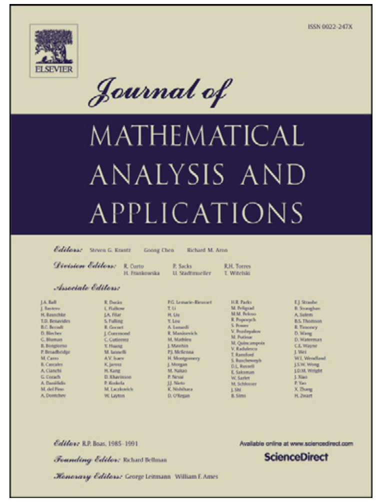
Critical groups at zero and multiple solutions for a quasilinear elliptic equation



Mingzheng Sun, Jiabao Su, Meiling Zhang

PII: S0022-247X(15)00257-7
DOI: http://dx.doi.org/10.1016/j.jmaa.2015.03.033
Reference: YJMAA 19321

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 25 September 2014

Please cite this article in press as: M. Sun et al., Critical groups at zero and multiple solutions for a quasilinear elliptic equation, *J. Math. Anal. Appl.* (2015), http://dx.doi.org/10.1016/j.jmaa.2015.03.033

Critical groups at zero and multiple solutions for a quasilinear elliptic equation

Mingzheng Sun[a*] Jiabao Su[b] Meiling Zhang[a]

[a]College of Sciences, North China University of Technology, 100144, Beijing, China

[b]School of Mathematical Sciences, Capital Normal University, 100048, Beijing, China

Abstract

In this paper, by Morse theory we will compute the critical groups at zero for a functional $I : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by setting

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F(x,u)dx,$$

where $p > 2$, Ω is a bounded domain in $\mathbb{R}^N$, $F(x,u) = \int_0^u f(x,t)dt$ and we assume that f is resonant at zero for the spectrum of $-\Delta$ in $W_0^{1,2}(\Omega)$. As an application of this critical groups estimates, some multiplicity results are also given.

Keywords: Quasilinear elliptic equations; Resonant; Morse theory
MR(2010) Subject Classification: 35J92, 35J35, 35B34

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$. We study the quasilinear elliptic problem

$$\begin{cases} -\Delta_p u - \Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

*Corresponding author. Phone: +86 13811722895. Fax: +86 01088803275. E-mail addresses: suncut@163.com (M. Sun), sujb@cnu.edu.cn (J. Su).

where $2 < p < \infty$, Δ_p denotes the p-Laplacian operator defined by $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$. This equation arises naturally in various contexts of physics, we refer to [2, 3] for details and further references.

In this paper, we assume that

(f_0) $f \in \mathcal{C}^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ with $f(x, 0) = 0$, and satisfies the following condition:

$$|f'(x,u)| \leq c(1 + |u|^{q-2}), \quad \forall\ u \in \mathbb{R},\ x \in \Omega,$$

for some constants $c > 0$ and $q \in [2, p^*)$, where $p^* = Np/(N-p)$ if $p < N$ and $p^* = +\infty$ if $N \leq p$, then it is well known that the weak solutions of equation (1.1) correspond to the critical points of the $\mathcal{C}^2$ functional $I : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F(x,u)dx,$$

where $F(x,u) = \int_0^u f(x,t)dt$, and $W_0^{1,p}(\Omega)$ is the Sobolev space endowed with the norm

$$\|u\| = \|\nabla u\|_p = (\int_\Omega |\nabla u|^p dx)^{1/p}.$$

In what follows, we denote by $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ the eigenvalues of $-\Delta$ in $W_0^{1,2}(\Omega)$, and let μ_1 be the first eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\Omega)$ (see [25]).

In recent years, there are a lot of literatures studying the existence of solutions for (1.1). For example, using the following conditions

$$\lambda_m < f'(x,0) < \lambda_{m+1},\ x \in \Omega,$$

and

$$F(x,u) < \frac{\mu_1}{p}|u|^p + C,\ x \in \Omega,\ u \in \mathbb{R},$$

where $m \geq 1$ and $C > 0$, the paper [6] proves that (1.1) has at least two nontrivial solutions by an extension of three critical point theorem. For the case of (1.1) with right-hand side having p-linear growth at infinity, i.e.,

$$\lim_{|u|\to\infty} \frac{f(x,u)}{|u|^{p-2}u} = \lambda \notin \sigma(-\Delta_p),$$

where $\sigma(-\Delta_p)$ is the spectrum of $-\Delta_p$ in $W_0^{1,p}(\Omega)$, the paper [10] gets the existence of one nontrivial solution. In the papers [15, 29], the authors study the problem (1.1) with concave and convex nonlinearities, and obtain the existence and multiple solutions. In [30], the existence of nodal solution to a quasilinear problem with (p,q)-Laplacian and reaction term that makes coercive the corresponding energy functional is investigated via variational methods besides truncation techniques.

The main aim of this paper is to give some results on the critical groups of an isolated critical point of I and its applications to the existence and multiplicity of solutions of (1.1) by Morse theory. Therefore, we need the following notions (see [7, 28]). Let u_0 be an isolated critical point of I with $I(u_0) = c \in \mathbb{R}$, and U be an isolated neighborhood of u_0, the group

$$C_*(I, u_0) = H_*(I^c \cap U, I^c \cap U \setminus \{u_0\}), \quad * = 0, 1, 2, \cdots$$

is called the $*$-th critical group of I at u_0, where $I^c = \{u \in W_0^{1,p}(\Omega) : I(u) \le c\}$, and $H_*(\cdot, \cdot)$ are the singular relative homological groups with a coefficient group $\mathbb{F}$. It follows from the excision property of the homology groups that the critical groups are independent of the choices of U, hence they are well defined.

