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Rigidity theorems on conformal class of compact manifolds with boundary

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ABSTRACT

Let M be a compact manifold with boundary. In this paper, we discuss some rigidity theorems on Riemannian metrics in a same conformal class that fix the boundary and satisfy certain integral condition on the scalar curvature and on the mean curvature along the boundary. As an application, we will state some rigidity theorems on the conformal class of static metrics.

1. Introduction

Let (M, g_0) be a compact n-dimensional Riemannian smooth manifold with $n \geq 2$ and nonempty smooth boundary ∂M (possibly non-connected). Let R_{g_0} denote the scalar curvature of (M, g_0) and let $h_{g_0} = \mathrm{div}_{g_0}\eta_{g_0}$ denote the mean curvature of ∂M in (M, g_0), in the direction of the exterior conormal $\eta = \eta_{g_0}$. If $n = 2$ then $K_{g_0} = R_{g_0}/2$ denotes the Gaussian curvature and $h_{g_0} = \kappa_{g_0}$ denotes the geodesic curvature of the curve ∂M with respect to g_0.

We recall that the conformal class of a metric g on M, say $[g]$, is the set of metrics of the form $\tilde{g} = \mu^2 g$, where μ is a positive smooth function defined on M. Escobar [4] had dealt with the following question:

$$\textit{Given a metric } g \in [g_0] \textit{ with } R_g = R_{g_0} \textit{ in } M, \textit{ and } h_g = h_{g_0} \textit{ on } \partial M, \textit{ when is } g = g_0?$$

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In Corollary 2 of [4], Escobar obtained the following result: *Let* $g \in [g_0]$ *satisfying* $R_g = R_{g_0} \leq 0$ *and* $h_g = h_{g_0} \leq 0$. *Then* $g = g_0$. In an opposite direction, in [3], Escobar also described the conformally flat metrics $g \in [\delta_{ij}]$ on the ball $B = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$, with $n \geq 3$, having constant scalar curvature and constant mean curvature on ∂B. By this classification theorem, there is a non-compact set of metrics $g \in [\delta_{ij}]$ with $R_g = 0$ and $h_g = 1$.

Min-Oo [9] conjectured the following: *Let* g *be a metric on the upper hemisphere* S^n_+ *satisfying the following properties: The scalar curvature* $R_g \geq n(n-1)$, *the induced metric on* ∂S^n_+ *agrees with the standard metric on* ∂S^n_+, *and the boundary* ∂S^n_+ *is totally geodesic in* S^n_+. *Then,* g *is isometric to the standard metric* $g_{S^n_+}$ *on* S^n_+. Despite Min-Oo's conjecture is false (see the counterexample due to Brendle, Marques and Neves [2]), Hang and Wang [6] proved Min-Oo's conjecture is true among metrics that are conformal to $g_{S^n_+}$. Namely, they proved the following

Theorem B. *(See Theorem 3.4 of [6].) Let* $g \in [g_{S^n_+}]$ *on* S^n_+. *Assume that the scalar curvature* $R_g \geq R_{g_{S^n_+}} = n(n-1)$ *and* $g = g_{S^n_+}$ *on the boundary* ∂S^n_+. *Then* $g = g_{S^n_+}$.

The upper hemisphere S^n_+ is a static manifold, that means there is a smooth function f satisfying the equation

$$\begin{cases} f\mathrm{Ric} - \nabla^2 f + (\Delta f)g = 0, \text{ in } M \setminus \partial M, \\ f > 0 \text{ in } M \setminus \partial M, \text{ and } f = 0, \text{ on } \partial M. \end{cases} \tag{1}$$

As a solution of (1), we take the height function $f(x) = x_{n+1}$, for all $x = (x_1, \ldots, x_{n+1}) \in S^n_+$. Taking the trace in (1), we see that static manifolds are solutions of $\mathcal{L}_{g_{S^n_+}} f = 0$, where

$$\mathcal{L}_g f = \Delta_g f + \frac{1}{n-1} R_g f, \tag{2}$$

for some smooth function f that is positive in M and vanishes on ∂M.

Our first theorem says the following:

Theorem 1. *Let* g_0 *be a metric on* M *and* $g = \mu^2 g_0$ *a metric in the conformal class* $[g_0]$ *such that* $g = g_0$ *on* ∂M. *Let* $f \in C^1(M) \cap C^2(M \setminus \partial M)$ *positive almost everywhere satisfying*

$$\int_M f(R_g - R_{g_0})\mathrm{dvol}_{g_0} + 2\int_{\partial M} f(h_g - h_{g_0})d\mathcal{H}^{n-1}_{g_0} \geq 0. \tag{3}$$

If $\int_M \mathcal{L}_{g_0} f(1-\mu^{-2})\mathrm{dvol}_{g_0} \geq 0$ *then* $g = g_0$.

