



# Harnack inequalities for supersolutions of fully nonlinear elliptic difference and differential equations



Nao Hamamuki

Department of Mathematics, Hokkaido University, Kita 10, Nishi 8, Kita-Ku, Sapporo, Hokkaido, 060-0810, Japan

## ARTICLE INFO

### Article history:

Received 22 February 2015

Available online 3 February 2016

Submitted by J. Guermond

### Keywords:

Harnack inequality

Fully nonlinear elliptic equations

Discrete solutions

Viscosity solutions

## ABSTRACT

We present a new Harnack inequality for non-negative discrete supersolutions of fully nonlinear uniformly elliptic difference equations on rectangular lattices. This estimate applies to all supersolutions and has the Harnack constant depending on the graph distance on lattices. For the proof we modify the proof of the weak Harnack inequality. Applying the same idea to elliptic equations in a Euclidean space, we also derive a Harnack type inequality for non-negative viscosity supersolutions.

© 2016 Elsevier Inc. All rights reserved.

## 1. Introduction

We consider fully nonlinear, non-homogeneous second order equations of the form

$$F(D^2u) = f(x) \quad (1.1)$$

with a uniformly elliptic operator  $F$ . A typical statement of the Harnack inequality is that there exists a constant  $C > 0$  such that the inequality

$$\max_U u \leq C \left\{ \min_U u + \|f\|_{L^n(V)} \right\} \quad (1.2)$$

holds for every non-negative solution  $u$  of (1.1) in  $V$ . Here  $V$  is a set which is (enough) larger than  $U$ , and  $n$  represents the dimension of space. One of well-known proofs of the Harnack inequality is a combination of a *weak Harnack inequality*, which asserts that, for some  $p > 0$ ,

$$\|u\|_{L^p(U)} \leq C \left\{ \min_U u + \|f\|_{L^n(V)} \right\}$$

E-mail address: [hnao@math.sci.hokudai.ac.jp](mailto:hnao@math.sci.hokudai.ac.jp).

holds for every non-negative supersolution  $u$ , and a *local maximum principle* (or a *mean value inequality*):

$$\max_U u \leq C \{ \|u\|_{L^q(U)} + \|f\|_{L^n(V)} \}$$

for subsolutions  $u$ , where  $q > 0$  is arbitrary. These estimates are well-known in the continuum case where (1.1) is studied as a partial differential equation in  $\mathbf{R}^n$ ; for instance, the reader is referred to [9, Chapter 9] for linear equations and to [3, Chapter 4] for fully nonlinear equations. The corresponding results are also obtained in the discrete case when we study (1.1) as a difference equation on lattices. In [13] the Harnack inequality for elliptic difference equations is derived via the weak Harnack inequality and the local maximum principle. See also [15] for the parabolic case and [16,17] for general meshes.

The main goal of this paper is to show that, in the discrete case, a modified proof of the weak Harnack inequality implies a new type of the Harnack inequality for discrete supersolutions to (1.1) on rectangular lattices (Theorem 4.1). Our proof is direct and simple in the sense that we do not need the local maximum principle. Accordingly, the resulting estimate is valid for every supersolution which is not necessarily a subsolution. This is a difference from the literature.

It seems that importance of analysis on various non Euclidean spaces has been increasing. In fact, one of motivations of this work comes from partial differential equations posed on a discrete space such as graphs, whose application include, e.g., stochastic controls, mean field games, social networks, and so on. (See [18] and references therein.) In the recent paper [18], nonlinear elliptic partial differential equations are studied on a finite graph, and uniqueness and existence results of solutions are obtained. Our setting on rectangular lattices is a special case of [18]. A flavor of analysis for graph spaces could be seen in the present paper; we use the graph distance  $\rho(x)$  and the ball  $B_r$  with respect to this distance (Section 3). They appear in the statement of our main theorem (Theorem 4.1). It is our surprise that Harnack estimate holds for supersolutions if we consider a discrete space. This enables us to control more functions since our result does not rely on a subsolution property of solutions.

It turns out that our Harnack constant,  $C$  in (1.2), depends on the graph distance on lattices. Due to this, passing to limits in our Harnack inequality does not imply the continuum Harnack inequality since the Harnack constant  $C$  goes to infinity when the mesh size tends to 0 (Remarks 3.4 and 4.2). Here it is worth mentioning that such reconstruction of the continuum Harnack inequality should not be possible since (1.2) does not hold even for the Laplace equation if we do not require  $u$  to be a subsolution; see Example 5.3 for the counter-example. Contribution of this paper is a discovery of a Harnack estimate for functions (supersolutions) belonging to a wider class which are excluded in a study of a convergent scheme. This is our first attempt to give a priori estimate for discrete solutions, and the extension of the result to more general lattices rather than rectangular lattices is one of interesting our future problems.

In the proof of the weak Harnack inequality for fully nonlinear equations of the continuum case ([2,3]), we take a radially symmetric and increasing supersolution  $\phi$  of the Pucci equation

$$\mathcal{P}^-(D^2\phi) = -\xi(x). \quad (1.3)$$

Here  $\mathcal{P}^-$  is a Pucci operator (see (5.2) or (2.2) for definition) and  $\xi$  is a non-negative, continuous function whose support is contained in a small ball centered at the origin. Such a function  $\phi$  is often called a *barrier function*. In the discrete case, we are able to construct the barrier function so that  $\xi$  is non-zero only at the origin (Lemma 3.1 and Remark 3.2). In other words, its support is only one point. This is a crucial difference from the continuous case, and this enables a pointwise estimate for supersolutions of difference equations. In our proof of the Harnack inequality, we translate the barrier function so that its minimum point, which originally lies at the origin, comes to a maximum point of the supersolution  $u$  of (1.1). As a result, we obtain the Harnack inequality without discussing the local maximum principle.

We apply the same idea involving translation of the barrier function to partial differential equations in  $\mathbf{R}^n$ . This gives a Harnack type estimate for viscosity supersolutions of elliptic equations (Theorem 5.7). To describe the result, we first note that (1.2) can be stated equivalently as

$$u(z) \leq C \left\{ \min_U u + \|f\|_{L^n(V)} \right\} \quad \text{for all } z \in U, \quad (1.4)$$

which is a pointwise estimate and does not hold for supersolutions. Employing the theory of viscosity solutions, we prove that, for a fixed  $\varepsilon > 0$ , there exists a constant  $C > 0$  depending on  $\varepsilon$  such that, for every  $z \in U$ , the minimum value of  $u$  over  $\{|x - z| \leq \varepsilon\}$  is dominated by the right-hand side of (1.4). In other words, our Harnack inequality needs further information of  $u$  around  $z$ . The barrier function  $\phi$  which we will use in the proof is chosen so that the support of  $\xi$  appearing in (1.3) is contained in  $\{|x| \leq \varepsilon\}$ . Also, around the origin,  $\phi$  is defined by using a modulus of continuity (from below) of  $u$  near  $z$ . The resulting estimate can be said to be a “very weak Harnack inequality” since the minimum of  $u$  over  $\{|x - z| \leq \varepsilon\}$  is controlled by its  $L^p$ -norm on  $U$ . Thus the method in this paper presents how a simple estimate is established by a simple argument without the Calderón–Zygmund decomposition appearing in the literature ([2,3]).

Our difference equation (2.1) below is “diagonal”; namely it does not include mixed directional derivatives. In a study of a finite graph ([18]) it is natural to consider such operators depending only on adjacent points, while a convergent scheme allowing mixed directional derivatives is developed in [14]. If we include mixed directional derivatives, a definition of a discrete ball  $B_r$  (Section 3) needs to be changed since difference operators depend on values on diagonal points. Also, Pucci operators (2.2) should be defined for symmetric matrices as in the continuum case (5.2). However, it seems non-trivial to construct a barrier function in the general case. To avoid a technical difficulty we consider (2.1) for a discrete problem. (The equation (5.1) in the continuum case is allowed to depend on mixed directional derivatives.)

This paper is organized as follows: Section 2 is devoted to preparation for studies of difference equations on rectangular lattices. In Section 3 we construct a barrier function  $\phi$  in (1.3) so that the support of  $\xi$  lies only at the origin. Then, in Section 4 we give a proof of the Harnack inequality for non-negative discrete supersolutions. Section 5 is concerned with the Harnack inequality in  $\mathbf{R}^n$  for viscosity supersolutions. We use a similar idea to the one presented in Section 4. In Appendix A we establish a unique existence of discrete solutions to Dirichlet problems of fully nonlinear uniformly elliptic difference equations. This unique solution is needed in Section 4 to derive the Harnack inequality with non-zero  $f$ .