Now, let us recall some results of the critical groups estimates of an isolated critical point u_0 for the functional I. Using a finite dimension reduction procedure, the authors in [8, 9] prove that if $I''(u_0)$ is injective, then the Morse index μ of u_0 is finite and

$$C_*(I, u_0) = \delta_{*,\mu}\mathbb{F},$$

where μ is defined as the supremum of the dimensions of the subspaces of $W_0^{1,p}(\Omega)$ on which $I''(u_0)$ is negative definite. Moreover, in the case in which $I''(u_0)$ is not injective, they have proved that the number of nontrivial critical groups of I in u_0 is finite, and the same result for $1 < p < 2$ can be found in [23]. For other qualitative results of the critical groups for the p-Laplacian equations, we refer to [11, 13] for details and further references. For the case of $u_0 = 0$, the author of [32] assumes that there exist $\alpha > 0$ and $m \ge 1$ such that

$$\frac{1}{2}\lambda_m u^2 < F(x, u) \le \frac{1}{2}\lambda_{m+1} u^2, \text{ for } x \in \Omega, \ 0 < |u| \le \alpha,$$

and proves that the functional I has a local linking at 0 and

$$C_{d_m}(I, 0) \ne 0,$$

here and in the sequel, we assume that

$$d_m = \dim\{\oplus_{i \le m} \ker(-\Delta - \lambda_i)\}.$$

Motivated by [9, 33], combining the minimax methods and Morse theory, we want to compute exactly the critical groups at zero for the resonant quasilinear elliptic equation (1.1). More specifically, we make the following assumptions:

(f_1) there exist $\alpha > 0$ and $m \geq 1$ such that

$$f'(x,0) = \lambda_m,\ F(x,u) \leq \frac{1}{2}\lambda_m u^2,\ \text{for } |u| \leq \alpha,\ x \in \Omega,$$

$(f_1)'$ there exist $\alpha > 0$ and $m \geq 1$ such that

$$f'(x,0) = \lambda_m,\ F(x,u) \geq \frac{1}{2}\lambda_m u^2 + C|u|^\theta,\ \text{for } |u| \leq \alpha,\ x \in \Omega,$$

where $C > 0$ and $2 < \theta < p$.

Without loss of generality, we can assume that $u = 0$ is an isolated critical point of equation (1.1). Our first result in this paper reads as follows.

Theorem 1.1. *Assume $2 < p < \infty$ and (f_0) holds.*

(i) If (f_1) holds, then

$$C_*(I,0) = \delta_{*,d_{m-1}}\mathbb{F}.$$

(ii) If $(f_1)'$ holds, then

$$C_*(I,0) = \delta_{*,d_m}\mathbb{F}.$$

Remark 1. (1) Assume that $f(x,u) = \lambda_m u + |u|^{\theta-1}$ with $2 < \theta < p$, we know that $(f_1)'$ is satisfied, but from [8] or [9, Lemma 2.2], we know that $I''(0)$ is not injective, then our result is new.

(2) Note that for the semilinear elliptic equation, i.e., $p = 2$, this result is due to the paper [33], now we can generalize the same result to the quasilinear equation (1.1) with $p > 2$. However, in our theorem there are many difficulties to get the critical group estimates for the functional I. For example, the space $W_0^{1,p}(\Omega)$ with $p > 2$ is not a Hilbert space, then we can not get a space decomposition according to the eigenfunctions which is the basis of linking theorem; Moreover, the second derivative of I in each critical point is not a Fredholm operator from $W_0^{1,p}(\Omega)$ to its dual space, so that the generalized Morse splitting lemma does not work. In spite of these difficulties, using the results in [9, 33] we are able to compute the critical groups at zero for the functional I.

As a byproduct of Theorem 1.1 we also obtain some multiplicity results. Before stating the results, let us recall the following notions. Let $\varphi_1 > 0$ be the eigenfunction of μ_1, and $\mu_2 = \inf\{\lambda \in \sigma(-\Delta_p) :\ \lambda > \mu_1\}$. If we assume $V_1 = \text{span}\{\varphi_1\}$, and denote by

$$V_1^\perp = \{u \in W_0^{1,p}(\Omega) :\ \int_\Omega (\varphi_1)^{p-1} u dx = 0\},$$

then we have

$$W_0^{1,p}(\Omega) = V_1 \oplus V_1^{\perp}. \tag{1.2}$$

From [21], we know that there exists $\mu_1 < \overline{\mu} < \mu_2$ such that

$$\int_\Omega |\nabla u|^p dx \geq \overline{\mu} \int_\Omega |u|^p dx, \text{ for any } u \in V_1^{\perp}. \tag{1.3}$$

Moreover, we make the following assumptions:

(f_2) there exist $M > 0$ and $\lambda < \frac{\lambda_1}{2}$ such that

$$F(x,u) - \frac{1}{p}\mu_1 |u|^p \leq \lambda |u|^2, \text{ for } |u| \geq M, \ x \in \Omega,$$

(f_3) there exists $\mu_1 < \eta < \overline{\mu}$ such that

$$\lim_{|u|\to\infty} \frac{f(x,u)}{|u|^{p-2}u} = \eta, \text{ for } x \in \Omega,$$

and our results read as follows.

Theorem 1.2. *Assume $2 < p < \infty$. If (f_0) and (f_2) hold, then equation (1.1) has at least four nontrivial solutions in each of the following cases:*

(i) (f_1) holds with $m \geq 3$;

(ii) $(f_1)'$ holds with $m \geq 2$.

Theorem 1.3. *Assume $2 < p < \infty$. If (f_0) and (f_3) hold, then equation (1.1) has at least one nontrivial solution in each of the following cases:*

(i) (f_1) holds with $m \neq 2$;

(ii) $(f_1)'$ holds with $m \neq 1$.

The proofs of our theorems are based on the critical groups estimates both at zero and at infinity for the functional I. By condition (f_2), we will prove that the functional I is coercive on $W_0^{1,p}(\Omega)$, and note that for the p-Laplacian operator, using the condition

$$\lim_{|u|\to\infty} (F(x,u) - \frac{1}{p}\mu_1 |u|^p) = -\infty, \ x \in \Omega, \tag{1.4}$$

the paper [24] also get the same result. Obviously, because of the existence of Laplacian operator, we only need (f_2) which is weaker than (1.4). For other results of the p-Laplacian equation we refer to [5, 12, 16, 17, 22] and references therein.

