



Invariance of polynomial inequalities under polynomial maps

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ABSTRACT

We prove the invariance under simple polynomial maps of a few useful (multivariate) polynomial inequalities. The proofs are elementary and based on an algebraic decomposition of any polynomial as a sum of polynomials, which we call q -coordinates, composed with the polynomial map q , balanced with monomials of bounded degree.

1 Introduction

Let E be a compact set in the complex plane and q a non constant polynomial. It is reasonable to expect to relate the analytic properties of polynomials on E and on $q^{-1}(E)$, the obvious observation being that, for any polynomial p , $p \circ q$ behave on $q^{-1}(E)$ like p on E . A natural strategy to study an arbitrary polynomial on $q^{-1}(E)$ would be to exhibit, in a certain sense, compositions by q in its expression. Such a decomposition is easily obtained. In fact, supposing that p is a polynomial of degree at most md where m is the degree of q , by performing sufficiently many euclidean divisions by q , we may write p as $p(z) = \sum_{j=0}^d Q_j(z)q^j(z)$ where each Q_j is a polynomial of degree at most d . Next, collecting the (polynomial) coefficients of the monomials z^j , $j = 0, \dots, m-1$, appearing in the quotients $Q_j(z)$, we obtain an expression of p of the form $p(z) = \sum_{j=0}^{m-1} z^j R_j(q(z))$ where each R_j is a polynomial of degree at most d . We will show that these polynomials R_j furnish the suitable connection for going from polynomials on $q^{-1}(E)$ to polynomials on E . We call them the q -coordinates of p . It is easily checked that, for every w , the values $R_j(w)$ form the coefficients on the standard monomial basis of the Lagrange interpolation polynomial of p at the roots of the equation $q(z) = w$. The purpose of this note is to study this construction, in particular the definition of q -coordinates in the higher dimensional case, when q is a sufficiently simple (so as to enable extending the above decomposition by elementary means) polynomial map from \mathbb{C}^n to \mathbb{C}^n . We apply our q -coordinates to the construction of admissible meshes (see section 3) on the pre-image of a compact set, to characterise Bernstein-Markov measures on the pre-image in terms of those on the original set, to show that the pre-image satisfies essentially the same Markov inequalities, as well as other similar inequalities, as its original set.

We use straightforward notation for spaces of functions, of polynomials ($\mathcal{P}(\mathbb{C}^n)$, $\mathcal{P}_d(\mathbb{C}^n)$), for monomials and multi-indices and, as usual, $\|h\|_E$ stands for the sup-norm $\max_{z \in E} |h(z)|$.

Finally, let us point out that several papers deal with pre-images under polynomial maps, see for instance [20], [24] and [26] in the univariate case and [7] in the several variables case.

2 q-Coordinates

2.1 Simple maps

We use mostly basic tools from computational algebraic geometry as presented in [9]. Recall that given a monomial order \succ , the *multi-degree* $\text{mdeg}(p)$ of a polynomial p is α where z^α is the *leading monomial* ($\text{LM}(p)$) of p with respect to \succ . We will consider a polynomial map $q : \mathbb{C}^n \rightarrow \mathbb{C}^n$, $q = (q_1, \dots, q_n)$, $\deg q_i = m$. Such a map must be proper (since $q^{-1}(E)$ needs to be compact). A useful sufficient (but not necessary) condition guaranteeing that q is proper is, see [18], that

$$\bigcap_{i=1}^n \hat{q}_i^{-1}(0) = \{0\}, \quad (1)$$

where \hat{q}_i denotes the homogeneous part of highest degree in q_i . In order to extend the algebraic decomposition used in the univariate case and sketched in the introduction, we must further restrict the generality on q , limiting ourselves to the case of *simple polynomial maps*.

Definition 1 *We say that a polynomial map $q : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is simple (of degree $m \geq 1$) if we have $\text{LM}(q_i) = z_i^m$, $i = 1, \dots, n$, where the leading monomial is computed with respect to the graded lexicographic order.*

Here, we use the graded lexicographic order defined by the conditions $z_1 \succ z_2 \succ \dots \succ z_n$. Of course, we might use without change any other graded lexicographic order (obtained from \succ by a permutation of the variables). Recall that a monomial order is said to be graded if $\alpha \succeq \beta$ implies that $|\alpha| \geq |\beta|$.

In fact, a map q is simple if and only if the homogeneous parts of highest degree are triangular in the sense that

$$\hat{q}_i(z) = z_i^m + \sum_{j=1}^m z_i^{m-j} h_j(z_{i+1}, \dots, z_n), \quad i = 1, \dots, n, \quad (2)$$

where h_j is a homogeneous polynomial of degree j (depending on i) in the lower variables z_{i+1}, \dots, z_n . In particular, $\hat{q}_n(z) = z_n^m$.

Lemma 2 *A simple map satisfies condition (1) and it is therefore proper.*

Proof. We take $z \in \bigcap_{i=1}^n \hat{q}_i^{-1}(0)$ and show that $z = 0$. From $0 = \hat{q}_n(z) = z_n^m$, we get $z_n = 0$ and, having shown $0 = z_n = \dots = z_{n-k+1}$, we use

$$0 = \hat{q}_{n-k}(z) = z_{n-k}^m + \sum_{j=1}^m z_{n-k}^{m-j} h_j(z_{n-k+1}, \dots, z_n).$$

Since $0 = z_n = \dots = z_{n-k+1}$ and the polynomials h_j are homogeneous, all terms in the above sum vanish so that $z_{n-k} = 0$. \square

Everything in this note but the above lemma remains true when \succ is replaced by a graded *non lexicographic* order. Thus, the statements below hold true when q is simple with respect to an arbitrary graded monomial order provided that, in addition, q is assumed to be proper.

2.2 Decomposition of polynomials

We denote by A_{m-1} the set of multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ such that $\alpha_i \leq m-1$ for $i = 1, \dots, n$ and $P_{m-1} = \text{span}\{z \rightarrow z^\alpha : \alpha \in A_{m-1}\}$ which is the tensor product of the n univariate spaces of polynomials in z_i , $i = 1, \dots, n$, of degree at most $m-1$.