## 2. Preliminaries

In this paper we consider an  $n$ -dimensional weighted lattice  $h\mathbf{Z}^n$  defined as

$$h\mathbf{Z}^n := \{(h_1 m_1, \dots, h_n m_n) \in \mathbf{R}^n \mid (m_1, \dots, m_n) \in \mathbf{Z}^n\}.$$

Here  $h_i$  is a fixed positive constant which represents a mesh size in the direction of  $x_i$ . We set  $h_{\max} := \max\{h_1, \dots, h_n\}$  and  $h_{\min} := \min\{h_1, \dots, h_n\}$ . For  $\Omega \subset h\mathbf{Z}^n$  we define its closure  $\overline{\Omega} \subset h\mathbf{Z}^n$  and its boundary  $\partial\Omega \subset h\mathbf{Z}^n$  as

$$\overline{\Omega} := \Omega \cup \{x \pm h_i e_i \mid x \in \Omega, i \in \{1, \dots, n\}\}, \quad \partial\Omega := \overline{\Omega} \setminus \Omega,$$

where  $\{e_i\}_{i=1}^n \subset \mathbf{R}^n$  is the standard orthogonal basis of  $\mathbf{R}^n$ , e.g.,  $e_1 = (1, 0, \dots, 0)$ .

We next introduce difference operators. Let  $u : h\mathbf{Z}^n \rightarrow \mathbf{R}$ ,  $x \in h\mathbf{Z}^n$  and  $i \in \{1, \dots, n\}$ . We define the second order difference operators as follows:

$$\delta_i^2 u(x) := \frac{u(x + h_i e_i) + u(x - h_i e_i) - 2u(x)}{h_i^2},$$

$$\vec{\delta^2}u(x) := (\delta_1^2u(x), \dots, \delta_n^2u(x)).$$

The difference equation we consider is

$$F(\vec{\delta^2}u(x)) = f(x), \quad (2.1)$$

where  $F : \mathbf{R}^n \rightarrow \mathbf{R}$  is uniformly elliptic (Definition 2.2),  $F(0) = 0$  and  $f : h\mathbf{Z}^n \rightarrow \mathbf{R}$ .

**Definition 2.1.** Let  $\Omega \subset h\mathbf{Z}^n$ . We say  $u : \overline{\Omega} \rightarrow \mathbf{R}$  is a *discrete subsolution* (resp. *supersolution*) of (2.1) in  $\Omega$  if  $F(\vec{\delta^2}u(x)) \leq f(x)$  (resp.  $\geq f(x)$ ) for all  $x \in \Omega$ . If  $u$  is both a discrete sub- and supersolution, it is called a *discrete solution*.

Throughout this paper we fix ellipticity constants  $0 < \lambda \leq \Lambda$ . To describe the uniform ellipticity of  $F$  in (2.1) we introduce *Pucci operators*  $\mathcal{P}^\pm : \mathbf{R}^n \rightarrow \mathbf{R}$ , which are defined as

$$\mathcal{P}^+(\vec{X}) := -\lambda \sum_{X_i > 0} X_i - \Lambda \sum_{X_i < 0} X_i, \quad \mathcal{P}^-(\vec{X}) := -\lambda \sum_{X_i < 0} X_i - \Lambda \sum_{X_i > 0} X_i \quad (2.2)$$

for  $\vec{X} = (X_1, \dots, X_n) \in \mathbf{R}^n$ . An easy computation shows that the Pucci operators satisfy  $\mathcal{P}^-(\vec{X} + \vec{Y}) \leq \mathcal{P}^+(\vec{X}) + \mathcal{P}^-(\vec{Y}) \leq \mathcal{P}^+(\vec{X} + \vec{Y})$  for all  $\vec{X}, \vec{Y} \in \mathbf{R}^n$ .

**Definition 2.2.** We say  $F : \mathbf{R}^n \rightarrow \mathbf{R}$  is *uniformly elliptic* if  $\mathcal{P}^-(\vec{X} - \vec{Y}) \leq F(\vec{X}) - F(\vec{Y}) \leq \mathcal{P}^+(\vec{X} - \vec{Y})$  for all  $\vec{X}, \vec{Y} \in \mathbf{R}^n$ .

Putting  $\vec{Y} = 0$ , we see that  $\mathcal{P}^-(\vec{X}) \leq F(\vec{X}) \leq \mathcal{P}^+(\vec{X})$  since  $F(0) = 0$ .

We next state the ABP maximum principle (ABP estimate). This is a pointwise estimate for subsolutions and supersolutions of elliptic equations, and it will be used in the proof of the Harnack inequality. We prepare some notations before stating the estimate. For  $a \in \mathbf{R}$  we set  $a^\pm := \max\{\pm a, 0\}$  ( $\geq 0$ ). Let  $\Omega \subset h\mathbf{Z}^n$  and  $u : \overline{\Omega} \rightarrow \mathbf{R}$ . We define  $\Gamma_\Omega[u]$ , an *upper contact set* of  $u$  on  $\Omega$ , as

$$\Gamma_\Omega[u] := \left\{ x \in \Omega \mid \begin{array}{l} \text{there exists some } p \in \mathbf{R}^n \text{ such that} \\ u(y) \leq \langle p, y - x \rangle + u(x) \text{ for all } y \in \overline{\Omega} \end{array} \right\}, \quad (2.3)$$

where  $\langle \cdot, \cdot \rangle$  is the standard Euclidean inner product in  $\mathbf{R}^n$ . The  $p$ -norm ( $p \in [1, \infty)$ ) of  $u$  over  $\Omega$  is given as  $\|u\|_{\ell^p(\Omega)} := (\sum_{x \in \Omega} h^n |u(x)|^p)^{1/p}$ , where  $h^n := h_1 \times \dots \times h_n$ . We only use the case  $p = n$  in this paper. The diameter of  $\Omega$  is  $\text{diam}(\Omega) := \max_{x \in \Omega, y \in \partial\Omega} |x - y|$ . Here  $|\cdot|$  stands for the standard Euclidean norm in  $\mathbf{R}^n$ .

**Theorem 2.3** (ABP maximum principle). *Let  $\Omega \subset h\mathbf{Z}^n$  be bounded. There exists a constant  $C_A = C_A(n, \lambda) > 0$  such that, for every discrete subsolution (resp. supersolution)  $u$  of (2.1) in  $\Omega$ , the estimate*

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+ + C_A \text{diam}(\Omega) \|f^+\|_{\ell^n(\Gamma_\Omega[u^+])} \quad (2.4)$$

$$(\text{resp. } \min_{\overline{\Omega}} u \geq \min_{\partial\Omega} (-u^-) - C_A \text{diam}(\Omega) \|f^-\|_{\ell^n(\Gamma_\Omega[u^-])}) \quad (2.5)$$

holds.

We do not give a proof of Theorem 2.3; see [13, Theorem 2.1], [10, Theorem 4.1].

### 3. Barrier function

In the proof of the Harnack inequality we use a barrier function, which is a radially increasing supersolution of  $\mathcal{P}^- = 0$  except at the origin. (See [3, Lemma 4.1] for the continuum case.)

For  $x \in h\mathbf{Z}^n$  given as  $x = (h_1 m_1, \dots, h_n m_n)$  with  $(m_1, \dots, m_n) \in \mathbf{Z}^n$ , we define  $\rho(x) := |m_1| + \dots + |m_n|$ . This represents the graph distance on  $h\mathbf{Z}^n$  between 0 and  $x$ , i.e., the number of edges in a shortest path connecting them. We emphasize that  $\rho(x)$  is different from the Euclidean distance between 0 and  $x$ . For instance, if  $x = (5h_1, 0, \dots, 0) \in h\mathbf{Z}^n$ , then  $\rho(x)$  is not  $5h_1$  but 5. Let  $k \in \mathbf{N} \cup \{0\}$ . We define a ball  $B_k \subset h\mathbf{Z}^n$  as  $B_k := \{x \in h\mathbf{Z}^n \mid \rho(x) \leq k\}$ , which is a diamond-shaped set. Note that the index  $k$  is not the Euclidean distance but the graph distance.

**Lemma 3.1** (*Barrier function*). *Let  $k \in \mathbf{N}$ . There exists a function  $\phi : \overline{B_k} \rightarrow \mathbf{R}$  such that*

$$\begin{cases} \mathcal{P}^-(\vec{\delta}^2 \phi) \geq 0 & \text{in } B_k \setminus \{0\}, & \text{(a)} \\ \phi = 0 & \text{on } \partial B_k, & \text{(b)} \\ \phi \leq -1 & \text{in } B_k. & \text{(c)} \end{cases} \quad (3.1)$$

**Proof.** We will construct a radially symmetric  $\phi$  satisfying (3.1) (a)–(c). Namely, we seek  $\phi$  of the form  $\phi(x) = a_m$  if  $\rho(x) = m \in \{0, 1, \dots, k+1\}$ . First we set  $a_{k+1} = 0$  and  $a_k = -1$ , which come from the conditions (3.1) (b) and (c). We next define the values  $a_{k-1}, \dots, a_1, a_0$  by induction so that  $a_{k+1} = 0 > a_k = -1 > a_{k-1} > \dots > a_1 > a_0$ . In the rest of the proof we show that, for given  $a_{m+1}$  and  $a_m$  such that  $a_{m+1} > a_m$ , we have  $\mathcal{P}^-(\vec{\delta}^2 \phi(x)) \geq 0$  for  $x$  with  $\rho(x) = m$  if we take  $a_{m-1}$  sufficiently small (i.e.,  $a_{m-1} \ll -1$ ).