This paper is organized as follows. In Section 2, some preliminaries on Morse theory are given. The proofs of Theorem 1.1–1.3 are given in Sections 3–5, respectively. In the sequel, the letter C will be used indiscriminately to denote a suitable positive constant whose value may change from line to line.

2 Preliminaries

In this paper, we will apply the minimax methods and the Morse theory to prove our theorem. Therefore, we recall some notions and results (see e.g., [7]). Let E be a real Banach space and $\Phi \in \mathcal{C}^1(E, \mathbb{R})$.

Definition 2.1. The functional Φ is said to satisfy Palais-Smale (for short (PS)) condition if every sequence $\{u_n\} \subset E$ with

$$\Phi(u_n) \text{ being bounded, } \Phi'(u_n) \to 0, \text{ as } n \to \infty, \tag{2.1}$$

possesses a convergent subsequence.

Let $K = \{u \in E : \ \Phi'(u) = 0\}$, for $a < \inf \Phi(K)$, the $*$-th critical group of Φ at infinity is defined by

$$C_*(\Phi, \infty) = H_*(E, \Phi^a), \quad * = 0, 1, 2, \cdots .$$

If we denote

$$P(u,t) = \sum_i rankC_*(\Phi, u)t^i, \quad P(\infty, t) = \sum_i rankC_*(\Phi, \infty)t^i,$$

then the Morse inequality for the functional Φ is as follows: there is a polynomial $Q(t)$ with nonnegative integer as its coefficients such that

$$\sum_{u_j \in K} P(u_j, t) = P(\infty, t) + (1 + t)Q(t). \tag{2.2}$$

We also need the following critical point theorems.

Proposition 2.2. *([26]) Assume that Φ satisfies (PS) condition, $\Phi(0) = 0$ and Φ has a local linking at 0 with respect to $E = V \oplus W$, i.e., there exists $\rho > 0$ such that*

$$\Phi(u) \leq 0,\ u \in V,\ \|u\| \leq \rho; \quad \Phi(u) > 0,\ u \in W,\ 0 < \|u\| \leq \rho,$$

then $C_k(\Phi, 0) \neq 0$, where $k = \dim(V) < \infty$.

Proposition 2.3. *([4]) Assume that $E = H^- \oplus H^+$ and Φ satisfies the (PS) condition. If Φ is bounded from below on H^+ and $\Phi(u) \to -\infty$ as $\|u\| \to \infty$ with $u \in H^-$, then $C_d(\Phi, \infty) \neq 0$, where $d = \dim(H^-) < \infty$. Moreover, there exists a solution u_0 such that $C_d(\Phi, u_0) \neq 0$.*

Next, we recall some results in [8, 9]. We assume that (f_0) holds and u_0 is an isolated critical point of the functional I. Since $p > 2$, the second order differential of I in u_0 is given by

$$\begin{aligned} \langle I''(u_0)v, w\rangle = &\int_\Omega (1 + |\nabla u_0|^{p-2})(\nabla v|\nabla w)dx \\ &+ \int_\Omega (p-2)|\nabla u_0|^{p-4}(\nabla u_0|\nabla v)(\nabla u_0|\nabla w)dx \\ &- \int_\Omega f'(x, u_0)vwdx, \end{aligned} \tag{2.3}$$

for any $v, w \in W_0^{1,p}(\Omega)$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing, and $(\cdot|\cdot)$ denotes the scalar product in $\mathbb{R}^N$.

Since $u_0 \in C^1(\overline{\Omega})$, we have

$$b(x) = |\nabla u_0|^{(p-4)/2}\nabla u_0 \in L^\infty(\Omega).$$

Let H_{u_0} be the closure of $C_0^\infty(\Omega)$ under the scalar product

$$(v, w)_{u_0} = \int_\Omega [(1 + |b|^2)(\nabla v|\nabla w) + (p-2)(b|\nabla v)(b|\nabla w)]dx,$$

then H_{u_0} is isomorphic to $W_0^{1,2}(\Omega)$, and for $u_0 = 0$ we get that $H_{u_0} = W_0^{1,2}(\Omega)$. Since $p > 2$, $W_0^{1,p}(\Omega) \subset H_{u_0}$ continuously. Using (f_0), $I''(u_0)$ can be extended to a Fredholm operator $L_{u_0} : H_{u_0} \to H_{u_0}^*$ defined by setting

$$\langle L_{u_0}v, w\rangle = (v, w)_{u_0} - \int_\Omega f'(x, u_0)vwdx, \tag{2.4}$$

for any $v, w \in H_{u_0}$. So we can consider the natural splitting $H_{u_0} = H^- \oplus H^0 \oplus H^+$, where H^-, H^0, H^+ are, respectively, the negative, null, and positive spaces, according to the spectral decomposition of L_{u_0} in $L^2(\Omega)$, and H^-, H^0 have finite dimensions. From standard regularity theory (see [19]),

$$H^- \oplus H^0 \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

If we set $W = H^+ \cap W_0^{1,p}(\Omega)$ and $V = H^- \oplus H^0$, then we get the splitting

$$W_0^{1,p}(\Omega) = V \oplus W. \tag{2.5}$$

Lemma 2.4. *(Lemma 4.6 of [8]) There exist $r \in (0, \delta)$ and $\rho \in (0, r)$ such that for any $v \in V \cap \overline{B}_\rho(0)$ there exists one and only one $\overline{w} \in W \cap B_r(0) \cap L^\infty(\Omega)$ such that for any $z \in W \cap \overline{B}_r(0)$ we have*

$$f(v + \overline{w} + u_0) \leq f(v + z + u_0).$$

Moreover $\overline{w}$ is the only element of $W \cap \overline{B}_r(0)$ such that

$$\langle f'(u_0 + v + \overline{w}), z\rangle = 0 \;\; \forall\; z \in W.$$

So we can introduce the map $\psi : V \cap \overline{B}_\rho(0) \to W \cap B_r(0)$ defined by $\psi(v) = \overline{w}$ and the function $\varphi : V \cap \overline{B}_\rho(0) \to \mathbb{R}$ defined by $\varphi(v) = I(u_0 + v + \psi(v))$, which is a continuous map by [8]. Moreover, we have that

