . We have

$$\Gamma^1(F; \Omega) = \max\{\text{Lip}(Df; \Omega), \Gamma^1(F; \partial\Omega)\}.$$

**Proof.** We have  $\Gamma^1(F; \Omega) = \Gamma^1(F; \overline{\Omega})$ . Thus

$$\Gamma^1(F; \Omega) \geq \Gamma^1(F; \partial\Omega). \quad (31)$$

Furthermore, we know that  $\Gamma^1(F; \Omega) \geq \text{Lip}(Df; \Omega)$ . Therefore,

$$\Gamma^1(F; \Omega) \geq \max\{\text{Lip}(Df; \Omega), \Gamma^1(F; \partial\Omega)\}.$$

Conversely, let us turn to the proof of the opposite inequality:

$$\Gamma^1(F; \Omega) \leq \max\{\text{Lip}(Df; \Omega), \Gamma^1(F; \partial\Omega)\}. \quad (32)$$

Let  $F|_{\partial\Omega}$  be the restriction of  $F$  to  $\partial\Omega$  and let  $G$  be a MLE of  $F|_{\partial\Omega}$  on  $\mathbb{R}^n \setminus \Omega$ . We have  $G = F$  on  $\partial\Omega$  and

$$\Gamma^1(G; \mathbb{R}^n \setminus \Omega) = \Gamma^1(F; \partial\Omega). \quad (33)$$

We define

$$H(x) \triangleq \begin{cases} F(x), & \text{if } x \in \Omega, \\ G(x), & \text{if } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

**Step 1.** We will prove that  $H \in \mathcal{F}^1(\mathbb{R}^n)$ . Indeed, let  $x, y \in \mathbb{R}^n$  ( $x \neq y$ ). We have three cases:

*Case 1.* If  $x, y \in \Omega$  ( $x \neq y$ ) then  $\Gamma^1(H; x, y) = \Gamma^1(F; x, y) \leq \Gamma^1(F; \Omega)$ .

*Case 2.* If  $x, y \in \mathbb{R}^n \setminus \Omega$  ( $x \neq y$ ) then by (31) and (33) we have

$$\Gamma^1(H; x, y) = \Gamma^1(G; x, y) \leq \Gamma^1(G; \mathbb{R}^n \setminus \Omega) = \Gamma^1(F; \partial\Omega) \leq \Gamma^1(F; \Omega).$$

*Case 3.* Assume  $x \in \Omega$  and  $y \in \mathbb{R}^n \setminus \Omega$ . Let  $H|_{\{x, y\}}$  be the restriction of  $H$  to  $\text{dom}(H|_{\{x, y\}}) = \{x, y\}$ . From [14, Proposition 2], there exists  $c \in B_{1/2}(x, y)$  such that:

$$\Gamma^1(H; x, y) \leq \max\{\Gamma^1(H; x, z), \Gamma^1(H; z, y)\}, \quad \text{for all } z \in [x, c] \cup [y, c].$$

Letting  $z \in ([x, c] \cup [y, c]) \cap \partial\Omega$ , we obtain

$$\Gamma^1(H; x, y) \leq \max\{\Gamma^1(H; x, z), \Gamma^1(H; z, y)\}.$$

Moreover, since  $x, z \in \overline{\Omega}$  we get

$$\Gamma^1(H; x, z) = \Gamma^1(F; x, z) \leq \Gamma^1(F; \overline{\Omega}) = \Gamma^1(F; \Omega),$$

and since  $y, z \in \mathbb{R}^n \setminus \Omega$  we get

$$\Gamma^1(H; z, y) = \Gamma^1(G; z, y) \leq \Gamma^1(G; \mathbb{R}^n \setminus \Omega) = \Gamma^1(F; \partial\Omega) \leq \Gamma^1(F; \Omega).$$

Therefore  $\Gamma^1(H; x, y) \leq \Gamma^1(F; \Omega)$ .

Combining these three cases we have  $\Gamma^1(H; \mathbb{R}^n) \leq \Gamma^1(F; \Omega) < +\infty$ . This implies that  $H \in \mathcal{F}^1(\mathbb{R}^n)$ .

**Step 2.** We will prove (32). Since  $H \in \mathcal{F}^1(\mathbb{R}^n)$ , we have  $\Gamma^1(H; \mathbb{R}^n) = \text{Lip}(\nabla h; \mathbb{R}^n)$  by [12, Proposition 2.4]. Thus

$$\Gamma^1(F; \Omega) = \Gamma^1(H; \Omega) \leq \Gamma^1(H; \mathbb{R}^n) = \text{Lip}(\nabla h; \mathbb{R}^n).$$

On the other hand,  $\text{Lip}(Df; \Omega) = \text{Lip}(\nabla h; \Omega)$  and

$$\Gamma^1(F; \partial\Omega) = \Gamma^1(G; \mathbb{R}^n \setminus \Omega) = \Gamma^1(H; \mathbb{R}^n \setminus \Omega) \geq \text{Lip}(\nabla h; \mathbb{R}^n \setminus \Omega).$$

Therefore, to prove (32), it suffices to show that

$$\text{Lip}(\nabla h; \mathbb{R}^n) \leq \max\{\text{Lip}(\nabla h; \Omega), \text{Lip}(\nabla h; \mathbb{R}^n \setminus \Omega)\}.$$

The final inequality is true from Lemma 10.  $\square$

#### 4. Sup–Inf explicit minimal Lipschitz extensions for 1-fields

In this section let  $\Omega$  be a subset of  $\mathbb{R}^n$  and  $F \in \mathcal{F}^1(\Omega)$ . To better understand Sections 4 and 6, let us recall some selected results in [24, Theorems 1, 2].

**Definition 14.** The real  $\kappa \in \mathbb{R}$ , with  $\kappa > 0$ , is allowable for  $F$  if  $\kappa$  satisfies the following inequality

$$f_y \leq f_x + \frac{1}{2} \langle D_x f + D_y f, y - x \rangle + \frac{\kappa}{4} \|x - y\|^2 - \frac{1}{4\kappa} \|D_x f - D_y f\|^2, \quad \forall x, y \in \Omega. \quad (34)$$

**Lemma 15.** The real  $\kappa$  is allowable for  $F$  iff  $\Gamma^1(F; \Omega) \leq \kappa$ .

**Proof.** Applying Proposition 4, we have  $\Gamma^1(F; \Omega) \leq \kappa$  if and only if

$$\sqrt{A_{x,y}^2(F) + B_{x,y}^2(F)} + A_{x,y}(F) \leq \kappa, \quad \forall x, y \in \Omega.$$

This inequality is equivalent to

$$\frac{B_{x,y}^2(F)}{2\kappa} + A_{x,y}(F) \leq \frac{\kappa}{2},$$

and hence it is equivalent to

$$f_y \leq f_x + \frac{1}{2} \langle D_x f + D_y f, y - x \rangle + \frac{1}{4\kappa} \|y - x\|^2 - \frac{1}{4\kappa} \|D_x f - D_y f\|^2, \quad \text{for any } x, y \in \Omega. \quad \square$$

**Definition 16.** Let  $\kappa$  be allowable for  $F$ . Denote by  $w^+(F, \Omega, \kappa)$  the Wells function which is described in [24, Theorem 1] when  $\Omega$  is finite and

$$w^+(F, \Omega, \kappa)(x) = \inf_{P \in \mathcal{P}(\Omega)} w^+(F, P, \kappa)(x),$$

when  $\Omega$  is infinite, where  $\mathcal{P}(\Omega)$  is the set of all finite subsets of  $\Omega$ .