Fix  $m \in \{1, \dots, k\}$  and  $x = (x_1, \dots, x_n) \in B_k \setminus \{0\}$  such that  $\rho(x) = m$ . Let us calculate  $\delta_i^2 \phi(x)$ . If  $x_i = 0$ , we observe

$$\delta_i^2 \phi(x) = \frac{a_{m+1} + a_{m+1} - 2a_m}{h_i^2} = \frac{2(a_{m+1} - a_m)}{h_i^2} > 0.$$

On the other hand, if  $x_i \neq 0$ , then

$$\delta_i^2 \phi(x) = \frac{a_{m+1} + a_{m-1} - 2a_m}{h_i^2},$$

which is negative when  $a_{m-1} \ll -1$ . Thus the definition of  $\mathcal{P}^-$  implies that

$$\mathcal{P}^-(\vec{\delta}^2 \phi(x)) = -\lambda \sum_{x_i \neq 0} \delta_i^2 \phi(x) - \Lambda \sum_{x_i = 0} \delta_i^2 \phi(x).$$

Now, there exists at least one index  $i$  such that  $x_i \neq 0$  since  $x \neq 0$ . Therefore

$$\begin{aligned} \mathcal{P}^-(\vec{\delta}^2 \phi(x)) &\geq -\lambda \frac{a_{m+1} + a_{m-1} - 2a_m}{h_{\max}^2} - \Lambda \frac{2(a_{m+1} - a_m)}{h_{\min}^2} (n-1) \\ &= -\frac{\lambda}{h_{\max}^2} \left( a_{m+1} + a_{m-1} - 2a_m + \frac{2\Lambda h_{\max}^2 (n-1)(a_{m+1} - a_m)}{\lambda h_{\min}^2} \right). \end{aligned} \quad (3.2)$$

This is non-negative if  $a_{m-1} \ll -1$ , and hence (3.1) (a) holds. The conditions (3.1) (b) and (c) are clear by construction.  $\square$

**Remark 3.2.** Using the barrier function  $\phi$  in Lemma 3.1, we define

$$\xi(x) := \begin{cases} -\mathcal{P}^-(\delta^2\phi(0)) & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

Then  $\phi$  is a supersolution of  $\mathcal{P}^- = -\xi$  in  $B_k$ . We also note that  $\xi(0) > 0$  since  $\delta_i^2\phi(0) = 2(a_1 - a_0)/h_i^2 > 0$  for all  $i = 1, \dots, n$ .

**Remark 3.3.** In view of the proof, we notice that  $\phi(0)$  depends on  $k, n, \Lambda/\lambda$  and  $h_{\max}/h_{\min}$ . The positive constant  $-\phi(0)$  will appear as the Harnack constant  $C_H$  in (4.2).

**Remark 3.4.** The quantity in parentheses of (3.2) is chosen to be non-positive, and so we have  $a_{m+1} - a_m < a_m - a_{m-1}$  for  $m \in \{1, \dots, k\}$ . This yields  $a_0 < -k - 1$  since  $a_{k+1} - a_k = 1$ . It thus follows that the value  $\phi(0) = a_0$  goes to  $-\infty$  as  $k \rightarrow \infty$ . This implies that we cannot obtain the continuum Harnack inequality as the limit of our discrete Harnack inequality; see Remark 4.2.

#### 4. Harnack inequality

We show the Harnack inequality for non-negative discrete supersolutions of

$$\mathcal{P}^+(\delta^2 u) = -f^-(x). \quad (4.1)$$

Note that a supersolution of (2.1) is also a supersolution of (4.1).

**Theorem 4.1** (Harnack inequality). *Let  $r \in \mathbf{N}$ . Then there exists a constant  $C_H = C_H(r, n, \Lambda/\lambda, h_{\max}/h_{\min}) > 0$  such that, for every non-negative discrete supersolution  $u : \overline{B_{3r}} \rightarrow [0, \infty)$  of (4.1) in  $B_{3r}$ , the estimate*

$$\max_{B_r} u \leq C_H \left\{ \min_{B_r} u + C_A \text{diam}(B_{3r}) \|f^-\|_{\ell^n(B_{3r})} \right\} \quad (4.2)$$

holds, where  $C_A$  is the constant in Theorem 2.3.

We first prove (4.2) in the case when  $f^- \equiv 0$ ; a crucial difference between the discrete case and the continuum case appears in this part. We translate  $\xi$  in Remark 3.2 so that its support comes to a maximum point of  $u$  and derive the estimate for  $u$  at the point. The proof for a general  $f$  is similar to the proof in the continuum case; see, e.g., [1, Proof of Theorem 1.11]. We employ a solution  $v$  of a Pucci equation and study  $u + v$  to apply (4.2) with  $f^- \equiv 0$ .

**Proof.** Case:  $f^- \equiv 0$ . 1. We take  $x_M, x_m \in B_r$  such that  $u(x_M) = \max_{B_r} u$  and  $u(x_m) = \min_{B_r} u$ . Our goal is to derive  $u(x_M) \leq C_H u(x_m)$ . Let  $\phi$  be the barrier function in Lemma 3.1 with  $k = 2r$ . Let  $\beta > u(x_m)$  ( $\geq 0$ ). We define  $\tilde{\phi}(x) := \beta\phi(x - x_M)$  and

$$\tilde{\xi}(x) := \begin{cases} -\mathcal{P}^-(\delta^2\tilde{\phi}(x_M)) & \text{if } x = x_M, \\ 0 & \text{if } x \neq x_M. \end{cases}$$

Set  $B := x_M + B_{2r}$ . Then  $B_r \subset B \subset B_{3r}$  since  $x_M \in B_r$ . By virtue of Lemma 3.1 and Remark 3.2, we have

$$\begin{cases} \mathcal{P}^-(\delta^2\tilde{\phi}) \geq -\tilde{\xi} & \text{in } B, & \text{(a)} \\ \tilde{\phi} = 0 & \text{on } \partial B, & \text{(b)} \\ \tilde{\phi} \leq -\beta & \text{in } B. & \text{(c)} \end{cases} \quad (4.3)$$

2. Let us study a function  $u + \tilde{\phi}$ . For every  $x \in B$  we deduce from (4.3) (a) that

$$\mathcal{P}^+(\tilde{\delta}^2 u(x) + \tilde{\delta}^2 \tilde{\phi}(x)) \geq \mathcal{P}^+(\tilde{\delta}^2 u(x)) + \mathcal{P}^-(\tilde{\delta}^2 \tilde{\phi}(x)) \geq 0 - \tilde{\xi}(x).$$

Namely,  $u + \tilde{\phi}$  is a supersolution of  $\mathcal{P}^+ = -\tilde{\xi}$  in  $B$ . Applying the ABP maximum principle (2.5) to  $u + \tilde{\phi}$ , we obtain

$$\min_B(u + \tilde{\phi}) \geq \min_{\partial B}\{-(u + \tilde{\phi})^-\} - C_A \text{diam}(B) \|\tilde{\xi}\|_{\ell^n(\Gamma_B[(u + \tilde{\phi})^-])}. \quad (4.4)$$

Since  $u$  is non-negative and (4.3) (b) holds, we have  $u + \tilde{\phi} \geq 0$  on  $\partial B$ , and thus  $\min_{\partial B}\{-(u + \tilde{\phi})^-\} = 0$ . As for the left-hand side of (4.4), using (4.3) (c), we compute

$$\min_B(u + \tilde{\phi}) \leq \min_B u - \beta \leq \min_{B_r} u - \beta < 0.$$

Therefore it follows from (4.4) that  $0 > -\|\tilde{\xi}\|_{\ell^n(\Gamma_B[(u + \tilde{\phi})^-])}$ . Since  $\tilde{\xi}$  is nonzero only at  $x_M$  by its definition, we must have

$$x_M \in \Gamma_B[(u + \tilde{\phi})^-]. \quad (4.5)$$

3. We claim  $(u + \tilde{\phi})(x_M) < 0$ . Suppose by contradiction that  $(u + \tilde{\phi})(x_M) \geq 0$ , i.e.,  $(u + \tilde{\phi})^-(x_M) = 0$ . Then, since  $(u + \tilde{\phi})^- = 0$  on  $\partial B$ , (4.5) implies

$$(u + \tilde{\phi})^- = 0 \quad \text{on } \overline{B}. \quad (4.6)$$

Indeed, by (4.5) there exists some  $p = (p_1, \dots, p_n) \in \mathbf{R}^n$  such that

$$0 \leq (u + \tilde{\phi})^-(y) \leq (u + \tilde{\phi})^-(x_M) + \langle p, y - x_M \rangle = \langle p, y - x_M \rangle \quad (4.7)$$

for all  $y \in \overline{B}$ . Fix  $i \in \{1, \dots, n\}$  and choose  $k_+, k_- \in \mathbf{N}$  such that  $x_M \pm k_{\pm} h_i e_i \in \partial B$ . (Such numbers  $k_{\pm}$  exist since  $B$  is bounded.) Taking  $y = x_M \pm k_{\pm} h_i e_i$  in (4.7), we observe

$$0 \leq \langle p, k_+ h_i e_i \rangle = k_+ h_i p_i, \quad 0 \leq \langle p, -k_- h_i e_i \rangle = -k_- h_i p_i,$$

which imply  $p_i = 0$ . Finally, applying  $p = 0$  to (4.7) yields (4.6). However, at a minimum point  $x_m$  we have  $(u + \tilde{\phi})(x_m) \leq u(x_m) - \beta < 0$ . This contradicts to (4.6).