Lemma 2.5. *([9, Lemma 2.2]) For any $v \in V \cap \overline{B}_\rho(0)$, $z \in V$, $w \in V$, we have*

$$\begin{aligned}
&\psi \in \mathcal{C}^1(\overline{\Omega}), \psi(0) = 0, \;\; \psi'(0) = 0,\\
&\langle \varphi'(v), z\rangle = \langle I'(u_0 + v + \psi(v)), z\rangle,\\
&\langle \varphi''(v)z, w\rangle = \langle I''(u_0 + v + \psi(v))(z + \psi'(v)(z)), w\rangle.
\end{aligned} \tag{2.6}$$

Lemma 2.6. *([8, p. 286]) If (f_0) holds, then*

$$C_*(I, u_0) = C_*(\varphi, 0), \quad * = 0, 1, 2, \cdots .$$

3 Critical groups at zero

Without loss of generality, we can assume that $u_0 = 0$ is an isolated critical point of equation (1.1). First, using (f_1) we will show that the functional I has a local linking at 0, and for the similar results with $p = 2$ we refer to [26, 27] for details and further references.

Lemma 3.1. *If* (f_0) *and* (f_1) *hold, then we have*

$$C_{d_{m-1}}(I,0) \neq 0. \tag{3.1}$$

Proof. We set

$$H_1 = \oplus_{i\leq m-1}\ker(-\Delta-\lambda_i),$$
$$H_2 = \overline{\oplus_{j\geq m}\ker(-\Delta-\lambda_j)},$$

and $W_1 = H_2 \cap W_0^{1,p}(\Omega)$. Since $p > 2$, the standard regularity theory (see [19]) gives

$$H_1 \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

then we get the splitting

$$W_0^{1,p}(\Omega) = H_1 \oplus W_1.$$

We first prove that there exists $\rho > 0$ such that

$$\begin{cases} I(u) \leq 0, & \text{for } u \in H_1,\ \|u\| \leq \rho, \\ I(u) > 0, & \text{for } u \in W_1,\ 0 < \|u\| \leq \rho. \end{cases}$$

(1) Let $\varepsilon > 0$ such that $\lambda_{m-1} < \lambda_m - \varepsilon$. By (f_0) and (f_1), there exists $p < \gamma < p*$ such that

$$F(x,u) \geq \frac{1}{2}(\lambda_m - \varepsilon)u^2 - C|u|^\gamma,\ \forall\ u \in \mathbb{R},\ x \in \Omega,$$

this together with the fact that H_1 is a finite dimensional space gives

$$\begin{aligned} I(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F(x,u)dx \\ &\leq C\|u\|^p + \frac{\lambda_{m-1}}{2}\int_\Omega |u|^2 dx - \frac{1}{2}(\lambda_m - \varepsilon)\int_\Omega |u|^2 dx + C\|u\|^\gamma \\ &\leq C\|u\|^p - C\|u\|^2 + C\|u\|^\gamma, \end{aligned}$$

then using $p > 2$ there exists $\rho > 0$ such that $I(u) \leq 0$ as $\|u\| \leq \rho$.

(2) By conditions (f_0) and (f_1), there exists $p < \nu < p*$ such that

$$F(x,u) \leq \frac{1}{2}\lambda_m u^2 + C|u|^\nu,\ \forall\ u \in \mathbb{R},\ x \in \Omega,$$

then for $u \in W_1$ we have

$$
\begin{aligned}
I(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F(x,u)dx \\
&\geq \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{2}\int_\Omega \lambda_m u^2 dx - C\int_\Omega |u|^\nu dx \\
&\geq \frac{1}{p}\|u\|^p - C\|u\|^\nu,
\end{aligned}
$$

which implies that there exists $\rho > 0$ such that $I(u) > 0$ as $0 < \|u\| \leq \rho$.

Now using the results in Proposition 2.2, we complete the lemma. □

Proof of (i) **in Theorem 1.1:** Using (f_1) and (2.3), in the particular case $u_0 = 0$, we have

$$
\begin{aligned}
\langle I''(0)v, w\rangle &= \int_\Omega (\nabla v|\nabla w)dx - \int_\Omega f'(x,0)vwdx \\
&= \int_\Omega (\nabla v|\nabla w)dx - \lambda_m \int_\Omega vwdx,
\end{aligned} \tag{3.2}
$$

where $v, w \in W_0^{1,p}(\Omega)$.

Therefore, the Fredholm operator $L_0 : H_0 \to H_0^*$ in (2.4) is defined as

$$
\begin{aligned}
\langle L_0 v, w\rangle &= (v,w)_0 - \int_\Omega f'(x,0)vwdx \\
&= \int_\Omega (\nabla v|\nabla w)dx - \lambda_m \int_\Omega vwdx,
\end{aligned} \tag{3.3}
$$

where $v, w \in H_0 = W_0^{1,2}(\Omega)$. Now, we consider the natural splitting

$$
W_0^{1,2}(\Omega) = H^- \oplus H^0 \oplus H^+,
$$

which H^-, H^0, H^+ are, respectively, the negative, null, and positive spaces, according to the spectral decomposition of L_0 in $L^2(\Omega)$. Obviously, by (3.3) we know that

$$
H^- = \oplus_{i\leq m-1}\ker(-\Delta - \lambda_i), \quad H^0 = \ker(-\Delta - \lambda_m), \tag{3.4}
$$

and

$$
H^+ = \overline{\oplus_{j\geq m+1}\ker(-\Delta - \lambda_j)},
$$

Moreover, H^-, H^0 have finite dimensions.