Theorem 1 requires no condition on the first eigenvalue $\lambda_1 = \lambda_1(\mathcal{L}_{g_0})$ of the operator $\mathcal{L}_{g_0}$. However, the first eigenvalue

$$\lambda_1 = \inf\{\int_M (|\nabla\varphi|^2 - \frac{R_{g_0}}{n-1}\varphi^2)\mathrm{dvol}_{g_0} \mid \varphi \in C^\infty_0(M), \int_M \varphi^2 \mathrm{dvol}_{g_0} = 1\}$$

satisfies $\mathcal{L}_{g_0}\varphi_1 + \lambda_1\varphi_1 = 0$, for some C^2 eigenfunction $\varphi_1 = \varphi_1(\mathcal{L}_{g_0})$ that is positive in M and vanishes along the boundary ∂M. Thus, by Theorem 1, we have

Corollary 2. *Let* $g = \mu^2 g_0 \in [g_0]$ *satisfying* $g = g_0$, *on* ∂M. *Let* φ_1 *be the eigenfunction corresponding to the first eigenvalue* $\lambda_1 = \lambda_1(\mathcal{L}_{g_0})$. *Assume* $\int_M \varphi_1(R_g - R_{g_0})\mathrm{dvol}_{g_0} \geq 0$ *and* $\lambda_1 \int_M \varphi_1(1-\mu^{-2})\mathrm{dvol}_{g_0} \leq 0$. *Then* $g = g_0$.

As a consequence of Corollary 2, since static metrics g on M satisfy $\lambda_1(\mathcal{L}_g) = 0$, it follows

Corollary 3. *Let g_0 be a static metric on M and $g \in [g_0]$ such that $g = g_0$ on ∂M. If $R_g \geq R_0$ then $g = g_0$.*

Another consequence of Theorem 1, using that $\mathcal{L}_{g_0}(1) = \frac{1}{n-1} R_{g_0}$, is

Corollary 4. *Let $g = \mu^2 g_0$ be a metric in the conformal class $[g_0]$ such that $g = g_0$ on ∂M. Assume that*

$$\int_M (R_g - R_{g_0}) d\mathrm{vol}_{g_0} + 2 \int_{\partial M} (h_g - h_{g_0}) d\mathcal{H}^{n-1}_{g_0} \geq 0. \tag{4}$$

If $\int_M R_{g_0}(1 - \mu^{-2}) d\mathrm{vol}_{g_0} \geq 0$ then $g = g_0$.

Araujo [1] studied the functional

$$F(g) = \int_M R_g \, d\mathrm{vol}_g + 2 \int_{\partial M} h_g \, d\mathcal{H}^{n-1}_g \tag{5}$$

restricted to the subset of metrics $\mathcal{M}_{ab} = \{g \mid a \, \mathrm{vol}_g(M) + b\mathcal{A}_g(\partial M) = 1\}$. Araujo [1] proved that the critical points of F are the Einstein metrics with umbilical boundary that satisfy $b(n-1)R_g = 2nah_g$. It is worthwhile to point out that assumption (4) of Corollary 4 does not imply $F(g) \geq F(g_0)$, since the volume and area elements $d\mathrm{vol}_{g_0}$, $d\mathcal{H}^{n-1}_{g_0}$ in (4) do not vary with the metric g.

By Gauss–Bonnet Theorem, Corollary 4 in dimension 2 can be rewritten as

Corollary 5. *Let (M, g_0) be a surface with smooth boundary ∂M. Let $u \in C^2(M)$ with $u = 0$ on ∂M and consider the metric $g = e^{2u} g_0$. Assume*

$$\int_M K_g \, d\mathrm{vol}_{g_0} + \int_{\partial M} \kappa_g d\mathcal{H}^1_{g_0} \geq 2\pi\chi(M).$$

If $\int_M K_{g_0}(1 - e^{-2u}) d\mathrm{vol}_{g_0} \geq 0$ then $u = 0$ in M.