The following corollaries are the direct consequences of Lemma 15, [12, Proposition 2.4] and [24, Theorems 1, 2].

**Corollary 17.**  $W^+(F, \Omega, \kappa)$  is an extension of  $F$  and  $\Gamma^1(W^+(F, \Omega, \kappa); \mathbb{R}^n) \leq \kappa$ . Moreover, for any extension  $G$  of  $F$  on  $\mathbb{R}^n$  which satisfies  $\Gamma^1(G; \mathbb{R}^n) \leq \kappa$  we have

$$g(x) \leq w^+(F, \Omega, \kappa)(x), \quad x \in \mathbb{R}^n.$$

**Corollary 18.** If  $\Omega \subset \Omega_1 \subset \Omega_2$ , then

$$w^+(F, \Omega_2, \kappa)(x) \leq w^+(F, \Omega_1, \kappa)(x), \quad x \in \mathbb{R}^n.$$

**Corollary 19.** If  $\kappa = \Gamma^1(F; \Omega)$ , then  $w^+(F, \Omega, \kappa)$  is an over minimal Lipschitz extension of  $F$ .

In the remainder of this section, we define  $\kappa \triangleq \Gamma^1(F; \Omega)$ . We will give two explicit formulas for extremal extension problem of  $F$  on  $\mathbb{R}^n$ .

**Definition 20.** For any  $a, b \in \Omega$  and  $x \in \mathbb{R}^n$ , we define

$$\begin{aligned} v_{a,b} &\triangleq \frac{1}{2}(D_a f + D_b f) + \frac{\kappa}{2}(b - a), \\ \alpha_{a,b} &\triangleq 2\kappa(f_a - f_b) + \kappa\langle D_a f + D_b f, b - a \rangle - \frac{1}{2}\|D_a f - D_b f\|^2 + \frac{\kappa^2}{2}\|a - b\|^2 \\ &\triangleq \left( \kappa A_{a,b}(F) - \frac{B_{a,b}(F)^2}{2} + \frac{\kappa^2}{2} \right) \|a - b\|^2, \\ \beta_{a,b}(x) &\triangleq \left\| \frac{1}{2}(D_a f - D_b f) + \frac{\kappa}{2}(2x - a - b) \right\|^2. \end{aligned}$$

From the proof of [12, Theorem 2.6] we know that  $\alpha_{a,b} \geq 0$  thus we can define

$$\begin{aligned} r_{a,b}(x) &\triangleq \sqrt{\alpha_{a,b} + \beta_{a,b}(x)}, \\ \Lambda_x &\triangleq \{v \in \mathbb{R}^n : \|v - v_{a,b}\| \leq r_{a,b}(x), \forall a, b \in \Omega\}. \end{aligned}$$

**Definition 21.** For any  $a \in \Omega$ ,  $x \in \mathbb{R}^n$  and  $v \in \Lambda_x$  we define

$$\begin{aligned} \Psi^+(F, x, a, v) &\triangleq f_a + \frac{1}{2}\langle D_a f + v, x - a \rangle + \frac{\kappa}{4}\|a - x\|^2 - \frac{1}{4\kappa}\|D_a f - v\|^2, \\ \Psi^-(F, x, a, v) &\triangleq f_a + \frac{1}{2}\langle D_a f + v, x - a \rangle - \frac{\kappa}{4}\|a - x\|^2 + \frac{1}{4\kappa}\|D_a f - v\|^2, \\ u^+(x) &\triangleq \sup_{v \in \Lambda_x} \inf_{a \in \Omega} \Psi^+(F, x, a, v). \end{aligned}$$

An important part of the proof of [12, Theorem 2.6] shows that  $\Lambda_x$  is non-empty for all  $x \in \mathbb{R}^n$ . This allows us to define  $u^+$ . Since  $\Psi^+$  is continuous, the map  $v \in \mathbb{R}^n \mapsto \inf_{a \in \Omega} \Psi^+(F, x, a, v)$  is an upper semicontinuous mapping therefore  $\sup_{v \in \Lambda_x} \inf_{a \in \Omega} \Psi^+(F, x, a, v)$  exists and the map  $u^+$  is well defined.

**Proposition 22.** Fix  $x \in \mathbb{R}^n$ . Then there exists a unique element  $v_x^+ \in \Lambda_x$  such that

$$u^+(x) = \inf_{a \in \Omega} \Psi^+(F, x, a, v_x^+).$$

**Proof.** Since  $\Lambda_x$  is compact and non-empty, there exists  $v_x^+ \in \Lambda_x$  such that

$$u^+(x) = \inf_{a \in \Omega} \Psi^+(F, x, a, v_x^+).$$

We will prove that  $v_x^+$  is uniquely determined. Indeed, for any  $a \in \Omega$  we define

$$g_a(v) = \Psi^+(F, x, a, v), \quad \text{for } v \in \Lambda_x.$$

Then for any  $t \in (0, 1)$  and  $(v_1, v_2) \in \mathbb{R}^n \times \mathbb{R}^n$ , we have

$$g_a(tv_1 + (1-t)v_2) = tg_a(v_1) + (1-t)g_a(v_2) + \frac{1}{4\kappa}t(1-t)\|v_1 - v_2\|^2.$$

Thus  $g_a$  is strictly concave. If we define  $g(v) = \inf_{a \in \Omega} g_a(v)$  for  $v \in \Lambda_x$ , then for any  $t \in (0, 1)$  and  $(v_1, v_2) \in \mathbb{R}^n \times \mathbb{R}^n$  we have

$$g(tv_1 + (1-t)v_2) \geq tg(v_1) + (1-t)g(v_2) + \frac{1}{4\kappa}t(1-t)\|v_1 - v_2\|^2.$$

Thus  $g$  is also strictly concave.

To prove  $v_x^+$  is uniquely determined, we need to prove that if  $g(v) = g(v_x^+)$  then  $v = v_x^+$ , where  $v \in \Lambda_x$ . Assume by a contradiction that there exists  $v \in \Lambda_x$  such that  $v \neq v_x^+$  and  $g(v) = g(v_x^+)$ . Since  $\Lambda_x$  is a convex subset of  $\mathbb{R}^n$ , we have  $tv + (1-t)v_x^+ \in \Lambda_x$  for  $t \in (0, 1)$ . Thus  $g(tv + (1-t)v_x^+) > tg(v) + (1-t)g(v_x^+) = g(v_x^+)$ , which contradicts the equality  $g(v_x^+) = \sup_{v \in \Lambda_x} g(v)$ .  $\square$

The previous proposition allows us to define the following 1-field.