Since $p > 2$, $W_0^{1,p}(\Omega) \subset W_0^{1,2}(\Omega)$ continuously. Similar to (2.5), if we define

$$V = H^- \oplus H^0, \quad W = H^+ \cap W_0^{1,p}(\Omega),$$

then we get the splitting

$$W_0^{1,p}(\Omega) = V \oplus W. \tag{3.5}$$

Now, using Lemma 2.5 we know that

$$\varphi : (H^- \oplus H^0) \cap \overline{B}_\rho(0) \to \mathbb{R}$$

is a $\mathcal{C}^2$ function. Moreover, for any $z, w \in H^- \oplus H^0$, by (2.6) and (3.2) we have

$$\begin{aligned}\langle \varphi''(0)z, w\rangle &= \langle I''(0)z, w\rangle \\ &= \int_\Omega (\nabla v|\nabla w)dx - \int_\Omega f'(x,0)vwdx \\ &= \int_\Omega (\nabla v|\nabla w)dx - \lambda_m \int_\Omega vwdx,\end{aligned}$$

then $\varphi''(0)$ is a Fredholm operator with kernel H^0. Meanwhile, Lemma 2.6 implies that

$$C_*(I,0) = C_*(\varphi,0), \quad * = 0,1,2,\cdots. \tag{3.6}$$

Next, for the rest of the proof, we follow the methods in [33]. From the Shifting theorem (see [7]), there exist a ball B centered at 0, and a $\mathcal{C}^1$ map $h : B \cap H^0 \to H^-$ such that

$$C_*(\varphi,0) = C_{*-d_{m-1}}(\widetilde{\varphi},0), \quad * = 0,1,2,\cdots, \tag{3.7}$$

where $\widetilde{\varphi}(u) = \varphi(u + h(u))$ for any $u \in H^0$, and $d_{m-1} = \dim(H^-)$.

By (3.7) and Lemma 3.1 we get

$$C_0(\widetilde{\varphi},0) = C_{d_{m-1}}(I,0) \neq 0,$$

which is equivalent to 0 being an isolated local minimum of $\widetilde{\varphi}$, so

$$C_*(\widetilde{\varphi},0) = \delta_{*,0}\mathbb{F},$$

this together with (3.6) and (3.7) gives

$$C_*(I,0) = \delta_{*,d_{m-1}}\mathbb{F}.$$

The proof is completed. $\square$

Next, we give the proof of $(f_1)'$.

Lemma 3.2. *If (f_0) and $(f_1)'$ hold, then by (3.4) we have*

$$C_{d_m}(I,0) \neq 0, \ \text{where } d_m = \dim(H^- \oplus H^0). \tag{3.8}$$

Proof. Similar to Lemma 3.1 above, we only need to prove that there exists $\rho > 0$ such that

$$\begin{cases} I(u) \leq 0, & \text{for } u \in H^- \oplus H^0, \ \|u\| \leq \rho, \\ I(u) > 0, & \text{for } u \in W, \ 0 < \|u\| \leq \rho. \end{cases}$$

(1) By (f_0) and $(f_1)'$, there exists $p < \gamma < p*$ such that

$$F(x,u) \geq \frac{1}{2}\lambda_m u^2 + C|u|^\theta - C|u|^\gamma, \ \forall\, u \in \mathbb{R}, \ x \in \Omega. \tag{3.9}$$

For $u \in H^- \oplus H^0$ which is a finite dimensional space, (3.9) implies that

$$\begin{aligned} I(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F(x,u)dx \\ &\leq \frac{1}{p}\int_\Omega |\nabla u|^p dx - C\int_\Omega (|u|^\theta - |u|^\gamma)dx \\ &\leq C\|u\|^p - C\|u\|^\theta + C\|u\|^\gamma, \end{aligned}$$

then using $2 < \theta < p < \gamma < p*$ there exists $\rho > 0$ such that $I(u) \leq 0$ as $\|u\| \leq \rho$.

(2) By the conditions (f_0) and $(f_1)'$, for $\varepsilon > 0$ satisfying $\lambda_{m+1} > \lambda_m + \varepsilon$, there is $p < \nu < p*$ such that

$$F(x,u) \leq \frac{1}{2}(\lambda_m + \varepsilon)u^2 + C|u|^\nu, \ \forall\, u \in \mathbb{R}, \ x \in \Omega,$$

which implies that, for $u \in W$,

$$\begin{aligned} I(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F(x,u)dx \\ &\geq \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{\lambda_{m+1}}{2}\int_\Omega u^2 dx - \frac{\lambda_m + \varepsilon}{2}\int_\Omega u^2 dx - C\int_\Omega |u|^\nu dx \\ &\geq \frac{1}{p}\|u\|^p - C\|u\|^\nu, \end{aligned}$$

then there exists $\rho > 0$ such that $I(u) > 0$ as $0 < \|u\| \leq \rho$. We complete the lemma. □

Proof of (ii) **in Theorem 1.1:** Set $d_0 = \dim(H^0)$. Similar to (3.6) and (3.7), we have

$$C_*(I,0) = C_*(\varphi,0) = C_{*-(d_m-d_0)}(\widetilde{\varphi},0), \quad * = 0,1,2,\cdots. \tag{3.10}$$

By Lemma 3.2 we get

$$C_{d_0}(\widetilde{\varphi},0) \neq 0,$$

which is equivalent to 0 being an isolated local maxmum of $\widetilde{\varphi}$, so

$$C_*(\widetilde{\varphi},0) = \delta_{*,d_0}\mathbb{F},$$

this together with (3.10) gives

$$C_*(I,0) = \delta_{*,d_m}\mathbb{F}.$$

The proof is completed. □

4 Proof of Theorem 1.2

Let

$$f_\pm(x,t) = \begin{cases} f(x,t) & \pm t \geq 0 \\ 0 & \pm t < 0, \end{cases}$$

and

$$I_\pm(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F_\pm(x,u)dx,$$

where $F_\pm(x,u) = \int_0^u f_\pm(x,s)ds$. It is well known that the critical points of $I_\pm$ are exactly the weak solutions of equation (1.1) (cf. [14]).