By the claim we have  $u(x_M) < -\tilde{\phi}(x_M) = -\beta\phi(0)$ , and sending  $\beta \rightarrow u(x_m)$  yields  $u(x_m) \leq C_H u(x_m)$  with  $C_H = -\phi(0)$ .

Case:  $f^- \not\equiv 0$ . 1. Let  $v$  be the discrete solution of

$$\begin{cases} \mathcal{P}^-(\tilde{\delta}^2 v) = f^- & \text{in } B_{3r}, \\ v = 0 & \text{on } \partial B_{3r}. \end{cases}$$

We will prove a unique existence of solutions in Appendix A (Theorem A.4) for more general Dirichlet problems. By the ABP maximum principles we see that  $v$  satisfies

$$\begin{aligned} \max_{B_{3r}} v &\leq \max_{\partial B_{3r}} v^+ + C_A \text{diam}(B_{3r}) \|(f^-)^+\|_{\ell^n(\Gamma_{B_{3r}}[v^+])} \\ &\leq 0 + C_A \text{diam}(B_{3r}) \|f^-\|_{\ell^n(B_{3r})} \end{aligned} \quad (4.8)$$

and

$$\begin{aligned} \min_{\overline{B_{3r}}} v &\geq \min_{\partial B_{3r}} (-v^-) - C_A \text{diam}(B_{3r}) \| (f^-)^- \|_{\ell^n(\Gamma_{B_{3r}}[v^-])} \\ &= 0 - 0. \end{aligned} \quad (4.9)$$

2. We now consider a function  $u + v$ . By the non-negativity of  $u$  and (4.9), we have  $u + v \geq 0$  in  $\overline{B_{3r}}$ . Next, for  $x \in B_{3r}$  we compute

$$\mathcal{P}^+(\delta^2 u(x) + \delta^2 v(x)) \geq \mathcal{P}^+(\delta^2 u(x)) + \mathcal{P}^-(\delta^2 v(x)) \geq -f^-(x) + f^-(x) = 0.$$

Thus  $u + v$  is a non-negative supersolution of  $\mathcal{P}^+ = 0$  in  $B_{3r}$ . From the Harnack inequality of the case  $f^- \equiv 0$  it follows that

$$\max_{B_r}(u + v) \leq C_H \min_{B_r}(u + v).$$

Finally, applying the estimates (4.8) and (4.9) to the right- and the left-hand side respectively, we obtain (4.2).  $\square$

**Remark 4.2.** Passing to limits in (4.2) as  $h \rightarrow 0$  does not imply the Harnack inequality in the continuum case. Indeed, to derive the continuum Harnack inequality on a bounded set  $K \subset \mathbf{R}^n$ , one needs to “cover”  $K$  by a discrete ball  $B_r \subset h\mathbf{Z}^n$ . When the mesh size goes to 0, the radius  $r \in \mathbf{N}$  must tend to infinity, and thus the value  $C_H = -\phi(0)$  goes to infinity as we observed in Remark 3.4.

## 5. Continuum case

We consider elliptic partial differential equations of the form

$$F(D^2 u(x)) = f(x), \quad (5.1)$$

where  $D^2 u(x) = (\partial_{ij}^2 u(x))_{ij}$  denotes the Hessian matrix,  $F \in C(\mathbf{S}^n)$  is uniformly elliptic (Definition 5.2),  $F(O) = 0$  and  $f \in C(\mathbf{R}^n)$ . Here  $\mathbf{S}^n$  is the set of real  $n \times n$  symmetric matrices. In this section, applying the idea of the proof of Theorem 4.1, we deduce a Harnack type inequality for supersolutions of (5.1).

We employ a notion of viscosity solutions to solve (5.1) since it is fully nonlinear.

**Definition 5.1.** Let  $\Omega \subset \mathbf{R}^n$  be open. We say that  $u \in C(\Omega)$  is a *viscosity subsolution* (resp. *supersolution*) of (5.1) in  $\Omega$  if  $F(D^2 \phi(x)) \leq f(x)$  (resp.  $\geq f(x)$ ) for all  $(x, \phi) \in \Omega \times C^2(\Omega)$  such that  $u - \phi$  attains a local maximum (resp. minimum) at  $x$ .

For given ellipticity constants  $0 < \lambda \leq \Lambda$  we define *Pucci operators*  $\mathcal{P}^\pm : \mathbf{S}^n \rightarrow \mathbf{R}$  as

$$\mathcal{P}^+(X) := -\lambda \sum_{\mu_i > 0} \mu_i - \Lambda \sum_{\mu_i < 0} \mu_i, \quad \mathcal{P}^-(X) := -\lambda \sum_{\mu_i < 0} \mu_i - \Lambda \sum_{\mu_i > 0} \mu_i, \quad (5.2)$$

where  $\mu_i$  ( $i = 1, \dots, n$ ) are the eigenvalues of  $X \in \mathbf{S}^n$ . It is easily seen that these operators satisfy  $\mathcal{P}^-(X + Y) \leq \mathcal{P}^+(X) + \mathcal{P}^-(Y) \leq \mathcal{P}^+(X + Y)$  for all  $X, Y \in \mathbf{S}^n$ . We also have

$$\begin{aligned} \mathcal{P}^+(X) &= \sup\{-\text{trace}(AX) \mid A \in \mathbf{S}^n, \lambda I \leq A \leq \Lambda I\}, \\ \mathcal{P}^-(X) &= \inf\{-\text{trace}(AX) \mid A \in \mathbf{S}^n, \lambda I \leq A \leq \Lambda I\}, \end{aligned}$$



i.e.,  $\mathcal{P}^\pm$  are Bellman type operators. Here  $I$  is the identity matrix, and for  $X, Y \in \mathbf{S}^n$  we write  $X \leq Y$  if  $\langle (Y - X)\xi, \xi \rangle \geq 0$  for all  $\xi \in \mathbf{R}^n$ .

**Definition 5.2.** We say  $F : \mathbf{S}^n \rightarrow \mathbf{R}$  is *uniformly elliptic* if  $\mathcal{P}^-(X - Y) \leq F(X) - F(Y) \leq \mathcal{P}^+(X - Y)$  for all  $X, Y \in \mathbf{S}^n$ .

Now, we shall give examples showing that the usual Harnack inequality does not hold in the continuum case if we require  $u$  to be only a non-negative supersolution. In this section  $B_r$  stands for the open ball  $\{|x| < r\}$  in  $\mathbf{R}^n$ . The closure of it in  $\mathbf{R}^n$  is denoted by  $\overline{B_r}$ . Also, set  $B_r(z) := \{|x - z| < r\}$ .

**Example 5.3.** We consider the Laplace equation  $-\Delta u = 0$  in  $\mathbf{R}^n$  when  $n \geq 3$ . Set  $u(x) = \min\{c|x|^{2-n}, 1\}$  with  $c > 0$ . As is known,  $|x|^{2-n}$  is the fundamental solution of the Laplace equation while any constant is trivially a solution. Since the minimum of two supersolutions is still a supersolution ([5, Lemma 4.2]),  $u$  is a viscosity supersolution. On the other hand,  $u$  is not a viscosity subsolution. Indeed, letting  $\phi(x) = -\varepsilon|x|^2$  for  $\varepsilon > 0$  small, we see that  $u - \phi$  takes a maximum at a point  $z$  such that  $c|z|^{2-n} = 1$ , but  $-\Delta\phi(z) = 2n\varepsilon > 0$ . Now, let us fix  $r > 0$ . We then have  $\max_{\overline{B_r}} u = 1$  and  $\min_{\overline{B_r}} u = cr^{2-n}$  for  $c$  small. Thus the ratio  $(\max_{\overline{B_r}} u)/(\min_{\overline{B_r}} u)$  tends to  $\infty$  as  $c \rightarrow 0$ . This implies that the Harnack inequality does not hold.