Lemma 3 *Let q be a simple polynomial map of degree m on \mathbb{C}^n . Every polynomial $p \in \mathcal{P}_{md}(\mathbb{C}^n)$, $d \in \mathbb{N}$, can be written as*

$$p(z) = \sum_{\alpha \in A_{m-1}} z^\alpha R_\alpha(q(z)), \quad z \in \mathbb{C}^n, \quad (3)$$

where $R_\alpha \in \mathcal{P}_d(\mathbb{C}^n)$, $\alpha \in A_{m-1}$.

Proof. By induction on d . If $d = 0$ then $\mathcal{P}_{md}(\mathbb{C}^n) = \mathcal{P}_0(\mathbb{C}^n)$ (i.e. constant polynomials). Hence $R_0 = \text{constant}$ and $R_\alpha = 0$ if $\alpha \neq 0$ will work. We now assume that the claim is proved for $\deg p = md$, $d \geq 0$ and we establish it for $\deg p = m(d+1)$. Using the division algorithm in $\mathcal{P}(\mathbb{C}^n)$, see [9, Theorem 3, p. 63], we may write

$$p(z) = \sum_{i=1}^n a_i(z) q_i(z) + r(z),$$

where $a_i, r \in \mathcal{P}(\mathbb{C}^n)$ and no term in $r(z)$ is divisible by any of the $\text{LM}(q_i) = z_i^m$, $i = 1, \dots, n$, so that $r(z)$ must belong to P_{m-1} . Moreover, the division algorithm ensures that $\text{mdeg}(p) \succeq \text{mdeg}(a_i q_i)$ and, since we work with a graded order, this implies $\deg(a_i q_i) \leq \deg p$, hence $\deg a_i \leq \deg p - m \leq md$. We may therefore apply the induction hypothesis to each a_i , to write, say,

$$a_i(z) = \sum_{\alpha \in A_{m-1}} z^\alpha R_\alpha^{(i)}(q(z)), \quad \text{with} \quad \deg R_\alpha^{(i)} \leq d.$$

It follows that

$$p(z) = r(z) + \sum_{i=1}^n q_i(z) \sum_{\alpha \in A_{m-1}} z^\alpha R_\alpha^{(i)}(q(z)) = r(z) + \sum_{\alpha \in A_{m-1}} z^\alpha \sum_{i=1}^n q_i(z) R_\alpha^{(i)}(q(z)).$$

Hence the polynomials $R_\alpha(z)$ corresponding to p are given by

$$R_\alpha(z) = \sum_{i=1}^n z_i R_\alpha^{(i)}(z) + (\text{coefficient of } z^\alpha \text{ in } r(z)),$$

whose degrees are at most $d + 1$ since each $R_\alpha^{(i)}$ is of degree at most d . This concludes the proof. \square

The lemma does not say that the polynomials R_α are unique. Uniqueness holds however, it is proved below.

Lemma 4 *Let q be a simple map of degree m and let $w \in \mathbb{C}^n$ such that $q(z) = w$ has m^n (pairwise distinct) solutions. Then these solutions form a unisolvent set for P_{m-1} .*

By unisolvent, we mean that, for every function f defined on $q^{-1}(w)$, there exists a unique polynomial $p \in P_{m-1}$ such that $p = f$ on $q^{-1}(w)$. Observe that, thanks to (1), $q(z) = w$ has no roots at infinity, hence, by Bézout's theorem, it has exactly m^n solutions taking multiplicity into account. Thus, the assumption of the lemma merely requires that all solutions be simple.

Proof. Let $X = q^{-1}(w)$. We have to prove that the linear map

$$\Psi : p \in P_{m-1} \rightarrow p|_X \in \mathcal{F}(X),$$

where $p|_X$ denotes the restriction of p to X and $\mathcal{F}(X)$ the space of functions on X , is an isomorphism. Since both spaces have the same dimension (here the assumption on the cardinality of X is essential), it suffices to show that Ψ is onto. So, letting $f \in \mathcal{F}(X)$, we look for $P \in P_{m-1}$ such that $P|_X = f$. We claim that we may assume that this function f is the restriction to X of a polynomial p (of high degree). Indeed, more generally, given any finite set of points-values $(u_i, v_i) \in \mathbb{C}^n \times \mathbb{C}$, $i \in I$, there exists a polynomial p such that $p(u_i) = v_i$ for $i \in I$. (Choose a linear form x such that the scalars $x(u_i)$ are pairwise distinct and, by analogy with the Lagrange interpolation formula, take

$$p(z) = \sum_{i \in I} v_i \ell_i(z) \quad \text{with} \quad \ell_i(z) = \prod_{j \in I \setminus \{i\}} \frac{x(z) - x(u_j)}{x(u_i) - x(u_j)}.$$

Now, since $f = p|_X$, the existence of $P \in P_{m-1}$ so that $P|_X = f$ follows from the previous lemma. We just need to take as P the polynomial defined with the help of p by

$$P(u) = \sum_{\alpha \in A_{m-1}} u^\alpha R_\alpha(w),$$

where the R_α are defined by the decomposition (3) for p . To show that P coincides with p , hence with f , on $X = q^{-1}(w)$, we observe that if $z \in q^{-1}(w)$ then

$$P(z) = \sum_{\alpha \in A_{m-1}} z^\alpha R_\alpha(w) = \sum_{\alpha \in A_{m-1}} z^\alpha R_\alpha(q(z)) = p(z). \quad (4)$$

\square

Lemma 5 *The polynomials R_α corresponding to p in Lemma 3 are unique.*

Proof. Suppose that p can be decomposed with polynomials R_α and polynomials \tilde{R}_α . We know that $R_\alpha(w) = \tilde{R}_\alpha(w)$ at every point w such that $q(z) = w$ admits exactly m^n solutions since, in that case, the previous lemma ensures there are the coordinates on z^α of the (unique) interpolation polynomial of P in P_{m-1} . This set of points w contains $\mathbb{C}^n \setminus V$ where V is the algebraic set defined by $\det(q') = 0$ (see next section). This implies that equality holds everywhere. \square

Definition 6 Let q be a simple polynomial map. The unique system of polynomials $R_\alpha = R_\alpha[p]$ such that decomposition (3) holds true are called the q -coordinates of p . If precision is needed, we say that R_α is the q -coordinate of p on z^α .

In view of Lemma 3, each $R_\alpha[p]$ is a polynomial of degree at most $\lceil \deg p / m \rceil$ where $\lceil s \rceil$ denotes the smallest integer that is greater or equal to s .