Llarull [8], confirming a Gromov's conjecture, proved that if g is any metric on the whole sphere S^n satisfying $g \geq g_0$ and $R_g \geq R_{g_{S^n}} = n(n-1)$ then $g = g_{S^n}$. For domains in S^n_+, Hang and Wang [7] proved the following

Theorem C. *(See Proposition 1 of [7].) Let Ω be a smooth domain in S^n_+ and let $g \in [g_{S^n_+}]$ in $\bar{\Omega}$, satisfying $R_g \geq n(n-1)$ and $g = g_{S^n_+}$ on $\partial\Omega$. Then, either $g = g_{S^n_+}$, in Ω, or $g > g_{S^n_+}$ and $h_g < h_{g_{S^n_+}}$.*

Our next theorem says the following:

Theorem 6. *Let g_0 be a metric on M satisfying $R_{g_0} \geq 0$ and $\mathcal{L}_{g_0} f \leq 0$, for some $f \in C^2(M \setminus \partial M) \cap C^1(M)$ positive almost everywhere. Let Ω be a smooth domain in M and let $g = \mu^2 g_0$, where $\mu \in C^2(\Omega) \cap C^0(\bar{\Omega})$ is positive with $\mu|_{\partial\Omega} = 1$. Assume further that $\chi_{\{\mu<1\}} R_g \geq \chi_{\{\mu<1\}} R_{g_0}$. Then, it holds*

$$\mu \geq 1 \ \text{in } \Omega, \ \text{and } h_g \leq h_{g_0} \ \text{in } \partial\Omega. \tag{6}$$

In addition, if $R_g \geq R_{g_0}$ in Ω then both inequalities in (6) are strict, unless $g = g_0$.

Theorem 6 above can be applied for static metrics as they satisfy $\lambda_1(\mathcal{L}_{g_0}) = 0$. More generally, as a consequence, we have

Corollary 7. *Let g_0 be a metric on M with $R_{g_0} \geq 0$ and $\lambda_1(\mathcal{L}_{g_0}) \geq 0$. Let $\Omega \subset M$ be a smooth domain and let $g = \mu^2 g_0$ be a metric in the conformal class $[g_0]$ satisfying $g = g_0$ on ∂M, where $\mu \in C^2(\Omega) \cap C^0(\bar{\Omega})$. Assume further that $\chi_{\{\mu<1\}} R_g \geq \chi_{\{\mu<1\}} R_{g_0}$. Then, it holds*

$$g \geq g_0 \text{ in } \Omega, \text{ and } h_g \leq h_{g_0} \text{ in } \partial\Omega. \tag{7}$$

In addition, if $R_g \geq R_{g_0}$ in Ω, then both inequalities in (7) are strict, unless $g = g_0$.

Finally, using Theorem 6 with $f = 1$, we have

Corollary 8. *Let g_0 be a metric on M with $R_{g_0} = 0$. Let $\Omega \subset M$ be a smooth domain. Let $g = \mu^2 g_0$ be a metric in the conformal class $[g_0]$ satisfying $g = g_0$ on ∂M, where $\mu \in C^2(\Omega) \cap C^0(\bar{\Omega})$. Assume $\chi_{\{\mu<1\}} R_g \geq 0$. Then, it holds*

$$g \geq g_0 \text{ in } \Omega, \text{ and } h_g \leq h_{g_0} \text{ in } \partial\Omega.$$

In addition, if $R_g \geq 0$ in Ω, then both inequalities above are strict, unless $g = g_0$.

2. Proof of Theorem 1

First, consider the case $n = 2$ and write $g = e^{2u} g_0$, with $u \in C^2(M \setminus \partial M) \cap C^1(M)$. Since $g = g_0$ on ∂M, one has $u|_{\partial M} = 0$. The geodesic curvatures κ_g, κ_{g_0} satisfy

$$\frac{\partial u}{\partial \eta} = \kappa_g e^u - \kappa_{g_0} = \kappa_g - \kappa_{g_0}, \quad \text{on } \partial M, \tag{8}$$

where $\eta = \eta_{g_0}$ is the outward unit normal vector of $(\partial M, g_0)$. Furthermore, the Gaussian curvatures K_g, K_{g_0} of (M, g_0) and (M, g), respectively, satisfy

$$\Delta_{g_0} u - K_{g_0} + K_g e^{2u} = 0, \quad \text{in } M. \tag{9}$$

Using that $u|_{\partial M} = 0$, by (8), (9) and integration by parts,[2] we obtain

$$\begin{aligned}
\int_M e^{-2u} \Delta_{g_0} f &= \int_M f \Delta_{g_0}(e^{-2u}) + \int_{\partial M} \left(e^{-2u} \frac{\partial f}{\partial \eta} - f \frac{\partial(e^{-2u})}{\partial \eta}\right) \\
&= \int_M f \Delta_{g_0}(e^{-2u}) + \int_{\partial M} \frac{\partial f}{\partial \eta} + 2 \int_{\partial M} f \frac{\partial u}{\partial \eta} \\
&= \int_M [-2f e^{-2u}(\Delta_{g_0} u - 2|Du|^2_{g_0}) + \Delta_{g_0} f] + 2 \int_{\partial M} f(\kappa_g - \kappa_{g_0}) \\
&= \int_M [-2f e^{-2u}(K_{g_0} - K_g e^{2u} - 2|Du|^2_{g_0}) + \Delta_{g_0} f] \\
&\quad + \int_{\partial M} 2f(\kappa_g - \kappa_{g_0}).
\end{aligned}$$

[2] Hereinafter, for aesthetics reasons, we will sometimes omit the volume and area elements in the integrals.