The functions  $u(x) = \min\{|x|^{2-n}, M\}$  with  $M > 0$  also show that the Harnack inequality does not hold by letting  $M \rightarrow \infty$ .

We state the ABP maximum principle for viscosity solutions. Let  $\Omega \subset \mathbf{R}^n$  be an open set and  $u : \overline{\Omega} \rightarrow \mathbf{R}$ . Similarly to the discrete case, we define an upper contact set  $\Gamma_\Omega[u]$  by (2.3). Set  $\|u\|_{L^n(\Omega)} := (\int_\Omega |u(x)|^n dx)^{1/n}$ .

**Theorem 5.4** (ABP maximum principle). *Let  $\Omega \subset \mathbf{R}^n$  be a bounded open set. There exists a constant  $C_A = C_A(n, \lambda) > 0$  such that, for every viscosity subsolution (resp. supersolution)  $u \in C(\overline{\Omega})$  of (5.1) in  $\Omega$ , the estimate*

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+ + C_A \text{diam}(\Omega) \|f^+\|_{L^n(\Gamma_\Omega[u^+])} \quad (5.3)$$

$$(\text{resp. } \min_{\overline{\Omega}} u \geq \min_{\partial\Omega} (-u^-) - C_A \text{diam}(\Omega) \|f^-\|_{L^n(\Gamma_\Omega[u^-])}) \quad (5.4)$$

holds.

For the proof see [4, Proposition 2.12, Appendix A] or [11, Proposition 6.2, Section 7.2].

To present a barrier function in the continuum case, we first prepare

**Lemma 5.5.** *Let  $0 < \rho < R$  and define  $\psi(x) := M_1 - M_2|x|^{-\alpha}$  with*

$$M_1 = \frac{R^{-\alpha}}{\rho^{-\alpha} - R^{-\alpha}}, \quad M_2 = \frac{1}{\rho^{-\alpha} - R^{-\alpha}}, \quad \alpha = \max \left\{ 1, \frac{(n-1)\Lambda}{\lambda} - 1 \right\}. \quad (5.5)$$

Then

$$\begin{cases} \mathcal{P}^-(D^2\psi) \geq 0 & \text{in } \mathbf{R}^n \setminus \{0\}, & \text{(a)} \\ \psi \geq 0 & \text{in } \mathbf{R}^n \setminus B_R, & \text{(b)} \\ \psi \leq -1 & \text{in } B_\rho. & \text{(c)} \end{cases} \quad (5.6)$$

**Proof.** The proof uses a similar technique to [3, Proof of Lemma 4.1]. Indeed, the choices of  $M_1$  and  $M_2$  guarantee (5.6) (b) and (c). Also, at a point  $z = (r, 0, \dots, 0)$  with  $r > 0$ , we have  $\partial_{ij}^2\psi(z) = 0$  ( $i \neq j$ ),  $\partial_{11}^2\psi(z) = -M_2\alpha(\alpha+1)r^{-\alpha-2}$  and  $\partial_{ii}^2\psi(z) = M_2\alpha r^{-\alpha-2}$  ( $i \geq 2$ ). Since  $\psi$  is radially symmetric, for  $x \neq 0$

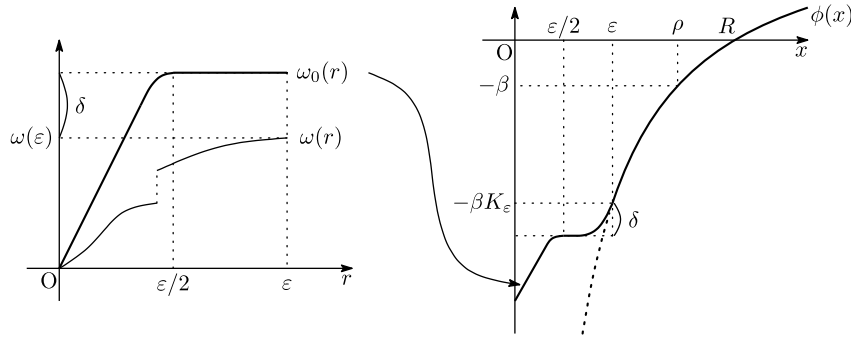


Fig. 1. The graphs of  $\omega_0$  and  $\phi$ .

$$\begin{aligned} \mathcal{P}^-(D^2\psi(x)) &= \lambda \cdot M_2\alpha(\alpha+1)|x|^{-\alpha-2} - \Lambda \cdot (n-1)M_2\alpha|x|^{-\alpha-2} \\ &= M_2\alpha|x|^{-\alpha-2}\{\lambda(\alpha+1) - \Lambda(n-1)\}, \end{aligned}$$

which is non-negative by the definition of  $\alpha$ . Thus (5.6) (a) is fulfilled.  $\square$

Let  $\varepsilon > 0$ . We say that a function  $\omega : [0, \varepsilon] \rightarrow [0, \infty)$  is a *modulus* on  $[0, \varepsilon]$  if  $\omega(0) = 0$ ,  $\lim_{r \rightarrow 0} \omega(r) = 0$  and  $\omega$  is non-decreasing on  $[0, \varepsilon]$ .

**Lemma 5.6.** *Let  $\omega$  be a modulus on  $[0, \varepsilon]$ . Let  $\delta > 0$ . Then there exists a modulus  $\omega_0$  on  $[0, \varepsilon]$  such that  $\omega_0 \in C^2(0, \varepsilon)$ ,  $\omega_0 > \omega$  on  $(0, \varepsilon]$  and  $\omega_0(r) = \omega(\varepsilon) + \delta$  for all  $r \in [\varepsilon/2, \varepsilon]$ .*

**Proof.** We set  $\omega_1(0) := 0$ ,  $\omega_1(\varepsilon) = \omega_1(\varepsilon/2) := \omega(\varepsilon) + \delta$  and  $\omega_1(\varepsilon/2^{j+1}) := \omega(\varepsilon/2^j)$  for  $j \in \mathbf{N}$ . On each interval  $[\varepsilon/2^j, \varepsilon/2^{j-1}]$  we interpolate  $\omega_1$  by a linear function. Then  $\omega_1$  is a modulus such that  $\omega_1 \geq \omega$  on  $[0, \varepsilon]$ . We next define  $\omega_2(r) := \min\{2\omega_1(r), \omega(\varepsilon) + \delta\}$ , which is again a piecewise linear modulus satisfying  $\omega_2 > \omega$  on  $(0, \varepsilon]$  and  $\omega_0(r) = \omega(\varepsilon) + \delta$  for all  $r \in [\varepsilon/2, \varepsilon]$ . Finally, mollifying  $\omega_2$  near each corner of the graph, we obtain the desired  $C^2$ -function  $\omega_0$ .  $\square$

A similar technique to make a smooth modulus can be found in [8, Lemma 2.1.9]. Using the above functions, let us construct a barrier function which will be used in the proof of our Harnack inequality. Let  $0 < \varepsilon < \rho < R$  and  $\omega$  be a modulus on  $[0, \varepsilon]$ . We also give positive constants  $\beta, \delta > 0$ . Set

$$K_\varepsilon := -(M_1 - M_2\varepsilon^{-\alpha}) > 0, \quad (5.7)$$

which will appear as the Harnack constant  $C_H$  in (5.11). We define

$$\phi(x) := \begin{cases} \beta\psi(x) & \text{if } |x| \geq \varepsilon, \\ \omega_0(|x|) - \beta K_\varepsilon - 2\delta - \omega(\varepsilon) & \text{if } |x| \leq \varepsilon/2. \end{cases}$$

On  $\{\varepsilon/2 \leq |x| \leq \varepsilon\}$  we extend  $\phi$  smoothly so that  $\phi \in C^2(\mathbf{R}^n \setminus \{0\})$  and  $-\beta K_\varepsilon - \delta \leq \phi(x) \leq -\beta K_\varepsilon$  if  $\varepsilon/2 \leq |x| \leq \varepsilon$ ; see Fig. 1. Then, by Lemma 5.5 and 5.6, the function  $\phi$  possesses the following properties:

$$\begin{cases} \mathcal{P}^-(D^2\phi) \geq 0 & \text{in } \mathbf{R}^n \setminus \overline{B_\varepsilon}, \\ \phi \geq 0 & \text{in } \mathbf{R}^n \setminus B_R, \\ \phi \leq -\beta & \text{in } B_\rho \end{cases} \quad (5.8)$$

and

$$\phi(x) - \phi(0) > \omega(|x|) \quad \text{if } 0 < |x| \leq \varepsilon. \quad (5.9)$$

Since  $\phi$  is not necessarily a  $C^2$ -function on the whole space, we next mollify it near the origin. For  $j \in \mathbf{N}$  we mollify  $\phi$  in  $B_{\varepsilon/2j}$  so that  $\phi_j \in C^2(\mathbf{R}^n)$ ,  $\phi_j \leq -\beta$  in  $B_\rho$  and  $\phi_j$  converges to  $\phi$  uniformly in  $\mathbf{R}^n$  as  $j \rightarrow \infty$ . Then each  $\phi_j$  satisfies the three properties in (5.8). We next define  $\xi_j(x) := |\mathcal{P}^-(D^2\phi_j(x))|$ . It then follows that

$$\mathcal{P}^-(D^2\phi_j) \geq -\xi_j(x) \quad \text{in } \mathbf{R}^n, \quad \text{supp}(\xi_j) \subset \overline{B_\varepsilon}.$$

Here  $\text{supp}(\xi_j) := \overline{\{x \in \mathbf{R}^n \mid \xi_j(x) \neq 0\}}$ .