**Definition 23.**

$$U^+ : x \in \mathbb{R}^n \mapsto U^+(x)(y) \triangleq u^+(x) + \langle D_x u^+; y - x \rangle, \quad y \in \mathbb{R}^n, \quad (35)$$

with

$$u^+(x) \triangleq \max_{v \in \Lambda_x} \inf_{a \in \Omega} \Psi^+(F, x, a, v), \quad D_x u^+ \triangleq \arg \max_{v \in \Lambda_x} \inf_{a \in \Omega} \Psi^+(F, x, a, v).$$

Using the proof of [12, Theorem 2.6], we can easily show the following proposition:

**Proposition 24.** Let  $x_0 \in \mathbb{R}^n$  and define  $\Omega_1 = \Omega \cup \{x_0\}$ . Let  $U$  be an extension of  $F$  on  $\Omega_1$ . Then the following conditions are equivalent:

- (i)  $U$  is a MLE of  $F$  on  $\Omega_1$ .
- (ii)

$$\sup_{a \in \Omega} \Psi^-(F, x, a, D_x u) \leq u(x) \leq \inf_{a \in \Omega} \Psi^+(F, x, a, D_x u), \quad \forall x \in \Omega_1.$$

Furthermore

$$\left[ \sup_{a \in \Omega} \Psi^-(F, x, a, D_x u) \leq \inf_{a \in \Omega} \Psi^+(F, x, a, D_x u) \right] \Leftrightarrow [D_x u \in \Lambda_x], \quad \forall x \in \Omega_1.$$

**Corollary 25.** Let  $\Omega_1$  be a subset of  $\mathbb{R}^n$  such that  $\Omega \subset \Omega_1$ . Let  $G$  be a MLE of  $F$  on  $\Omega_1$ . For all  $x \in \Omega_1$ , we have  $D_x g \in \Lambda_x$  and

$$\sup_{a \in \Omega} \Psi^-(F, x, a, D_x g) \leq g(x) \leq \inf_{a \in \Omega} \Psi^+(F, x, a, D_x g) \leq u^+(x), \quad \forall x \in \Omega_1.$$

**Proof.** The proof is immediate from Proposition 24.  $\square$

**Theorem 26.** The 1-field  $U^+$  is the unique over extremal extension of  $F$ .

**Proof.** Applying Corollary 19,  $W^+(F; \Omega, \kappa)$  is an over extremal extension of  $F$  on  $\mathbb{R}^n$ . Let  $w^+$  be an over extremal extension of  $F$  on  $\mathbb{R}^n$ . Let  $W^+$  be the 1-field canonical associated to  $w^+$ . We will prove  $U^+ = W^+$  on  $\mathbb{R}^n$ .

**Step 1.** Let  $x \in \Omega$ . Since  $W^+$  is an extension of  $F$  we have  $W^+(x) = F(x)$ . Note that  $\Lambda_x$  has a unique element to be  $D_x f$  (since  $x \in \Omega$  and from the definition of  $\Lambda_x$ ). So that  $D_x u^+ = D_x f$  and

$$u^+(x) = \inf_{a \in \Omega} \Psi^+(F, x, a, D_x f).$$

From [Proposition 24](#) we have

$$\Psi^+(F, x, a, D_x f) \geq f(x), \quad \text{for any } a \in \Omega.$$

Furthermore, when  $a = x$  we have  $\Psi^+(F, x, x, D_x f) = f(x)$ . Therefore  $u^+(x) = f(x)$ . Thus we have for all  $x \in \Omega$ ,  $U^+(x) = W^+(x) = F(x)$ .

**Step 2.** Let  $x \in \mathbb{R}^n \setminus \Omega$ . We first prove that  $u^+(x) \geq w^+(x)$ . Since  $W^+$  is a MLE of  $F$  on  $\mathbb{R}^n$ , we can apply [Proposition 24](#) to obtain  $D_x w \in \Lambda_x$  and

$$w^+(x) \leq \inf_{a \in \Omega} \Psi^+(F, x, a, D_x w) \leq u^+(x).$$

Conversely, we will prove that  $u^+(x) \leq w^+(x)$ . Applying [Proposition 22](#),  $D_x u^+$  is the unique element in  $\Lambda_x$  such that  $u^+(x) = \inf_{a \in \Omega} \Psi^+(F, x, a, D_x u^+)$ . We define the 1-field  $G$  of domain  $\Omega \cup \{x\}$  as

$$G(y) \triangleq F(y), \quad y \in \Omega \quad \text{and} \quad G(x) \triangleq U^+(x).$$

Since  $D_x g = D_x u^+ \in \Lambda_x$ , we can apply [Proposition 24](#) to have

$$g(x) = u^+(x) = \inf_{a \in \Omega} \Psi^+(F, x, a, D_x g) \geq \sup_{a \in \Omega} \Psi^-(F, x, a, D_x g). \quad (36)$$

From (36) and [Proposition 24](#), we have  $G$  to be a MLE of  $F$  on  $\Omega \cup \{x\}$ . By applying [\[12, Theorem 2.6\]](#) there exists  $\tilde{G}$  to be a MLE of  $G$  on  $\mathbb{R}^n$ . Since  $\text{dom}(F) \subset \text{dom}(G)$ ,  $\tilde{G}$  is also a MLE of  $F$  on  $\mathbb{R}^n$ . Since  $W^+$  is over extremal extension of  $F$  on  $\mathbb{R}^n$ , we have  $\tilde{g}(x) \leq w^+(x)$  for all  $x \in \mathbb{R}^n$ . Thus  $u^+(x) = g(x) = \tilde{g}(x) \leq w^+(x)$ . Combining this with  $w^+(x) \leq u^+(x)$  we have  $u^+(x) = w^+(x)$ . Finally, using [Proposition 24](#) and the previous equality we have

$$u^+(x) = w^+(x) \leq \inf_{a \in \Omega} \Psi^+(F, x, a, D_x w^+) \leq u^+(x) = \inf_{a \in \Omega} \Psi^+(F, x, a, D_x u^+).$$

Thus we obtain the following equality

$$\inf_{a \in \Omega} \Psi^+(F, x, a, D_x w^+) = \inf_{a \in \Omega} \Psi^+(F, x, a, D_x u^+).$$

Therefore  $D_x w^+ = D_x u^+$  by [Proposition 22](#). Thus we have for all  $x \in \mathbb{R}^n$ ,  $U^+(x) = W^+(x)$ .