Thus, since $\Delta_{g_0} f = \mathcal{L}_{g_0} f - 2K_{g_0} f$, we obtain

$$\int_M e^{-2u}(\Delta_{g_0} f + 2K_{g_0} f) = \int_M (\mathcal{L}_{g_0} f + 4fe^{-2u}|Du|^2_{g_0}) + \int_M 2f(K_g - K_{g_0}) + \int_{\partial M} 2f(\kappa_g - \kappa_{g_0}).$$

Hence,

$$\int_M \mathcal{L}_{g_0} f(e^{-2u} - 1) = \int_M 4fe^{-2u}|Du|^2_{g_0} + \int_M 2f(K_g - K_{g_0}) + \int_{\partial M} 2f(\kappa_g - \kappa_{g_0}).$$

By hypothesis, $\int_M \mathcal{L}_{g_0} f(1 - e^{-2u}) d\mathrm{vol}_{g_0} \geq 0$ and $\int_M 2f(K_g - K_{g_0}) + \int_{\partial M} 2f(\kappa_g - \kappa_{g_0}) \geq 0$. Hence, $Du = 0$, which together the fact that $u|_{\partial M} = 0$, imply that $g = g_0$.

Now, we assume $n \geq 3$ and write $g = u^{\frac{4}{n-2}} g_0$, for some $u \in C^2(M \setminus \partial M) \cap C^1(M)$, positive in M and with $u = 1$ on ∂M. The mean curvatures $h_{g_0} = \mathrm{div}_{g_0} \eta_{g_0}$ and $h_g = \mathrm{div}_g \eta_g$ satisfy

$$\frac{\partial u}{\partial \eta} = \frac{n-2}{2(n-1)}(h_g u^{\frac{n}{n-2}} - h_{g_0}) = \frac{n-2}{2(n-1)}(h_g - h_{g_0}), \text{ on } \partial M, \tag{10}$$

where $\eta = \eta_{g_0}$. Furthermore, the scalar curvatures R_g and R_{g_0} satisfy

$$\Delta_{g_0} u - \frac{n-2}{4(n-1)} R_{g_0} u + \frac{n-2}{4(n-1)} R_g u^{\frac{n+2}{n-2}} = 0, \text{ in } M. \tag{11}$$

Let λ be a constant to be chosen soon. Using that $u|_{\partial M} = 1$, integrating by parts we obtain

$$\begin{aligned}
\int_M u^\lambda \Delta_{g_0} f &= \int_M f \Delta_{g_0} u^\lambda + \int_{\partial M} (u^\lambda \frac{\partial f}{\partial \eta} - f \frac{\partial (u^\lambda)}{\partial \eta}) \\
&= \int_M (f\lambda u^{\lambda-1} \Delta_{g_0} u + f\lambda(\lambda - 1)u^{\lambda-2}|Du|^2_{g_0}) + \int_{\partial M} \frac{\partial f}{\partial \eta} - \lambda f \frac{\partial u}{\partial \eta} \\
&= \int_M [(f\lambda u^{\lambda-1} \Delta_{g_0} u + f\lambda(\lambda - 1)u^{\lambda-2}|Du|^2_{g_0}) + \Delta_{g_0} f] \\
&\quad - \frac{(n-2)}{2(n-1)} \lambda \int_{\partial M} f(h_g - h_{g_0}) \\
&= \int_M [f\lambda u^{\lambda-1}(\frac{n-2}{4(n-1)}(R_{g_0} u - R_g u^{\frac{n+2}{n-2}}) + f\lambda(\lambda - 1)u^{\lambda-2}|Du|^2_{g_0})] \\
&\quad + \int_M \Delta_{g_0} f - \frac{(n-2)}{2(n-1)} \lambda \int_{\partial M} f(h_g - h_{g_0}).
\end{aligned}$$