We now derive the Harnack inequality for viscosity supersolutions of

$$\mathcal{P}^+(D^2u) = -f^-(x). \quad (5.10)$$

A viscosity supersolution of (5.1) is always a supersolution of (5.10).

**Theorem 5.7** (Harnack type inequality). *Let  $r > 0$  and  $0 < \varepsilon < 2r$ . Then there exists a constant  $C_H = C_H(r, \varepsilon, n, \Lambda/\lambda) > 0$  such that, for every non-negative viscosity supersolution  $u \in C(\overline{B_{4r}})$  of (5.10) in  $B_{4r}$ , the estimate*

$$\min_{\overline{B_\varepsilon(z)}} u \leq C_H \left\{ \min_{\overline{B_r}} u + C_A \text{diam}(\Omega) \|f^-\|_{L^n(B_{4r})} \right\} \quad (5.11)$$

holds for all  $z \in \overline{B_r}$ , where  $C_A$  is the constant in Theorem 5.4. Moreover, the constant  $C_H$  is explicitly given as  $K_\varepsilon$  in (5.7), i.e.,

$$C_H = K_\varepsilon = -(M_1 - M_2\varepsilon^{-\alpha}), \quad (5.12)$$

where  $M_1, M_2$  and  $\alpha$  are the constants in (5.5) with  $\rho = 2r$  and  $R = 3r$ .

**Proof.** Case:  $f^- \equiv 0$ . 1. Fix any  $z \in \overline{B_r}$ . For  $t \in [0, \varepsilon]$  we define  $\omega(t) := u(z) - \min_{\overline{B_t(z)}} u$ . It is easily seen that  $\omega$  is a modulus on  $[0, \varepsilon]$ . Choose  $x_m$  as a minimum point of  $u$  over  $\overline{B_r}$ , i.e.,  $u(x_m) = \min_{\overline{B_r}} u$ , and let  $\beta > u(x_m)$  ( $\geq 0$ ). We take  $\phi, \phi_j$  and  $\xi_j$  as the functions given after Lemma 5.6 with  $\rho = 2r$  and  $R = 3r$ , where  $\omega$  and  $\beta$  are chosen as above. Define  $\tilde{\phi}(x) := \phi(x - z)$ ,  $\tilde{\phi}_j(x) := \phi_j(x - z)$  and  $\tilde{\xi}(x) := \xi(x - z)$ . We furthermore set  $B' := B_{2r}(z)$  and  $B := B_{3r}(z)$ , so that we have  $B_r \subset B' \subset B \subset B_{4r}$ .

By (5.9) we see that  $u + \tilde{\phi}$  attains its strict minimum at  $z$  over  $\overline{B_\varepsilon(z)}$ . Indeed, if  $0 < |x - z| \leq \varepsilon$ , we compute

$$u(x) + \tilde{\phi}(x) > \{u(z) - \omega(|x - z|)\} + \{\tilde{\phi}(z) + \omega(|x - z|)\} = u(z) + \tilde{\phi}(z).$$

We let  $z_j$  be a minimum point of  $u + \tilde{\phi}_j$  over  $\overline{B_\varepsilon(z)}$ . Then, since  $\tilde{\phi}_j$  uniformly converges to  $\tilde{\phi}$ , it follows that  $z_j \rightarrow z$  as  $j \rightarrow \infty$ .

2. We show that  $u + \tilde{\phi}_j$  is a viscosity supersolution of  $\mathcal{P}^+ = -\tilde{\xi}_j$  in  $B$ . Assume that  $u + \tilde{\phi}_j - \psi$  attains a local minimum at  $x \in B$  for  $\psi \in C^2(B)$ . Since  $u + \tilde{\phi}_j - \psi = u - (\psi - \tilde{\phi}_j)$  and  $u$  is a viscosity supersolution of (5.10), we observe

$$\begin{aligned} 0 &\leq \mathcal{P}^+(D^2\psi(x) - D^2\tilde{\phi}_j(x)) \leq \mathcal{P}^+(D^2\psi(x)) - \mathcal{P}^-(D^2\tilde{\phi}_j(x)) \\ &\leq \mathcal{P}^+(D^2\psi(x)) + \tilde{\xi}_j(x), \end{aligned}$$

which implies the assertion. Therefore the ABP maximum principle (5.4) implies

$$\min_{\overline{B}} (u + \tilde{\phi}_j) \geq \min_{\partial B} \{-(u + \tilde{\phi}_j)^-\} - C_A \text{diam}(B) \|\tilde{\xi}_j\|_{L^n(\Gamma_B[(u + \tilde{\phi}_j)^-])}.$$

Similarly to the discrete case, we have  $\min_{\partial B} \{-(u + \tilde{\phi}_j)^-\} = 0$  and  $\min_{\overline{B}}(u + \tilde{\phi}_j) < 0$  by the properties of  $\tilde{\phi}_j$ , and hence  $\|\tilde{\xi}_j\|_{L^n(\Gamma_B[(u + \tilde{\phi}_j)^-])} > 0$ . Since  $\text{supp}(\tilde{\xi}_j) \subset \overline{B_\varepsilon(z)}$ , we see that the set  $\overline{B_\varepsilon(z)} \cap \Gamma_B[(u + \tilde{\phi}_j)^-]$  is not empty.

3. Choose an arbitrary  $y \in \overline{B_\varepsilon(z)} \cap \Gamma_B[(u + \tilde{\phi}_j)^-]$ . Then we see  $(u + \tilde{\phi}_j)(y) < 0$  by a similar argument to the discrete case. Since  $u + \tilde{\phi}_j$  attains its minimum at  $z_j$  over  $\overline{B_\varepsilon(z)}$ , it follows that

$$u(z_j) + \tilde{\phi}_j(z_j) \leq u(y) + \tilde{\phi}_j(y) < 0.$$

Letting  $j \rightarrow \infty$ , we have

$$u(z) \leq -\tilde{\phi}(z) = -\phi(0) = \beta K_\varepsilon + 2\delta + \omega(\varepsilon).$$

By the definition of  $\omega$ , this gives

$$\min_{\overline{B_\varepsilon(z)}} u \leq \beta K_\varepsilon + 2\delta.$$

Finally, sending  $\beta \rightarrow u(x_m)$  and  $\delta \rightarrow 0$  yield  $\min_{\overline{B_\varepsilon(z)}} u \leq C_H u(x_m)$  with  $C_H = K_\varepsilon$ .

Case:  $f^- \not\equiv 0$ . 1. Let  $\{f_j\}_{j=1}^\infty \subset C^\infty(\mathbf{R}^n)$  be a sequence of smooth functions such that  $f_j \geq f^-$  in  $\overline{B_{4r}}$  for all  $j \in \mathbf{N}$  and that  $f_j$  converges to  $f^-$  uniformly on  $\overline{B_{4r}}$  as  $j \rightarrow \infty$ . We consider the Dirichlet problem

$$\begin{cases} \mathcal{P}^-(D^2 v_j) = f_j & \text{in } B_{4r}, \\ v_j = 0 & \text{on } \partial B_{4r} \end{cases} \quad (5.13)$$

and denote by  $v_j \in C^2(B_{4r}) \cap C(\overline{B_{4r}})$  the solution of (5.13). The existence of smooth solutions is due to the classical results by Evans–Krylov for convex/concave (or Bellman type) equations. See [6,7,12] or [11, Section 7.3]. The ABP maximum principles, (5.3) and (5.4), yield

$$0 \leq v_j \leq C_A \text{diam}(\Omega) \|f_j\|_{L^n(B_{4r})} \quad \text{on } \overline{B_{4r}}. \quad (5.14)$$

2. Now, it is easy to see that  $u + v_j$  is a viscosity solution of  $\mathcal{P}^+ = 0$  in  $B_{4r}$ . Since  $u + v_j$  is non-negative on  $\overline{B_{4r}}$  by (5.14), the Harnack inequality (5.11) with  $f^- \equiv 0$  implies

$$\min_{\overline{B_\varepsilon(z)}} (u + v_j) \leq C_H \min_{\overline{B_r}} (u + v_j).$$

Applying the first and the second inequality in (5.14) to the left- and the right-hand side of the above estimate respectively, we obtain

$$\min_{\overline{B_\varepsilon(z)}} u \leq C_H \left\{ \min_{\overline{B_r}} u + C_A \text{diam}(\Omega) \|f_j\|_{L^n(B_{4r})} \right\}.$$

Sending  $j \rightarrow \infty$  gives (5.11).  $\square$

**Remark 5.8.** The estimate (5.11) we established can be written as

$$u(z) \leq C_H \left\{ \min_{\overline{B_r}} u + C_A \text{diam}(\Omega) \|f^-\|_{L^n(B_{4r})} \right\} + \omega(\varepsilon),$$

which gives a pointwise estimate for  $u$ , but the right-hand side involves a modulus of continuity from below of  $u$  around  $z$ .