2.3 The q -coordinates as functions of p

The following proposition which is the main tool used of this note shows that the linear maps $p \rightarrow R_\alpha[p]$ are continuous when the domain is endowed with the sup-norm on $q^{-1}(E)$ and the range with the sup-norm on E . Here q' denotes the total derivative of the map q and $\det q'$ is its Jacobian.

Proposition 7 Let q be a simple map on \mathbb{C}^n and E a compact subset of \mathbb{C}^n . If $\det q'$ does not vanish on the compact set $q^{-1}(E)$ then there exists a constant $C(E, q)$ depending only on E and q such that

$$|R_\alpha(w)| \leq C(E, q) \|p\|_{q^{-1}(w)}, \quad w \in E, \quad p \in \mathcal{P}(\mathbb{C}^n).$$

In particular,

$$\|R_\alpha\|_E \leq C(E, q) \|p\|_{q^{-1}(E)}.$$

Proof. Let $p \in \mathcal{P}(\mathbb{C}^n)$, $w \in E$. Using the q -coordinates R_α of p , we already showed, see (4), that

$$L(z) = \sum_{\alpha \in A_{m-1}} z^\alpha R_\alpha(w)$$

is the (unique) interpolation polynomial of p in P_{m-1} for the interpolation points $q^{-1}(w)$. An expression for the coefficient $R_\alpha(w)$ is therefore given by Cramer's rule. Letting $\mathbf{vdm}(q^{-1}(w))$ denote the Vandermonian determinant $\det(z_\beta^\alpha)$ where $\alpha \in A_{m-1}$ (ordered according to \succ) and z_β , $\beta \in A_{m-1}$, are the m^n elements of $q^{-1}(w)$ (arranged in some fixed order), we have

$$|R_\alpha(w)| = \frac{|\mathbf{vdm}(q^{-1}(w) [z_\beta^\alpha \leftarrow p(z_\beta)])|}{|\mathbf{vdm}(q^{-1}(w))|}, \quad \alpha \in A_{m-1}, \quad (5)$$

where the notation in the numerator means that we replace the α -th column by the column formed with the values of p at the points z_β . Note that the presence of the absolute value makes the denominator a well defined function of w , i.e., does not depend on the way we ordered the roots

z_β . Note also that the fact, proved in Lemma 4, that $q^{-1}(w)$ is unisolvent for P_{m-1} translates into the non vanishing of $\mathbf{vdm}(q^{-1}(w))$. Returning to (5), by expanding the numerator by the α -th column, we see that it is bounded by $C\|p\|_{q^{-1}(w)}$ where C depends only on the geometry of $q^{-1}(E)$, hence only on E and q . To prove the lemma, it therefore suffices to show that $\min_{w \in E} |\mathbf{vdm}(q^{-1}(w))| > 0$. Since, the function $w \rightarrow |\mathbf{vdm}(q^{-1}(w))|$ never vanishes on E , the claim will be a consequence of its continuity on the compact set E . Now, its continuity follows from the fact that, thanks to the assumption on q' , the map q is locally invertible around each root of $q(z) = w$, and the roots are therefore continuous functions of w and so is $|\mathbf{vdm}(q^{-1}(w))|$. \square

The point in the above proposition is that we have a bound that is independent of both p and w . However, it is worth observing that the same reasoning provides a bound depending on w (but, of course, not on p) under the much weaker assumption that the equation $q(z) = w$ admits exactly m^n (pairwise distinct) roots.

2.4 A counter-example

We show with a counter-example that a bound as in Proposition 7 cannot be expected without the assumption on q' .

Let $E = [0, 1] \subset \mathbb{R} \subset \mathbb{C}$ and let $q(z) = z^2$ so that $q'(0) = 0$ and $q^{-1}(E) = [-1, 1]$. The decomposition (3) always holds for any polynomial p and $R_0(z^2)$ is the even part of p whereas $R_1(z^2)$ is its odd part divided by z . Yet, as we saw, when w is a non zero element of E (and $\{\sqrt{w}, -\sqrt{w}\} = q^{-1}(w)$), the q -coordinate $R_1(w)$ of p on z is the coefficient of z in the Lagrange interpolation polynomial of p at the points \sqrt{w} and $-\sqrt{w}$. Hence, $R_1(w)$ is the divided difference of p at \sqrt{w} and $-\sqrt{w}$,

$$R_1(w) = \frac{p(\sqrt{w}) - p(-\sqrt{w})}{2\sqrt{w}}. \quad (6)$$

So, if there exists a constant $C(E, q)$ such that $|R_1(w)| \leq C(E, q)\|p\|_{q^{-1}(E)}$ for any polynomial p then, in view of (6), by letting $w \rightarrow 0$, we would obtain $|p'(0)| \leq C(E, q)\|p\|_{[-1, 1]}$ for any p which is impossible.

3 Application to the construction of admissible meshes

Definition 8 Let E be a compact set in \mathbb{C}^n . An E -admissible mesh is a sequence of finite subsets $\Lambda = (\Lambda_d : d \in \mathbb{N})$ in E such that the cardinality of Λ_d grows at most sub-exponentially and there is a constant $M = M(\Lambda)$ such that

$$\|P\|_E \leq M\|P\|_{\Lambda_d}, \quad p \in \mathcal{P}_d(\mathbb{C}^n), \quad d \in \mathbb{N}.$$

When M is allowed to depend on d , so that $M = M_d$, we speak of E -weakly admissible mesh provided that $\limsup_{d \rightarrow \infty} (M_d)^{1/d} = 1$.

Such meshes were introduced in [8] for studying approximation by discrete least squares polynomial and subsequently proved to be useful for several questions in approximation theory. We

refer to [21] for a recent example. One of the main practical advantage of (weakly) admissible meshes is their adaptation to set operations: unions, projections and Cartesian products of admissible meshes are still admissible meshes. Likewise, the direct image of an admissible mesh under any polynomial map is immediately seen to be an admissible mesh. We prove the invariance property for pre-images under polynomial maps. This was the original motivation of our work.