Lemma 4.1. *If* (f_0) *and* (f_2) *hold, then*
(1) I *and* $I_\pm$ *are coercive on* $W_0^{1,p}(\Omega)$*,*
(2) I *and* $I_\pm$ *satisfy the* (PS) *condition.*

Proof. (1) For the functional I, by contradiction, we assume that there is a sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ such that

$$I(u_n) \leq C, \text{ as } \|u_n\| \to \infty. \tag{4.1}$$

If we set $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$ and there is a $v_0 \in W_0^{1,p}(\Omega)$ such that

$$\begin{cases} v_n \rightharpoonup v_0, & \text{weakly in } W_0^{1,p}(\Omega), \\ v_n \to v_0, & \text{strongly in } L^p(\Omega), \\ v_n(x) \to v_0(x), & \text{a.e. } x \in \Omega. \end{cases} \tag{4.2}$$

From the condition (f_2), we get

$$F(x,u) - \frac{1}{p}\mu_1|u|^p \leq \lambda|u|^2 + C, \text{ for } u \in \mathbb{R},\ x \in \Omega, \tag{4.3}$$

this together with (4.1) gives

$$\frac{C}{\|u_n\|^p} \geq \frac{I(u_n)}{\|u_n\|^p} \geq \frac{1}{p}\int_\Omega (|\nabla v_n|^p - \mu_1|v_n|^p)dx - \frac{C + C\|u_n\|^2}{\|u_n\|^p},$$

then by $2 < p$ we obtain

$$\limsup_{n\to\infty} \int_\Omega |\nabla v_n|^p dx \leq \mu_1 \int_\Omega |v_0|^p dx.$$

On the other hand, we get that

$$\mu_1 \int_\Omega |v_0|^p dx \leq \int_\Omega |\nabla v_0|^p dx \leq \liminf_{n\to\infty} \int_\Omega |\nabla v_n|^p dx,$$

which implies that

$$\int_\Omega |\nabla v_0|^p dx = \mu_1 \int_\Omega |v_0|^p dx, \text{ and } v_n \to v_0 \text{ in } W_0^{1,p}(\Omega).$$

Then we get $\|v_0\| = 1$, and by (4.2) we have

$$|u_n(x)| \to +\infty, \text{ a.e. } x \in \Omega. \tag{4.4}$$

Using the Fatou lemma, (4.3) and (4.4) give

$$\begin{aligned} I(u_n) &\geq \frac{1}{2}\int_\Omega |\nabla u_n|^2 dx - \int_\Omega (F(x,u_n) - \frac{\mu_1}{p}|u_n|^p)dx \\ &\geq \frac{\lambda_1}{2}\int_\Omega |u_n|^2 dx - \lambda \int_\Omega |u_n|^2 dx - C \\ &\to +\infty, \quad \text{as } n \to \infty, \end{aligned}$$

this is a contradiction. The case of I_+ (I_-) is similar.

(2) The (PS) condition follows from the Theorem 3 in [6]. □

Lemma 4.2. *Let $e_1 > 0$ be the eigenfunction associated with λ_1. If (f_1) (or $(f_1)'$) holds, then there exists $t > 0$ such that $I_\pm(\pm te_1) < 0$.*

Proof. By (f_1), for $\varepsilon > 0$ small there exists $p < \nu \leq p*$ such that

$$F(x,u) \geq \frac{1}{2}(\lambda_m - \varepsilon)u^2 - C|u|^\nu, \ \forall \ u \in \mathbb{R}, \ x \in \Omega.$$

Then by $m \geq 3$, we get

$$\begin{aligned} I_+(te_1) = & \frac{|t|^p}{p}\int_\Omega |\nabla e_1|^p dx + \frac{\lambda_1 |t|^2}{2}\int_\Omega |e_1|^2 dx \\ & - \frac{(\lambda_m - \varepsilon)t^2}{2}\int_\Omega |e_1|^2 dx + C|t|^\nu \int_\Omega |e_1|^\nu dx \\ \leq & C|t|^p - C|t|^2 + C|t|^\nu, \\ < & 0, \quad \text{as } t > 0 \text{ small.} \end{aligned}$$

The other cases are similar. □

Proof of Theorem 1.2. Since the functional $I, I_\pm$ are coercive on $W_0^{1,p}(\Omega)$, Lemma 4.1 implies that

$$C_*(I,\infty) = \delta_{*,0}\mathbb{F}. \tag{4.5}$$

Using Lemma 4.2, the functional I has a positive critical point u_1 and a negative critical point u_2 which are all at negative energy such that

$$C_*(I,u_1) = C_*(I,u_2) = \delta_{*,0}\mathbb{F}. \tag{4.6}$$

Using the Mountain pass lemma in [1], we know that equation has a solution u_3 such that (see [7])

$$C_1(I,u_3) \neq 0.$$

The next claim can be found in [31, p. 412], and for the reader's convenience we sketch a proof of it.

Claim :

$$C_*(I,u_3) = \delta_{*,1}\mathbb{F}. \tag{4.7}$$

Using (2.4), for the isolated critical point u_3 we can define $V = H^- \oplus H^0 \subset H_{u_3}$, and Lemma 2.6 implies that there exists

$$\varphi : V \cap \overline{B}_\rho(0) \to \mathbb{R}$$

such that

$$C_*(I,u_3) = C_*(\varphi, 0), \quad * = 0,1,2,\cdots, \tag{4.8}$$

and

$$C_1(\varphi,0) = C_1(I,u_3) \neq 0. \tag{4.9}$$

Set m =dimH^- and n =dimH^0, we know that $m \leq 1$.

If $n = 0$, then 0 is a nondegenerate critical point of φ (see [7]), and

$$C_*(\varphi, 0) = \delta_{*,m}\mathbb{F},$$

which implies that (4.7) holds.

If $n \neq 0$, then 0 is a degenerate critical point of φ, and from the Shifting theorem (see [7]), we have

$$C_*(\varphi, 0) = C_{*-m}(\widetilde{\varphi}, 0), \quad * = 0, 1, 2, \cdots, \tag{4.10}$$

where $\widetilde{\varphi}(u) = \varphi|_{H^0}$.