**Remark 5.9.** The functions in Example 5.3 shows that the Harnack constant  $C_H$  in (5.11) must go to infinity when  $\varepsilon \rightarrow 0$ . Indeed, our Harnack constant (5.12) tends to infinity as  $\varepsilon \rightarrow 0$ .

As a byproduct of this observation, we see non-existence of a radially symmetric function  $\psi \in C(\mathbf{R}^n) \cap C^2(\mathbf{R}^n \setminus \{0\})$  satisfying the three conditions (5.6) (a)–(c). (Note that the function  $\psi$  in Lemma 5.5 does not belong to  $C(\mathbf{R}^n)$ .) If there were such a  $\psi$ , by a similar argument to the proof of Theorem 5.7 we would have the Harnack inequality (5.11) with  $C_H$  which is less than  $-\psi(0)$ , a contradiction.

**Remark 5.10.** The result in Theorem 5.7 still holds for  $L^p$ -viscosity solutions, although we do not give the details in this paper. In the theory of  $L^p$ -viscosity solutions,  $f$  is just assumed to be in  $L^p(\Omega)$  and solutions are defined by test functions belonging to  $W_{\text{loc}}^{2,p}$ . In this case, we do not need to approximate  $f^-$  by smooth functions  $f_j$  in the proof of the Harnack inequality because the Dirichlet problem (5.13) with  $f^-$  instead of  $f_j$  admits a solution in  $W_{\text{loc}}^{2,p}$ . Also, it is not difficult to extend the result to more general equations of the form

$$\mathcal{P}^+(D^2u) + \mu|Du| = -f^-(x)$$

with  $\mu \geq 0$ . See [4] and [11, Sections 6 and 7] for the theory of  $L^p$ -viscosity solutions and the above generalized equation.

## Acknowledgments

The author thanks the anonymous referees for their careful reading of the manuscript and valuable comments. The work of the author was supported by Grant-in-Aid for JSPS Fellows No. 23-4365 and No. 26-30001.

## Appendix A. A well-posedness of uniformly elliptic equations

We prove that the Dirichlet problem

$$F(\vec{\delta}^2 u(x)) = f(x) \quad \text{in } \Omega, \tag{A.1}$$

$$u(x) = g(x) \quad \text{on } \partial\Omega \tag{A.2}$$

has a unique discrete solution. Here  $\Omega \subset h\mathbf{Z}^n$  is a bounded set,  $F : \mathbf{R}^n \rightarrow \mathbf{R}$  is uniformly elliptic,  $F(0) = 0$ ,  $f : \Omega \rightarrow \mathbf{R}$  and  $g : \partial\Omega \rightarrow \mathbf{R}$  is a given boundary datum. The uniqueness easily follows from the ABP maximum principle. The existence of solutions to elliptic difference equations is more or less known even when the equation is degenerate; for example, the fixed point theorem is one of powerful tools to show the existence. However, we present it here to make the paper self-contained and to give the proof based on *Perron's method*, which cannot be found much in discrete problems.

For  $\vec{X}, \vec{Y} \in \mathbf{R}^n$  given as  $\vec{X} = (X_1, \dots, X_n)$  and  $\vec{Y} = (Y_1, \dots, Y_n)$ , we write  $\vec{X} \leq \vec{Y}$  if  $X_i \leq Y_i$  for all  $i \in \{1, \dots, n\}$ . By the uniform ellipticity of  $F$ , we have  $F(\vec{X}) \geq F(\vec{Y})$  if  $\vec{X} \leq \vec{Y}$ . This is a degenerate ellipticity ([5, (0.3)]).

From the ABP maximum principle (Theorem 2.3) we immediately deduce a comparison principle for a discrete sub- and supersolution of (A.1). This implies a uniqueness of solutions.

**Corollary A.1** (*Comparison principle*). *Let  $u$  and  $v$  be, respectively, a discrete subsolution and supersolution of (A.1). If  $u \leq v$  on  $\partial\Omega$ , then  $u \leq v$  in  $\Omega$ .*

**Proof.** Since  $F$  is uniformly elliptic, we observe

$$\mathcal{P}^-(\vec{\delta}^2 u(x) - \vec{\delta}^2 v(x)) \leq F(\vec{\delta}^2 u(x)) - F(\vec{\delta}^2 v(x)) \leq f(x) - f(x) = 0$$

for all  $x \in \Omega$ . Therefore  $u - v$  is a discrete subsolution of the Pucci equation  $\mathcal{P}^- = 0$  in  $\Omega$ . We now apply the ABP maximum principle to obtain  $\max_{\overline{\Omega}}(u - v) \leq \max_{\partial\Omega}(u - v) \leq 0$ .  $\square$

We turn to an existence problem. To construct discrete solutions, we employ the idea of Perron's method for viscosity solutions ([5, Section 4]).

**Proposition A.2** (Perron's method). *Let  $v$  and  $V$  be, respectively, a discrete sub- and supersolution of (A.1) such that  $v \leq V$  on  $\overline{\Omega}$ . Let*

$$\mathcal{S} := \left\{ w : \overline{\Omega} \rightarrow \mathbf{R} \mid \begin{array}{l} w \text{ is a discrete subsolution of (A.1)} \\ \text{such that } v \leq w \leq V \text{ on } \overline{\Omega} \end{array} \right\}.$$

Then  $u(x) := \sup_{w \in \mathcal{S}} w(x)$  is a discrete solution of (A.1).

**Proof.** 1. We first prove that  $u$  is a discrete subsolution. Fix  $x \in \Omega$  and  $\varepsilon > 0$ . By the definition of  $u$  there exists some  $w_\varepsilon \in \mathcal{S}$  such that  $u(x) - \varepsilon \leq w_\varepsilon(x) \leq u(x)$ . We then observe

$$\begin{aligned} \delta_i^2 u(x) &= \frac{u(x + h_i e_i) + u(x - h_i e_i) - 2u(x)}{h_i^2} \\ &\geq \frac{w_\varepsilon(x + h_i e_i) + w_\varepsilon(x - h_i e_i) - 2(w_\varepsilon(x) + \varepsilon)}{h_i^2} = \delta_i^2 w_\varepsilon(x) - \frac{2\varepsilon}{h_i^2} \end{aligned}$$

for each  $i \in \{1, \dots, n\}$ . Thus

$$\bar{\delta}^2 u(x) \geq \bar{\delta}^2 w_\varepsilon(x) - \frac{2\varepsilon}{h_{\min}^2}(1, \dots, 1).$$

From the uniform ellipticity of  $F$  it follows that

$$\begin{aligned} F(\bar{\delta}^2 u(x)) &\leq F\left(\bar{\delta}^2 w_\varepsilon(x) - \frac{2\varepsilon}{h_{\min}^2}(1, \dots, 1)\right) \\ &\leq F(\bar{\delta}^2 w_\varepsilon(x)) - \mathcal{P}^-\left(\frac{2\varepsilon}{h_{\min}^2}(1, \dots, 1)\right) \\ &= F(\bar{\delta}^2 w_\varepsilon(x)) + \Lambda \frac{2\varepsilon n}{h_{\min}^2}. \end{aligned} \tag{A.3}$$

We now have  $F(\bar{\delta}^2 w_\varepsilon(x)) \leq f(x)$  since  $w_\varepsilon \in \mathcal{S}$ . Applying this to (A.3) and then letting  $\varepsilon \rightarrow 0$ , we obtain  $F(\bar{\delta}^2 u(x)) \leq f(x)$ . This implies that  $u$  is a discrete subsolution.

2. We next show that  $u$  is a discrete supersolution. Suppose that this were false. Then we could find some  $y \in \Omega$  such that

$$F(\bar{\delta}^2 u(y)) < f(y). \tag{A.4}$$

For such  $y$  and  $\delta > 0$  we define

$$U(x) := \begin{cases} u(y) + \delta & \text{if } x = y, \\ u(x) & \text{if } x \neq y. \end{cases}$$

We claim that  $U \in \mathcal{S}$  for a sufficiently small  $\delta > 0$ . Showing this claim yields a contradiction since  $U$  is strictly larger than  $u$  at  $y$ .