Proposition 9 *Let q be a simple polynomial map of degree m and E be a compact subset of \mathbb{C}^n such that $\det q'$ does not vanish on $q^{-1}(E)$. If $\Lambda = (\Lambda_d : d \in \mathbb{N})$ is an E -admissible mesh of constant M then*

$$q^{-1}(\Lambda) := (q^{-1}(\Lambda_{\lceil d/m \rceil})) : d \in \mathbb{N}$$

is a $q^{-1}(E)$ -admissible mesh of constant $Mc(E, q)$ where $c(E, q)$ depends only on E and q .

Of course, a similar statement holds for weakly admissible meshes. The proposition is an immediate consequence of the following lemma (apply it with $s = \lceil d/m \rceil$, observing that $d \leq ms$).

Lemma 10 *Let E and q as above. If Δ is a subset of E such that there exists a constant $M(\Delta)$ with*

$$\|p\|_E \leq M(\Delta) \|p\|_\Delta, \quad p \in \mathcal{P}_d(\mathbb{C}^n), \quad (7)$$

then there exists a constant $c(E, q)$ depending only on E and q such that

$$\|p\|_{q^{-1}(E)} \leq c(E, q) M(\Delta) \|p\|_{q^{-1}(\Delta)}, \quad p \in \mathcal{P}_{ms}(\mathbb{C}^n).$$

Proof. Let $z \in q^{-1}(E)$ and $p \in \mathcal{P}_{ms}$. Using the q -coordinates of p and $|z| = \max_{1 \leq i \leq n} |z_i|$ for $z = (z_1, \dots, z_n)$, we have

$$|p(z)| \leq \sum_{\alpha \in A_{m-1}} |z^\alpha| |R_\alpha(w)| \leq \sum_{\alpha \in A_{m-1}} |z|^{|\alpha|} \|R_\alpha\|_E, \quad (8)$$

where $w = q(z) \in E$. Since $R_\alpha \in \mathcal{P}_s(\mathbb{C}^n)$, we may use (7) to get $\|R_\alpha\|_E \leq M(\Delta) \|R_\alpha\|_\Delta$, but, in view of Proposition 7,

$$\max_{a \in \Delta} |R_\alpha(a)| \leq C(E, q) \max_{a \in \Delta} \|p\|_{q^{-1}(a)} = C(E, q) \|p\|_{q^{-1}(\Delta)}.$$

Hence,

$$|p(z)| \leq \sum_{\alpha \in A_{m-1}} M(\Delta) C(E, q) |z|^{|\alpha|} \|p\|_{q^{-1}(\Delta)},$$

and the lemma follows on taking $c(E, q) = C(E, q) \sum_{\alpha \in A_{m-1}} T^{|\alpha|}$ with $T = \max\{|z|, z \in q^{-1}(E)\}$. \square

4 Application to the construction of Bernstein-Markov measures

4.1 What is a Bernstein-Markov measure ?

Roughly speaking, Bernstein-Markov measures (on E) are those for which the L^2 -norm of any polynomial is relatively close to its supremum norm (on E).

Definition 11 *We say that a positive measure μ on E is Bernstein-Markov (**BM**) when, for every $\Lambda > 0$, there exists a constant $C = C_\Lambda$ such that*

$$\|p\|_E \leq C(1 + \Lambda)^{\deg p} \sqrt{\int_E |p|^2 d\mu}, \quad p \in \mathcal{P}(\mathbb{C}^n). \quad (9)$$

Such measures are useful in complex approximation and (pluri)potential theory. For instance, with some assumption of regularity on E , the sequence of usual orthonormal polynomials with respect to μ converges to the Siciak extremal function (the exponential of the pluricomplex Green function) and the Fourier expansion of any holomorphic functions on a neighbourhood of E converges maximally, i.e., with the same geometric speed as the polynomial of best approximation of f on E . This was first observed by Zeriahi [27]. Such measures were studied by Bloom and Levenberg in [6] where, for instance, one can find a density criterion and examples of **BM** measures with discrete support.

4.2 A push forward transformation of measures

We use our decomposition in q -coordinates to prove a natural relation between **BM** measures on E and on $q^{-1}(E)$. Given a proper polynomial map q of degree m satisfying (1), we define the operator q_* on $\mathcal{C}(q^{-1}(E))$ with values in $\mathcal{C}(E)$ (where \mathcal{C} is used for standard Banach spaces of continuous functions) by the relation

$$q_* f(w) = \frac{1}{m^n} \sum_{z \in q^{-1}(w)} f(z), \quad w \in E,$$

where any root is repeated according to its multiplicity. A sufficient condition for $q_* f$ to be an element of $\mathcal{C}(E)$ is that the roots of $q^{-1}(w)$ continuously depend on w . As we already observed, this is the case when $\det q'$ does not vanish on $q^{-1}(E)$.

Definition 12 *Given a measure μ on E and a polynomial map satisfying (1) with $\det q' \neq 0$ on $q^{-1}(E)$, we define a (positive) measure $q_* \mu$ on $q^{-1}(E)$ by the relation*

$$\int_{q^{-1}(E)} f dq_* \mu = \int_E (q_* f) d\mu, \quad f \in \mathcal{C}(q^{-1}(E)).$$

To see that the definition is correct, it suffices to observe that the right hand side is a positive continuous linear form on $\mathcal{C}(q^{-1}(E))$.

4.3 Invariance of the BM property

Proposition 13 *Let q be a simple polynomial map of degree m and E be a compact subset of \mathbb{C}^n such that $\det q'$ does not vanish on $q^{-1}(E)$. A (positive) measure μ on E is **BM** if and only if the measure $q_*\mu$ is **BM** on $q^{-1}(E)$.*

We will see that the measure transformation $\mu \rightarrow q_*\mu$ naturally comes into play in several other questions.