Case A. If $m = 1$, then $C_0(\widetilde{\varphi}, 0) \neq 0$, which is equivalent to 0 being an isolated local minimum of $\widetilde{\varphi}$, so

$$C_*(\widetilde{\varphi}, 0) = \delta_{*,0}\mathbb{F},$$

then (4.7) holds.

Case B. If $m = 0$, then (4.10) implies that

$$C_*(\varphi, 0) = C_*(\widetilde{\varphi}, 0), \quad * = 0, 1, 2, \cdots. \tag{4.11}$$

Next, we show $n = 1$. For $\ker \varphi''(0)$ to be nontrivial it amounts to saying that 1 is the first eigenvalue of the following linear eigenvalue problem

$$-\mathrm{div}((1 + |b|^2)\nabla u + (p - 2)(b, \nabla u)_{\mathbb{R}^N} b) = \lambda f'(x, u_3)u, \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0.$$

From [18] or [20, Sect. 6.1], this first eigenvalue is simple, then $n = 1$, which implies that

$$C_*(I, u_3) = C_*(\varphi, 0) = C_*(\widetilde{\varphi}, 0) = \delta_{*,1}\mathbb{F}.$$

Moreover, using (f_1) with $m \geq 3$ or $(f_1)'$ with $m \geq 2$, Theorem 1.1 gives that there exists $d \geq 2$ such that

$$C_*(I, 0) = \delta_{*,d}\mathbb{F},$$

which implies that $u_3 \neq 0$.

Now, Morse inequality (2.2) for I gives that

$$(-1)^d + (-1)^0 + (-1)^0 + (-1)^1 = (-1)^0,$$

this is a contradiction. So equation (1.1) has at least four nontrivial solutions. The proof is completed. □

5 Proof of Theorem 1.3

Now, we give the proof of Theorem 1.3. First we deduce by standard arguments that the functional satisfies the (PS) condition (cf. [9]).

Lemma 5.1. *If* (f_0) *and* (f_3) *hold, then* I *satisfies the* (PS) *condition.*

Proof. Using (f_0), we need only to prove that if there is a sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ satisfying (2.1), then $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Arguing by contradiction, we assume $\|u_n\| \to \infty$ as $n \to \infty$. Let $z_n = \frac{u_n}{\|u_n\|}$, then there exists $z \in W_0^{1,p}(\Omega)$ such that, passing if necessary to a subsequence,

$$\begin{cases} z_n \rightharpoonup z & \text{weakly in } W_0^{1,p}(\Omega), \\ z_n \to z & \text{strongly in } L^p(\Omega), \\ z_n \to z & \text{a.e. } x \in \Omega. \end{cases}$$

For any $v \in W_0^{1,p}(\Omega)$, from (2.1) we have

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla v dx + \int_\Omega \nabla u_n \nabla v dx - \int_\Omega f(x, u_n) v dx = o(1)\|u_n\|. \quad (5.1)$$

Now dividing (5.1) by $\|u_n\|^{p-1}$, and then taking $v = z_n - z$, we derive by the assumption (f_3) that

$$\lim_{n\to\infty} \int_\Omega |\nabla z_n|^{p-2}\nabla z_n \nabla(z_n - z) dx = 0,$$

then from the fact that $-\Delta_p$ is of type S^+ (see [14]), we conclude that $z_n \to z$ in $W_0^{1,p}(\Omega)$ with $\|z\| = 1$.

Moreover, dividing (5.1) by $\|u_n\|^{p-1}$, from (f_3) we infer that

$$-\Delta_p z = \eta |z|^{p-2} z, \;\; z \in W_0^{1,p}(\Omega). \quad (5.2)$$

Using $\mu_1 < \eta < \overline{\mu}$, equation (5.2) has zero as the only solution, thus we conclude $z = 0$, which is a contradiction. The proof is completed. □

Lemma 5.2. *If* (f_0) *and* (f_3) *hold, then form (1.2) we have*
(1) $I(u) \to -\infty$, *as* $u \in V_1$ *and* $\|u\| \to \infty$,
(2) $I(u) \to +\infty$, *as* $u \in V_1^\perp$ *and* $\|u\| \to \infty$.

Proof. (1) Using (f_0) and (f_3), for $\varepsilon > 0$ small we have

$$F(x,u) \geq \frac{\mu_1 + \varepsilon}{p}|u|^p - C, \ \forall\ u \in \mathbb{R},\ x \in \Omega,$$

which implies that

$$\begin{aligned} I(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F(x,u)dx \\ &\leq \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\mu_1+\varepsilon}{p}\int_\Omega |u|^p dx + C \\ &\leq \frac{1}{p}(1 - \frac{\mu_1+\varepsilon}{\mu_1})\|u\|^p + C\|u\|^2 + C \\ &\to -\infty,\ \text{as } \|u\| \to \infty. \end{aligned}$$

(2) Similarly, for $\varepsilon > 0$ small we have

$$F(x,u) \leq \frac{\overline{\mu} - \varepsilon}{p}|u|^p + C, \ \forall\ u \in \mathbb{R},\ x \in \Omega,$$

then by (1.3) it implies that

$$\begin{aligned} I(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F(x,u)dx \\ &\geq \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\overline{\mu}-\varepsilon}{p}\int_\Omega |u|^p dx - C \\ &\geq \frac{1}{p}(1 - \frac{\overline{\mu}-\varepsilon}{\overline{\mu}})\|u\|^p - C,, \end{aligned}$$

then $I(u) \to +\infty$, as $\|u\| \to \infty$. The proof is completed. □

Proof of Theorem 1.3. From Lemma 5.1, we know that I satisfies the (PS) condition, and using Lemma 5.2 and Proposition 2.3, we get that there exists a solution u_0 of equation (1.1) such that

$$C_1(I, u_0) \neq 0.$$

Using (f_1) with $m \neq 2$ or $(f_1)'$ with $m \neq 1$, Theorem 1.1 gives that

$$C_*(I, 0) = \delta_{*,d}\mathbb{F},\ \text{where } d \neq 1,$$

which implies that equation (1.1) has at least one nontrivial solution u_0. The proof is completed. □

Acknowledgements

The authors would like to thank the referee for many valuable discussions and suggestions. This paper is supported by the BNSF (1132012) and NSFC (61272026,11271264, 11301010).