Hence,

$$\int_M u^\lambda(\Delta_{g_0} f - \lambda f \frac{n-2}{4(n-1)} R_{g_0}) = -\lambda \frac{n-2}{4(n-1)} \int_M f R_g u^{\lambda-1+\frac{n+2}{n-2}}$$
$$+ \int_M \Delta_{g_0} f + \lambda(\lambda-1) \int_M f u^{\lambda-2} |Du|^2_{g_0}$$
$$- \frac{(n-2)}{2(n-1)} \lambda \int_{\partial M} f(h_g - h_{g_0}).$$

Now, we choose $\lambda = 1 - \frac{n+2}{n-2} = \frac{-4}{n-2}$. We obtain

$$\int_M u^{\frac{-4}{n-2}} \mathcal{L}_{g_0} f = \int_M f \frac{R_g}{n-1} + \int_M (\mathcal{L}_{g_0} f - \frac{R_{g_0}}{n-1} f)$$
$$+ \frac{4(n+2)}{(n-2)^2} \int_M f u^{\frac{-2n}{n-2}} |Du|^2_{g_0} + \frac{2}{n-1} \int_{\partial M} f(h_g - h_{g_0}).$$

Therefore, we have

$$\int_M \mathcal{L}_{g_0} f(u^{\frac{-4}{n-2}} - 1) = \frac{4(n+2)}{(n-2)^2} \int_M f u^{\frac{-2n}{n-2}} |Du|^2_{g_0}$$
$$+ \frac{1}{n-1} \int_M f(R_g - R_{g_0}) + \frac{2}{n-1} \int_{\partial M} f(h_g - h_{g_0}).$$

By hypothesis, $g = u^{\frac{4}{n-2}} g_0$ satisfies (3) and $\int_M \mathcal{L}_{g_0} f(1 - u^{\frac{-4}{n-2}}) \geq 0$. Thus, it follows that $Du = 0$. Since $u|_{\partial M} = 1$, one has $u = 1$ in M; hence $g = g_0$. Theorem 1 is proved.

3. Proof of Theorem 6

First, consider the case $n = 2$ and rewrite $g = e^{2u} g_0$ with $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$. Since $u|_{\partial\Omega} = 0$ the geodesic curvatures κ_g, κ_{g_0} satisfy

$$\frac{\partial u}{\partial \eta} = \kappa_g e^u - \kappa_{g_0} = \kappa_g - \kappa_{g_0}, \text{ in } \partial\Omega, \tag{12}$$

where $\eta = \eta_{g_0}$ is the outward unit normal vector of $(\partial\Omega, g_0)$. Furthermore, the Gaussian curvatures K_g, K_{g_0} of (Ω, g_0) and (Ω, g), respectively, satisfy

$$\Delta_{g_0} u = K_{g_0} - K_g e^{2u}. \tag{13}$$

Let $\bar{u} = \min\{u, 0\}$. It turns out that $\bar{u}$ is continuous and, in the sense of distributions, it holds $\Delta_{g_0} \bar{u} \leq \chi_{\{u<0\}} \Delta_{g_0} u$. One can see it by observing that $\bar{u} = \lim_{\epsilon\to 0} u_\epsilon$, where $u_\epsilon = \frac{1}{2}(u - \sqrt{u^2 + \epsilon^2})$. Hence, in the sense of distributions, $\Delta_{g_0} \bar{u} \leq \chi_{\{u<0\}} \Delta_{g_0} u = \chi_{\{u<0\}}(K_{g_0} - K_g e^{2u}) \leq \chi_{\{u<0\}} K_{g_0}(1 - e^{2u})$, since $\chi_{\{u<0\}} K_g \geq \chi_{\{u<0\}} K_{g_0}$.

Let $A_u = \chi_{\{u<0\}} K_{g_0}(1 - e^{2u})$. It holds that $A_u = A_{\bar{u}}$ is a nonnegative continuous function, and

$$\Delta_{g_0} \bar{u} \leq A_{\bar{u}}, \text{ in } \Omega,$$

in the sense of distributions. The function $A_{\bar{u}}$ is Lipschitz in $\bar{\Omega}$. In fact, given $x, x_0 \in \bar{\Omega}$, if either $x, x_0 \in \{u < 0\}$, or $x, x_0 \in \{u \geq 0\}$, we have $|A_{\bar{u}}(x) - A_{\bar{u}}(x_0)| = \chi_{\{u<0\}} |K_0(x) e^{2u(x)} - K_0(x_0) e^{2u(x_0)}| \leq c\, d_{g_0}(x, x_0)$, for some $c > 0$, since $K_0 e^{2u} \in C^1(\bar{\Omega})$. Thus, we assume that $u(x) < 0$ and $u(x_0) \geq 0$. In this case, $|A_{\bar{u}}(x) - A_{\bar{u}}(x_0)| = |A_{\bar{u}}(x)| = |K_{g_0}(x)|(1 - e^{2u}) \leq (\max |K_{g_0}|)(e^{2u(x_0)} - e^{2u(x)}) \leq c\, d_{g_0}(x, x_0)$, for some $c > 0$, since $u \in C^1(\bar{\Omega})$.