The uniqueness of over extremal extension of  $F$  arises since  $W^+$  is an arbitrary over extremal extension of  $F$ .  $\square$

Thanks to [Corollary 46](#), we can define an under extremal extension of  $F$ . That is:

**Definition 27.** For any  $a \in \Omega$ ,  $x \in \mathbb{R}^n$  and  $v \in \Lambda_x$  we define

$$u^-(x) \triangleq \inf_{v \in \Lambda_x} \sup_{a \in \Omega} \Psi^-(F, x, a, v).$$

Using the strict convexity of the map  $v \mapsto \Psi^-(F, x, a, v)$  and the compactness of  $\Lambda_x$  as in the proof of [Proposition 22](#) when concavity is replaced with convexity, we obtain the following proposition:

**Proposition 28.** Let  $x \in \mathbb{R}^n$ . Then there exists a unique element  $v_x^- \in \Lambda_x$  such that  $u^-(x) = \sup_{a \in \Omega} \Psi^-(F, x, a, v_x^-)$ .

This allows us to define the following 1-field:

**Definition 29.**

$$U^- : x \in \mathbb{R}^n \mapsto U^-(x)(y) \triangleq u^-(x) + \langle D_x u^-; y - x \rangle, \quad y \in \mathbb{R}^n, \quad (37)$$

with

$$u^-(x) \triangleq \min_{v \in \Lambda_x} \sup_{a \in \Omega} \Psi^-(F, x, a, v), \quad D_x u^- \triangleq \arg \min_{v \in \Lambda_x} \sup_{a \in \Omega} \Psi^-(F, x, a, v).$$

**Theorem 30.** *The 1-field  $U^-$  is the unique under extremal extension of  $F$ .*

**Proof.** Using [Corollary 46](#) and [Proposition 28](#), the proof uses similar arguments as in the proof of [Theorem 26](#).  $\square$

In conclusion we have the following corollary

**Corollary 31.** *For all minimal Lipschitz extensions  $G$  of  $F$  we have*

$$u^-(x) \leq g(x) \leq u^+(x), \quad \forall x \in \mathbb{R}^n.$$

## 5. Sup–Inf explicit minimal Lipschitz extensions for functions from $\mathbb{R}^m$ maps to $\mathbb{R}^n$

Now, we propose to use the results of the previous section to produce formulas comparable to those Bauschke and Wang have found; see [\[7\]](#). Let us define  $\mathcal{Q}_0$  as the problem of the minimum extension for Lipschitz continuous functions and  $\mathcal{Q}_1$  as the problem of the minimum extension for 1-fields. Curiously, we will show that the problem  $\mathcal{Q}_0$  is a sub-problem of the problem  $\mathcal{Q}_1$ . As a consequence, we obtain two explicit formulas that solve the problem  $\mathcal{Q}_0$ .

More specifically, fix  $n, m \in \mathbb{N}^*$  and  $\omega \subset \mathbb{R}^m$ . We suppose that  $\omega$  has at least two elements. Let  $u$  be a function from  $\omega$  maps to  $\mathbb{R}^n$ . Suppose  $\text{Lip}(u; \omega) < +\infty$  and define  $l \triangleq \text{Lip}(u; \omega)$ . Let us define

$$\Omega \triangleq \{(x, 0) \in \mathbb{R}^m \times \mathbb{R}^n : x \in \omega\}.$$

A current element  $x$  of  $\mathbb{R}^{m+n}$  is denoted by  $x := (x^{(m)}, x^{(n)}) \in \mathbb{R}^{m+n}$ , with  $x^{(m)} \in \mathbb{R}^m$  and  $x^{(n)} \in \mathbb{R}^n$ . For each function  $u$  of domain  $\omega$  we associate the 1-field  $F$  from  $\Omega \subset \mathbb{R}^{n+m}$  maps to  $\mathcal{P}^1(\mathbb{R}^{n+m}, \mathbb{R})$  as the following

$$f_{(x,0)} \triangleq 0, \quad \text{and} \quad D_{(x,0)} f \triangleq (0, u(x)), \quad \text{for all } x \in \omega. \quad (38)$$

Let  $a, b \in \omega$ , with  $a \neq b$ . Observing that

$$f_{(a,0)} = f_{(b,0)} = 0, \quad \text{and} \quad \langle D_{(a,0)} f + D_{(b,0)} f, (b - a, 0) \rangle = 0,$$

and applying [Proposition 4](#) we have

$$\Gamma^1(F, (a, 0), (b, 0)) = \frac{\|D_{(a,0)} f - D_{(b,0)} f\|}{\|b - a\|}.$$

Therefore

$$\Gamma^1(F, \Omega) = \text{Lip}(u, \omega). \quad (39)$$

Let  $G$  be a minimal Lipschitz extension of  $F$ . We have  $G \in \mathcal{F}^1(\mathbb{R}^{m+n})$  and

$$\Gamma^1(F, \Omega) = \Gamma^1(G, \mathbb{R}^{m+n}). \quad (40)$$

Using [12, Proposition 2.4] we have

$$\Gamma^1(G; \mathbb{R}^{m+n}) = \text{Lip}(Dg; \mathbb{R}^{m+n}). \quad (41)$$

Now we define the map  $\tilde{u}$  from  $\mathbb{R}^m$  to  $\mathbb{R}^n$  as the following

$$\tilde{u}(x) \triangleq D_{(x,0)}g^{(n)}, \quad x \in \mathbb{R}^m, \quad (42)$$

where  $D_{(x,0)}g \triangleq (D_{(x,0)}g^{(m)}, D_{(x,0)}g^{(n)}) \in \mathbb{R}^m \times \mathbb{R}^n$ .

We will show that  $\tilde{u}$  is a minimal Lipschitz extension of  $u$ . Let  $x \in \omega$ . Since  $G$  is an extension of  $F$  and by construction of  $F$  we have

$$\tilde{u}(x) = (D_{(x,0)}g)^{(n)} = u(x).$$

Thus  $\tilde{u}$  is an extension of  $u$ . Let  $x, y \in \mathbb{R}^m$  with  $x \neq y$ . Using (39), (40) and (41) we have

$$\begin{aligned} \text{Lip}(\tilde{u}; x, y) &= \frac{\|D_{(x,0)}g^{(n)} - D_{(y,0)}g^{(n)}\|}{\|x - y\|} \\ &\leq \frac{\|D_{(x,0)}g - D_{(y,0)}g\|}{\|x - y\|} \\ &\leq \Gamma^1(G; \mathbb{R}^{m+n}) \\ &= \text{Lip}(u, \omega). \end{aligned} \quad (43)$$

This implies that  $\tilde{u}$  is a minimal Lipschitz extension of  $u$ . Therefore, we obtain another proof of Kirszbraun's theorem (see [17]).