References

[1] A. Ambrosetti, P. H. Rabinowitz, Dual variational methods in critical points theory and applications, J. Funct. Anal. 14 (1973) 349–381.

[2] V. Benci, D. Fortunato, L. Pisani, Soliton-like solutions of a Lorentz invariant equation in dimension 3, Math. Phys. 3 (1998) 315–344.

[3] V. Benci, P. D'Avenia, D. Fortunato, L. Pisani, Solitons in several space dimensions: a Derrick's problem and infinitely many solutions, Arch. Rat. Mech. Anal. 154 (2000) 297–324.

[4] T. Bartsch, S.-J. Li, Critical point theory for asymptotically quadratic functionals and applications to problems with resonance, Nonlinear Anal. 28 (1997) 419–441.

[5] T. Bartsch, Z.-L. Liu, T. Weth, Nodal solutions of a p-Laplacian equation, Proc. London. Math. Soc. 91 (2005) 129–152.

[6] K.-C. Chang, Morse theory on Banach space and its applications to partial differential equations, Chinese Ann. Math. Ser. B 4 (1983) 381–399.

[7] K.-C. Chang, Infinite Dimensional Morse Theory and Multiple Solution Problems, Birkhäuser, Boston, 1993.

[8] S. Cingolani, G. Vannella, Critical groups computations on a class of Sobolev Banach spaces via Morse index, Ann. Inst. H. Poincaré Anal. Non Linéaire 20 (2003) 271–292.

[9] S. Cingolani, G. Vannella, Marino-Prodi perturbation type results and Morse indices of minimax critical points for a class of functionals in Banach spaces, Ann. Mat. Pura Appl. 186 (2007) 155–183.

[10] S. Cingolani, M. Degiovanni, Nontrivial solutions for p-Laplace equations with right-hand side having p-linear growth at infinity, Comm. Partial Differential Equations 30 (2005) 1191–1203.

[11] S. Cingolani, M. Degiovanni, On the Poincare-Hopf theorem for functionals defined on Banach spaces, Adv. Nonlinear Stud. 9 (2009) 679–699.

[12] S. Carl, D. Motreanu, Constant-sign and sign-changing solutions for nonlinear eigenvalue problems, Nonlinear Anal. 68 (2008) 2668–2676.

[13] M. Degiovanni, Critical groups of finite type for functionals defined on Banach spaces, Commun. Appl. Anal. 13 (2009) 395–410.

[14] G. Dinca, P. Jebelean, J. Mawhin, Variational and topological methods for Dirichlet problems with p-Laplacian, Port. Math. (N.S.) 58 (2001) 339–378.

[15] F.O. de Paiva, J.M. do Ó, E.S. de Medeiros, Multiplicity results for some quasilinear elliptic problems, Topol. Methods Nonlinear Anal. 34 (2009) 77–89.

[16] N. Dancer, K. Perera, Some remarks on the Fucik spectrum of the p-Laplacian and critical groups, J. Math. Anal. Appl. 254 (2001) 164–177.

[17] J.M. do Ó, Existence of solutions for quasilinear elliptic equations, J. Math. Anal. Appl. 207 (1997) 104–126.

[18] D. de Figueiredo, Positive solutions of semilinear elliptic problems, In: Differential Equations, Sao Paulo, 1981. Lecture Notes in Math., vol. 957, pp. 34–87. Springer, Berlin, 1982.

[19] D. Gilbarg, N. S. Trudinger, Elliptic Partial Differential Equation of Second Order, Springer-Verlag, Berlin, 1998.

[20] L. Gasinski, N.S. Papageorgiou, Nonlinear Analysis, Chapman, Hall/CRC, Boca Raton, 2006.

[21] G.L. Garza, A.J. Rumbos, Existence and multiplicity for a resonance problem for the p-Laplacian on bounded domain in $\mathbb{R}^N$, Nonlinear Anal. 70 (2009) 1193–1208.

[22] M.-Y. Jiang, Critical groups and multiple solutions of the p-Laplacian equations, Nonlinear Anal. 59 (2004) 1221–1241.

[23] M.-Y. Jiang, M. Sun, Some qualitative results of the critical groups for the p-Laplacian equations, Nonlinear Anal. 75 (2012) 1778–1786.

[24] Q.-S. Jiu, J. Su, Existence and multiplicity results for Dirichlet problems with p-Laplacian, J. Math. Anal. Appl. 281 (2003) 587–601.

[25] P. Lindqvist, On the equation $\mathrm{div}(|\nabla u|^{p-2}\nabla u)+\lambda|u|^{p-2}u = 0$, Proc. Amer. Math. Soc. 109 (1990) 157–164.

[26] J.-Q. Liu, A Morse index for a saddle point, Systems Sci. Math. Sci. 2 (1989) 32–39.

[27] S.-J. Li, M. Willem, Applications of local linking to critical point theory, J. Math. Anal. Appl. 189 (1995) 6–32.

[28] J. Mawhin, M. Willem, Critical point theory and Hamiltonian systems, Springer, Berlin, 1989.

[29] E. Medeiros and K. Perera, Multiplicity of solutions for a quasilinear elliptic problem via the cohomological index, Nonlinear Anal. 71 (2009) 3654–3660.

[30] S.A. Marano, N.S. Papageorgiou, Constant-sign and nodal solutions of coercive (p,q)-Laplacian problems, Nonlinear Anal. 77 (2013) 118–129.

[31] N.S. Papageorgiou, V.D. Rădulescu, Qualitative phenomena for some classes of quasilinear elliptic equations with multiple resonance, Appl. Math. Optim. 69 (2014) 393–430.

[32] M. Sun, Multiplicity of solutions for a class of the quasilinear elliptic equations at resonance, J. Math. Anal. Appl. 386 (2012) 661–668.

[33] J. Su, Semilinear elliptic boundary value problems with double resonance between two consecutive eigenvalues, Nonlinear Anal. 48 (2002) 881–895.