Now, let $\bar{v} : \bar{\Omega} \to \mathbb{R}$ be a solution of the Dirichlet problem:

$$\Delta_{g_0} \bar{v} = A_{\bar{u}}, \text{ in } \Omega, \text{ and } \bar{v}|_{\partial\Omega} = 0.$$

Since $A_{\bar{u}}$ is Lipschitz in $\bar{\Omega}$ we have $v \in C^2(\bar{\Omega})$ (see Theorem 8.34, p. 211, of [5]). Furthermore, since $\Delta_{g_0}(\bar{u} - \bar{v}) \leq 0$ and $(\bar{u} - \bar{v})|_{\partial\Omega} = 0$, one has $\bar{v} \leq \bar{u} \leq 0$. This implies $1 - e^{2\bar{u}} \leq 1 - e^{2\bar{v}}$ and $\chi_{\{\bar{u}<0\}} \leq \chi_{\{\bar{v}<0\}}$, hence $A_{\bar{u}} \leq A_{\bar{v}}$ in $\bar{\Omega}$, since $K_{g_0} \geq 0$. Hence,

$$\Delta_{g_0} \bar{v} \leq A_{\bar{v}}, \text{ in } \Omega, \text{ and } \bar{v}|_{\partial\Omega} = 0.$$

Let $v : M \to \mathbb{R}$ be defined by

$$v(x) = \begin{cases} \bar{v}(x), \text{ if } x \in \bar{\Omega}; \\ 0, \text{ if } x \in M \setminus \bar{\Omega}. \end{cases}$$

We have A_v is Lipschitz, since $\bar{v} \in C^2(\bar{\Omega})$, and v satisfies

$$\Delta_{g_0} v \leq A_v, \text{ in } M,$$

in the sense of distributions, and $v|_{\partial M} = 0$. Let ω be a solution of the Dirichlet problem

$$\Delta_{g_0} \omega = A_v, \text{ in } M, \text{ and } \omega|_{\partial M} = 0.$$

Since Ω is a domain in M, it follows $v = 0$ in a neighborhood $\mathcal{U}$ of ∂M in M, hence $A_v = 0$ in $\mathcal{U}$, hence $\omega \in C^2(M)$. Furthermore, we have $\Delta_{g_0}(v - \omega) \leq 0$, $(v - \omega)|_{\partial M} = 0$ and $v = 0$ in $\mathcal{U}$. These imply $\omega \leq v \leq 0$ and

$$\frac{\partial \omega}{\partial \eta} \geq \frac{\partial v}{\partial \eta} = 0 \text{ on } \partial M. \tag{14}$$

In addition, we also have $A_v \leq A_\omega$, since $K_{g_0} \geq 0$ in M. Hence, $\Delta_{g_0} \omega \leq A_\omega$. Thus, the metric $\tilde{g} = e^{2\omega} g_0$ satisfies

$$\begin{aligned} K_{\tilde{g}} &= e^{-2\omega}(K_{g_0} - \Delta_{g_0} \omega) \\ &\geq e^{-2\omega}(K_{g_0} - A_\omega) = e^{-2\omega}(K_{g_0} - \chi_{\{\omega<0\}} K_{g_0}(1 - e^{2\omega})) \\ &= e^{-2\omega}(K_{g_0}(1 - \chi_{\{\omega<0\}}) + \chi_{\{\omega<0\}} K_{g_0} e^{2\omega}) \\ &= K_{g_0}. \end{aligned}$$

The last equality follows just analyzing the cases $\omega < 0$ and $\omega = 0$. Furthermore, by (14), one has $k_{\tilde{g}} = \frac{\partial \omega}{\partial \eta} + k_{g_0} \geq k_{g_0}$.

Since $\omega \leq 0$ and, by hypothesis, $\mathcal{L}_{g_0} f \leq 0$, for some $f \in C^2(M \setminus \partial M) \cap C^1(M)$ positive a.e., we obtain $\mathcal{L}_{g_0} f(1 - e^{-2\omega}) \geq 0$. By Theorem 1, one has $\tilde{g} = g_0$, hence $\omega = 0$. This implies $v = \bar{v} = \bar{u} = 0$, hence $u \geq 0$. Hence, $g \geq g_0$. Moreover, using $u \geq 0$ and $u|_{\partial\Omega} = 0$, one has $\frac{\partial u}{\partial \eta} \leq 0$, hence, by (8), it follows that $\kappa_g \leq \kappa_{g_0}$.