Proof. We assume that μ is **BM** on E and prove that $q_*\mu$ is **BM** on $q^{-1}(E)$. Decomposing a polynomial p with its q -coordinates, we get

$$\|p\|_{q^{-1}(E)} \leq \left\| \sum_{\alpha \in A_{m-1}} z^\alpha (R_\alpha \circ q) \right\|_{q^{-1}(E)} \leq K_1 \sum_{\alpha \in A_{m-1}} \|R_\alpha\|_E. \quad (10)$$

Since μ is **BM**, for every $\Lambda > 0$ there exists $C = C(\Lambda)$ such (9) holds, hence, estimating each $\|R_\alpha\|_E$ in the above inequality, we have

$$\|p\|_{q^{-1}(E)} \leq K_1 \sum_{\alpha \in A_{m-1}} C(1 + \Lambda)^{\lceil \deg p / m \rceil} \sqrt{\int_E |R_\alpha(w)|^2 d\mu(w)}. \quad (11)$$

Observe that we do not necessarily have $\deg R_\alpha = \lceil \deg p / m \rceil$. Yet, if (9) holds for $\deg R_\alpha$, it is a fortiori true with a bigger integer. A use of Proposition 7 now gives

$$|R_\alpha(w)|^2 \leq C^2(E, q) \max_{z \in q^{-1}(w)} |p(z)|^2 \leq C^2(E, q) \sum_{z \in q^{-1}(w)} |p(z)|^2.$$

Using this in (11), we obtain

$$\begin{aligned} \|p\|_{q^{-1}(E)} &\leq K_2 \sum_{\alpha \in A_{m-1}} C(1 + \Lambda)^{\lceil \deg p / m \rceil} \sqrt{\int_E \sum_{z \in q^{-1}(w)} |p(z)|^2 d\mu(w)} \\ &\leq K_2 \sum_{\alpha \in A_{m-1}} C(1 + \Lambda)^{\lceil \deg p / m \rceil} \sqrt{\int_E m^n q_*(|p|^2) d\mu} \leq K_3 (1 + \Lambda)^{\deg p} \sqrt{\int_E |p|^2 dq_*\mu}, \end{aligned}$$

which proves that $q_*\mu$ is **BM** on $q^{-1}(E)$.

Conversely, to see that μ is **BM** on E when $q_*\mu$ is **BM** on $q^{-1}(E)$, taking $\Delta > 0$ it suffices to apply the Bernstein-Markov inequality to $p \circ q$ and $\Delta' = (1 + \Delta)^{1/m} - 1 > 0$ observing that $q_*(|p \circ q|^2) = |p|^2$ so that

$$\int_{q^{-1}(E)} |p \circ q|^2 dq_*\mu = \int_E q_*(|p \circ q|^2) d\mu = \int_E |p|^2 d\mu.$$

Indeed, for some $C = C(\Delta')$, we have

$$\|p\|_E = \|p \circ q\|_{q^{-1}(E)} \leq C(1 + \Delta')^{m \deg p} \int_{q^{-1}(E)} |p \circ q|^2 dq_*(\mu) \leq C(1 + \Delta)^{\deg p} \int_E |p|^2 d\mu.$$

This concludes the proof of the proposition. \square

5 Application to Markov inequality

5.1 The Markov inequality

Definition 14 (see [4]) *A compact set $E \subset \mathbb{C}^n$ is said to satisfy a Markov inequality of exponent β if there exists a constant $B = B(\beta)$ such that*

$$\|\nabla p\|_E \leq B(\deg p)^\beta \|p\|_E, \quad p \in \mathcal{P}(\mathbb{C}^n) \quad (12)$$

where $\nabla p(z) = (\partial_1 p(z), \dots, \partial_n p(z))$ is the gradient of p at z and $\|\nabla p\|_E$ is the supremum of the euclidean norm of $\nabla p(z)$ for $z \in E$, i.e.,

$$\|\nabla p\|_E = \max_{z \in E} \|\nabla p(z)\| = \max_{z \in E} \left(\sum_{i=1}^n |\partial_i p(z)|^2 \right)^{1/2}. \quad (13)$$

Sets admitting the above property are said to be Markov sets. The Markov exponent $\mathfrak{M}(E)$ of E is the infimum of all exponents $\beta > 0$ such that there exists $B = B(\beta)$ for which (12) holds true. In particular, if E is not a Markov set, $\mathfrak{M}(E) = \infty$.

We may also define the Markov constant $M_d(E)$ for the compact set E (see e.g. [23] or [25]) as

$$M_d(E) = \sup \left\{ \frac{\|\nabla p\|_E}{\|p\|_E} : p \in \mathcal{P}_d(\mathbb{C}^n), p|_E \not\equiv 0 \right\}, \quad d \in \mathbb{N}. \quad (14)$$

This gives a non decreasing sequence satisfying $M_d(E) \geq d$ (to see that, we may take $p(z) = z_i^d$ when $E \neq \{0\}$ and it is obvious otherwise); hence $M_d(E)$ tends to ∞ as $d \rightarrow \infty$ and

$$\mathfrak{M}(E) = \limsup_{d \rightarrow \infty} \frac{\ln M_d(E)}{\ln d}. \quad (15)$$

It turned out that inequality (12) needs not be fulfilled with $\beta = \mathfrak{M}(E)$, see [3] in the multivariate case and [16] for univariate sets.

Observe that, although the value of $M_d(E)$ depends on the norm we use for the gradient, the value of $\mathfrak{M}(E)$ does not depend on this norm.

5.2 Invariance of the Markov inequality

Baran and Pleśniak showed in [4] that for a regular analytic map f the Markov exponents satisfy the inequality $\mathfrak{M}(f(E)) \leq \mathfrak{M}(E)$ (see also [3] and [22]). In a very recent paper [22], Pierzchała discussed a problem of the invariance of the Markov inequality under polynomial maps. In particular, he presented some specific examples to indicate a variety of situations. Here, we show that the Markov exponent is invariant under pre-images of regular simple polynomial maps.

Proposition 15 *Let q be a simple polynomial map of degree m and E be a compact subset of \mathbb{C}^n such that $\det q'$ does not vanish on $q^{-1}(E)$. There exist constants k and K depending only on E and q such that*

$$k M_{\lfloor d/m \rfloor}(E) \leq M_d(q^{-1}(E)) \leq K M_{\lceil d/m \rceil}(E). \quad (16)$$

In particular,

$$\mathfrak{M}(q^{-1}(E)) = \mathfrak{M}(E).$$

Note that $\lfloor \cdot \rfloor$ (the integer part) rather than $\lceil \cdot \rceil$ is used for the lower bound.