**Theorem 32.** *Let  $u$  be a function from  $\omega \subset \mathbb{R}^m$  to  $\mathbb{R}^n$ . Suppose that  $u$  is a Lipschitz continuous function. Let  $F$  be the 1-field defined by the formula (38). Let  $G$  be any minimal Lipschitz extension of  $F$ . Then the extension  $\tilde{u}$  defined by the formula (42) is a minimal Lipschitz extension of  $u$ .*

If we replace the 1-field  $G$  of Theorem 32 by  $U^-$  and  $U^+$ , which are defined by Definitions 23 and 29, we obtain two explicit formulas which solve the problem  $\mathcal{Q}_0$ . Now we will describe these formulas. Letting  $a, b \in \omega$  and  $x \in \mathbb{R}^m$  we define

$$\begin{aligned} v_{a,b} &\triangleq \left( \frac{l}{2}(b-a), \frac{1}{2}(u(a) + u(b)) \right), \\ \alpha_{a,b} &\triangleq -\frac{1}{2}\|u(a) - u(b)\|^2 + \frac{l^2}{2}\|a - b\|^2, \\ \beta_{a,b}(x) &\triangleq l^2 \left\| x - \frac{a+b}{2} \right\|^2 + \left\| \frac{u(a) - u(b)}{2} \right\|^2, \\ r_{a,b}(x) &\triangleq \sqrt{\alpha_{a,b} + \beta_{a,b}(x)}, \\ \Lambda_x &\triangleq \{v \in \mathbb{R}^{m+n} : \|v - v_{a,b}\| \leq r_{a,b}(x), \forall a, b \in \omega\}. \end{aligned}$$

For  $v = (v^{(m)}, v^{(n)}) \in \mathbb{R}^m \times \mathbb{R}^n$ , we define

$$\Phi^+(u, x, a, v) \triangleq \frac{1}{2} \langle v^{(m)}, x - a \rangle + \frac{l}{4} \|a - x\|^2 - \frac{1}{4l} (\|v^{(m)}\|^2 + \|u(a) - v^{(n)}\|^2), \quad (44)$$

$$\Phi^-(u, x, a, v) \triangleq \frac{1}{2} \langle v^{(m)}, x - a \rangle - \frac{l}{4} \|a - x\|^2 + \frac{1}{4l} (\|v^{(m)}\|^2 + \|u(a) - v^{(n)}\|^2). \quad (45)$$

Now using the previous notations, we define two maps from  $\mathbb{R}^m$  to  $\mathbb{R}^n$  as the following

$$k^+(x) \triangleq \left( \arg \max_{v \in \Lambda_x} \inf_{a \in \omega} \Phi^+(u, x, a, v) \right)^{(n)}, \quad x \in \mathbb{R}^m, \quad (46)$$

and

$$k^-(x) \triangleq \left( \arg \min_{v \in \Lambda_x} \sup_{a \in \omega} \Phi^-(u, x, a, v) \right)^{(n)}, \quad x \in \mathbb{R}^m. \quad (47)$$

**Theorem 33.** *The maps  $k^+$  and  $k^-$  defined by the formulas (46) and (47) are minimal Lipschitz extensions of  $u$ .*

**Remark 34.** If  $\omega$  is finite using the previous transformation  $u \rightarrow F$  then the Wells explicit construction of  $u^+$  or  $u^-$  allows to compute  $u^+$  and  $u^-$ .

We know that the proof of Kirszbraun–Valentine’s theorem and the proof of [12, Theorem 2.6] use Zorn’s lemma. Noticing that, the proof which allows that  $\Lambda_x$  is non-empty set, does not use Zorn’s lemma. Thus the proofs of Theorem 26, 30 and 33 do not use Zorn’s lemma. This is also true in [7] and [24, Theorem 2].

## 6. Absolutely minimal Lipschitz extensions

In this section let  $\Omega$  be a subset of  $\mathbb{R}^n$  ( $n \geq 2$ ) and  $F \in \mathcal{F}^1(\Omega)$  and  $\kappa = \Gamma^1(F, \Omega)$ . Let  $w^\pm = w^\pm(F, \Omega, \kappa)(x)$  where  $w^\pm(F, \Omega, \kappa)$  are defined by Definition 16 and Definition 42.

### 6.1. Finite domain

**Proposition 35.** *Suppose  $\Omega$  is finite, then  $W^+$  and  $W^-$  are AMLEs of  $F$  on  $\mathbb{R}^n$ .*

**Proof.** By Corollaries 19 and 45,  $W^\pm$  are two MLEs of  $F$  and

$$\Gamma^1(W^\pm; \mathbb{R}^n) = \kappa. \quad (48)$$

Let  $V$  be bounded open satisfying  $\bar{V} \subset \mathbb{R}^n \setminus \Omega$ . We need to prove that  $\Gamma^1(W^+, V) = \Gamma^1(W^+, \partial V)$ . Using the same notations like in the proof of [24, Theorem 1] and using [24, Lemma 17], there exist  $S \in K$  and  $x, y \in \partial V$  with  $x \neq y$  such that  $x, y \in T_S$ . Applying [24, Lemma 21] we have

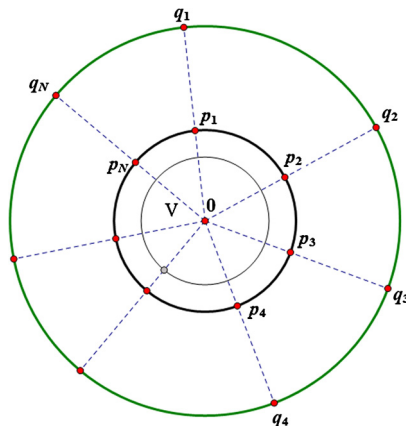
$$\|\nabla w^+(x) - \nabla w^+(y)\| = \kappa \|x - y\|.$$

Thus  $\text{Lip}(\nabla w^+; x, y) = \kappa$ . Using (48) and the previous equality, we obtain

$$\Gamma^1(W^+, V) \leq \kappa = \text{Lip}(\nabla w^+; x, y) \leq \text{Lip}(\nabla w^+; \partial V) \leq \Gamma^1(W^+, \partial V) \leq \Gamma^1(W^+, V). \quad (49)$$

Thus  $\Gamma^1(W^+, V) = \Gamma^1(W^+, \partial V)$ . Therefore  $W^+$  is an AMLE of  $F$ .

The proof for  $W^-$  is similar by using Proposition 41 and Lemma 43.  $\square$

Fig. 2. Illustration of subset  $A_n$  of  $\Omega$ .

**Corollary 36.** Suppose  $\Omega$  is finite and  $w^+ \neq w^-$ , then there exists an infinite number of AMLEs of  $F$  on  $\mathbb{R}^n$ .