Now, assume further $K_g \geq K_{g_0}$ in Ω. Using (13), one has $\Delta_{g_0} u \leq 0$. Since $\frac{\partial u}{\partial \eta} \leq 0$ and $u|_{\partial\Omega} = 0$, by interior maximum principle and Hopf Lemma, it follows that $u = 0$ in $\bar{\Omega}$, provided $u = 0$ somewhere in Ω or $\frac{\partial u}{\partial \eta} = 0$ somewhere on $\partial\Omega$.

Now, let us consider the case $n \geq 3$. We rewrite $g = u^{\frac{4}{n-2}} g_0$, for $u \in C^2(\bar{\Omega})$ positive and satisfying $u = 1$ on $\partial\Omega$. Let $\bar{u} = \min\{1, u\}$. The function $\bar{u}$ is continuous in $\bar{\Omega}$ and satisfies $\bar{u}|_{\partial\Omega} = 1$. Furthermore, $\Delta_{g_0}\bar{u} \leq \chi_{\{u<1\}}\Delta_{g_0} u$, in the sense of distributions. One can see it, by observing that $\bar{u} = \lim_{\epsilon\to 0}(\frac{u+1}{2} - \varphi_\epsilon(\frac{u-1}{2}))$, where $\varphi_\epsilon(t) = \sqrt{t^2 + \epsilon^2}$. Thus, using $\chi_{\{u<1\}} R_g \geq \chi_{\{u<1\}} R_{g_0} \geq 0$, by (11), we obtain

$$\begin{aligned} \Delta_{g_0}\bar{u} &\leq \frac{n-2}{4(n-1)}\chi_{\{u<1\}}(R_{g_0} u - R_g u^{\frac{n+2}{n-2}}) \\ &\leq \frac{n-2}{4(n-1)}\chi_{\{\bar{u}<1\}} R_{g_0}(\bar{u} - \bar{u}^{\frac{n+2}{n-2}}) \\ &= A_{\bar{u}}\,\bar{u}, \text{ in } \Omega, \end{aligned} \tag{15}$$

in the sense of distributions, where $A_{\bar{u}} = \frac{n-2}{4(n-1)}\chi_{\{\bar{u}<1\}} R_{g_0}(1 - \bar{u}^{\frac{4}{n-2}})$. As in the two-dimensional case we observe that $A_{\bar{u}} \geq 0$ is Lipschitz in $\bar{\Omega}$.

Let $\bar{v} \in C^2(\bar{\Omega})$ be a solution of the Dirichlet problem

$$\Delta_{g_0}\bar{v} - A_{\bar{u}}\,\bar{v} = 0 \text{ and } \bar{v}|_{\partial\Omega} = 1$$

(see Theorem 8.34, p. 211, of [5]). By the strong maximum principle, one has $\bar{v} > 0$ in Ω. Furthermore, since $\Delta_{g_0}(\bar{v} - \bar{u}) - A_{\bar{u}}(\bar{v} - \bar{u}) \geq 0$ and $(\bar{u} - \bar{v})|_{\partial\Omega} = 0$, also by the strong maximum principle, we have that $\bar{v} \leq \bar{u} \leq 1$. We obtain that $\chi_{\{\bar{v}<1\}} \geq \chi_{\{\bar{u}<1\}}$ and $1 - \bar{v}^{\frac{4}{n-2}} \geq 1 - \bar{u}^{\frac{4}{n-2}}$. This implies $A_{\bar{v}} \geq A_{\bar{u}}$, since $R_{g_0} \geq 0$. Hence,

$$\Delta_{g_0}\bar{v} - A_{\bar{v}}\,\bar{v} \leq 0, \text{ in } M, \text{ and } \bar{v}|_{\partial\Omega} = 1.$$

Let $v : M \to \mathbb{R}$ be defined by

$$v(x) = \begin{cases} \bar{v}(x), \text{ if } x \in \bar{\Omega}; \\ 1, \text{ if } x \in M \setminus \bar{\Omega}. \end{cases}$$

Note that $v \leq 1$ in M and A_v is Lipschitz. Furthermore, it holds

$$\Delta_{g_0} v - A_v v \leq 0, \text{ in } M, \tag{16}$$

in the sense of distributions.