Proof. We first prove the upper bound. For a fixed positive integer d we assume that $M_{\lceil d/m \rceil}(E)$ is finite for otherwise the claim is obvious. Let $p \in \mathcal{P}_d(\mathbb{C}^n)$ and $z \in q^{-1}(E)$. Differentiating the decomposition of p into its q -coordinates (3), we obtain

$$\nabla p(z) = \sum_{\alpha \in A'_{m-1}} R_\alpha(q(z)) \nabla_\alpha(z) + \sum_{\alpha \in A'_{m-1}} z^\alpha (q'(z))^t (\nabla R_\alpha(q(z))), \quad (17)$$

where A'_{m-1} is the set of all α in A_{m-1} such that R_α is not the zero polynomial, $\nabla_\alpha(z) = \nabla(u \rightarrow u^\alpha)(z)$ and $(q'(z))^t$ denotes the transpose of $q'(z)$. Using the euclidean norm (and the corresponding operator norm), for every $\alpha \in A'_{m-1}$, we have

$$\|(q'(z))^t (\nabla R_\alpha(q(z)))\| \leq \|(q'(z))^t\| \|\nabla R_\alpha(q(z))\| \quad (18)$$

$$\leq C_1 \|\nabla R_\alpha\|_E \leq C_1 M_{\lceil d/m \rceil}(E) \|R_\alpha\|_E, \quad (19)$$

where we applied the definition of $M_{\lceil d/m \rceil}(E)$ taking into account that $\deg R_\alpha \leq \lceil d/m \rceil$. However, to do that, we must be sure that no non zero R_α vanishes on E . In fact, if $R_\alpha = 0$ on E without being zero then E is included in an algebraic hypersurface and we may therefore choose a polynomial π of minimal positive degree such that $E \subset \{\pi = 0\}$. The minimality of π implies that $\|\nabla \pi\|_E \neq 0$ and $\deg \pi \leq \deg R_\alpha$. Considering the polynomials $\pi_s = \pi + s$ with $s > 0$ we have $M_{\deg(\pi)}(E) \geq \|\nabla \pi_s\|_E / \|\pi_s\|_E = \|\nabla \pi\|_E / s$ which, as $s \rightarrow 0$, shows $M_{\deg(\pi)}(E) = \infty$ and so $M_{\lceil d/m \rceil}(E) = \infty$ which has been previously excluded.

Returning to (17), we now have

$$\|\nabla p(z)\| \leq \sum_{\alpha \in A'_{m-1}} \|R_\alpha\|_E \|\nabla_\alpha\|_{q^{-1}(E)} + C_1 M_{\lceil d/m \rceil}(E) \sum_{\alpha \in A'_{m-1}} \|z^\alpha\|_E \|R_\alpha\|_E,$$

and a use of Proposition 7 for estimating $\|R_\alpha\|_E$ yields

$$\|\nabla p(z)\| \leq C_2 \|p\|_{q^{-1}(E)} + C_3 M_{\lceil d/m \rceil}(E) \|p\|_{q^{-1}(E)}.$$

Hence,

$$M_d(q^{-1}(E)) \leq C_2 + C_3 M_{\lceil d/m \rceil}(E),$$

and the upper bound follows since $M_{\lceil d/m \rceil}(E) \rightarrow \infty$ as $d \rightarrow \infty$.

The lower bound is easily proved. Starting from $p \in \mathcal{P}_{[d/m]}(\mathbb{C}^n)$ such that $\|p\|_E \neq 0$, we consider $p \circ q \in \mathcal{P}_d(\mathbb{C}^n)$ and remark that this polynomial does not coincide with 0 on $q^{-1}(E)$. Hence, $p \circ q$ is a competitor for $M_d(q^{-1}(E))$ so that

$$M_d(q^{-1}(E)) \geq \frac{\|\nabla(p \circ q)\|_{q^{-1}(E)}}{\|p \circ q\|_{q^{-1}(E)}} \geq \frac{1}{c} \frac{\|\nabla p\|_E}{\|p\|_E}, \quad (20)$$

where c denotes the maximum of the (euclidean operator) norm of the inverse of the transpose of $q'(z)$ over all $z \in q^{-1}(E)$. To obtain the last inequality, setting $U_z = (q'(z))^t$, we observe that

$$(\nabla p)(q(z)) = U_z^{-1}(\nabla(p \circ q)(z)) \implies \|\nabla p\|_E \leq c \|\nabla(p \circ q)\|_{q^{-1}(E)}.$$

Since the estimate (20) is valid for every p used in the definition of $M_d(E)$, the lower bound is established. The consequence on the Markov exponents is immediate since $\ln d \sim \ln[d/m] \sim \ln[d/m]$. \square

Of course the proof also shows that if E satisfies a Markov inequality with exponent β as in (12) then so does $q^{-1}(E)$.

5.3 A L^r version

We may study a version of the Markov inequality if the sup norm is replaced by a L^r norm with respect to a (probability) measure μ supported on a compact set $E \subset \mathbb{C}^n$ which we assume is not included in an algebraic hypersurface, so that

$$\|p\|_{r,\mu} = \left(\int_E |p|^r d\mu \right)^{1/r} > 0, \quad p \in \mathcal{P}(\mathbb{C}^n), \quad p \neq 0.$$

Definition 11 is readily extended. For instance, corresponding to (14), for $r \geq 1$, we define

$$M_d(\mu, r, E) = \sup \left\{ \frac{\|\nabla p\|_{r,\mu}}{\|p\|_{r,\mu}} : p \in \mathcal{P}_d(\mathbb{C}^n), p \neq 0 \right\}, \quad d \in \mathbb{N}, \quad (21)$$

where we set, compare to (13),

$$\|\nabla p\|_{r,\mu} = \left(\int_E \|\nabla p\|^r d\mu \right)^{1/r}, \quad \|\nabla p\| = \left(\sum_{i=1}^n |\partial_i p|^2 \right)^{1/2}. \quad (22)$$

On taking $p = z_1^d$ and observing $\|z_1^d\|_{\mu,r} \leq \|z_1\|_E^r \|z_1^{d-1}\|_{\mu,r}$, we see that

$$M_d(\mu, r, E) \gtrsim d \quad (d \rightarrow \infty). \quad (23)$$

Likewise, we set

$$\mathfrak{M}(\mu, r, E) = \limsup_{d \rightarrow \infty} \frac{\ln M_d(\mu, r, E)}{\ln d}.$$

For instance, Goetgheluck ([12]) proved that if E is a locally Lipschitzian subset of \mathbb{R}^n and μ is the Lebesgue measure, then $M_d(\mu, r, E) = O(d^2)$ where the constant involved in O depends only on E and r , see also [17]. Other results concerning this kind of polynomial inequality can be found e.g. in [10], [14] or [2].