**Proof.** Let  $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$  such that  $w^+(x_0) \neq w^-(x_0)$ . Note that

$$W_\tau = \tau W^+ + (1 - \tau)W^-$$

is MLE of  $F$  on  $\mathbb{R}^n$ , for all  $\tau \in [0, 1]$ . Thus there exists an infinite number of MLE of  $F$  on  $\Omega \cup \{x_0\}$ . Let  $G$  be a MLE of  $F$  on  $\omega \cup \{x_0\}$ . By the same argument as in the proof of Proposition 35, we have that  $w^+(G, \Omega \cup \{x_0\}, \kappa)$  and  $w^-(G, \Omega \cup \{x_0\}, \kappa)$  are two AMLEs of  $F$  on  $\mathbb{R}^n$ . Therefore, there exists an infinite number of AMLE of  $F$  on  $\mathbb{R}^n$ .  $\square$

## 6.2. Infinite domain

From Section 6.1, we know that if  $\Omega$  is a finite set then the functions  $W^+$  and  $W^-$  are two AMLEs of  $F$  on  $\mathbb{R}^n$ . When  $\Omega$  is infinite, we give an example that shows the opposite.

**Proposition 37.** Suppose  $n = 2$ , then there exist  $\Omega$  and  $F \in \mathcal{F}^1(\Omega)$  such that  $W^+$  and  $W^-$  are not AMLEs of  $F$  on  $\mathbb{R}^n$ .

**Proof.** Let us define  $\Omega_1 = \partial B(0; 1)$ ,  $\Omega_2 = \partial B(0; 2)$  and  $\Omega = \Omega_1 \cup \Omega_2$ . We define  $F \in \mathcal{F}^1(\Omega)$  as the following

$$f_x = 0, \quad \text{for } x \in \Omega_1, \quad f_x = 1, \quad \text{for } x \in \Omega_2, \quad \text{and} \quad D_x f = 0, \quad \text{for } x \in \Omega.$$

We will prove that  $W^+$  is not an AMLE of  $F$  on  $\mathbb{R}^n$ . To do this, we need to find an open set  $V \subset \subset \mathbb{R}^2/\Omega$  such that  $\Gamma^1(W^+; V) \neq \Gamma^1(W^+; \partial V)$ . Let us define  $V = \{x \in \mathbb{R}^2 : \|x\| < 3/4\} \subset \subset \mathbb{R}^2/\Omega$ ; we will prove that

$$\Gamma^1(W^+; V) \neq \Gamma^1(W^+; \partial V).$$

Applying Proposition 4, we have  $\kappa = \Gamma^1(F; \Omega) = 4$ .

Let  $\mathcal{A}$  be the set of all finite subsets  $A_N$  of  $\Omega$  of the form  $A_N = \{p_1, \dots, p_N\} \cup \{q_1, \dots, q_N\}$ ,  $N \in \mathbb{N}^*$ , which satisfies  $\{p_1, p_2, \dots, p_N\} \subset \Omega_1$ ,  $\{q_1, q_2, \dots, q_N\} \subset \Omega_2$  and  $p_i \in [0, q_i]$  for all  $i \in 1, \dots, N$  (see Fig. 2). By construction, we have  $A_N \in \mathcal{A}$ . For brevity let us denote the functions  $w^+(F, A_N, \kappa)$  by  $w_{A_N}^+$ . Applying Proposition 4, we have

$$\kappa_N = \Gamma^1(W_{A_N}^+; A_N) = 4 = \kappa, \quad \text{for all } A_N \in \mathcal{A}.$$

As in [24, Theorem 1] and using the same notation, we define  $T_S$  associated with  $A = A_N$ . We can check that

$$\left[ \frac{p_i}{2}, \frac{p_i + q_i}{2} \right] \subset T_{p_i},$$

and from the definition of  $w_{A_N}^+$ , we have

$$w_{A_N}^+(x) = \frac{\kappa}{2}(x - p_i)^2 = \frac{\kappa}{2}d^2(x; \partial\Omega_1), \quad \text{for all } x \in \left[ \frac{p_i}{2}, \frac{p_i + q_i}{2} \right].$$

Let us define  $\mathcal{D}_{\alpha, \beta} \triangleq \{y \in \mathbb{R}^n \mid \alpha \leq \|y\| \leq \beta\}$ ,  $\alpha \leq \beta$  and  $\mathcal{P}(\Omega)$  to be the set of all finite subsets of  $\Omega$ .

We will prove that

$$w^+(x) = \frac{\kappa}{2}d^2(x; \partial\Omega_1) \quad \text{for all } x \in \mathcal{D}_{1/2, 3/2}.$$

Indeed, letting  $x \in \mathcal{D}_{1/2, 3/2}$ , there exists  $A_N \in \mathcal{A}$  such that  $x \in [0, q_1]$  where  $q_1 \in A_N \cap \Omega_2$ . From the definition of  $w^+$  (see 16), we have  $w^+(x) \leq w_{A_N}^+(x) = \frac{\kappa}{2}d^2(x; \partial\Omega_1)$ . Conversely, for all  $P \in \mathcal{P}(\Omega)$ , there exists  $A_N \in \mathcal{A}$  such that  $P \subset A_N$  and  $x \in [0, q_2]$  where  $q_2 \in A_N \cap \Omega_2$ . Applying Corollary 18, we have  $w^+(F, P, \kappa) \geq w_{A_N}^+(x) = \frac{\kappa}{2}d^2(x; \partial\Omega_1)$ . Hence

$$w^+(x) = \inf_{P \in \mathcal{P}(\Omega)} w^+(F, P, \kappa)(x) \geq \frac{\kappa}{2}d^2(x; \partial\Omega_1).$$

Therefore  $w^+(x) = \frac{\kappa}{2}d^2(x; \partial\Omega_1)$  for all  $x \in \mathcal{D}_{1/2, 3/2}$ . Noticing that

$$w^+(x) = \frac{\kappa}{2}d^2(x; \partial\Omega_1) = \frac{\kappa}{2}(\|x\| - 1)^2, \quad \text{for all } x \in \mathcal{D}_{1/2, 3/4},$$

for all  $x, y \in \mathcal{D}_{1/2, 3/4}$  such that  $x \in [0, y]$ , we have

$$\|\nabla w^+(x) - \nabla w^+(y)\| = \left\| \kappa(\|x\| - 1) \frac{x}{\|x\|} - \kappa(\|y\| - 1) \frac{y}{\|y\|} \right\| = \kappa\|x - y\|.$$

Hence  $\kappa \geq \Gamma^1(W^+; \mathbb{R}^n) \geq \Gamma^1(W^+; V) \geq \text{Lip}(\nabla w^+; V) \geq \kappa$ , and hence

$$\Gamma^1(W^+; V) = \kappa = 4. \quad (50)$$

On the other hand, for all  $x, y \in \partial V$ , we have  $\|x\| = \|y\| = 3/4$ , so that

$$\|\nabla w^+(x) - \nabla w^+(y)\| = \left\| \kappa(\|x\| - 1) \frac{x}{\|x\|} - \kappa(\|y\| - 1) \frac{y}{\|y\|} \right\| = \frac{\kappa}{3}\|x - y\|.$$