We first prove  $u(y) < V(y)$ . Suppose  $u(y) = V(y)$ . Then, noting that  $u(x) \leq V(x)$  for  $x \neq y$ , we would have  $\delta^2 u(y) \leq \delta^2 V(y)$ . Since  $V$  is a supersolution, it would follow that  $F(\delta^2 u(y)) \geq F(\delta^2 V(y)) \geq f(y)$ , a contradiction to (A.4). Thus  $v \leq U \leq V$  on  $\overline{\Omega}$  if we take  $\delta \leq V(y) - u(y)$ .

Let us show that  $U$  is a subsolution. Let  $x \in \Omega$ . It is easily seen that  $\delta^2 U(x) = \delta^2 u(x)$  if  $x \notin \overline{\{y\}}$  and that  $\delta^2 U(x) \geq \delta^2 u(x)$  if  $x \in \overline{\{y\}} \setminus \{y\}$ . Therefore the ellipticity of  $F$  implies that  $F(\delta^2 U(x)) \leq F(\delta^2 u(x)) \leq f(x)$  for  $x \neq y$ . We next consider the case  $x = y$ . Then

$$\delta_i^2 U(y) = \frac{u(y + h_i e_i) + u(y - h_i e_i) - 2(u(y) + \delta)}{h_i^2} = \delta_i^2 u(y) - \frac{2\delta}{h_i^2},$$

and so the same calculation as in Step 1 yields

$$F(\delta^2 U(y)) \leq F(\delta^2 u(y)) + \Lambda \frac{2\delta n}{h_{\min}^2} = f(y) + \Lambda \frac{2\delta n}{h_{\min}^2} - \{f(y) - F(\delta^2 u(y))\}.$$

In view of (A.4), it follows that  $F(\delta^2 U(y)) \leq f(y)$  if  $\delta \leq h_{\min}^2 \{f(y) - F(\delta^2 u(y))\} / (2\Lambda n)$ . Summarizing the above argument, we conclude that  $U \in \mathcal{S}$ , and hence  $u$  is a discrete supersolution.  $\square$

The remaining thing is to construct  $v$  and  $V$  in Proposition A.2 which attain a given boundary datum on  $\partial\Omega$ . For this purpose, we prepare quadratic functions on lattices. We will use these functions to make such  $v$  and  $V$ .

**Example A.3.** Let  $A = (A_1, \dots, A_n) \in \mathbf{R}^n$ . We define a quadratic function  $q = q_A : h\mathbf{Z}^n \rightarrow \mathbf{R}$  as  $q(x) := \sum_{j=1}^n A_j x_j^2$  for  $x = (x_1, \dots, x_n) \in h\mathbf{Z}^n$ . Then  $\delta_i^2 q$  is a constant for each  $i \in \{1, \dots, n\}$ . Indeed, we observe

$$\begin{aligned} \delta_i^2 q(x) &= \frac{q(x + h_i e_i) + q(x - h_i e_i) - 2q(x)}{h_i^2} \\ &= \frac{A_i(x_i + h_i)^2 + A_i(x_i - h_i)^2 - 2A_i x_i^2}{h_i^2} = 2A_i \end{aligned}$$

for all  $x \in h\mathbf{Z}^n$ . In particular, if we take  $A_i = -c/(2\lambda n)$  with  $c \geq 0$ , then

$$\mathcal{P}^-(\delta^2 q(x)) = -\lambda \cdot \frac{-c}{\lambda n} \cdot n = c,$$

i.e.,  $q$  is a discrete solution of the above Pucci equation in  $h\mathbf{Z}^n$ .

**Theorem A.4** (Unique solvability). *The Dirichlet problem (A.1) and (A.2) admits a unique discrete solution.*

**Proof.** The uniqueness is a consequence of the comparison principle, Corollary A.1. To show the existence we construct  $v$  and  $V$  in the statement of Proposition A.2 such that  $v = V = g$  on  $\partial\Omega$ ; then Proposition A.2 ensures that  $u := \sup_{w \in \mathcal{S}} w$  is a discrete solution of (A.1) and (A.2). To construct such  $v$  and  $V$  we use quadratic functions in Example A.3. Let  $q_A$  be the quadratic function in Example A.3 with  $A_i = -(\max_{\Omega} |f|)/(2\lambda n)$ , so that

$$\mathcal{P}^-(\delta^2 q_A(x)) = \max_{\Omega} |f| \quad \text{in } \Omega. \quad (\text{A.5})$$

Choose  $k \geq 0$  such that  $q_A + k \geq \max_{\partial\Omega} g$  on  $\overline{\Omega}$ , and define

$$V(x) := \begin{cases} q_A(x) + k & \text{if } x \in \Omega, \\ g(x) & \text{if } x \in \partial\Omega. \end{cases}$$

Then  $V$  is a discrete supersolution of (A.1). Indeed, since  $V \leq q_A + k$  on  $\partial\Omega$ , we have  $\bar{\delta}^2 V(x) \leq \bar{\delta}^2(q_A + k)(x) = \bar{\delta}^2 q_A(x)$  for  $x \in \Omega$ . Therefore it follows from ellipticity that  $F(\bar{\delta}^2 V(x)) \geq F(\bar{\delta}^2 q_A(x)) \geq \mathcal{P}^-(\bar{\delta}^2 q_A(x))$ . By virtue of (A.5) we conclude that  $V$  is a discrete supersolution of (A.1).

Similarly, using a suitable quadratic function, we are able to construct a discrete subsolution  $v$  which satisfies  $v = g$  on  $\partial\Omega$  and  $v \leq \min_{\partial\Omega} g$  in  $\Omega$ . The proof is now complete since  $v \leq \min_{\partial\Omega} g \leq \max_{\partial\Omega} g \leq V$  in  $\Omega$ .  $\square$

## References

- [1] X. Cabré, On the Alexandroff–Bakel’man–Pucci estimate and the reversed Hölder inequality for solutions of elliptic and parabolic equations, *Comm. Pure Appl. Math.* 48 (5) (1995) 539–570.
- [2] L.A. Caffarelli, Interior a priori estimates for solutions of fully nonlinear equations, *Ann. of Math.* (2) 130 (1) (1989) 189–213.
- [3] L.A. Caffarelli, X. Cabré, *Fully Nonlinear Elliptic Equations*, Amer. Math. Soc. Colloq. Publ., vol. 43, American Mathematical Society, Providence, RI, 1995.
- [4] L.A. Caffarelli, M.G. Crandall, M. Kocan, A. Święch, On viscosity solutions of fully nonlinear equations with measurable ingredients, *Comm. Pure Appl. Math.* 49 (4) (1996) 365–397.
- [5] M.G. Crandall, H. Ishii, P.L. Lions, User’s guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc. (N.S.)* 27 (1) (1992) 1–67.
- [6] L.C. Evans, Classical solutions of fully nonlinear, convex, second-order elliptic equations, *Comm. Pure Appl. Math.* 35 (3) (1982) 333–363.
- [7] L.C. Evans, Classical solutions of the Hamilton–Jacobi–Bellman equation for uniformly elliptic operators, *Trans. Amer. Math. Soc.* 275 (1) (1983) 245–255.
- [8] Y. Giga, *Surface Evolution Equations: A Level Set Approach*, Monogr. Math., vol. 99, Birkhauser Verlag, Basel, 2006, xii+264 pp.
- [9] D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Reprint of the 1998 edition, *Classics Math.*, Springer-Verlag, Berlin, 2001.
- [10] N. Hamamuki, A discrete isoperimetric inequality on lattices, *Discrete Comput. Geom.* 52 (2) (2014) 221–239.
- [11] S. Koike, *A Beginner’s Guide to the Theory of Viscosity Solutions*, MSJ Mem., vol. 13, Mathematical Society of Japan, Tokyo, 2004, viii+123 pp.
- [12] N.V. Krylov, Boundedly nonhomogeneous elliptic and parabolic equations, *Izv. Akad. Nauk SSSR Ser. Mat.* 46 (1982) 487–523 (Russian). English translation in *Math. USSR, Izv.* 20 (1983) 459–492.
- [13] H.-J. Kuo, N.S. Trudinger, Linear elliptic difference inequalities with random coefficients, *Math. Comp.* 55 (191) (1990) 37–53.
- [14] H.-J. Kuo, N.S. Trudinger, Discrete methods for fully nonlinear elliptic equations, *SIAM J. Numer. Anal.* 29 (1) (1992) 123–135.
- [15] H.-J. Kuo, N.S. Trudinger, Local estimates for parabolic difference operators, *J. Differential Equations* 122 (2) (1995) 398–413.
- [16] H.-J. Kuo, N.S. Trudinger, Positive difference operators on general meshes, *Duke Math. J.* 83 (2) (1996) 415–433.
- [17] H.-J. Kuo, N.S. Trudinger, Evolving monotone difference operators on general space-time meshes, *Duke Math. J.* 91 (3) (1998) 587–607.
- [18] J.J. Manfredi, A.M. Oberman, A.P. Sviridov, Nonlinear elliptic partial differential equations and  $p$ -harmonic functions on graphs, *Differential Integral Equations* 28 (1–2) (2015) 79–102.