Proposition 16 *Let q be a simple polynomial map of degree m . Let E be a compact subset of \mathbb{C}^n not included in an algebraic hypersurface such that $\det q'$ does not vanish on $q^{-1}(E)$ and μ a probability measure on E . For any $r \geq 1$, there exist constants c and C depending only on E , μ , r and q such that*

$$c M_{\lfloor \frac{d}{m} \rfloor}(\mu, r, E) \leq M_d(q_*\mu, r, q^{-1}(E)) \leq C M_{\lceil \frac{d}{m} \rceil}(\mu, r, E). \quad (24)$$

In particular,

$$\mathfrak{M}(q_*(\mu), r, q^{-1}(E)) = \mathfrak{M}(\mu, r, E).$$

Proof. We prove the upper bound in (24). The lower bound is proved using the same idea as in the proof of Proposition 15.

In view of Definition 12, we have

$$\|\nabla p\|_{r, q_*\mu}^r = \frac{1}{m^n} \int_E \sum_{z \in q^{-1}(w)} \|\nabla p(z)\|^r d\mu(w).$$

A use of relation (17), for $q(z) = w$, readily yields

$$\|\nabla p(z)\| \leq C_1 \sum_{\alpha \in A'_{m-1}} (|R_\alpha(w)| + \|\nabla R_\alpha(w)\|),$$

where C_1 depends on E and q . Since $(x_1 + \dots + x_l)^r \leq D(x_1^r + \dots + x_l^r)$ for $x_1, \dots, x_l \geq 0$ with D independent of x_1, \dots, x_l , we have

$$\begin{aligned} \frac{1}{m^n} \sum_{z \in q^{-1}(w)} \|\nabla p(z)\|^r &\leq \frac{1}{m^n} \sum_{z \in q^{-1}(w)} \sum_{\alpha \in A'_{m-1}} C_1^r (|R_\alpha(w)| + \|\nabla R_\alpha(w)\|)^r \\ &\leq C_2 \frac{1}{m^n} \sum_{z \in q^{-1}(w)} \sum_{\alpha \in A'_{m-1}} |R_\alpha(w)|^r + \|\nabla R_\alpha(w)\|^r \leq C_2 \sum_{\alpha \in A'_{m-1}} |R_\alpha(w)|^r + \|\nabla R_\alpha(w)\|^r. \end{aligned}$$

By integrating, we obtain

$$\|\nabla p\|_{r, q_*\mu}^r \leq C_2 \sum_{\alpha \in A'_{m-1}} \|R_\alpha\|_{\mu, r}^r + \|\nabla R_\alpha\|_{\mu, r}^r.$$

Using the definition of $M_d(\mu, r, E)$ to bound $\|\nabla R_\alpha\|_{\mu, r}^r$, we get

$$\|\nabla p\|_{r, q_*\mu}^r \leq C_2 \sum_{\alpha \in A'_{m-1}} \|R_\alpha\|_{\mu, r}^r + M_{\lceil d/m \rceil}^r(\mu, r, E) \|R_\alpha\|_{\mu, r}^r \quad (25)$$

$$\leq C_2 \sum_{\alpha \in A'_{m-1}} (1 + M_{\lceil d/m \rceil}^r(\mu, r, E)) \|R_\alpha\|_{\mu, r}^r \quad (26)$$

$$\leq C_3 M_{\lceil d/m \rceil}^r(\mu, r, E) \sum_{\alpha \in A'_{m-1}} \|R_\alpha\|_{\mu, r}^r, \quad (27)$$

where we used that $1 + M_{\lceil d/m \rceil}(\mu, r, E) = O(M_{\lceil d/m \rceil}(\mu, r, E))$ (see (23)) on the last line. Now, by Proposition 7, we have

$$|R_\alpha(w)|^r \leq C^r(E, q) \max_{z \in q^{-1}(w)} |p(z)|^r \leq C^r(E, q) \sum_{z \in q^{-1}(w)} |p(z)|^r,$$

and, by integration, $\|R_\alpha\|_{\mu, r}^r \leq m^n C^r(E, q) \|p\|_{q_*\mu, r}^r$. Plugging this into (27), we obtain

$$\|\nabla p\|_{r, q_*\mu}^r \leq m^{2n} C_3 M_{\lceil d/m \rceil}^r(\mu, r, E) C^r(E, q) \|p\|_{q_*\mu, r}^r$$

and the upper bound follows. \square

5.4 Further inequalities

We mention a few other inequalities to which our method applies. The list is certainly far from being exhaustive. We omit the proofs; they use the same reasoning as the above.

5.4.1 The Bernstein inequality

The original *Bernstein inequality* states (or, depending on the version, implies) that for $a < c < d < b$ and $p \in \mathcal{P}_d(\mathbb{R})$, we have

$$\|p'\|_{[c, d]} \leq \frac{d}{\sqrt{(c-a)(b-c)}} \|p\|_{[a, b]}.$$

Given $F \subset E \subset \mathbb{C}^n$, $E \neq F$ and $d \in \mathbb{N}$, we may define the *Bernstein constant*

$$B(E, F, d) = \max\{\|\nabla p\|_F / \|p\|_E, p \in \mathcal{P}_d(\mathbb{C}^n), \|p\|_E \neq 0\}$$

and the corresponding (Bernstein) exponent, see (15), $\mathfrak{B}(E, F)$. When q is a simple map with $\det q' \neq 0$ on $q^{-1}(E)$, using \asymp to indicate that a double inequality as in (16) holds true, we have

$$B(E, F, d) \asymp B(q^{-1}(E), q^{-1}(F), \lceil d/m \rceil) \quad \text{and} \quad \mathfrak{B}(E, F) = \mathfrak{B}(q^{-1}(E), q^{-1}(F)). \quad (28)$$

The (Bernstein) constant $B(E, F, d)$ is known to grow like d when E is a convex set and F is a natural geometric subset of E , see [1], [19]. Observe that (28) furnishes examples of non-convex sets whose Bernstein constants grow like d .