Hence

$$B_{x,y}(W^+) = \frac{\|\nabla w^+(x) - \nabla w^+(y)\|}{\|x - y\|} = \frac{\kappa}{3}.$$

Moreover, since  $w^+(x) = w^+(y)$  and  $(\nabla w^+(x) + \nabla w^+(y))$  is perpendicular to  $(y - x)$ , we have

$$A_{x,y}(W^+) = \frac{2(w^+(x) - w^+(y)) + \langle \nabla w^+(x) + \nabla w^+(y), y - x \rangle}{\|x - y\|^2} = 0.$$

Applying [Proposition 4](#), we have

$$\Gamma^1(W^+; \partial V) = \sup_{x, y \in \partial V} \left( \sqrt{A_{a,b}^2 + B_{a,b}^2} + |A_{a,b}| \right) = \frac{\kappa}{3} = \frac{4}{3}. \quad (51)$$

From [\(50\)](#) and [\(51\)](#) we have  $\Gamma^1(W^+; V) \neq \Gamma^1(W^+; \partial V)$ . And therefore  $W^+$  is not an AMLE of  $F$  on  $\mathbb{R}^n$ .

The proof for  $W^-$  is similar.  $\square$

**Remark 38.** With the same notation as in [Proposition 37](#), by computing directly  $w^+$  and  $w^-$ , we obtain

$$\begin{aligned} w^+(x) &= -w^-(x) = 1 - \frac{\kappa}{2}x^2, \quad \forall x \in \mathcal{D}_{0, \frac{1}{2}}, \\ w^+(x) &= -w^-(x) = \frac{\kappa}{2}d^2(x, \partial\Omega_1), \quad \forall x \in \mathcal{D}_{\frac{1}{2}, 1}, \\ w^+(x) &= w^-(x) = \frac{\kappa}{2}d^2(x, \partial\Omega_1), \quad \forall x \in \mathcal{D}_{1, \frac{3}{2}}, \\ w^+(x) &= w^-(x) = 1 - \frac{\kappa}{2}d^2(x, \partial\Omega_2), \quad \forall x \in \mathcal{D}_{\frac{3}{2}, 2}, \\ w^+(x) &= 1 + \frac{\kappa}{2}d^2(x, \partial\Omega_2), \quad \forall x \in \mathcal{D}_{2, +\infty}, \\ w^-(x) &= 1 - \frac{\kappa}{2}d^2(x, \partial\Omega_2), \quad \forall x \in \mathcal{D}_{2, +\infty}, \end{aligned}$$

where  $\kappa = \Gamma^1(F; \Omega) = 4$ .

We see that  $W = \frac{W^+ + W^-}{2}$  is an AMLE of  $F$  on  $\mathbb{R}^2$  (although  $W^+$  and  $W^-$  are not AMLEs of  $F$  on  $\mathbb{R}^2$ ) and  $w \notin \mathcal{C}^2(\mathbb{R}^2, \mathbb{R})$ . Moreover, all MLEs of  $F$  coincide on  $\{x \in \mathbb{R}^2 : 1 \leq \|x\| \leq 2\}$  (because  $W^+ = W^-$  on  $\{x \in \mathbb{R}^2 : 1 \leq \|x\| \leq 2\}$ ).

From [Proposition 37](#), we know that  $W^+$  is not an AMLE of  $F$  on  $\mathbb{R}^n$  in general case. But in some case, we have  $W^+$  to be an AMLE of  $F$  on  $\mathbb{R}^n$ . We give an example:

**Proposition 39.** Let  $\Omega$  be a subset of  $\mathbb{R}^n$  ( $n \geq 2$ ). Let  $F \in \mathcal{F}^1(\Omega)$  such that  $\tilde{\Omega} = \{p - \frac{D_p f}{\kappa} : p \in \Omega\}$  is a subset of an  $(n-1)$ -dimensional hyperplane  $H$ , where  $\kappa = \Gamma^1(F, \Omega)$ . Then  $W^+$  is an AMLE of  $F$  on  $\mathbb{R}^n$ .

**Proof.** We prove that  $W^+$  is an AMLE of  $F$  on  $\mathbb{R}^n$ . From [Corollary 19](#), we have  $W^+$  to be a MLE of  $F$  on  $\mathbb{R}^n$ . Let  $V \subset \mathbb{R}^n \setminus \Omega$ . We need to prove that  $\Gamma^1(W^+; V) = \Gamma^1(W^+; \partial V)$ . Indeed, the inequality  $\Gamma^1(W^+; V) \geq \Gamma^1(W^+; \partial V)$  is clear, so that we only need to prove that  $\Gamma^1(W^+; \partial V) \geq \Gamma^1(W^+; V)$ . We have

$$\kappa = \Gamma^1(W^+; \mathbb{R}^n) \geq \Gamma^1(W^+; V) \geq \Gamma^1(W^+; \partial V) \geq \text{Lip}(\nabla w^+, \partial V).$$

Thus it suffices to show that  $\text{Lip}(\nabla w^+, \partial V) \geq \kappa$ .

Let  $x_0, y_0 \in \partial V$ ,  $x_0 \neq y_0$  such that  $(x_0 - y_0)$  is perpendicular to the hyperplane  $H$ . Let  $\mathcal{P}$  be the set of all finite subsets of  $\Omega$ . For any  $P \in \mathcal{P}$ , we have the corresponding function  $W^+(F, P, \kappa)$  (see [Definition 16](#)) or  $W_P^+$  for short. We define  $K$  and  $T_S$  for  $S \in K$  the same notations like in [\[24, Theorem 1\]](#) with the corresponding definition for the finite set  $A = P$  (see [Fig. 3](#)). Put  $\kappa_P = \Gamma^1(W_P^+; \mathbb{R}^n)$ . Since  $\tilde{\Omega} = \{p - \frac{D_p f}{\kappa} : p \in \Omega\}$  is a subset of  $H$  and  $(x_0 - y_0)$  is perpendicular to the hyperplane  $H$ , there exists  $S \in K$  such that  $x_0, y_0 \in T_S$ . Applying [\[24, Lemma 21\]](#), we have

$$\|\nabla w_P^+(x_0) - \nabla w_P^+(y_0)\| = \kappa_P \|x_0 - y_0\|.$$

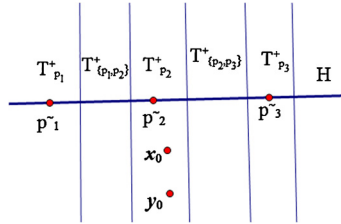


Fig. 3. The partitioning of  $\mathbb{R}^n$  into regions  $T_S$ .