Let $w \in C^2(M)$ be a solution of the Dirichlet problem

$$\Delta_{g_0} w - A_v\, w = 0, \text{ in } M, \text{ and } w|_{\partial M} = 1. \tag{17}$$

Since $A_v \geq 0$, by the strong maximum principle, $-w$ cannot achieve a nonnegative maximum in $M \setminus \partial M$, unless w is constant. Hence $w > 0$, since $w|_{\partial M} = 1$. Furthermore, by (16) and (17), we have $\Delta_{g_0}(w - v) - A_v(w - v) \geq 0$, in M, in the sense of distributions, and $w - v = 0$ in ∂M. Again by the strong maximum principle, we obtain $w \leq v \leq 1$ in M, hence $A_w \geq A_v$. Thus, by (17),

$$\Delta_{g_0} w - A_w\, w \leq 0, \text{ in } M, \text{ and } w|_{\partial M} = 1. \tag{18}$$

Consider the metric $\tilde{g} = w^{\frac{4}{n-2}} g_0$. By (11), the scalar curvatures $R_{\tilde{g}}$ and R_{g_0} satisfy

$$\begin{aligned}R_{\tilde{g}} &= w^{-\frac{n+2}{n-2}}(R_{g_0}w - \frac{4(n-1)}{n-2}\Delta_{g_0}w)\\ &\geq w^{-\frac{n+2}{n-2}}(R_{g_0}w - \frac{4(n-1)}{n-2}A_w\, w)\\ &= w^{-\frac{n+2}{n-2}}((1-\chi_{\{w<1\}})R_{g_0}w + \chi_{\{w<1\}}R_{g_0}w^{\frac{n+2}{n-2}})\\ &= R_{g_0}.\end{aligned}$$

The last equality follows just by analyzing the cases $w < 1$ and $w = 1$. Furthermore, since $w \leq 1$ and $w|_{\partial M} = 1$, we have $\frac{\partial w}{\partial \eta} \geq 0$ on ∂M. By (10), the mean curvatures h_{g_0} and $h_{\tilde{g}}$ satisfy

$$h_{\tilde{g}} = h_{g_0} + \frac{2(n-1)}{n-2}\frac{\partial w}{\partial \eta} \geq h_{g_0}.$$

Since $1 - w^{\frac{-4}{n-2}} \leq 0$, and, by hypothesis, there exists $f \in C^2(M \setminus \partial M) \cap C^1(M)$ positive a.e. such that $\mathcal{L}_{g_0} f \leq 0$, it follows that $\mathcal{L}_{g_0} f(1 - w^{\frac{-4}{n-2}}) \geq 0$. By Theorem 1, it holds $\tilde{g} = g_0$, hence $w = 1$. This implies that $v = \bar{v} = \bar{u} = 1$, hence $u \geq 1$. Thus, $g \geq g_0$. Moreover, since $u \geq 1$ and $u|_{\partial M} = 1$, we also have $\frac{\partial u}{\partial \eta} \leq 0$, hence $h_g \leq h_{g_0}$.

Now, we assume further $R_g \geq R_{g_0}$ in Ω. Since $u \geq 1$ and $R_{g_0} \geq 0$, by (11), one has $\Delta_{g_0} u \leq 0$. Thus, if $u = 1$, somewhere in Ω, or $\frac{\partial u}{\partial \eta} = 0$, somewhere in $\partial\Omega$, then, by interior maximum principle or Hopf Lemma, it holds $u = 1$ in Ω. Theorem 6 is proved.

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References

[1] H. Araujo, Critical points of the total scalar curvature plus total mean curvature functional, Indiana Univ. Math. J. 52 (2003) 85–107.
[2] S. Brendle, F.C. Marques, A. Neves, Deformations of the hemisphere that increase scalar curvature, Invent. Math. 185 (2011) 175–197.
[3] J.F. Escobar, Uniqueness theorems on conformal deformation of metrics, Sobolev inequalities, and an eigenvalue estimate, Comm. Pure Appl. Math. 43 (1990) 857–883.
[4] J.F. Escobar, Uniqueness and non-uniqueness of metrics with prescribed scalar and mean curvature on compact manifolds with boundary, J. Funct. Anal. 202 (2) (2003) 424–442.
[5] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Classics Math., ISSN 1431-0821, 2001.
[6] F. Hang, X. Wang, Rigidity and non-rigidity results on the sphere, Comm. Anal. Geom. 14 (2006) 91–106.
[7] F. Hang, X. Wang, Rigidity theorems for compact manifolds with boundary and positive Ricci curvature, J. Geom. Anal. 19 (2009) 628–642.
[8] M. Llarull, Sharp estimates and the Dirac operator, Math. Ann. 310 (1998) 55–71.
[9] M. Min-Oo, Scalar curvature rigidity of the hemisphere, 1995, unpublished.