5.4.2 The Division (Nikolski) inequality

Here the starting point is probably the *Schur inequality* which states that for $p \in \mathcal{P}_d(\mathbb{R})$,

$$\|p\|_{[-1, 1]} \leq (d+1) \|xp\|_{[-1, 1]}.$$

In general, given a compact set E and a continuous function ω on E , one may define the ω -division constant

$$D(E, \omega, d) = \max\{\|p\|_E / \|\omega p\|_E, p \in \mathcal{P}_d(\mathbb{C}^n), \|p\|_E \neq 0\}$$

and the corresponding exponent $\mathfrak{D}(E, \omega)$. In [11], Goetgheluck exhibited a general class of compact sets E for which $D(E, \omega, d)$ grows like a power of d . Again, here, if q is a simple map with $\det q' \neq 0$ on $q^{-1}(E)$, then

$$D(E, \omega, d) \asymp D(q^{-1}(E), \omega \circ q, \lceil d/m \rceil) \quad \text{and} \quad \mathfrak{D}(E, \omega) = \mathfrak{D}(q^{-1}(E), \omega \circ q). \quad (29)$$

We may equally well consider the division inequality when the sup-norm is replaced by a L^r norm with respect to some measure μ as is done in subsection 5.3. In that case the ω -division constant is $D(E, \omega, \mu, r, d)$, the ω -division exponent $\mathfrak{D}(E, \omega, \mu, r)$ and the invariance relations reads as

$$D(E, \omega, \mu, r, d) \asymp D(q^{-1}(E), \omega \circ q, q_*\mu, r, \lceil d/m \rceil) \quad \text{and} \quad \mathfrak{D}(E, \omega, \mu, r) = \mathfrak{D}(q^{-1}(E), \omega \circ q, q_*\mu, r), \quad (30)$$

where, as expected, the transformation $\mu \rightarrow q_*\mu$ comes again into play. For known results on $D(E, \omega, \mu, r, d)$, see [15], [13] or [5].

6 More about the univariate case

When we work in \mathbb{C} , the interpolation polynomials that came into play in section 2 are the well-known Lagrange interpolation polynomials for which simple formulas are available. The use of such formulas leads to more explicit and precise bounds. We now briefly illustrate this point.

Relation (3) reduces to $p(z) = \sum_{i=0}^{m-1} z^i R_i(q(z))$ and the value $R_i(w)$ is the coefficient of z^i of the Lagrange interpolation polynomial $L = L(p)$ of p at $q^{-1}(w) = \{z_1, z_2, \dots, z_m\}$ (the z_i are pairwise distinct since we assume that q' does not vanish on $q^{-1}(E)$). Lagrange interpolation formula gives

$$L(z) = \sum_{k=1}^m \frac{p(z_k)}{q'(z_k)} \frac{q(z) - w}{z - z_k},$$

and

$$R_j(w) = \frac{1}{j!} L^{(j)}(0) = \frac{1}{j!} \sum_{k=1}^m \frac{p(z_k)}{q'(z_k)} W_k^{(j)}(0),$$

where

$$W_k(z) = \frac{q(z) - w}{z - z_k} = \frac{q(z) - q(z_k)}{z - z_k}.$$

By Cauchy integral formula,

$$|R_j(w)| \leq \frac{1}{j!} \sum_{k=1}^m \left| \frac{p(z_k)}{q'(z_k)} \right| \frac{j!}{r^j} \|W_k\|_{D(0,r)} \leq \frac{m}{r^j} \frac{\|p\|_{q^{-1}(w)}}{\delta_w} \|q'\|_{D(0,r)}, \quad r > \rho_w$$

where $\delta_w := \min\{|q'(z)| : z \in q^{-1}(w)\}$, $\rho_w := \max\{|z| : z \in q^{-1}(w)\}$ and $D(0, r) = \{z \in \mathbb{C} : |z| \leq r\}$. We have proved the following proposition.

Proposition 17 *Let q be a non constant univariate polynomial of degree m and $w \in \mathbb{C}$ such that q' does not vanish on $q^{-1}(w)$ (i.e., the roots of the equation $q(z) = w$ are simple). Using the above notation, we have*

$$|R_j(w)| \leq \frac{m}{r^j} \frac{\|q'\|_{D(0,r)}}{\delta_w} \|p\|_{q^{-1}(w)}, \quad j = 0, \dots, m-1, \quad \text{for any } r \geq \rho_w. \quad (31)$$

In particular, if q' does not vanish on $q^{-1}(E)$, where E is a univariate compact set, the constant $C(E, q)$ in Proposition 7 can be taken as

$$C(E, q) = \frac{m}{\min\{1, \rho^{m-1}\}} \frac{\|q'\|_{D(0,\rho)}}{\delta}, \quad (32)$$

where $\delta := \min\{|q'(z)| : z \in q^{-1}(E)\}$, $\rho := \max\{|z| : z \in q^{-1}(E)\}$.

Proposition 18 *Let E be a compact set in the complex plane and q be a polynomial of degree m such that q' does not vanish on $q^{-1}(E)$. Then*

$$\delta M_{[d/m]}(E) \leq M_d(q^{-1}(E)) \leq \frac{m^2}{\delta} \left(\frac{m}{\rho} + \|q'\|_{q^{-1}(E)} M_{[d/m]}(E) \right) \|q'\|_{D(0,\rho)}, \quad d \in \mathbb{N},$$

where δ and ρ are defined in the previous proposition.

Proof. The principle of the proof is the same as in the multivariate case. We only detail the change in the upper bound estimate (in the generic case). We use

$$\|p'\|_{q^{-1}(E)} \leq \sum_{j=1}^{m-1} j \rho^{j-1} \|R_j\|_E + \sum_{j=0}^{m-1} \rho^j \|R'_j\|_E \|q'\|_{q^{-1}(E)},$$

and, by definition of Markov constants,

$$\begin{aligned} \|p'\|_{q^{-1}(E)} &\leq \sum_{j=1}^{m-1} j \rho^{j-1} \|R_j\|_E + \sum_{j=0}^{m-1} \rho^j M_{[d/m]}(E) \|R_j\|_E \|q'\|_{q^{-1}(E)} \\ &= \sum_{j=0}^{m-1} \rho^j \left(\frac{j}{\rho} + \|q'\|_{q^{-1}(E)} M_{[d/m]}(E) \right) \|R_j\|_E. \end{aligned}$$

We take $r = \rho \geq \rho_w$ (for any $w \in E$) in inequality (31) and then we use it to obtain the upper bound. \square

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