Moreover, for any  $\varepsilon > 0$ , there exists  $P \in \mathcal{P}$  such that  $\|\nabla w_P^+(x_0) - \nabla w^+(x_0)\| \leq \varepsilon$ ,  $\|\nabla w_P^+(y_0) - \nabla w^+(y_0)\| \leq \varepsilon$  (by [24, Proposition 3]) and  $\kappa_P > \kappa - \varepsilon$ . Therefore

$$\begin{aligned} \text{Lip}(\nabla w, \partial V) &\geq \frac{\|\nabla w^+(x_0) - \nabla w^+(y_0)\|}{\|x_0 - y_0\|} \\ &\geq -\frac{\|\nabla w^+(x_0) - \nabla w_P^+(x_0)\|}{\|x_0 - y_0\|} - \frac{\|\nabla w^+(y_0) - \nabla w_P^+(y_0)\|}{\|x_0 - y_0\|} \\ &\quad + \frac{\|\nabla w_P^+(x_0) - \nabla w_P^+(y_0)\|}{\|x_0 - y_0\|} \\ &\geq \frac{-2\varepsilon}{\|x_0 - y_0\|} + \kappa - \varepsilon. \end{aligned}$$

Hence  $\text{Lip}(\nabla w, \partial V) \geq \kappa$ .  $\square$

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## Appendix A

### A.1. The constructions of $w^-$

Let  $\Omega$  be a subset of  $\mathbb{R}^n$ , containing at least two elements, and  $F$  be a 1-field in  $\mathcal{F}^1(\Omega)$ . Suppose that  $\kappa$  is allowable for  $F$  that satisfies (34).

In a similar way to [24, Theorem 1], we will describe an under extension of 1-field  $F$  and we will give the properties used for the understanding of this article.

**Case 1.** Suppose that  $\Omega$  is finite. For  $p \in \Omega$ , we define:

$$\tilde{d}_p^-(x) \triangleq f_p + \frac{1}{2}\|D_p f\|^2/\kappa - \frac{1}{4}\kappa\|x - \tilde{p}^-\|^2,$$

and for any  $S \subset \Omega$ ,

$$d_S^-(x) \triangleq \sup_{p \in S} d_p^-(x),$$

$$\tilde{S}^- \triangleq \{\tilde{p}^- : p \in S\},$$

$$\hat{S}^- \triangleq \text{convex hull of } \tilde{S}^-,$$

$$S_H^- \triangleq \text{smallest hyperplane containing } \tilde{S}^-,$$

$$\begin{aligned}
S_E^- &\triangleq \{x : d_p^-(x) = d_{p'}^-(x) \text{ for all } p, p' \in S\}, \\
S_*^- &\triangleq \{x : d_p^-(x) = d_{p'}^-(x) \geq d_{p''}^-(x) \text{ for all } p, p' \in S, p'' \in \Omega\}, \\
K^- &\triangleq \{S : S \subset \Omega \text{ and for some } x \in S_*, d_S^-(x) > d_{\Omega-S}^-(x)\}.
\end{aligned}$$

**Definition 40.** Let us define

$$T_S^- \triangleq \left\{ x : x = \frac{1}{2}(y + z) \text{ for some } y \in \widehat{S}^- \text{ and } z \in S_*^- \right\}, \quad \text{for all } S \in K^-.$$

**Proposition 41.** For all  $S \in K^-$  and for all  $x \in T_S^-$  there exists a unique couple  $(y, z) \in \widehat{S}^- \times S_*^-$  such that  $x = \frac{1}{2}(y + z)$ .

We have

$$\begin{aligned}
\bigcup_{S \in K^-} T_S^- &= \mathbb{R}^n, \\
\mathring{T}_S^- \cap \mathring{T}_{S'}^- &= \emptyset, \quad \text{and} \quad \mathring{T}_S^- \neq \emptyset, \quad \text{for all } S, S' \in K^-, \text{ with } S \neq S'.
\end{aligned}$$

For all  $S \in K^-$ , the set  $S_E^- \cap S_H^-$  contains a single point denoted by  $S_C^-$ .

For all  $S \in K^-$ , hyperplanes  $S_E^-, S_H^-$  are orthogonal.

**Definition 42.** Let us define

$$w_S^-(x) \triangleq d_S(S_C^-) - \frac{1}{2}\kappa d^2(x, S_H^-) + \frac{1}{2}\kappa d^2(x, S_E^-), \quad \text{for all } S \in K^- \text{ and } x \in T_S^-,$$

and

$$w^-(F, \Omega, \kappa)(x) \triangleq w_S^-(x) \quad \text{for all } x \in T_S^-.$$

**Lemma 43.** For all  $S \in K^-$  and  $x \in T_S^-$ , we have

$$\nabla w^-(F, \Omega, \kappa)(x) = \frac{\kappa}{2}(y - z), \quad \text{where } x = \frac{1}{2}(y + z), \text{ with } (y, z) \in \widehat{S}^- \times S_*^-,$$

and

$$\|\nabla w^-(F, \Omega, \kappa)(x) - \nabla w^-(F, \Omega, \kappa)(x')\| = \kappa \|x - x'\|, \quad \text{for all } S \in K^- \text{ and } x, x' \in T_S^-. \quad (52)$$

**Proposition 44.** We have  $w^-(F, \Omega, \kappa) \in \mathcal{C}^{1,1}(\mathbb{R}^n, \mathbb{R})$  and  $\text{Lip}(\nabla w^-(F, \Omega, \kappa); \mathbb{R}^n) = \kappa$ . Furthermore,  $w^-(F, \Omega, \kappa)$  is an extension of  $F$  and for all  $g \in \mathcal{C}^{1,1}(\mathbb{R}^n, \mathbb{R})$  extension of  $F$  such that  $\text{Lip}(\nabla g; \mathbb{R}^n) \leq \kappa$ , we have

$$w^-(F, \Omega, \kappa)(x) \leq g(x), \quad x \in \mathbb{R}^n.$$

**Corollary 45.** If  $\kappa = \Gamma^1(F; \Omega)$ , then  $W^-(F, \Omega, \kappa)$  is an under extremal extension of  $F$ .

**Case 2.** Suppose that  $\Omega$  is not finite. Denote by  $\mathcal{P}(\Omega)$  the set of all finite subsets of  $\Omega$ . Since for any  $x \in \mathbb{R}^n$ , and for any  $P, P' \in \mathcal{P}(\Omega)$  satisfying  $P \subset P'$  we have

$$w^-(F, P, \kappa)(x) \leq w^-(F, P', \kappa)(x),$$

we can define

$$w^-(F, \Omega, \kappa)(x) = \sup_{P \in \mathcal{P}(\Omega)} w^-(F, P, \kappa)(x).$$

**Corollary 46.** *Suppose that  $\Omega$  is any subset of  $\mathbb{R}^n$  and  $F \in \mathcal{F}^1(\Omega)$ . If  $\kappa = \Gamma^1(F; \Omega)$ , then  $W^-(F, \Omega, \kappa)$  is an under extremal extension of  $F$ .*

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